Meromorphic Continuation of the Goldbach generating function

Gautami Bhowmik, Jan-Christoph Schlage-Puchta

MEROMORPHIC CONTINUATION OF THE GOLDBACH GENERATING FUNCTION

GAUTAMI BHOWMIK AND JAN-CHRISTOPH SCHLAGE-PUCHTA

Abstract. We consider the Dirichlet series associated to the number of representations of an integer as a sum of primes. Assuming certain reasonable hypotheses on the distribution of the zeros of the Riemann zeta function we obtain the domain of meromorphic continuation of this series.

1. Introduction and Results

In this paper we consider the number $G_r(n)$ of representations of an integer n as the sum of r primes. One possible way to obtain information is the use of complex integration. To do so Egami and Matsumoto[4] introduced the generating function

$$\Phi_r(s) = \sum_{k_1=1}^\infty \cdots \sum_{k_r=1}^\infty \frac{\Lambda(k_1)\dots\Lambda(k_r)}{(k_1+k_2+\dots+k_r)^s} = \sum_{n=1}^\infty \frac{G_r(n)}{n^s}.$$

This series is absolutely convergent for $\Re\, s > r$, and has a simple pole at $s = r$. By Perron's formula we have

$$\sum_{n\leq x} G_r(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \Phi_r(s)\frac{x^s}{s}\,ds + \mathcal{O}(\frac{x^{r+\epsilon}}{T}).$$

To shift the path of integration to the left, one needs at least meromorphic continuation to some half-plane $\Re\, s > r-\delta$ as well as some information on the growth and the distribution of the poles of Φ_r. Assuming the Riemann hypothesis, Egami and Matsumoto[4] described the behavior for the case $r = 2$. In addition to the RH, parts of their results depend on unproved assumptions on the distribution of the imaginary parts of zeros of ζ. Denote by Γ the set of imaginary parts of non-trivial zeros of ζ. While the assumption that the positive elements in Γ are rationally independent appears to be folklore, Fujii[5] drew attention to the following special case:

Conjecture 1. *Suppose that $\gamma_1+\gamma_2 = \gamma_3+\gamma_4 \neq 0$ with $\gamma_i \in \Gamma$. Then $\{\gamma_1,\gamma_2\} = \{\gamma_3,\gamma_4\}$.*

Egami and Matsumoto used an effective version of this conjecture, i.e.

Conjecture 2. *There is some $\alpha < \frac{\pi}{2}$, such that for $\gamma_1,\dots,\gamma_4 \in \Gamma$ we have either $\{\gamma_1,\gamma_2\} = \{\gamma_3,\gamma_4\}$, or*

$$|(\gamma_1+\gamma_2)-(\gamma_3+\gamma_4)| \geq \exp\big(-\alpha(|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|)\big).$$

Obviously, Conjecture 2 implies Conjecture 1. In [4] it is proven that:

2000 *Mathematics Subject Classification.* 11P32, 11P55, 30B40, 11M41.
Key words and phrases. Goldbach numbers, circle method, meromorphic continuation.

Theorem 1. *Suppose the Riemann hypothesis holds true. Then $\Phi_2(s)$ can be meromorphically continued into the half-plane $\Re s > 1$ with an infinitude of poles on the line $\frac{3}{2}+it$. If in addition Conjecture 2 holds true, then the line $\Re\, s = 1$ is the natural boundary of Φ_2. More precisely, the set of points $1+i\kappa$ with $\lim_{\sigma\searrow 1} |\Phi_2(\sigma+\kappa)| = \infty$ is dense on $\mathbb{R}$.*

The above mentioned authors conjectured that under the same assumptions the domain of meromorphic continuation of Φ_r should be the half-plane $\Re s > r-1$. In this direction we show that

Theorem 2. *Let $\mathcal{D}_r \subseteq \mathbb{C}$ be the domain of meromorphic continuation of $\Phi_r(s)$.*

