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# Upper bound on the density of Ruelle resonances for Anosov flows.

Frédéric Faure\*, Johannes Sjöstrand†

24 February 2010

## Abstract

Using a semiclassical approach we show that the spectrum of a smooth Anosov vector field  $V$  on a compact manifold is discrete (in suitable anisotropic Sobolev spaces) and then we provide an upper bound for the density of eigenvalues of the operator  $(-i)V$ , called Ruelle resonances, close to the real axis and for large real parts.

## Résumé

Par une approche semiclassique on montre que le spectre d'un champ de vecteur d'Anosov  $V$  sur une variété compacte est discret (dans des espaces de Sobolev anisotropes adaptés). On montre ensuite une majoration de la densité de valeurs propres de l'opérateur  $(-i)V$ , appelées résonances de Ruelle, près de l'axe réel et pour les grandes parties réelles.

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# 1 Introduction

Chaotic behavior of certain dynamical systems is due to hyperbolicity of the trajectories. This means that the trajectories of two initially close points will diverge in the future or in the past (or both) [9, 28]. As a result the behavior of an individual trajectory appears to be complicated and unpredictable. However evolution of a cloud of points seems more simple: it will spread and equidistribute according to an invariant measure, called an equilibrium measure (or S.R.B. measure). Also from the physical point of view, a distribution reflects the unavoidable lack of knowledge about the initial point. Following this idea, D. Ruelle in the 70' [39, 40], has shown that instead of considering individual trajectories, it is much more natural to consider evolution of densities under a linear operator called the Ruelle transfer operator or the Perron Frobenius operator.

For dynamical systems with strong chaotic properties, such as uniformly expanding maps or uniformly hyperbolic maps, Ruelle, Bowen, Fried, Rugh and others, using symbolic dynamics techniques (Markov partitions), have shown that the transfer operator has a discrete spectrum of eigenvalues. This spectral description has an important meaning for the dynamics since each eigenvector corresponds to an invariant distribution (up to a time factor). From this spectral characterization of the transfer operator, one can derive other specific properties of the dynamics such as decay of time correlation functions, central limit theorem, mixing, etc. In particular a spectral gap implies exponential decay of correlations.

This spectral approach has recently (2002-2005) been improved by M. Blank, S. Gouëzel, G. Keller, C. Liverani [6, 22, 31, 10], V. Baladi and M. Tsujii [3, 4] (see [4] for some historical remarks) and in [17], through the construction of functional spaces adapted to the dynamics, independent of every symbolic dynamics.

The case of flows i.e. dynamical systems with continuous time is more delicate (see [18] for historical remarks). This is due to the direction of time flow which is neutral (i.e. two nearby points on the same trajectory will not diverge from one another). In 1998 Dolgopyat [13, 14] showed the exponential decay of correlation functions for certain Anosov flows, using techniques of oscillatory integrals and symbolic dynamics. In 2004 Liverani [30] adapted Dolgopyat's ideas to his functional analytic approach, to treat the case of contact Anosov flows. In 2005 M. Tsujii [49] obtained an explicit estimate for the spectral gap for the suspension of an expanding map. In 2008 M. Tsujii [50] obtained an explicit estimate for the spectral gap, in the case of contact Anosov flows.

**Semiclassical approach for transfer operators:** It also appeared recently [16, 17, 15] that for hyperbolic dynamics on a manifold  $X$ , the study of transfer operators is naturally a semiclassical problem in the sense that a transfer operator can be considered as a “Fourier integral operator” and using standard tools of semiclassical analysis, some of its spectral properties can be obtained from the study of “the associated classical symplectic dynamics”, namely the initial hyperbolic dynamics on  $X$  lifted to the cotangent space  $T^*X$  (the phase space).

The simple idea behind this, crudely speaking, is that a transfer operator transports a “wave packet” (i.e. localized both in space and in Fourier space) into another wave packet,

and this is exactly the characterization of a Fourier integral operator. A wave packet is characterized by a point in phase space (its position and its momentum), hence one is naturally led to study the dynamics in phase space. Moreover, since every function or distribution can be decomposed into a linear superposition of wave packets, the dynamics of wave packets characterizes completely the transfer operators.

Following this approach, in the papers [16, 17] we studied hyperbolic diffeomorphisms. The aim of the present paper is to show that semiclassical analysis is also well adapted to hyperbolic systems with a neutral direction since it induces a natural semiclassical parameter ( $\alpha$  in Theorem 15 page 26), the Fourier component in the neutral direction. In the paper [15] one of us has considered a partially expanding map and showed that a spectral gap develops in the limit of large oscillations in the neutral direction (which is a semiclassical limit). In this paper we consider a hyperbolic flow on a manifold  $X$  generated by a vector field  $V$ . In Section 2 we recall the definition of a hyperbolic flow and some of its properties. In Section 3 we describe the dynamics induced on the cotangent space  $T^*X$ . In particular we construct an “escape function” which expresses the fact that the trajectories escape towards infinity in  $T^*X$  except on a specific subspace called the **trapped set**  $K$ . The vector field is considered as a partial differential operator of order 1 acting on smooth functions  $C^\infty(X)$  and can be extended to the space of distributions. In Section 4 using a semiclassical approach (with escape function on phase space) we establish a first result, in Theorem 12, which shows that the operator  $\hat{H} = -iV$  has a discrete spectrum of resonances in specific anisotropic Sobolev spaces. This discrete spectrum is intrinsic to the vector field  $V$  in the sense that we get the same spectrum in an overlap region for two different Sobolev spaces constructed according to some general principles. This result has already been obtained by O. Butterley and C. Liverani in [10, Theorem 1]. The novelty here is to show that this resonance spectrum fits with the general theory of semiclassical resonances developed by B. Helffer and J. Sjöstrand [25] and initiated by Aguilar, Baslev, Combes [1, 5]. Our main new result is Theorem 15 page 26 which provides an upper bound  $o(\alpha^{n-1/2})$  (with  $n = \dim X$ ), for the number of resonances in the spectral domain  $\Re(\lambda) \in [\alpha, \alpha + \sqrt{\alpha}]$ ,  $\Im(\lambda) > -\beta$  (all  $\beta$ ) in the semiclassical limit  $|\alpha| \rightarrow \infty$ .

The use of escape functions on phase space for resonances has been introduced by B. Helffer and J. Sjöstrand [25] and used in many situations [46, 42, 45, 44, 54, 24, 35]. In particular in [24], the authors consider the geodesic flow associated to Schottky groups and provide an upper bound for the density of Ruelle resonances (see also [8]).

In this paper as well as in [17], one aim is to make more precise the connection between the spectral study of Ruelle resonances and the spectral study in quantum chaos [34, 53], in particular to emphasize the importance of the symplectic properties of the dynamics in the cotangent space  $T^*X$  on the spectral properties of the transfer operator, and long time behavior of the dynamics.

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## 2 Anosov flows

Let  $X$  be a  $n$ -dimensional smooth compact connected manifold, with  $n \geq 3$ . Let  $\phi_t$  be the flow on  $X$  generated by a **smooth** vector field  $V \in C^\infty(X; TX)$ :

$$V(x) = \frac{d}{dt}(\phi_t(x))_{/t=0} \in T_x X, \quad x \in X. \quad (1)$$

We assume that the flow  $\phi_t$  is **Anosov**. We recall the definition (see [28] page 545, or [36] page 8)

**Definition 1.** On a smooth Riemannian manifold  $(X, g)$ , a vector field  $V$  generates an **Anosov flow**  $(\phi_t)_{t \in \mathbb{R}}$  (or **uniformly hyperbolic flow**) if:

- For each  $x \in X$ , there exists a decomposition

$$T_x X = E_u(x) \oplus E_s(x) \oplus E_0(x), \quad (2)$$

where  $E_0(x)$  is the one dimensional subspace generated by  $V(x)$ .

- The decomposition (2) is invariant by  $\phi_t$  for every  $t$ :

$$\forall x \in X, \quad (D_x \phi_t)(E_u(x)) = E_u(\phi_t(x)) \quad \text{and} \quad (D_x \phi_t)(E_s(x)) = E_s(\phi_t(x)).$$

- There exist constants  $c > 0$ ,  $\theta > 0$  such that for every  $x \in X$

$$\begin{aligned} |D_x \phi_t(v_s)|_g &\leq c e^{-\theta t} |v_s|_g, & \forall v_s \in E_s(x), \quad t \geq 0 \\ |D_x \phi_t(v_u)|_g &\leq c e^{-\theta|t|} |v_u|_g, & \forall v_u \in E_u(x), \quad t \leq 0, \end{aligned} \quad (3)$$

meaning that  $E_s$  is the stable distribution and  $E_u$  the unstable distribution for positive time.

### 2.1 Remarks:

The remarks of this Section give more information on Anosov flows but are not necessary for the rest of this paper.

#### 2.1.1 General remarks

- Standard examples of Anosov flows are **suspensions of Anosov diffeomorphisms** (see [36] p.8), or **geodesic flows on manifolds  $M$  with sectional negative curvature** (see [36] p.9, or [28] p.549, p.551). Notice that in this case, the geodesic flow is Anosov on the unit cotangent bundle  $X = T_1^* M$ .

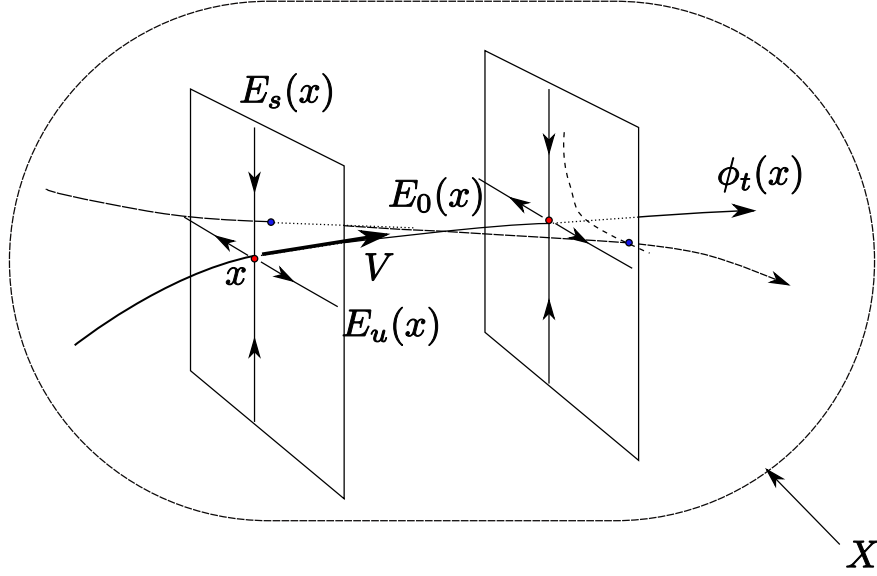


Figure 1: Picture of an Anosov flow in  $X$  and instability of trajectories.

- The global hyperbolic structure of Anosov flows or Anosov diffeomorphisms is a very strong geometric property, so that manifolds carrying such dynamics satisfy strong topological conditions and the list of known examples is not so long. See [7] for a detailed discussion and references on that question.
- Let

$$d_u = \dim E_u(x), \quad d_s = \dim E_s(x),$$

(they are independent of  $x \in X$ ). Eq.(2) implies  $d_u + d_s + 1 = \dim X = n$ . For every  $d_u, d_s \geq 1$  one may construct an example of an Anosov flow: one considers a suspension of a hyperbolic diffeomorphism of  $SL_{n-1}(\mathbb{Z})$  on  $\mathbb{T}^{n-1}$ , with  $n = d_u + d_s + 1$ , such that there are  $d_s$  eigenvalues with modulus  $|\lambda| < 1$ , and  $d_u$  eigenvalues with modulus  $|\lambda| > 1$ .

### 2.1.2 Constructive expressions for $E_u, E_s$

In the case where  $d_u = d_s = 1$ , there is a formula which gives the distributions  $E_u, E_s$  [2]. Let  $[v_0] \in C^\infty(\mathbb{P}(TX))$  be a global smooth section of the projective tangent bundle, such that at every point  $x \in X$ ,  $[v_0]_x \notin (E_0(x) \oplus E_s(x))$  (it is sufficient that the direction  $[v_0]_x$  is close enough to the unstable direction  $E_u(x)$ ). Then for every  $x \in X$ ,

$$E_u(x) = \lim_{t \rightarrow +\infty} D\phi_t \left( [v_0]_{\phi_{-t}(x)} \right). \quad (4)$$

Similarly if  $[v_0]_x \notin (E_0(x) \oplus E_u(x))$ ,  $\forall x \in X$ , then for every  $x \in X$ ,

$$E_s(x) = \lim_{t \rightarrow -\infty} D\phi_t \left( [v_0]_{\phi_{-t}(x)} \right).$$

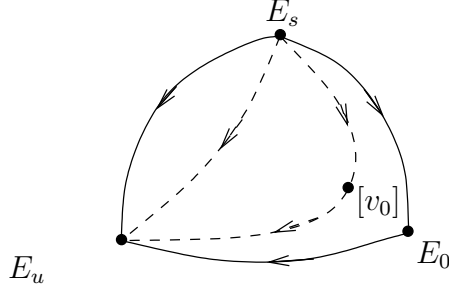


Figure 2: Picture of the flow on the projective tangent bundle  $\mathbb{P}(TX)$  induced by the hyperbolic vector field  $V$ . A global section  $[v_0]$  will converge towards  $E_u$  or  $E_s$  for  $t \rightarrow \pm\infty$  as explained in Eq.(4).

See figure 2.

In the general case, for every  $d_u, d_s$ , there exists a similar construction. Let  $[v_0] \in C^\infty(\text{Gr}_{d_u, n}(TX))$  be a global smooth and non vanishing section of the Grassmanian bundle, such that at every point  $x \in X$ , the linear space  $[v_0]_x$  does not intersect  $E_0(x) \oplus E_s(x)$ . Then for every  $x \in X$ ,

$$E_u(x) = \lim_{t \rightarrow +\infty} D\phi_t \left( [v_0]_{\phi_{-t}(x)} \right).$$

Similarly

$$E_s(x) = \lim_{t \rightarrow -\infty} D\phi_t \left( [v_0]_{\phi_{-t}(x)} \right),$$

when  $[v_0] \in C^\infty(\text{Gr}_{d_s, n}(TX))$  does not intersect  $E_0 \oplus E_u$ .

### 2.1.3 Anosov one form $\alpha$ and regularity of the distributions $E_u(x)$ , $E_s(x)$

The distribution  $E_0(x)$  is smooth since  $V(x)$  is assumed to be smooth. The distributions  $E_u(x)$ ,  $E_s(x)$  and  $E_u(x) \oplus E_s(x)$  are only Hölder continuous in general (see [36] p.15, [21] p.211). Smoothness can be present with additional hypothesis or with particular models. See the discussion below, section 2.1.3 page 7.

The above hypothesis on the flow implies that there is a particular continuous one form on  $X$ , denoted  $\alpha \in C^0(T^*X)$  called the **Anosov 1-form** and defined by

$$\ker(\alpha(x)) = E_u(x) \oplus E_s(x), \quad (\alpha(x))(V(x)) = 1, \quad \forall x \in X. \quad (5)$$

Since  $E_u$  and  $E_s$  are invariant by the flow then  $\alpha$  is invariant as well,  $\phi_t^*(\alpha) = \alpha$  for every  $t \in \mathbb{R}$ , and therefore (in the sense of distributions)

$$\mathcal{L}_V(\alpha) = 0, \quad (6)$$

where  $\mathcal{L}_V$  denotes the Lie derivative.

We discuss now some known results about the smoothness of the distributions  $E_u(x)$ ,  $E_s(x)$  in some special cases.



- In the case of a **geodesic flow** on a smooth Riemannian negative curvature manifold  $M$  (with  $X = T_1^*M$ ),  $E_u(x) \oplus E_s(x)$  is orthogonal<sup>2</sup> to  $E_0(x)$ , therefore  $E_u(x) \oplus E_s(x)$  is  $C^\infty$ . The distributions  $E_u(x)$ ,  $E_s(x)$  are  $C^1$  individually (see [20] p.252).
- More generally, the flow  $\phi_t$  is a **contact flow** (or **Reeb vector field**, see [33] p.106, [41, p.55]) if the associated one form  $\alpha$  defined in Eq.(5) is  $C^\infty$  and if

$$d\alpha|_{(E_u \oplus E_s)} \text{ is non degenerate (i.e. symplectic)} \quad (7)$$

meaning that  $\alpha$  is a **contact one form**. Equivalently,  $dx := \alpha \wedge (d\alpha)^n$  is a volume form on  $X$  with  $n := \frac{1}{2} \dim(E_u \oplus E_s)$ . Notice that (6) implies that the volume form is invariant by the flow:

$$\mathcal{L}_V(dx) = 0. \quad (8)$$

In that case,  $E_u(x) \oplus E_s(x) = \ker(\alpha)$  is  $C^\infty$  and  $\alpha$  determines  $V$  by  $d\alpha(V) = 0$  and  $\alpha(V) = 1$ .

- (From [21] p.211) Hurder and Katok in [27] showed that if  $\phi_t$  is an Anosov flow on  $X$ ,  $\dim X = 3$ , and if  $\alpha$  is a contact form of class  $C^1$  then  $\alpha$  is  $C^\infty$  in fact, and  $E_u(x)$ ,  $E_s(x)$  are  $C^{2-\varepsilon}$  for every  $\varepsilon > 0$ . Moreover if  $E_u(x)$ ,  $E_s(x)$  are  $C^2$  then they are  $C^\infty$  in fact and  $C^\infty$  conjugated to an algebraic flow [20].

## 3 Transfer operator and the dynamics lifted on $T^*X$

### 3.1 The transfer operator

The flow  $\phi_t$  acts in a natural manner on functions by pull back and this defines the transfer operator:

**Definition 2.** The **Ruelle transfer operator**  $\widehat{M}_t : C^\infty(X) \rightarrow C^\infty(X)$ ,  $t \in \mathbb{R}$  is defined by:

$$\widehat{M}_t \varphi = \varphi \circ \phi_{-t}, \quad \varphi \in C^\infty(X), \quad (9)$$

$\widehat{M}_t$  can be expressed in terms of the vector field  $V$  as

$$\widehat{M}_t = \exp(-tV) = \exp(-it\widehat{H}) \quad (10)$$

with the **generator**

$$\widehat{H} := -iV. \quad (11)$$

---

<sup>2</sup>Proof: since the metric  $g$  is preserved by the flow, if  $v_u \in E_u$ , and  $V \in E_0$ ,

$$g(v_u, V) = g((D\phi_t)v_u, (D\phi_t)V) = g((D\phi_t)v_u, V),$$

goes to zero as  $t \rightarrow -\infty$ , from Eq.(3). Therefore  $g(E_u, E_0) = 0$ . Similarly  $g(E_s, E_0) = 0$ .

**Remarks:**

- If  $dx$  is a smooth density on  $X$ , then  $\widehat{M}_t$  can be extended to  $L^2(X, dx)$ . In this space,  $\widehat{M}_t$  is a bounded operator. The adjoint of  $\widehat{H}$  is ([47] prop.(2.4) p.129)

$$\widehat{H}^* = -iV - i \operatorname{div}(V) = \widehat{H} - i \operatorname{div}(V). \quad (12)$$

Hence ([47] def.(2.1) p.125)

$$\begin{aligned} \operatorname{div}(V) = 0 &\Leftrightarrow \mathcal{L}_V(dx) = 0 \Leftrightarrow \phi_t \text{ preserves } dx \\ &\Leftrightarrow \widehat{H} \text{ is self-adjoint on } L^2(X, dx) \\ &\Leftrightarrow \widehat{M}_t \text{ is a unitary operator on } L^2(X, dx). \end{aligned} \quad (13)$$

This is the case for the geodesic flow, which preserves the Liouville measure, or more generally for a contact flow from (8). But for a generic Anosov flow there does not exist any smooth invariant measure.

- From the probabilistic point of view, it is natural to consider the **Perron Frobenius transfer operator**  $\widehat{T}_t$ ,  $t \in \mathbb{R}$  whose generator is the adjoint  $\widehat{H}^*$ , Eq.(12):

$$\widehat{T}_t := e^{-i\widehat{H}^*t} = \left(\widehat{M}_{(-t)}\right)^*. \quad (14)$$

The reason is that one has the following relation which is interpreted as a conservation of the total probability measure:

$$\int_X \left(\widehat{T}_t \psi\right) dx = \int_X \psi dx, \quad \psi \in C^\infty(X).$$

*Proof.*  $\int_X \left(\widehat{T}_t \psi\right) dx = \left(1 | \widehat{T}_t \psi\right) = \left(\widehat{M}_{(-t)} 1 | \psi\right) = \int_X \psi dx$  since  $\widehat{M}_{(-t)} 1 = 1$  by (9).  $\square$

- We introduce the antilinear operator of complex conjugation  $\widehat{C} : C^\infty(X) \rightarrow C^\infty(X)$  by

$$\widehat{C}\varphi := \overline{\varphi}, \quad \varphi \in C^\infty(X), \quad (15)$$

which can be extended to  $\mathcal{D}'(X)$  by duality: for  $\psi \in \mathcal{D}'(X)$ ,  $\varphi \in C^\infty(X)$ ,

$$\langle \widehat{C}\psi | \varphi \rangle := \overline{\langle \psi | \widehat{C}\varphi \rangle}.$$

We have the following relation

$$\widehat{H}\widehat{C} + \widehat{C}\widehat{H} = 0 \quad (16)$$

(it will imply a symmetry for the Ruelle resonance spectrum, see Proposition 14 page 26)

*Proof.* Since  $V$  is real, for every  $\varphi \in C^\infty(X)$  one has  $\widehat{H}\widehat{C}\varphi = -iV(\overline{\varphi}) = \overline{iV(\varphi)} = -\overline{(-iV(\varphi))} = -\widehat{C}\widehat{H}\varphi$ .  $\square$

## 3.2 Generators of transfer operators are pseudo-differential operators

The generator  $\widehat{H}$  defined in (11) is a differential operator hence a pseudodifferential operator. This allows us to use the machinery of semiclassical analysis in order to study the spectral properties [48, chap. 7]. In particular it allows us to view  $\widehat{M}_t = \exp(-it\widehat{H})$  as a Fourier integral operator. See proposition 4 below.

### 3.2.1 Symbols and pseudodifferential operators

In this Section we recall how pseudodifferential operators are defined from their symbols on a manifold  $X$ . We first recall [23, 48, p.2] that:

**Definition 3.** The **symbol class**  $S^\mu$  with order  $\mu \in \mathbb{R}$  consist of  $C^\infty$  functions  $p(x, \xi)$  on  $\mathbb{R}^n \times \mathbb{R}^n$  such that

$$\forall \alpha, \beta, \quad |\partial_\xi^\alpha \partial_x^\beta p| \leq C_{\alpha, \beta} \langle \xi \rangle^{\mu - |\alpha|}, \quad \text{with } \langle \xi \rangle := \sqrt{1 + \xi^2}. \quad (17)$$

The value of  $\mu$  governs the increase (or decrease) of  $p(x, \xi)$  as  $|\xi| \rightarrow \infty$ .