(1) *If the RH holds true, then $\Phi_r(s)$ has a natural boundary at $\Re s = r-1$ for all $r \geq 2$ if and only if $\Phi_2(s)$ has a natural boundary at $\Re s = 1$.*
(2) *If the RH and Conjecture 1 hold true, then $\Phi_2(s)$ has a natural boundary at $\Re s = 1$.*
(3) *If the RH holds true, then Φ_2 has a singularity at $2\rho_1$, where $\rho_1 = \frac{1}{2} + 14.1347\dots i$ is the first root of ζ. Moreover,*

$$\lim_{\sigma\searrow 0}(\sigma-1)|\phi_2(2\rho_1+\sigma)| > 0. \tag{1}$$

(4) *If Conjecture 2 holds true, then $\{s : \Re s > 2\sigma_0\} \subseteq \mathcal{D}_2 \subseteq \{s : \Re s > 1\}$, where σ_0 is the infimum over all real numbers σ, such that ζ has only finitely many zeros in the half-plane $\Re s > \sigma_0$.*

The existence of a natural boundary already implies an Ω-theorem (confer [2]). Here we can do a little better because of (1). We set

$$H_r(x) = -r\sum_\rho \frac{x^{r-1+\rho}}{\rho(1+\rho)\dots(r-1+\rho)}$$

where the summation runs over all non-trivial zeros of ζ and we obtain the following corollary.

Corollary 1. *Suppose that RH holds true. Then we have*

$$\sum_{n\leq x} G_r(n) = \frac{1}{r!}x^r + H_r(x) + \Omega(x^{r-1}).$$

We note that without (1) the omega term would still be $\Omega(x^{r-1-\epsilon})$. The corresponding $\mathcal{O}$-result for $r=2$ is [3]

Theorem 3. *Suppose that the RH is true. Then we have*

$$\sum_{n\leq x} G_2(n) = \frac{1}{2}x^2 + H_2(x) + \mathcal{O}(x\log^5 x).$$

It extends easily to $r \geq 3$. One might expect that the quality of the error term would improve with r increasing, however this is not the case.

Part (1) of Theorem 2 follows immediately from the assertion that the analytic behavior of Φ_r is completely determined by the behavior of Φ_2. More precisely

Theorem 4. *Suppose the RH. Then for any $r \geq 3$ there exist rational functions $f_{1,r}, \dots, f_{4,r}(s)$, such that*

$$\Phi_r(s) = f_{1,r}(s)\zeta(s-r+1) + f_{2,r}(s)\zeta(s-r+2) + f_{3,r}(s)\frac{\zeta'}{\zeta}(s-r+1) + f_{4,r}(s)\Phi_2(s-r+2) + R(s),$$

where $R(s)$ is holomorphic in the half-plane $\Re s > r-1-1/10$ and uniformly bounded in each half-strip of the form $\Re s > r-1-1/10+\epsilon$, $T < \Im s < T+1$, with $T > 0$.

The constant $1/10$ can be improved, however, since we believe that Φ_2 has $\Re\ s = 1$ as natural boundary, we saw no point in doing so.

Our proof expresses the function $\Phi_r(s)$ using the circle method. This approach is the main novelty of this paper.

2. Proof of Theorem 4

In this section we prove Theorem 4 by computing the function using the circle method. We use the standard notation.

Fix a large integer x, set $e(\alpha) = e^{2\pi i\alpha}$,

$$\begin{aligned} S(\alpha) &= \sum_{n\leq x} \Lambda(n)e(\alpha n), \\ T(\alpha) &= \sum_{n\leq x} e(\alpha n), \\ T_3(\alpha) &= \sum_{|n|\leq x} (x-|n|)^2 e(n\alpha), \\ R(\alpha) &= S(\alpha) - T(\alpha). \end{aligned}$$