On a manifold  $X$  with a given system of coordinates (more precisely a chart system and a related partition of unity, see [48, p.30]), the symbol  $p$  determines a **pseudodifferential operator (PDO for short)** denoted  $\widehat{P} = \text{Op}(p)$  acting on  $u \in C^\infty(X)$  and defined locally by

$$\widehat{P} : u \rightarrow (\widehat{P}u)(x) = \int e^{i\xi(x-y)} p(x, \xi) u(y) dy d\xi. \quad (18)$$

Conversely  $p = \sigma(\widehat{P})$  is called the **symbol** of the PDO  $\widehat{P}$ . Notation: if  $p \in S^\mu$  we say that  $\widehat{P} \in \text{Op}(S^\mu)$ .

The value of the order  $\mu$  is independent on the choice of coordinates, but the symbol  $p$  of a given PDO  $\widehat{P}$  depends on a choice of a chart and a choice of a partition of unity ([48, p.30]). The symbol has not a “geometrical meaning”. However it appears that the change of coordinate systems changes the symbol only at a subleading order  $S^{\mu-1}$ . In other words, the **principal symbol**  $p_{ppal} = p \bmod S^{\mu-1}$  is a well defined function on the manifold  $X$  (independently of the charts).

Concerning the operator  $\widehat{H} = -iV$  given in Eq.(11), one easily checks [47, p.2] that it is obtained by  $\widehat{H} = \text{Op}(H)$  with the symbol

$$H(x, \xi) = V(\xi) \in S^1.$$

Notice that this symbol does not depend on the chart. This is very particular to differential operators of order 1.

The quantization formula (18) is sometimes called the **left-quantization** or ordinary quantization of differential operators. There are plenty of other quantization formulae which differ at subleading order  $\text{Op}(S^{\mu-1})$  so the principal symbol of a PDO is the same for the different quantizations. Some have interesting properties. For example the **Weyl quantization of a symbol**  $p_W$  [48, (14.5) p.60] denoted by  $\hat{P} = \text{Op}_W(p_W)$  is defined by:

$$\hat{P} : u \rightarrow (\hat{P}u)(x) = \int e^{i\xi(x-y)} p_W\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi. \quad (19)$$

In this quantization, a *real* symbol  $p_W$  is quantized as a formally self-adjoint operator.

In our example Eq.(11),  $\hat{H} = -iV$ , the Weyl symbol is<sup>3</sup>

$$H_W(x, \xi) = V(\xi) + \frac{i}{2} \text{div}(V). \quad (20)$$

Notice that this symbol does not depend on the choice of coordinates systems provided the volume form is expressed by<sup>4</sup>  $dx = dx_1 \dots dx_n$ . The term  $\frac{i}{2} \text{div}(V)$  in (20) belongs to  $S^0$  and is called the **subprincipal symbol**.

For general symbols and with the Weyl quantization, a change of coordinate systems preserving the volume form changes the symbol at a subleading order  $S^{\mu-2}$  only. In other words, on a manifold with a fixed smooth density  $dx$ , the Weyl symbol  $p_W$  of a given PDO  $\hat{P}$  is well defined modulo terms in  $S^{\mu-2}$ .

In this paper we will also use a **Toeplitz quantization** (or F.B.I. quantization) for the proof of Lemma 25 page 44.

### 3.2.2 Induced dynamics on $T^*X$

Recall that the canonical symplectic two form on  $T^*X$  is ([33] p.90)<sup>5</sup>

$$\omega := d\xi \wedge dx. \quad (21)$$

The following well known proposition shows that the flow on the cotangent bundle  $T^*X$  obtained by lifting the flow  $\phi_t$  is naturally associated to the Ruelle transfer operator we are interesting in.

---

<sup>3</sup>Indeed from [48, (14.7) p.60], in a given chart where  $V = V(x) \frac{\partial}{\partial x}$ ,

$$H_W(x, \xi) = \exp\left(\frac{i}{2} \partial_x \partial_\xi\right) (V(x) \cdot \xi) = V(x) \cdot \xi + \frac{i}{2} \partial_x V = V(\xi) + \frac{i}{2} \text{div}(V)$$

and  $\text{div}(V)$  depends only on the choice of the volume form, see [47, p.125].

<sup>4</sup>On a manifold there always exists charts such that a given volume form is expressed as  $dx_1 \dots dx_n$

<sup>5</sup>We take the convention of the “semiclassical analysis community” with  $\omega := d\xi \wedge dx$ . The opposite convention  $\omega := dx \wedge d\xi$  is more usual in the “symplectic geometry community”.

**Proposition 4.** *The symbol of the differential operator  $\widehat{H}$  defined in Eq.(11) belongs to the symbol class  $S^1$ . Its principal symbol is equal to:*

$$H_0(x, \xi) = V(\xi) \in S^1. \quad (22)$$

The Ruelle transfer operator  $\widehat{M}_t$ , defined in Eq.(10) is a semi-classical **Fourier integral operator** (FIO), whose associated canonical map denoted by  $M_t : T^*X \rightarrow T^*X$  is the canonical lift of the diffeomorphism  $\phi_t$  on  $T^*X$  (linear in the fibers). See figure 3. More precisely if  $x \in X$ ,  $x' = \phi_t(x)$  then

$$M_t : \begin{cases} T^*X & \rightarrow T^*X \\ x & \mapsto x' = \phi_t(x) \\ \xi \in T_x^*X & \mapsto \xi' = (D_{x'}\phi_{-t})^t \xi \in T_{x'}^*X \end{cases} \quad (23)$$

$M_t$  is also the Hamiltonian flow generated by the vector field  $\mathbf{X} \in C^\infty(T^*X, T(T^*X))$  defined by

$$\omega(., \mathbf{X}) = dH_0. \quad (24)$$

The vector field  $\mathbf{X}$  is the canonical lift of  $V$  on  $T^*X$ .

*Proof.* For (22), see [48] p.2. From (10),  $\widehat{M}_t$  is defined by

$$i \frac{d\widehat{M}_t}{dt} = \widehat{H} \widehat{M}_t, \quad \widehat{M}_{t=0} = Id.$$

For (23) see [32, ex.2 p.152] or [47] Eq.(14.20) p. 77 and Eq.(14.15) p. 76, or [33], ex. 3.12 p.92.  $\square$

#### Remarks:

- The Hamiltonian vector field  $\mathbf{X} \in C^\infty(T^*X, T(T^*X))$  can be expressed as usual with respect to a coordinate system by ([47, p.74])

$$\mathbf{X} = \frac{\partial H_0}{\partial \xi} \frac{\partial}{\partial x} - \frac{\partial H_0}{\partial x} \frac{\partial}{\partial \xi}.$$

- From (12) the operators  $\widehat{H}$  and  $\widehat{H}^*$  have the same real principal symbol  $H_0(x, \xi) = V(\xi)$ . Therefore the canonical transform associated to the Perron Frobenius operator  $\widehat{T}_t$ , Eq.(14), is also  $M_t$ .

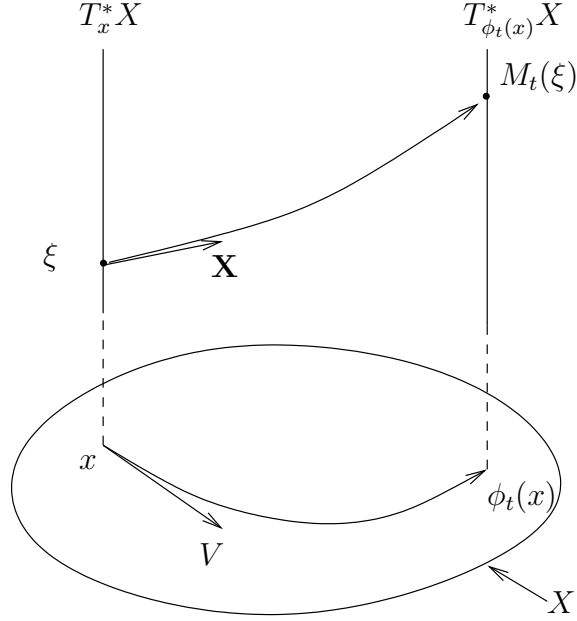


Figure 3: The flow  $\phi_t$  generated by the vector field  $V$  on  $X$  is lifted on the cotangent bundle  $T^*X$ . This gives a Hamiltonian flow  $M_t$  generated by a vector field  $\mathbf{X}$ .

### 3.2.3 Dual decomposition

Let

$$\boxed{T_x^*X = E_u^*(x) \oplus E_s^*(x) \oplus E_0^*(x)} \quad (25)$$

be the decomposition dual to (2) in the sense that

$$\begin{aligned} (E_0^*(x))(E_u(x) \oplus E_s(x)) &= 0, \\ (E_u^*(x))(E_u(x) \oplus E_0(x)) &= 0, \quad (E_s^*(x))(E_s(x) \oplus E_0(x)) = 0. \end{aligned} \quad (26)$$

**Remarks:**

- Let us remark that we have exchanged  $E_u^*$  and  $E_s^*$  with respect to the usual definition of dual spaces. Our choice of notations will be justified by Eq.(29).
- One has

$$\begin{aligned} \dim E_0^*(x) &= \dim E_0(x) = 1, \\ \dim E_u^*(x) &= \dim E_s(x) = d_s, \quad \dim E_s^*(x) = \dim E_u(x) = d_u. \end{aligned}$$

- From (5) and (26), one deduces that  $E_0^*$  is spanned by the one form  $\alpha$ :

$$E_0^*(x) = \mathbb{R} \alpha(x). \quad (27)$$

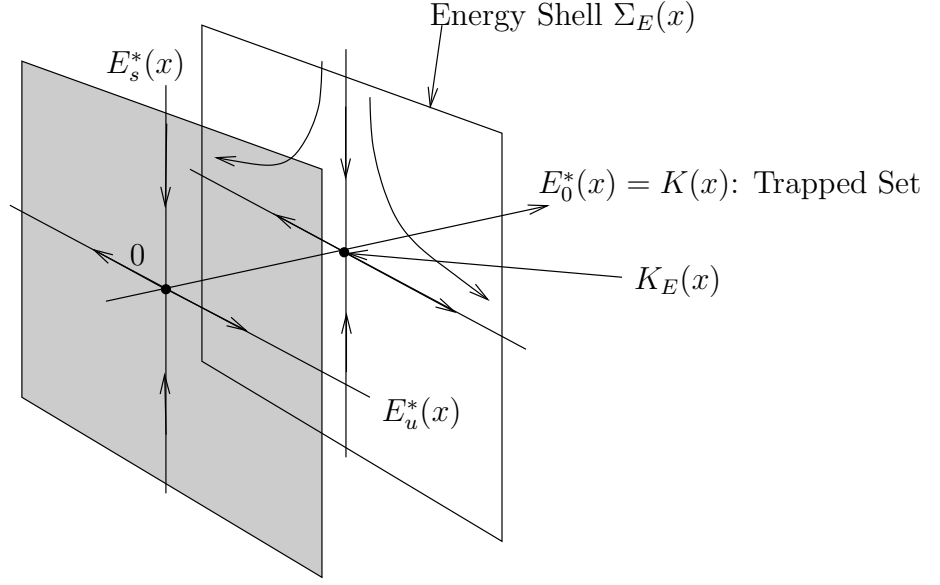


Figure 4: Picture of  $T_x^*X$  for a given  $x \in X$ , and a given  $E \in \mathbb{R}$ .

- Notice that since  $E_u(x) \oplus E_s(x)$  is not smooth in general (see remarks page 5), the same holds for  $E_0^*(x)$ . However  $(E_u^*(x) \oplus E_s^*(x))$  is smooth since  $E_0(x)$  is smooth.

**Definition 5.** For  $E \in \mathbb{R}$ , let

$$\Sigma_E := H_0^{-1}(E) \subset T^*X \quad (28)$$

be the “**energy shell**” (where the function  $H_0(x, \xi) = V(\xi)$  has been defined in (22)).

$\Sigma_E$  is a smooth hyper-surface in  $T^*X$ . More precisely for every  $x \in X$ ,

$$\Sigma_E(x) := \Sigma_E \cap T_x^*X$$

is an affine hyperplane in  $T_x^*X$ , parallel<sup>6</sup> to  $E_u^*(x) \oplus E_s^*(x)$  and therefore transverse to  $E_0^*(x)$ . See Figure 4.

---

<sup>6</sup>Since  $E_0(E_u^* \oplus E_s^*) = 0$  and  $V \in E_0$  then

$$H_0(E_u^*(x) \oplus E_s^*(x)) = V(E_u^*(x) \oplus E_s^*(x)) = 0.$$

**Proposition 6.** *The decomposition (25) is invariant by the flow  $M_t$  and there exists  $c > 0, \theta > 0$  such that*

$$\begin{aligned} |M_t(\xi_s)| &\leq ce^{-\theta t} |\xi_s|, & \forall \xi_s \in E_s^*, \quad \forall t \geq 0, \\ |M_t(\xi_u)| &\leq ce^{-\theta|t|} |\xi_u|, & \forall \xi_u \in E_u^*, \quad \forall t \leq 0. \end{aligned} \quad (29)$$

For every  $E \in \mathbb{R}$ , the energy shell  $\Sigma_E$  is invariant by the flow  $M_t$ . In the energy shell  $\Sigma_E$  the **trapped set**  $K_E$  is defined by

$$K_E := \{K_E(x) = (\Sigma_E(x) \cap E_0^*(x)), \quad x \in X\}. \quad (30)$$

$K_E$  is a global continuous section of the cotangent bundle  $T^*X$  given in terms of the associated one form (5) by:

$$K_E = E\alpha.$$

In general  $K_E(x)$  is not smooth but only Hölder continuous (as  $E_0^*(x)$ ).  $K_E$  is globally invariant under the flow  $M_t$ .

*Proof.* By duality with what happens in  $TX$  described in (3). Since  $\alpha(V) = 1$  then  $H_0(\alpha) = V(\alpha) = 1$ . Therefore  $\alpha \in \Sigma_{E=1}$ . Also  $\alpha \in E_0^*$  therefore  $\alpha = K_{E=1}$ .  $\square$

#### Remarks:

- In general the **trapped set**  $K$  is defined by:

$$K := \{(x, \xi) \in T^*X, \quad \exists C > 0; \forall t \in \mathbb{R}, |M_t(x, \xi)| \leq C\},$$

i.e.  $K$  contains trajectories which do not escape towards infinity neither in the future nor in the past. We have:

$$K = \bigcup_{E \in \mathbb{R}} K_E = E_0^*.$$

- Notice that for every  $E \in \mathbb{R}$  the trapped set  $K_E$  is a sub-manifold of  $T^*X$  homeomorphic to  $X$  hence compact. This observation is at the origin of the method to prove the existence of discrete resonance spectrum below (Theorem 12). The dynamics of  $M_t$  restricted to  $K_E$  is conjugated to the dynamics of  $\phi_t$  on  $X$  (it is a lift of  $\phi_t$  on  $K_E$ ).
- For the special case of a **contact flow** on  $X$  with a  $C^\infty$  contact 1 form  $\alpha$  then  $K_{E=1} = \alpha$  and therefore  $K_E = E\alpha$  is a  $C^\infty$  section. The restriction of the canonical



two form (11) to this section (seen as a submanifold of  $T^*X$ ) is<sup>7</sup>

$$\omega_{/K_E} = E(\pi^*(d\alpha))_{/K_E}, \quad (31)$$

where  $\pi : T^*X \rightarrow X$  is the bundle projection. We observe that  $K \setminus \{\xi = 0\}$  is a smooth symplectic submanifold of  $T^*X$  (far from being a Lagrangian submanifold of  $T^*X$ ),  $(K_E, E\xi dx)$  is a contact manifold for  $E \neq 0$  isomorphic to  $(X, E\alpha)$ .

### 3.3 The escape function

In this section we construct a smooth function  $G_m$  on the cotangent space  $T^*X$  called the escape function. We will denote  $\frac{\xi}{|\xi|}$  the direction of a cotangent vector  $\xi$  and  $S^*X := (T^*X \setminus \{0\})/\mathbb{R}^+$  the **cosphere bundle** which is the bundle of directions of cotangent vectors  $\xi/|\xi|$ .  $S^*X$  is a compact space. The images of  $E_u^*, E_s^*, E_0^* \subset T^*X$  by the projection  $T^*X \setminus \{0\} \rightarrow S^*X$  are denoted respectively  $\tilde{E}_u^*, \tilde{E}_s^*, \tilde{E}_0^* \subset S^*X$ , see Figure 5(a) page 18.

---

<sup>7</sup>proof: for a contact flow with contact 1-form  $\alpha$  then  $K_E = E\alpha$ , with  $E \in \mathbb{R}$ . The restriction of the canonical symplectic 1-form  $\eta := \xi dx$  is then  $\eta_{/K_E} = (\pi^*(K_E))_{/K_E} = E(\pi^*(\alpha))_{/K_E}$ , therefore  $\omega_{/K_E} = d\eta_{/K_E} = E(\pi^*(d\alpha))_{/K_E}$ .

**Lemma 7.** *Let  $u, n_0, s \in \mathbb{R}$  with  $u < n_0 < s$ . There exists a smooth function  $m(x, \xi) \in C^\infty(T^*X)$  called an “**order function**”, taking values in the interval  $[u, s]$ , and an “**escape function**” on  $T^*X$  defined by:*

$$G_m(x, \xi) := m(x, \xi) \log \sqrt{1 + (f(x, \xi))^2}, \quad (32)$$

where  $f \in C^\infty(T^*X)$  and for  $|\xi| \geq 1$ ,  $f > 0$  is positively homogeneous of degree 1 in  $\xi$ .  $f(x, \xi) = |\xi|$  in a conical neighborhood of  $E_u^*$  and  $E_s^*$ .  $f(x, \xi) = H_0(x, \xi)$  in a conical neighborhood of  $E_0^*$ , such that:

1. For  $|\xi| \geq 1$ ,  $m(x, \xi)$  depends only on the direction  $\frac{\xi}{|\xi|} \in S_x^*X$  and takes the value  $u$  (respect.  $n_0, s$ ) in a small neighborhood of  $E_u^*$  (respect.  $E_0^*, E_s^*$ ). See figure 5(a).
2.  $G_m$  decreases strictly and uniformly along the trajectories of the flow  $M_t$  in the cotangent space, except in a conical vicinity  $\tilde{N}_0$  of the neutral direction  $E_0^*$  and for small  $|\xi|$ : there exists  $R > 0$  such that

$$\forall x \in X, \forall |\xi| \geq R, \frac{\xi}{|\xi|} \notin \tilde{N}_0 \quad \mathbf{X}(G_m)(x, \xi) < -C_m < 0, \quad (33)$$

with

$$C_m := c \min(|u|, s) \quad (34)$$

and  $c > 0$  independent of  $u, n_0, s$ .

3. More generally

$$\forall x \in X, \forall |\xi| \geq R, \quad \mathbf{X}(G_m)(x, \xi) \leq 0. \quad (35)$$

See figure 5(b).

## Remarks

- It is important to notice that we can choose  $m$  such that the value of  $C_m$  is **arbitrarily large** (by making  $s, |u| \rightarrow \infty$ ) and that the neighborhood  $\tilde{N}_0$  is **arbitrarily small**.
- The value of  $n_0$  could be chosen to be  $n_0 = 0$  to simplify. But it is interesting to observe that letting  $n_0, s \rightarrow +\infty$ , the order function  $m(x, \xi)$  can be made arbitrarily large for  $|\xi| \geq 1$ , outside a small vicinity of  $E_u^*$ . We will use this in the proof of Theorem 13 in order to show that the wavefront of the eigen-distributions are included in  $E_u^*$ .
- Inspection of the proof shows that with an adapted norm  $|\xi|$  obtained by averaging,  $c$  can be chosen arbitrarily close to  $\theta$ , defined in (29).

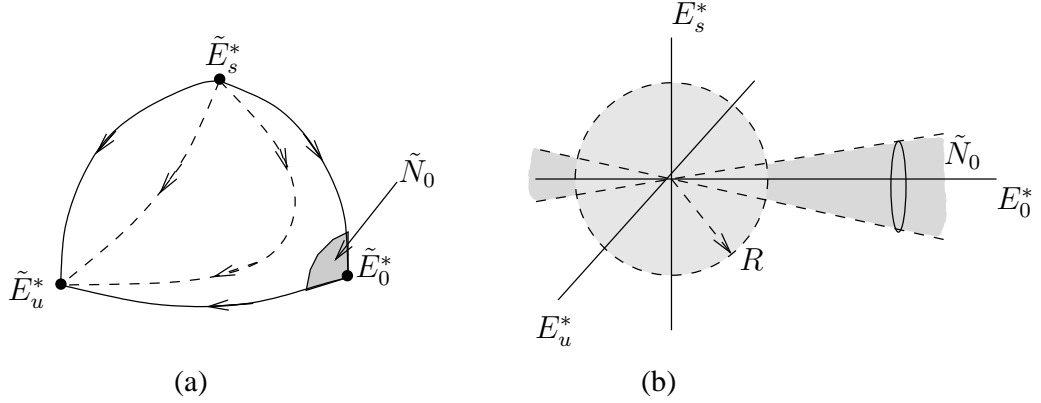


Figure 5: (a) The induced flow  $\widetilde{M}_t$  on the cosphere bundle  $S^*X := (T^*X \setminus \{0\})/\mathbb{R}^+$  which is the bundle of directions of cotangent vectors  $\xi/|\xi|$ . (Here the picture is restricted to a fiber  $S_x^*X$ ).

(b) Picture in the cotangent space  $T_x^*X$  which shows in grey the sets outside of which the escape estimate (33) holds.

- The constancy of  $m$  in the vicinity of the stable/unstable/neutral directions allows us to have a smooth escape function  $G_m$  although the distributions  $E_s^*(x), E_u^*(x), E_0^*(x)$  have only Hölder regularity in general.

### 3.3.1 Proof of Lemma 7

We first define a function  $m(x, \xi)$  called the **order function** following closely [17] Section 3.1 (and [19] p.196).

**The function  $m$ .** The following Lemma is useful for the construction of escape functions. Let  $M$  be a compact manifold and let  $v$  be a smooth vector field on  $M$ . We denote  $\exp(tv) : M \rightarrow M$  the flow at time  $t$  generated by  $v$ . Let  $\Sigma_u, \Sigma_s$  be compact disjoint subsets of  $M$  such that

$$\begin{aligned} \text{dist}(\exp(tv)(\rho), \Sigma_s) &\rightarrow 0, \quad t \rightarrow +\infty \text{ when } \rho \in M \setminus \Sigma_u \\ \text{dist}(\exp(tv)(\rho), \Sigma_u) &\rightarrow 0, \quad t \rightarrow -\infty \text{ when } \rho \in M \setminus \Sigma_s. \end{aligned}$$

**Lemma 8.** *Let  $V_u, V_s \subset M$  be open neighborhoods of  $\Sigma_u$  and  $\Sigma_s$  respectively and let  $\varepsilon > 0$ . Then there exist  $W_u \subset V_u, W_s \subset V_s, m \in C^\infty(M; [0, 1]), \eta > 0$  such that  $v(m) \geq 0$  on  $M$ ,  $v(m) > \eta > 0$  on  $M \setminus (W_u \cup W_s)$ ,  $m(\rho) > 1 - \varepsilon$  for  $\rho \in W_s$  and  $m(\rho) < \varepsilon$  for  $\rho \in W_u$ .*

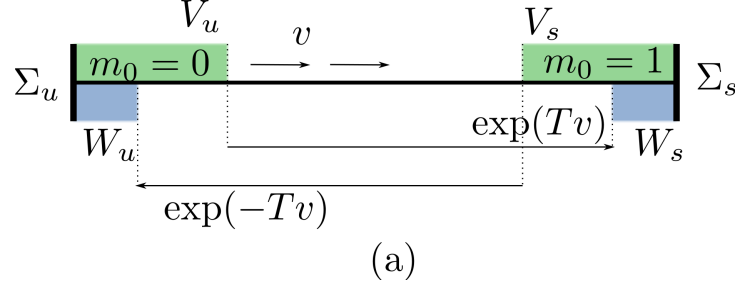


Figure 6: Illustration for the proof of Lemma 8. The horizontal axis is a schematic picture of  $M$  and this shows the construction and properties of the sets  $V_u, V_s$  and  $W_u, W_s$ .