Lemma 1. *Under the Riemann hypothesis we have*

$$R(\alpha) \ll x^{1/2}\log^2 x + \alpha x^{3/2}\log^2 x.$$

Proof. The Riemann hypothesis is equivalent to the estimate $\Psi(x) = x + \mathcal{O}(x^{1/2}\log^2 x)$, hence,

$$\begin{aligned} R(\alpha) &= \sum_{n\leq x} (\Lambda(n)-1)e(\alpha n) \\ &= (\Psi(x)-x)e(\alpha x) - \sum_{n\leq x}(\Psi(n)-n)(e(\alpha n+\alpha)-e(\alpha n)) \\ &\ll x^{1/2}\log^2 x + \alpha x^{3/2}\log^2 x, \end{aligned}$$

and our claim follows. □

The next statement is a consequence of partial summation.

Lemma 2. *Let a_n be a sequence of complex numbers, set $A_n = \sum_{\nu\leq n} a_n$, $d(s) = \sum_n a_n n^{-s}$, and $D(s) = \sum_n A_n n^{-s}$. Suppose that $D(s)$ is absolutely convergent for $\Re\ s > \sigma_0$ and has meromorphic continuation to $\Re\ s > \sigma_1$. Then $d(s)$ has* *meromorphic continuation to* $\Re s > \sigma_1 - 1$*, and there exist polynomials* Q_i*,* $0 \leq i \leq \sigma_0 - \sigma_1$*, such that*

$$d(s) = \sum_{i=0}^{\lfloor \sigma_0 - \sigma_1 \rfloor} Q_i(s) D(s+1+i) + R(s),$$

where R *is holomorphic on* $\Re s > \sigma_1 - 1$*, and continuous on* $\Re s \geq \sigma_1 - 1$*.*

Proof. We have

$$\begin{aligned} d(s) &= \sum_n A_n \big(n^{-s} - (n+1)^{-s}\big) \\ &= \sum_n A_n \sum_{\nu=1}^{N} \frac{(-1)^{\nu+1} s(s+1)\dots(s+\nu-1)}{\nu!} n^{-s-\nu} + R(s) \\ &= \sum_{\nu=1}^{N} \frac{(-1)^{\nu+1} s(s+1)\dots(s+\nu-1)}{\nu!} D(s+\nu) + R(s), \end{aligned}$$

where $R(s)$ is holomorphic on $\Re s > \sigma_0 - N$. Choosing $N > \sigma_0 - \sigma_1 + 1$ our claim follows. □

We can now establish Theorem 4.

Proof of Theorem 4. Define the sequence of functions A_r^k by $A_r^0(n) = G_r(n)$, and $A_r^{k+1}(n) = \sum_{\nu \leq n} A_r^k(\nu)$.

We compute $A_r^3(x)$ using the circle method. We have

$$\begin{aligned} A_r^3(x) &= \int_0^1 S^r(\alpha) T_3(\alpha) d\alpha \\ &= \sum_{k=0}^{r} \binom{r}{k} \int_0^1 T(\alpha)^{r-k} R^k(\alpha) T_3(\alpha) d\alpha \\ &= \sum_{k=0}^{r} \binom{r}{k} B_{r,k}(x), \end{aligned}$$

say. Our aim is to show that $B_{r,0}(x), B_{r,1}(x), B_{r,2}(x)$ are quite regular and have main terms corresponding to the Dirichlet-series explicitly mentioned in Theorem 4, and that $B_{r,k}(x)$ for $k \geq 3$ is of order $\mathcal{O}(x^{r-1/10})$. We collect the contribution of the coefficients $B_{r,k}(x)$ into a Dirichlet-series, which converges uniformly in any half-plane of the form $\Re\ s > r + 1 - 1/10 + \epsilon$, which can be absorbed into $R(s)$. Once we have shown these facts, Theorem 4 follows.