*Proof.* After shrinking  $V_u, V_s$  we may assume that  $V_u \cap V_s = \emptyset$  and

$$t \geq 0 \Rightarrow \exp(tv)(V_s) \subset V_s, \text{ and } t \leq 0 \Rightarrow \exp(tv)(V_u) \subset V_u. \quad (36)$$

Let  $T > 0$  and let  $W_s := M \setminus \exp(Tv)(V_u) = \exp(Tv)(M \setminus V_u)$  and  $W_u := M \setminus \exp(-Tv)(V_s) = \exp(-Tv)(M \setminus V_s)$ . See figure 6.

If  $T$  is large enough one has  $W_u \subset V_u$ ,  $W_s \subset V_s$  and  $W_s \cap W_u = \emptyset$ . Let  $m_0 \in C^\infty(M; [0, 1])$  be equal to 1 on  $V_s$  and equal to 0 on  $V_u$ . Put

$$m = \frac{1}{2T} \int_{-T}^T m_0 \circ \exp(tv) dt. \quad (37)$$

Then

$$v(m)(\rho) = \frac{1}{2T} (m_0(\exp(Tv)(\rho)) - m_0(\exp(-Tv)(\rho))). \quad (38)$$

- Let  $\rho \in M \setminus (W_u \cup W_s)$ . From (38) we see that  $v(m)(\rho) = \frac{1}{2T}(1 - 0) = \frac{1}{2T} > 0$ .

For  $\rho \in M$  let

$$\mathcal{I}(\rho) := \{t \in \mathbb{R}, \exp(tv)(\rho) \in M \setminus (V_u \cup V_s)\}.$$

This is a closed connected interval by (36) and moreover its length is uniformly bounded:

$$\exists \tau > 0, \quad \forall \rho \in M, \quad |\max(\mathcal{I}(\rho)) - \min(\mathcal{I}(\rho))| \leq \tau.$$

In other words,  $\tau$  is an upper bound for the travel time in the domain  $M \setminus (V_u \cup V_s)$ .

To prove the Lemma, we have to consider two more cases:

- Let  $\rho \in W_u$ . If  $t \leq T - \tau$  then  $m_0(\exp(tv)(\rho)) = 0$  and

$$m(\rho) = \frac{1}{2T} \left( \underbrace{\int_{-T}^{T-\tau} m_0(\exp(tv)(\rho)) dt}_{=0} + \underbrace{\int_{T-\tau}^T m_0(\exp(tv)(\rho)) dt}_{\leq 1} \right) \leq \frac{\tau}{2T} < \varepsilon,$$

where the last inequality holds if one chooses  $T$  large enough. One has  $m_0(\exp(-Tv)(\rho)) = 0$  therefore (38) implies that  $v(m)(\rho) \geq 0$ .

- Let  $\rho \in W_s$ . One shows similarly that

$$m(\rho) = \frac{1}{2T} \left( \int_{-T}^{-T+\tau} \underbrace{m_0(\exp(tv)\rho)}_{\geq 0} dt + \int_{-T+\tau}^T \underbrace{m_0(\exp(tv)\rho)}_{=1} dt \right) \geq \frac{2T-\tau}{2T} > 1-\varepsilon,$$

for  $T$  large enough, and  $v(m)(\rho) \geq 0$ .

□

We now apply Lemma 8 to the case when  $M = S^*X$  and  $v$  is the image  $\tilde{\mathbf{X}}$  on  $S^*X$  of our Hamilton field  $\mathbf{X}$ . See figure 7.

- We first take  $\Sigma_u = \Sigma_u^1 = \tilde{E}_s^*$  and let  $\Sigma_s = \Sigma_s^1 \subset M$  be the set of limit points  $\lim_{j \rightarrow +\infty} \exp t_j v(\rho)$ , where  $\rho \in M \setminus \Sigma_u^1$  and  $t_j \rightarrow +\infty$ .  $\Sigma_s^1$  is the union of  $\tilde{E}_0^*$ ,  $\tilde{E}_u^*$  and all trajectories  $\exp(\mathbb{R}v)(\rho)$  where  $\rho$  has the property that  $\exp tv(\rho)$  converges to  $\tilde{E}_0^*$  when  $t \rightarrow -\infty$  and to  $\tilde{E}_u^*$  when  $t \rightarrow +\infty$ . Equivalently,  $\Sigma_s^1$  is the image  $\widetilde{E_u^* \oplus E_0^*}$  in  $S^*X$  of  $E_u^* \oplus E_0^*$ . Applying the Lemma, we get  $m_1 = m \in C^\infty(M; [0, 1])$  such that  $m_1 < \varepsilon$  outside an arbitrarily small neighborhood  $W_u^1$  of  $\Sigma_u^1 = \tilde{E}_s^*$ ,  $m_1 > 1 - \varepsilon$  outside an arbitrarily small neighborhood  $W_s^1$  of  $\Sigma_s^1 = \widetilde{E_u^* \oplus E_0^*}$  and  $\tilde{\mathbf{X}}(m_1) \geq 0$  everywhere with strict inequality  $\tilde{\mathbf{X}}(m_1) > \eta > 0$  outside  $W_s^1 \cup W_u^1$ .
- Similarly, we can find  $m_2 = m \in C^\infty(M; [0, 1])$ , such that  $m_2 < \varepsilon$  outside an arbitrarily small neighborhood  $W_u^2$  of  $\Sigma_u^2 = \widetilde{E_s \oplus E_0^*}$ ,  $m_2 > 1 - \varepsilon$  outside an arbitrarily small neighborhood  $W_s^2$  of  $\Sigma_s^2 = \tilde{E}_u^*$  and  $\tilde{\mathbf{X}}(m_2) \geq 0$  everywhere with strict inequality  $\tilde{\mathbf{X}}(m_2) > \eta > 0$  outside  $W_s^2 \cup W_u^2$ .

Let  $u < n_0 < s$  and put

$$\begin{aligned} \tilde{m} &:= s + (n_0 - s)m_1 + (u - n_0)m_2, \\ \tilde{N}_s &:= W_u^1 \cap W_u^2, \quad \tilde{N}_0 := W_s^1 \cap W_u^2, \quad \tilde{N}_u := W_s^1 \cap W_s^2. \end{aligned}$$

Then

- on  $S^*X \setminus (\tilde{N}_s \cup \tilde{N}_0 \cup \tilde{N}_u) = (S^*X \setminus (W_u^1 \cup W_s^1)) \cup (S^*X \setminus (W_u^2 \cup W_s^2))$  we have  $\tilde{\mathbf{X}}(m_1) > \eta$  or  $\tilde{\mathbf{X}}(m_2) > \eta$  therefore

$$\begin{aligned} \tilde{\mathbf{X}}(\tilde{m}) &= (n_0 - s)\tilde{\mathbf{X}}(m_1) + (u - n_0)\tilde{\mathbf{X}}(m_2) \\ &< -\eta \min(|n_0 - s|, |u - n_0|). \end{aligned} \tag{39}$$

- on  $\tilde{N}_s = W_u^1 \cap W_u^2$  we have  $m_1 < \varepsilon$  and  $m_2 < \varepsilon$  therefore

$$\begin{aligned} \tilde{m} &> s + (n_0 - s)\varepsilon + (u - n_0)\varepsilon \\ &= s(1 - \varepsilon) + u\varepsilon > \frac{s}{2}, \end{aligned} \tag{40}$$

where the last inequality holds if  $\varepsilon$  is chosen small enough.

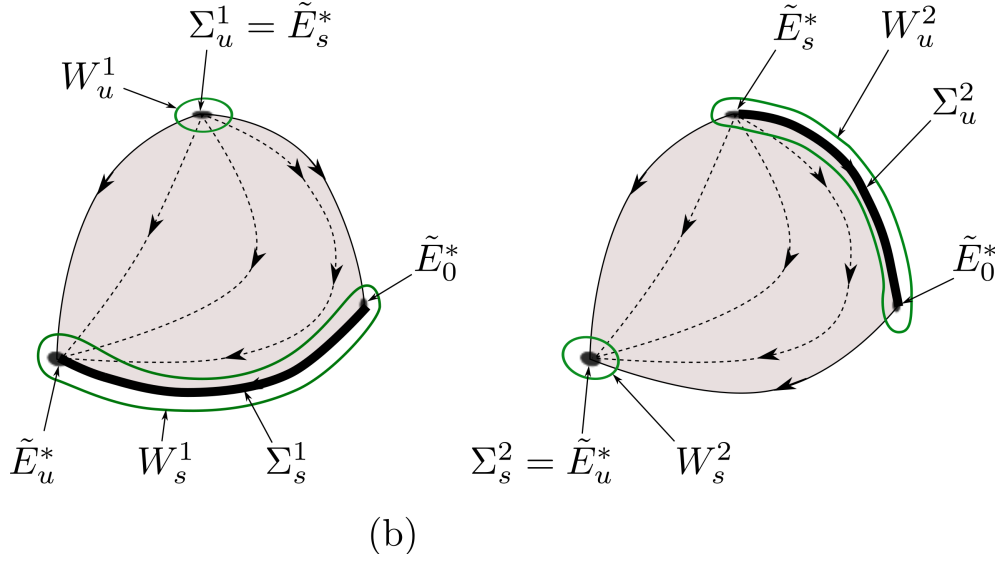


Figure 7: Representation of different sets on  $S^*X$  used in the proof.

- on  $\tilde{N}_u = W_s^1 \cap W_s^2$  we have  $m_1 > 1 - \varepsilon$  and  $m_2 > 1 - \varepsilon$  therefore

$$\begin{aligned} \tilde{m} &< s + (n_0 - s)(1 - \varepsilon) + (u - n_0)(1 - \varepsilon) \\ &= \varepsilon s + u(1 - \varepsilon) < \frac{u}{2}, \end{aligned} \quad (41)$$

where the last inequality holds if  $\varepsilon$  is chosen small enough.