We first show that terms with $k \geq 3$ are negligible. Note that $T_3(\alpha) \ll \min(x^3, \alpha^{-3})$. We split the integral into the range $[-\beta, \beta]$ and $[\beta, 1-\beta]$. In the former range, we use Lemma 1 to bound all occurring values of R, whereas in the latter we use the estimate $\int_0^1 |R(\alpha)|^2\ d\alpha \ll x \log^2 x$. By symmetry it suffices to consider the integral over $[0,1/2]$, we begin with the case of small α. We have

$$\begin{aligned}\int\limits_0^\beta |T(\alpha)|^{r-k}|R(\alpha)|^k|T_3(\alpha)|d\alpha &\ll \int\limits_0^\beta \min(x^{r+3-k},\alpha^{k-3-r})(x^{1/2}+\alpha x^{3/2})^k x^\epsilon\, d\alpha\\ &\ll x^{r+2-k/2+\epsilon}+\int\limits_{x^{-1}}^\beta \alpha^{2k-2-r}x^{3k/2+\epsilon}\,d\alpha\\ &\ll x^{r+2-k/2+\epsilon}+\begin{cases}\beta^{2k-1-r}x^{3k/2+\epsilon}, & 2k-1-r>0\\ x^{r+2-k/2}, & 2k-1-r\leq 0\end{cases}\end{aligned}$$

Using $k\geq 3$ we see that in the second case the integral is bounded by $x^{r+1/2}$, which is sufficient. In the first case we take $\beta=x^{-1/2}$ and obtain that the integral is bounded by $x^{(k+r+1)/2}$, since $k\leq r$, this is also of order $x^{r+1/2}$, and therefore admissible.

For the remainder of the integral we use the L^2-estimate $\int_0^1|R(\alpha)|^2d\alpha\ll x^{1+\epsilon}$ and the trivial bound $|R(\alpha)|\ll x^{1+\epsilon}$ and obtain

$$\int\limits_{x^{-1/2}}^{1/2} |T(\alpha)|^{r-k}|R(\alpha)|^k|T_3(\alpha)|d\alpha \ll x^{1+\epsilon}\max_{x^{-1/2}\leq\alpha\leq 1/2}\alpha^{k-3-r}x^{k-2}=x^{(k+r+1)/2},$$

Using $k\leq r$ again we see that this is also $\mathcal{O}(x^{r+1/2+\epsilon})$. Hence, we find that the Dirichlet-series with coefficients $B_{r,k}(n)$ converge absolutely for $\sigma>r+3/2$.

Next, we explicitly compute the contribution of the terms $k\leq 2$. We have $B_{r,0}(x)=\int_0^1 T(\alpha)^r T_3(\alpha)\,d\alpha$, that is,

$$\begin{aligned}B_{r,0}(x) &= \sum_{n\leq x}(x-n)\#\{n_1+\dots+n_r=n\}\\ &= \sum_{n\leq x}(x-n)^2\binom{n+r-1}{r-1}\\ &= P_r(x)\end{aligned}$$

for some polynomial P_r of degree $r+2$. Hence, the Dirichlet-series with coefficients $B_{r,0}$ can be expressed as a linear combination of the functions $\zeta(s),\zeta(s-1),\dots,\zeta(s-r-2)$.

The corresponding computations for $B_{r,1}$ and $B_{r,2}$ are simplified by observing that

$$\int\limits_0^1 T(\alpha)^{r-1}R(\alpha)T_3(\alpha)\,d\alpha=\int\limits_0^1 T(\alpha)^{r-1}S(\alpha)T_3(\alpha)\,d\alpha-\int\limits_0^1 T(\alpha)^r T_3(\alpha)\,d\alpha$$

and

$$\int_0^1 T(\alpha)^{r-2}R(\alpha)^2T_3(\alpha)\,d\alpha = \int_0^1 T(\alpha)^{r-2}S(\alpha)^2T_3(\alpha)\,d\alpha - 2\int_0^1 T(\alpha)^{r-1}S(\alpha)T_3(\alpha)\,d\alpha + \int_0^1 T(\alpha)^rT_3(\alpha)\,d\alpha.$$

To evaluate these integrals we transform them back into counting problems. We have

$$\int_0^1 T(\alpha)^rT_3(\alpha)\,d\alpha = \sum_{\substack{n_1+\dots+n_r+m=0\\ 0\leq n_i\leq x,\ |m|<x}} (x-|m|)^2$$

$$= \sum_{0\leq m\leq x}(x-m)^2\binom{m+r-1}{r-1} = P(x),$$

where P is a polynomial of degree $r+2$. Hence, the generating function of $\int_0^1 T(\alpha)^rT_2(\alpha)\,d\alpha$ is a linear combination of $\zeta(s),\zeta(s-1),\dots,\zeta(s-r-2)$.

Similarly,

$$\begin{aligned}\int_0^1 T(\alpha)^{r-1}S(\alpha)T_3(\alpha)\,d\alpha &= \sum_{\substack{n_1+\dots+n_r+m=0\\ 0\leq n_i\leq x,\ |m|<x}} \Lambda(m)(x-|m|)^2\\ &= \sum_{0\leq m\leq x}(x-m)\Lambda(m)\binom{m+r-1}{r-1}\\ &= \sum_{0\leq m\leq x}\Lambda(m)P_1(m) + x\sum_{0\leq m\leq x}\Lambda(m)P_2(m),\end{aligned}$$

where P_1 is a polynomial of degree $r+2$, and P_2 a polynomial of degree $r+1$. The generating function of $\Lambda(m)P_1(m)$ is a linear combination of $\frac{\zeta'}{\zeta}(s), \frac{\zeta'}{\zeta}(s-1),\dots,\frac{\zeta'}{\zeta}(s-r-1)$, applying partial summation we find that the generating function of $\sum_{0\leq m\leq x}\Lambda(m)P_1(m)$ is a linear combination with rational coefficients plus a remainder, which is holomorphic in the half-plane $\Re\ s>0$, the same argument applies to the second sum.

Finally,

$$\int_0^1 T(\alpha)^{r-2}S(\alpha)^2T_3(\alpha)\,d\alpha = \sum_{\substack{n_1+\dots+n_{r-1}+m=0\\ 0\leq n_i\leq x,\ |m|<x}} G_2(n_{r-1})(x-|m|)$$

and as for the previous integral we find that the generating function with coefficients $\int_0^1 T(\alpha)^{r-2}S(\alpha)^2T_3(\alpha)\,d\alpha$ is a linear combination of $\Phi_2(s),\ \dots,\ \Phi_2(s-r-1)$ with rational coefficients, plus a function which is holomorphic in the half-plane $\Re\ s>0$.

Combining this observation with Lemma 2 we find that $\Phi_r(s)$ can be written as a linear combination of the functions $\zeta(s),\ \dots,\ \zeta(s-r+1)$, $\frac{\zeta'}{\zeta}(s),\ \dots,\ \frac{\zeta'}{\zeta}(s-r+1)$,

$\Phi_2(s)$, $\ldots$, $\Phi_2(s-r+2)$ with rational coefficients plus a remainder $R(s)$ which is holomorphic in the half-plane $\Re\, s > r-3/2$. But among these functions only $\zeta(s-r+2)$, $\zeta(s-r+1)$, $\frac{\zeta'}{\zeta}(s-r+2)$, $\frac{\zeta'}{\zeta}(s-r+1)$,$\Phi_2(s-r+3)$ and $\Phi_2(s-r+2)$ are not holomorphic in the half-plane $\Re\, s > r-2$, hence, all but this six functions can be subsumed under R. Moreover, since we work under the Riemann hypothesis, $\frac{\zeta'}{\zeta}(s-r+2)$ and $\Phi_2(s-r+3)$ are holomorphic in $\Re\, s > r-3/2$ with the exception of a pole at $s = r-1$, hence, we can replace these functions by $\zeta(s-r+2)$. Hence, the claim of the theorem follows. □

3. Proof of Theorem 2

Part (1) of the theorem follows from Theorem 4 because in the half plane $\Re(s) > r-1-1/10$ only $\Phi_2(s-r+2)$ has essential singularities.