- on  $S^*X$  we have

$$\tilde{\mathbf{X}}(\tilde{m}) = (n_0 - s)\tilde{\mathbf{X}}(m_1) + (u - n_0)\tilde{\mathbf{X}}(m_2) \leq 0. \quad (42)$$

We construct a smooth function  $m$  on  $T^*M$  satisfying

$$\begin{aligned} m(x, \xi) &= \tilde{m}\left(\frac{\xi}{|\xi|}\right), & \text{if } |\xi| \geq 1, \\ &= 0 & \text{if } |\xi| \leq 1/2. \end{aligned}$$

**The symbol  $G_m$ .** Let

$$G_m(x, \xi) := m(x, \xi) \log \sqrt{1 + (f(x, \xi))^2}$$

with  $f \in C^\infty(T^*X)$  such that for  $|\xi| \geq 1$ ,  $f > 0$  is positively homogeneous of degree 1 in  $\xi$ , and

$$\frac{\xi}{|\xi|} \in \tilde{N}_u \cup \tilde{N}_s \Rightarrow f(x, \xi) := |\xi|$$

$$\frac{\xi}{|\xi|} \in \tilde{N}_0 \Rightarrow f(x, \xi) := H_0(x, \xi).$$

The consequences of these choices are:

- Since  $\mathbf{X}(H_0) = 0$  then  $\mathbf{X}\left(\log \sqrt{1 + (f(x, \xi))^2}\right) = 0$  for  $\frac{\xi}{|\xi|} \in \tilde{N}_0$ .
- Since  $E_s^*$  is the stable direction and  $E_u^*$  the unstable one,

$$\exists C > 0, \quad \frac{\xi}{|\xi|} \in \tilde{N}_s \Rightarrow \mathbf{X}(\log \langle \xi \rangle) < -C, \quad \frac{\xi}{|\xi|} \in \tilde{N}_u \Rightarrow \mathbf{X}(\log \langle \xi \rangle) > C. \quad (43)$$

Notice that by averaging, the norm  $|\xi|$  can be chosen such that for  $|\xi|$  large enough,  $C$  is arbitrarily closed to  $\theta$  defined in (29).

- In general  $\left| \mathbf{X}\left(\log \sqrt{1 + f^2}\right) \right|$  is bounded:

$$\exists C_2 > 0, \quad \forall \xi \in T^*X, \quad \left| \mathbf{X}\left(\log \sqrt{1 + (f(x, \xi))^2}\right) \right| < C_2.$$

We will show now the uniform escape estimate Eq.(33) page (17). One has

$$\mathbf{X}(G_m) = \mathbf{X}(m) \log \sqrt{1 + f^2} + m \mathbf{X}\left(\log \sqrt{1 + f^2}\right). \quad (44)$$

We will first consider each term separately assuming  $|\xi| \geq 1$ .

- If  $\tilde{\xi} \in S^*X \setminus (\tilde{N}_s \cup \tilde{N}_u \cup \tilde{N}_0)$  then using (39) and the fact that  $\left| \mathbf{X}\left(\log \sqrt{1 + f^2}\right) \right|$  and  $m$  are bounded, one has for  $|\xi|$  large enough

$$\mathbf{X}(G_m)(x, \xi) < -c \min(s, |u|)$$

with  $c > 0$  independent of  $u, n_0, s$ .

- If  $\tilde{\xi} \in \tilde{N}_u$  then from (43) and (41) there exists  $c > 0$  such that

$$\mathbf{X}(G_m) = \underbrace{\mathbf{X}(m)}_{\leq 0} \underbrace{\log \langle \xi \rangle}_{\geq 0} + \underbrace{m}_{< \frac{s}{2}} \underbrace{\mathbf{X}(\log \langle \xi \rangle)}_{> C} < -c|u| < 0.$$

- If  $\tilde{\xi} \in \tilde{N}_s$  then from (43) and (40) there exists  $c > 0$  such that

$$\mathbf{X}(G_m) = \underbrace{\mathbf{X}(m)}_{\leq 0} \underbrace{\log \langle \xi \rangle}_{\geq 0} + \underbrace{m}_{> \frac{s}{2}} \underbrace{\mathbf{X}(\log \langle \xi \rangle)}_{< -C} < -cs < 0.$$

We have obtained the uniform escape estimate Eq.(33) page 17. Finally for  $\tilde{\xi} \in \tilde{N}_0$ , we have

$$\mathbf{X}(G_m) = \underbrace{\mathbf{X}(m)}_{\leq 0} \underbrace{\log \sqrt{1 + f^2}}_{\geq 0} + \underbrace{m \mathbf{X}\left(\log \sqrt{1 + f^2}\right)}_{=0} \leq 0,$$

and we deduce (35) page 17. We have finished the proof of Lemma 7 page 17.

## 4 Spectrum of resonances

In this Section we give our main results about the spectrum of the generator  $\widehat{H}$ , Eq(11), in specific Sobolev spaces. We first define these Sobolev spaces.

### 4.1 Anisotropic Sobolev spaces

#### 4.1.1 Symbol classes with variable order

The escape function  $G_m$  defined in Lemma 7 has some regularity expressed by the fact that it belongs to some symbol classes  $S^\mu$ . This will allow us to perform some semiclassical calculus. In this section, we describe these symbol classes.

**Lemma 9.** *The order function  $m(x, \xi)$  defined in in Lemma 7 belongs to  $S^0$  ( definition 3 page 10). The escape function  $G_m$  defined in (32) belongs to the symbol class  $S^\mu$  for every  $\mu > 0$ . For short, we will write  $G_m \in S^{+0}$ .*

In the paper [17, Appendix] we have shown that the order function  $m(x, \xi) \in S^0$  can be used to define the class  $S_\rho^{m(x, \xi)}$  of symbols of variable order  $m(x, \xi)$ . We recall the definition:

**Definition 10.** Let  $m(x, \xi) \in S^0$  and  $\frac{1}{2} < \rho \leq 1$ . A function  $p \in C^\infty(T^*X)$  belongs to the **class  $S_\rho^{m(x, \xi)}$  of variable order** if for every trivialization  $(x, \xi) : T^*X|_U \rightarrow \mathbb{R}^{2n}$ , for every compact  $K \subset U$  and all multi-indices  $\alpha, \beta \in \mathbb{N}^n$ , there is a constant  $C_{K, \alpha, \beta}$  such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{K, \alpha, \beta} \langle \xi \rangle^{m(x, \xi) - \rho|\alpha| + (1-\rho)|\beta|} \quad (45)$$

for every  $(x, \xi) \in T^*X|_U$ .

We refer to [17, Section A.2.2] for a precise description of semiclassical theorems related to symbols with variable orders.



**Proposition 11.** *The operator*

$$\widehat{A}_m := \text{Op}(\exp(G_m)) \quad (46)$$

*is a PDO whose symbol belongs to the class  $S_\rho^{m(x,\xi)}$  for every  $\rho < 1$  (we write  $S_{1-0}^{m(x,\xi)}$  for short). Its principal symbol is*

$$A_m(x, \xi) = e^{G_m(x,\xi)} = \langle \xi \rangle^{m(x,\xi)}.$$

*The symbol  $A_m$  can be modified at a subleading order (i.e.  $S_\rho^{m(x,\xi)-(2\rho-1)}$ ) such that the operator becomes formally **self-adjoint** and **invertible** on  $C^\infty(X)$ .*

Remark: one can also show that  $A_m \in S_1^{m(x,\xi)+0}$  but this is less precise than  $A_m \in S_{1-0}^{m(x,\xi)}$ .

*Proof.* We refer to the appendix in the paper [17, Lemma 6]. □

#### 4.1.2 Anisotropic Sobolev spaces

For every order function  $m$  as in Lemma 7, we define the **anisotropic Sobolev space**  $H^m$  to be the space of distributions (included in  $\mathcal{D}'(X)$ ):

$$\boxed{H^m := \widehat{A}_m^{-1}(L^2(X))}. \quad (47)$$

Some basic properties of the space  $H^m$ , such as embedding properties, are given in [17, section 3.2].

The generator  $\widehat{H} = -iV$ , Eq.(11), is defined by duality on the distribution space  $\mathcal{D}'(X)$  and we can therefore consider its restriction to the anisotropic Sobolev space  $H^m$ .

## 4.2 Main results on the spectrum of Ruelle resonances

The following theorem 12 has been obtained in [10, Theorem 1] (with the slight difference that the authors use Banach spaces). In particular we refer to this paper for results and discussions concerning the SRB measure. We provide a new proof below, based on semiclassical analysis in the spirit of the paper [17].

**Theorem 12. “Discrete spectrum”.** Let  $m$  be a function which satisfies the hypothesis of Lemma 7 page 17. The generator  $\widehat{H} = -iV$ , Eq.(11), defines by duality an unbounded operator on the anisotropic Sobolev space  $H^m$ , Eq.(47),

$$\widehat{H} : H^m \rightarrow H^m$$

in the sense of distributions with domain given by

$$\mathcal{D}(\widehat{H}) := \left\{ \varphi \in H^m, \quad \widehat{H}\varphi \in H^m \right\}.$$

It coincides with the closure of  $(-iV) : C^\infty \rightarrow C^\infty$  in the graph norm for operators. For  $z \in \mathbb{C}$  such that  $\Im(z) > -(C_m - C)$  with  $C_m$  defined in (34) and some  $C$  independent of  $m$ , the operator  $(\widehat{H} - z) : \mathcal{D}(\widehat{H}) \cap H^m \rightarrow H^m$  is a Fredholm operator with index 0 depending analytically on  $z$ . Recall that  $C_m$  is arbitrarily large. Consequently the operator  $\widehat{H}$  has a **discrete spectrum** in the domain  $\Im(z) > -(C_m - C)$ , consisting of eigenvalues  $\lambda_i$  of finite algebraic multiplicity. See Figure 8. Moreover,  $\widehat{H}$  has no spectrum in the half plane  $\Im(z) > 0$ .

Concerning Fredholm operators we refer to [12, p.122] or [25, Appendix A p.220]. The proof of Theorem 12 is given page 27.

The next Theorem show that the spectrum is intrinsic and describes the wavefront of the eigenfunctions associated to  $\lambda_i$ . The wavefront of a distribution has been introduced by Hörmander. See for instance [23, p.77] of [48, p.27] for the definition. The wavefront corresponds to the directions in  $T^*X$  where the distribution is not  $C^\infty$ .

**Theorem 13. “The discrete spectrum is intrinsic to the Anosov vector field”.**

More precisely, let  $\tilde{m}, \tilde{f}, \tilde{G}_m = \tilde{m} \log \sqrt{1 + \tilde{f}^2}$  be another set of functions as in Lemma 7 so that Theorem 12 applies and  $\widehat{H} : H^{\tilde{m}} \rightarrow H^{\tilde{m}}$  has discrete spectrum in the set  $\Im(z) > -(\tilde{C}_{\tilde{m}} - \tilde{C})$ . Then in the set  $\Im(z) > -\min((C_m - C), (\tilde{C}_{\tilde{m}} - \tilde{C}))$  the eigenvalues of  $\widehat{H} : H^m \rightarrow H^m$  counted with their multiplicity and their respective eigenspaces coincide with those of  $\widehat{H} : H^{\tilde{m}} \rightarrow H^{\tilde{m}}$ .

The eigenvalues  $\lambda_i$  are called the **Ruelle Resonances** and we denote the set by  $\text{Res}(\widehat{H})$ . The **wavefront** of the associated generalized eigenfunctions is contained in the unstable direction  $E_u^*$ .

The resolvent  $(z - \widehat{H})^{-1}$  viewed as an operator  $C^\infty(X) \rightarrow \mathcal{D}'(X)$  has a meromorphic extension from  $\Im(z) \gg 1$  to  $\mathbb{C}$ . The poles of this extension are the Ruelle resonances.

The proof of Theorem 13 is given page 33.

The following proposition is a very simple observation.