We now indicate the proof of part (4) which is closely related to the one given under the Riemann hypothesis by Egami and Matsumoto. The following serves as a substitute for [4, Lemma 4.1]

Lemma 3. *let D be the closure of the set $\{\rho_1+\rho_2 : \zeta(\rho_i) = 0, \Re\rho_i > 0\}$. Then $\mathbb{C}\setminus D$ is not connected, and, denoting by $\mathcal{D}$ the component containing the half-plane $\Re s > 2$, we have*

$$\{s : \Re s > 3/2\} \subseteq \mathcal{D} \subseteq \{s : \Re s > 1\}.$$

Proof. Let $\epsilon > 0$ and $t_0 \in \mathbb{R}$ be given. We show that there are zeros ρ_1, ρ_2 of ζ such that $\rho_1+\rho_2$ is within the square $1 \leq \Re s < 1+\epsilon$, $t_0 < \Im s < t_0+\epsilon$, which implies the claim. Let $N(T,\sigma)$ be the number of zeros ρ of ζ with $\Re\rho > \sigma$ and $0 < \Im\rho < T$. Call a real number t good, if there is a zero ρ of ζ with $\frac{1}{2} <\leq \Re s < \frac{1}{2}+\epsilon/2$, $t_0 < \Im s < t_0+\epsilon/2$, and let $\mathcal{T}$ be the set of good numbers. We have to show that there exists good numbers t_1, t_2 with $t_1+t_2 = t_0$. This in turn would follow if we show that asymptotically almost all real numbers are good. To do so, we use the estimate $N(T,\sigma) \ll T^{\frac{3(1-\sigma)}{2-\sigma}} \log^5 T$ due to Ingham and the fact that the distance between consecutive abscissae of zeros tends to zero, proven by Littlewood. The second statement shows that every sufficiently large real number t is good, unless there is a zero of ζ in the domain $\Re s > \frac{1}{2}+\epsilon/2$, $t < \Im s < t+\epsilon/2$. Hence, we obtain

$$|\mathcal{T}\cap[0,T]| \geq T - C(\epsilon) - N(T, \frac{1+\epsilon}{2}) \sim T,$$

that is, for T sufficiently large the measure of $\mathcal{T}\cap[0,T]$ supersedes $T/2$, hence, we find real numbers $t_1, t_2 \in \mathcal{T}$ with $t_1+t_2 = t_0$. □

It follows from [4, Lemma 4.2], that under Conjecture 2 every complex number of the form $\rho_1+\rho_2$, $\zeta(\rho_1) = \zeta(\rho_2) = 0$ is a singularity of Φ_2. The proof of the fact that Φ_2 is meromorphic in the half plane $\Re s > 2\sigma_0$ runs parallel to the proof of [4, Theorem 2.1] and need not be repeated here.

Finally we prove parts (2) and (3).

Let ρ_1, ρ_2 be zeros of ζ. Our aim is to show that either there are zeros ρ_3, ρ_4 with $\rho_1+\rho_2-\rho_3-\rho_4 = 0$ and $|\Im\rho_3|+|\Im\rho_4| \leq 5(|\Im\rho_1|+|\Im\rho_2|)$, or $|\Phi_2(\rho_1+\rho_2+\eta)| \gg \frac{1}{\eta}$ for $\eta \searrow 0$. Our proof starts similar to the proof by Egami and Matsumoto (confer [4, section 4]).