**Proposition 14. "Symmetry".** *The order function  $m$  can be chosen such that  $m(x, -\xi) = m(x, \xi)$ . Then the conjugation operator  $\widehat{C}$  defined in (15) leaves the space  $H^m$  invariant. If  $\widehat{H}\psi = \lambda\psi$ ,  $\psi \in H^m$  then  $\widetilde{\psi} := \widehat{C}\psi \in H^m$  is also an eigenfunction with eigenvalue  $\widetilde{\lambda} = -\overline{\lambda}$ . The spectrum of Ruelle resonances is therefore symmetric with respect to the imaginary axis.*

*Proof.* of Proposition 14. We first have to show that the space  $H^m(X) = \widehat{A}^{-1}(L^2(X))$  is invariant by  $\widehat{C}$ , equivalently that  $L^2(X)$  is invariant by  $\widehat{A}\widehat{C}\widehat{A}^{-1}$ . Notice that  $\widehat{C}$  is an "anti-linear FIO" whose associated transformation is  $C : (x, \xi) \rightarrow (x, -\xi)$ , which is anti-canonical since  $C^*\omega = -\omega$ . The symbol  $A(x, \xi)$  is invariant under the map  $C : (x, \xi) \rightarrow (x, -\xi)$ . One can therefore construct  $\widehat{A}$  such that  $\widehat{C}\widehat{A}\widehat{C} = \widehat{A}$ . Since  $\widehat{C}\widehat{A}\widehat{C} = \widehat{A} \Leftrightarrow \widehat{A}\widehat{C}\widehat{A}^{-1} = \widehat{C}$  and since the space  $L^2(X)$  is invariant under  $\widehat{C}$ , we conclude that  $L^2(X)$  is invariant by  $\widehat{A}\widehat{C}\widehat{A}^{-1}$ . Finally if  $\widehat{H}\psi = \lambda\psi$ ,  $\psi \in H^m$ , let  $\widetilde{\psi} = \widehat{C}\psi \in H^m$ . Then using (16),  $\widehat{H}\widetilde{\psi} = \widehat{H}\widehat{C}\psi = -\widehat{C}\widehat{H}\psi = -\overline{\lambda}\widetilde{\psi}$ .  $\square$

Here is the new result of this paper:

**Theorem 15. "Semiclassical upper bound for the density of resonances".** *For every  $E \in \mathbb{R} \setminus \{0\}$ , every  $\beta > 0$ , in the semiclassical limit  $\alpha \rightarrow +\infty$  we have*

$$\sharp \left\{ \lambda \in \text{Res} \left( \widehat{H} \right), |\Re(\lambda) - E\alpha| \leq \sqrt{\alpha}, \Im(\lambda) > -\beta \right\} \leq o(\alpha^{n-1/2}), \quad (48)$$

with  $n = \dim X$ .

#### Remarks:

- Notice that by a simple scaling in  $\alpha$  we can reduce the values of  $E$  to  $E = \pm 1$  in Theorem 15.
- The case  $E = 0$  is excluded in Theorem 15 because the vicinity of the origin  $\xi = 0$  is excluded in (35). If one were able to construct an escape function such that in addition  $\mathbf{X}(G_m)(x, \xi) \leq C$ ,  $\forall (x, \xi)$ , with some  $C$  independent of  $m$  then  $E = 0$  would not be excluded.

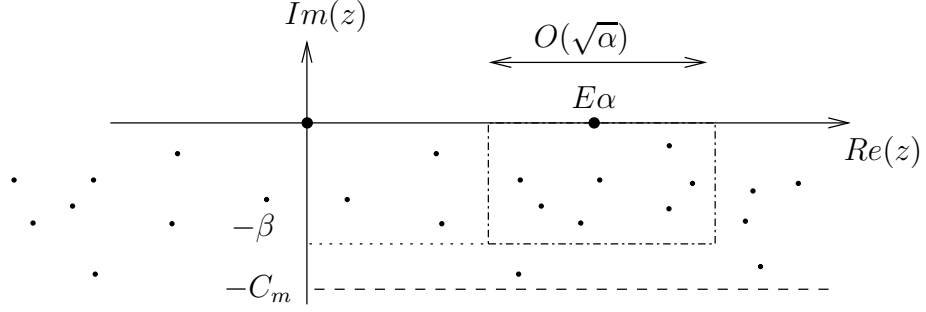


Figure 8: Spectrum of Ruelle resonances of  $\hat{H} = -iV$ . From Theorem 15 the number of eigenvalues in the rectangle is  $o(\alpha^{n-1/2})$  for  $\alpha \rightarrow \infty$ .

- We recall a simple and well known result (which follows from the property that  $\|\widehat{M}_t\|_\infty = 1$ ), that there is no eigenvalue in the upper half plane and no Jordan block on the real axis .
- The upper bound given in (48) results from our method and choice of escape function  $A(x, \xi)$ . In the proof,  $o(\alpha^{n-1/2})$  comes from a symplectic volume  $\mathcal{V}$  in phase space which contains the trapped set  $\Sigma_E$  and which is of order  $\mathcal{V} \simeq \delta \alpha^{-1/2}$ , with  $\delta$  arbitrarily small. Using Weyl inequalities we obtain an upper bound of order  $\alpha^n \mathcal{V} \simeq \delta \alpha^{n-1/2}$  in (48). It is expected that a better choice of the escape function could improve this upper bound. For specific models, e.g. geodesic flows on a surface with constant negative curvature, it is known that the upper bound is  $O(\alpha^{\frac{n}{2}})$  (see [29]). We reasonably expect this in general.
- From the upper bound (48), one can deduce upper bounds in larger spectral domains. For example: for every  $\beta > 0$ , in the semiclassical limit  $\alpha \rightarrow +\infty$  we have

$$\# \left\{ \lambda \in \text{Res}(\hat{H}), \Re(\lambda) \in [-\alpha, \alpha], \Im(\lambda) > -\beta \right\} \leq o(\alpha^n),$$

with  $n = \dim X$ .

### 4.3 Proof of theorem 12 about the discrete spectrum of resonances

Here are the different steps that we will follow in the proof.

1. The operator  $\hat{H}$  on the Sobolev space  $H^m = \hat{A}_m^{-1}(L^2(X))$  is unitarily isomorphic to the operator  $\hat{P} := \hat{A}_m \hat{H} \hat{A}_m^{-1}$  on  $L^2(X)$ . We will show that  $\hat{P}$  is a pseudo-differential operator. We will compute the symbol  $P(x, \xi)$  of  $\hat{P}$  in Lemma 16. The important fact is that the derivative of the escape function appears in the imaginary part of the symbol  $P(x, \xi)$ .

2. For  $\Im(z) \gg 1$ , using the Gårding inequality, we will show that  $(\widehat{P} - z)$  is invertible and therefore that  $\widehat{P}$  has no spectrum in the domain  $\Im(z) \gg 1$ .
3. Using the Gårding inequality again for a modified operator and analytic Fredholm theory we will show that  $(\widehat{P} - z)$  is invertible for  $\Im(z) > -(C_m - C)$  for some constant  $C$  independent of  $m$ , except for a discrete set of points  $z = \lambda_i$  with finite multiplicity.

#### 4.3.1 Conjugation by the escape function and unique closed extension of $\widehat{P}$ on $L^2(X)$

Let us define

$$\boxed{\widehat{P} := \widehat{A}_m \widehat{H} \widehat{A}_m^{-1}} \quad (49)$$

The following commuting diagram shows that the operator  $\widehat{P}$  on  $L^2(X)$  is unitarily equivalent to  $\widehat{H}$  on  $H^m$ .

$$\begin{array}{ccccc} L^2(X) & \xrightarrow{\widehat{P}} & L^2(X) \\ \downarrow \widehat{A}_m^{-1} & \circlearrowleft & \downarrow \widehat{A}_m^{-1} \\ H^m & \xrightarrow{\widehat{H}} & H^m \end{array}$$

The definitions of symbol classes  $S^\mu$  and  $S_\rho^m$  are given in Sections 3.2.1 and 4.1.1. In the following Lemma, the notation  $\mathcal{O}_m(S^{-1+0})$  means that the term is a symbol in  $S^{-1+0}$ . We add the index  $m$  to emphasize that it depends on the escape function  $m$  whereas  $\mathcal{O}(S^0)$  means that the term is a symbol in  $S^0$  which *does not* depend on  $m$ .

**Lemma 16.** *The operator  $\widehat{P}$  defined in (49) is a PDO in  $\text{Op}(S^1)$ . With respect to every given system of coordinates its symbol is equal to*

$$P(x, \xi) = H(x, \xi) + i(\mathbf{X}(G_m))(x, \xi) + \mathcal{O}_m(S^{-1+0}), \quad (50)$$

where  $H(x, \xi)$  is the symbol of  $\widehat{H}$ :

$$H(x, \xi) = V(\xi) + \mathcal{O}(S^0),$$

with principal symbol  $V(\xi) \in S^1$ , see Eq.(22), and  $\mathbf{X}(G_m) \in S^{+0}$ .  $\mathbf{X}$  is the Hamiltonian vector field of  $H$  defined in (24).

*Proof.* The proof consists in making the following two lines precise and rigorous:

$$\begin{aligned}\hat{P} &= \hat{A}\hat{H}\hat{A}^{-1} = \text{Op}(e^{G_m}) \hat{H} (\text{Op}(e^{G_m}))^{-1} \simeq (1 + \text{Op}(G_m) + \dots) \hat{H} (1 - \text{Op}(G_m) + \dots) \\ &= \hat{H} + [\text{Op}(G_m), \hat{H}] + \dots = \text{Op}(H - i\{G_m, H\} + \dots) = \text{Op}(H + i\mathbf{X}(G_m) + \dots).\end{aligned}$$

In order to avoid to work with exponentials of operators, let us define

$$\hat{A}_{m,t} := \text{Op}(e^{tG_m}) = \text{Op}(e^{G_{tm}}) = \hat{A}_{tm}, \quad 0 \leq t \leq 1,$$

and

$$\hat{H}_{m,t} := \hat{A}_{m,t} \hat{H} \hat{A}_{m,t}^{-1}$$

which interpolates between  $\hat{H} = \hat{H}_{m,0}$  and  $\hat{P} = \hat{H}_{m,1}$ . We have seen in Lemma 9 that  $\hat{G}_m \in \text{Op}(S^{+0})$ , in Proposition 11 that  $\hat{A}_{m,t} \in \text{Op}(S^{tm(x,\xi)+0})$ ,  $\hat{A}_{m,t}^{-1} \in \text{Op}(S^{-tm(x,\xi)+0})$  and in Eq.(22) that  $\hat{H} \in \text{Op}(S^1)$ . We deduce that<sup>8</sup>  $\hat{H}_{m,t} \in \text{Op}(S^{1+0})$ . Then

$$\begin{aligned}\frac{d\hat{A}_{m,t}}{dt} &= \text{Op}(G_m e^{tG_m}) = \text{Op}(G_m) \text{Op}(e^{tG_m}) + \mathcal{O}_m(\text{Op}(S^{tm-1+0})) \\ \left(\frac{d\hat{A}_{m,t}}{dt}\right) \hat{A}_{m,t}^{-1} &= -\hat{A}_{m,t} \left(\frac{d\hat{A}_{m,t}^{-1}}{dt}\right) = \text{Op}(G_m + r_{m,t})\end{aligned}$$

with  $r_{m,t} \in S^{-1+0}$  and<sup>9</sup>

$$\begin{aligned}\frac{d}{dt} \hat{H}_{m,t} &= \left(\frac{d\hat{A}_{m,t}}{dt} \hat{A}_{m,t}^{-1}\right) \hat{A}_{m,t} \hat{H} \hat{A}_{m,t}^{-1} + \hat{A}_{m,t} \hat{H} \hat{A}_{m,t}^{-1} \left(\hat{A}_{m,t} \frac{d\hat{A}_{m,t}^{-1}}{dt}\right) \\ &= [\text{Op}(G_m + r_{m,t}), \hat{H}_{m,t}] \in \text{Op}(S^{+0}).\end{aligned}$$

Therefore  $\hat{H}_{m,t} - \hat{H} = \left(\int_0^t \frac{d}{ds} \hat{H}_{m,s} ds\right) \in \text{Op}(S^{+0})$  and

$$\begin{aligned}\frac{d}{dt} \hat{H}_{m,t} &= [\text{Op}(G_m), \hat{H}] + [\text{Op}(r_{m,t}), \hat{H}] + [\text{Op}(G_m + r_{m,t}), \hat{H}_{m,t} - \hat{H}] \\ &= [\text{Op}(G_m), \hat{H}] + \mathcal{O}_m(\text{Op}(S^{-1+0})).\end{aligned}$$

---

<sup>8</sup>From the Theorem of composition of pseudodifferential operators (PDO), see [48, Prop.(3.3) p.11], if  $A \in S_\rho^{m_1}$  and  $B \in S_\rho^{m_2}$  then

$$\text{Op}(A) \text{Op}(B) = \text{Op}(AB) + \mathcal{O}\left(\text{Op}(S_\rho^{m_1+m_2-(2\rho-1)})\right)$$

i.e. the symbol of  $\text{Op}(A) \text{Op}(B)$  is the product  $AB$  and belongs to  $S_\rho^{m_1+m_2}$  modulo terms in  $S_\rho^{m_1+m_2-(2\rho-1)}$ .