Lemma 4. *Put $M(s) = -\frac{\zeta'}{\zeta}(s)$. Then we have*

$$\begin{aligned}\Phi_2(s) &= \frac{M(s-1)}{s-1} - \sum_\rho \frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)} M(s-\rho) - M(s)\log 2\pi \\ &\quad + \frac{1}{2\pi i}\int\limits_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z)M(z)dz \\ &= -\frac{1}{\Gamma(s)}\sum_{\rho,\rho'} \frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} + R(z),\end{aligned}$$

where R is meromorphic in the whole complex plane

Proof. This follows from [4, (2.2)] and[4, (4.2)]. □

Now suppose that $\rho_1 = \frac{1}{2} + i\gamma_1$, $\rho_2 = \frac{1}{2} + i\gamma_2$ are zeros of ζ. We want to show that in a small neighourhood of $1 + i(\gamma_1 + \gamma_2)$ the behaviour of $\Phi_2(s)$ is dominated by the summand coming from ρ_1, ρ_2. To do so we estimate the contribution of different ranges for ρ, ρ' in different ways.

Consider pairs ρ, ρ' with $|\rho + \rho' - \rho_1 - \rho_2| > \frac{1}{4}$. For $|s - \rho_1 - \rho_2| < \frac{1}{8}$, the sum can be bounded as

$$\begin{aligned}\sum_{\rho,\rho'} \frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} &= \sum_{\substack{\rho,\rho' \\ |\rho_1+\rho_2-\rho-\rho'|>\rho'/2}} \frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} \\ &\quad + \sum_{\substack{\rho,\rho' \\ |\rho_1+\rho_2-\rho-\rho'|\leq\rho'/2}} \frac{\Gamma(s+1-\rho)\Gamma(\rho)}{(s-\rho-\rho')\rho'} \\ &= \sum\nolimits_1 + \sum\nolimits_2,\end{aligned}$$

say. We have

$$\sum\nolimits_1 \ll \sum_{\substack{\rho,\rho' \\ |\rho_1+\rho_2-\rho-\rho'|>\rho'/2}} \frac{\Gamma(\rho)}{\rho'^2} \ll \sum_{\substack{\rho,\rho' \\ |\rho_1+\rho_2-\rho-\rho'|>\rho'/2}} \frac{\Gamma(\rho)}{\rho'^2}$$

and

$$\sum\nolimits_2 \ll \sum_{\substack{\rho,\rho' \\ |\rho_1+\rho_2-\rho-\rho'|\leq\rho'/2}} \Gamma(s+1-\rho)\Gamma(\rho) \ll \sum_\rho \Gamma(\rho)N(2|\rho|),$$

Since $N(T) \ll T\log T$, and $\Gamma(\sigma+it) \ll e^{-ct}$, we see that this sum converges uniformly in the open ball $B_{\frac{1}{8}}(\rho_1+\rho_2)$.

Now consider pairs of zeros with $|\rho| + |\rho'| > 5(|\rho_1| + |\rho_2|)$ and $|\rho + \rho' - \rho_1 - \rho_2| \leq 1/4$. For $s \in B_{\frac{1}{8}}(\rho_1+\rho_2)$ we have

$$s + 1 - \rho = 1 + \rho_1 + \rho_2 - \rho + \theta/8 = 1 + \rho' + 3\theta/8,$$

where θ is a complex number of absolute value ≤ 1, hence, the sum taken over all ρ, ρ' in this range is bounded by

$$\begin{aligned}\sum_{\rho,\rho'}\left|\frac{\Gamma(\rho)\max_{|s-\rho'|\leq 3/8}|\Gamma(1+s)|}{(s-\rho-\rho')\rho'}\right| &< \frac{1}{\eta}\sum_{\substack{|\rho|+|\rho'|>5(|\rho_1|+|\rho_2|)\\ |\rho+\rho'-\rho_1-\rho_2|<1}}|\rho'\Gamma(\rho)\Gamma(\rho')|\\ &< \frac{1}{\eta}\left(\sum_{\rho>2(|\rho_1|+|\rho_2|)-1}|\rho\Gamma(\rho)|\right)^2.\end{aligned}$$

To transform the sum over zeros into a sum over integers, we use the following bound, which follows from a more precise result by Backlund[1].

Lemma 5. *We have $N(T+1)-N(T)<\log T$ for $T\geq 1$.*

Using this bound together with the estimate $\Gamma(\sigma+it)<e^{-\frac{\pi}{4}t}$ we obtain that the contribution of zeros of the form under consideration is bounded by

$$\frac{1}{\eta}\left(\sum_{n\geq 2(|\rho_1|+|\rho_2|)-2} n\log n e^{-\frac{\pi}{4}n}\right)^2 < \frac{270}{\eta}e^{-\frac{12}{5}(|\rho_1|+|\rho_2|)},$$

where we used the fact that $|\rho_1|+|\rho_2|>28$.

On the other hand, the pair ρ_1,ρ_2 itself contributes

$$\frac{1}{\eta}\left(\frac{\Gamma(\rho_1+1+\eta)\Gamma(\rho_2)}{\rho_1}+\frac{\Gamma(\rho_2+1+\eta)\Gamma(\rho_1)}{\rho_2}\right)\sim\frac{2}{\eta}\Gamma(\rho_1)\Gamma(\rho_2),$$

hence, the contribution of these zeros cannot be canceled by the contribution of zeros satisfying $|\rho|+|\rho'|>5(|\rho_1|+|\rho_2|)$ since

$$2e^{-\frac{4}{5}(|\rho_1|+|\rho_2|)}>270e^{-\frac{12}{5}(|\rho_1|+|\rho_2|)}.$$

In fact the above inequality follows from $|\rho_1|+|\rho_2|>28$.

The arguments used till now were under the assumption of RH. Now we separate the arguments for parts (2) and (3).

If in addition to the RH we assume Conjecture 1, then the finitely many pairs ρ,ρ' different from ρ_1,ρ_2 which we have not yet dealt with define a function meromorphic on $\mathbb{C}$ without poles on the line $\Im s=\gamma_1+\gamma_2$. In some neighborhood of $\rho_1+\rho_2$ this function is bounded and the proof of part (2) is done.

We did not actually use the full strength of Conjecture 1 but only the non-existence of linear relations $\gamma_1+\gamma_2=\gamma_3+\gamma_4\neq 0$ for $|\gamma_1|+|\gamma_2|\leq 5(|\gamma_3|+|\gamma_4|)$. Even though this condition seems as unreachable as Conjecture 1, for each fixed pair γ_1,γ_2 it can be verified. Taking the minimal value 14.1347... for γ_1,γ_2 there are only 39 zeros with imaginary part at most 142 it is easy to check that no other pair adds up to $\gamma_1+\gamma_2$. hence $2\rho_1$ is a singularity of Φ_2 and we are done.

We now prove the Corollary. Set

$$\Delta_r(x)=\sum_{n\leq x}G_r(n)-\frac{1}{r!}x^r-H_r(x).$$

In view of Lemma 2 and Theorem 2, part (3) the generating Dirichlet-series $D(s)$ of Δ_r has a singularity at $2\rho_1+r-1$. Moreover,

$$\lim_{\sigma\searrow 0}\sigma|D(\sigma+2\rho_1+r-1)|>0$$

and if we have $\Delta_r(x) = o(x^{r-1})$, the last limit is zero. Thus our claim follows.

References

[1] R.J. Backlund, Über die Nullstellen der Riemannschen Zetafunktion, *Acta Math.* **41** (1918), 345–375.

[2] G. Bhowmik, J-C. Schlage-Puchta, Natural Boundaries of Dirichlet series, *Func. Approx. Comment. Math.* **XXXVII.1** (2007), 17–29.

[3] G. Bhowmik, J-C. Schlage-Puchta, Mean representation number of integers as the sum of primes, *Nagoya Math. Journal* to appear.

[4] S. Egami, K. Matsumoto, Convolutions of von Mangoldt functions and related Dirichlet series, Proceedings of the 4th China-Japan Seminar held at Shandong, 1–23, S. Kanemitsu and J.-Y. Liu eds., World Sci. Publ., Hackensack, NJ, 2007.

[5] A. Fujii, An additive problem of prime numbers, *Acta Arith.* **58** (1991), 173–179.