<sup>9</sup>From [48, Eq.(3.24)(3.25) p.13], if  $A \in S_\rho^{m_1}$  and  $B \in S_\rho^{m_2}$  then the symbol of  $[\text{Op}(A), \text{Op}(B)]$  is the Poisson bracket  $-i\{A, B\}$  and belongs to  $S_\rho^{m_1+m_2-(2\rho-1)}$  modulo  $S_\rho^{m_1+m_2-2(2\rho-1)}$ . We also recall [47, (10.8) p.43] that  $\{A, B\} = -\mathbf{X}_B(A)$  where  $\mathbf{X}_B$  is the Hamiltonian vector field generated by  $B$ .

We deduce that

$$\widehat{P} = \widehat{H} + \left( \int_0^1 \frac{d}{dt} \widehat{H}_{m,t} dt \right) = \widehat{H} + [\text{Op}(G_m), \widehat{H}] + \mathcal{O}_m(\text{Op}(S^{-1+0})).$$

Since

$$[\text{Op}(G_m), \widehat{H}] = \text{Op}(i(\mathbf{X}(G_m))(x, \xi) + \mathcal{O}_m(S^{-1+0})),$$

we get

$$\widehat{P} = \widehat{H} + \text{Op}(i(\mathbf{X}(G_m))(x, \xi) + \mathcal{O}_m(S^{-1+0})).$$

Finally since  $\widehat{H} = \text{Op}(V(\xi) + \mathcal{O}(S^0))$  with a remainder in  $S^0$  which depends on the quantization (see discussion in Section 3.2.1) but which is independent of the escape function  $m$ , we get (50). Notice that with Weyl quantization, Eq. (20), the term  $\mathcal{O}(S^0)$  is precisely equal to  $\frac{i}{2} \text{div}(V)$ .  $\square$

We have shown that  $\widehat{P}$  is a unbounded PDO of order 1 that we may first equip with the domain  $C^\infty(X)$  which is dense in  $L^2(X)$ . Lemma 29 page 49 shows that  $\widehat{P}$  has a unique closed extension as an unbounded operator  $\widehat{P}$  on  $L^2(X)$ . The adjoint  $\widehat{P}^*$  is also a PDO of order 1 and it is the unique closed extension from  $C^\infty(X)$ .

#### 4.3.2 $\widehat{P}$ has empty spectrum for $\Im(z) \gg 1$ .

Let us write

$$\widehat{P} = \widehat{P}_1 + i\widehat{P}_2$$

with  $\widehat{P}_1 := \frac{1}{2}(\widehat{P} + \widehat{P}^*)$ ,  $\widehat{P}_2 := \frac{i}{2}(\widehat{P}^* - \widehat{P})$  self-adjoint. From (50) and (35), the symbol of the operator  $\widehat{P}_2$  is

$$P_2(x, \xi) = \mathbf{X}(G_m)(x, \xi) + \mathcal{O}(S^0) + \mathcal{O}_m(S^{-1+0}) \quad (51)$$

belongs to  $S^{+0}$  and satisfies

$$\exists C_0, \forall (x, \xi), \quad \Re(P_2(x, \xi)) \leq C_0.$$

From the sharp Gårding inequality (95) page 50 applied here with order  $\mu = 1$  (since  $P_2 \in S^{+0} \subset S^1$ ) we deduce that there exists  $C > 0$  such that  $(\widehat{P}_2 u | u) \leq (C_0 + C) \|u\|^2$  which writes:

$$(\widehat{P}_2 - (C_0 + C)u | u) \leq 0. \quad (52)$$

**Lemma 17.** *From the inequality (52) we deduce that for every  $z \in \mathbb{C}$ ,  $\Im(z) > C + C_0$ , the resolvent  $(\widehat{P} - z)^{-1}$  exists. Therefore  $\widehat{P}$  has **empty spectrum** for  $\Im(z) > C + C_0$ .*

*Proof.* Let  $\varepsilon = \Im(z) - (C_0 + C) > 0$ . Then for  $u \in C^\infty(X)$ ,

$$\Im\left(\left(\widehat{P} - z\right)u|u\right) = \left(\left(\widehat{P}_2 - (C_0 + C)\right)u|u\right) - (\Im(z) - (C_0 + C))\|u\|^2 \leq -\varepsilon\|u\|^2.$$

Using Cauchy-Schwarz inequality,

$$\left\|\left(\widehat{P} - z\right)u\right\|\|u\| \geq \left|\left(\left(\widehat{P} - z\right)u|u\right)\right| \geq \left|\Im\left(\left(\widehat{P} - z\right)u|u\right)\right| \geq \varepsilon\|u\|^2.$$

Hence for  $u \in C^\infty(X)$

$$\left\|\left(\widehat{P} - z\right)u\right\| \geq \varepsilon\|u\|. \quad (53)$$

By density this extends to all  $u \in \mathcal{D}(\widehat{P})$  and it follows that  $\widehat{P} - z$  is injective with closed range  $\mathcal{R}(\widehat{P} - z)$ .

The same argument for the adjoint  $\widehat{P}^* = \widehat{P}_1 - i\widehat{P}_2$  gives

$$\left\|\left(\widehat{P}^* - \bar{z}\right)u\right\| \geq \varepsilon\|u\|, \quad \forall u \in \mathcal{D}(\widehat{P}^*), \quad (54)$$

so  $\widehat{P}^* - \bar{z}$  is also injective. If  $u \in L^2(X)$  is orthogonal to  $\mathcal{R}(\widehat{P} - z)$  then  $u$  belongs to the kernel of  $\widehat{P}^* - \bar{z}$  which is 0. Hence  $\mathcal{R}(\widehat{P} - z) = L^2(X)$  and  $\widehat{P} - z : \mathcal{D}(\widehat{P}) \rightarrow L^2(X)$  is bijective with bounded inverses.  $\square$

#### 4.3.3 The spectrum of $\widehat{P}$ is discrete on $\Im(z) \geq -(C_m - C)$ with some $C \geq 0$ independent of $m$ .

As usual [38, p.113], in order to obtain a discrete spectrum for the operator  $\widehat{P}$ , we need to construct a relatively compact perturbation  $\widehat{\chi}$  of the operator such that  $(\widehat{P} - i\widehat{\chi})$  has no spectrum on  $\Im(z) \geq -(C_m - C)$ .

Let  $\chi_0 : T^*X \rightarrow \mathbb{R}^+$  be a smooth non negative function with  $\chi_0(x, \xi) = C_m > 0$  for  $(x, \xi) \in \widetilde{N}_0$  and  $\chi_0(x, \xi) = 0$  outside a neighborhood of  $\widetilde{N}_0$  where  $R$  and  $\widetilde{N}_0$  are defined in Eq.(33) page 17. See also figure 5 (b). We can assume that  $\chi_0 \in S^0$ .

Let  $\widehat{\chi}_0 := \text{Op}(\chi_0)$ . We can assume that  $\widehat{\chi}_0$  is self-adjoint. From Eq.(33), for every  $(x, \xi) \in T^*X$ ,  $|\xi| \geq R$ ,

$$(\mathbf{X}(G_m)(x, \xi) - \chi_0(x, \xi)) \leq -C_m,$$

hence (51) gives for every  $(x, \xi) \in T^*X$ :

$$P_2(x, \xi) - \chi_0(x, \xi) \leq -C_m + C + \mathcal{O}_m(S^{-1+0}),$$

with some  $C \in \mathbb{R}$  independent of  $m$ , coming from the  $\mathcal{O}(S^0)$  term in (51). Notice that the remainder term  $\mathcal{O}_m(S^{-1+0})$  could be bounded but by a constant which depends on  $m$ .



Since  $P_2 \in S^\mu$  with every order  $0 < \mu < 1$ , the sharp Gårding inequality (95) page 50 implies that for every  $u \in C^\infty(X)$  there exists  $C_\mu > 0$  such that

$$\left( \left( \widehat{P}_2 - \widehat{\chi}_0 + (C_m - C) \right) u | u \right) \leq C_\mu \|u\|_{H^{\frac{\mu-1}{2}}}^2.$$

The right hand side can be written  $C_\mu \|u\|_{H^{\frac{\mu-1}{2}}}^2 = C_\mu \left( \left\langle \widehat{\xi} \right\rangle^{\mu-1} u | u \right) = \Im(i\widehat{\chi}_1 u | u)$  with  $\widehat{\chi}_1 = \text{Op}(\chi_1)$ ,  $\chi_1 = C_\mu \langle \xi \rangle^{\mu-1} \in S^{\mu-1}$  and can be absorbed on the left by defining

$$\chi := \chi_0 + \chi_1, \quad \widehat{\chi} := \text{Op}(\chi).$$

We can assume that  $\widehat{\chi}$  is self-adjoint. We obtain:

$$\left( \left( \widehat{P}_2 - \widehat{\chi} + (C_m - C) \right) u | u \right) \leq 0.$$

As in the proof of Lemma 17 page 30 we obtain that the resolvent  $\left( \widehat{P} - i\widehat{\chi} - z \right)^{-1}$  exists for  $\Im(z) > -(C_m - C)$ . The following lemma is the central observation for the proof of Theorem 12.

**Lemma 18.** *For every  $z \in \mathbb{C}$  such that  $\Im(z) > -(C_m - C)$ , the operator  $\widehat{\chi} \left( \widehat{P} - i\widehat{\chi} - z \right)^{-1}$  is compact.*

*Proof.* On the cone  $\widetilde{N}_0$ , the operator  $\left( \widehat{P} - i\widehat{\chi} - z \right)$  is elliptic of order 1. We can therefore invert it micro-locally on  $\widetilde{N}_0$ , namely construct  $E \in S^{-1}$  and  $R_1, R_2 \in S^0$  such that

$$\left( \widehat{P} - i\widehat{\chi} - z \right) \widehat{E} = 1 + \widehat{R}_1, \quad \widehat{E} \left( \widehat{P} - i\widehat{\chi} - z \right) = 1 + \widehat{R}_2, \quad (55)$$

$$\forall j = 1, 2, \quad \text{WF}(\widehat{R}_j) \cap \widetilde{N}_0 = \emptyset.$$

In particular  $\text{WF}(\chi \widehat{R}_j) = \emptyset$  therefore  $\widehat{\chi} \widehat{R}_2$  is a compact operator. Also  $\widehat{E}$  is a compact operator (since  $E \in S^{-1}$ ). Then from (55), we write:

$$\begin{aligned} \left( \widehat{P} - i\chi - z \right)^{-1} &= \widehat{E} - \widehat{R}_2 \left( \widehat{P} - i\widehat{\chi} - z \right)^{-1}, \\ \widehat{\chi} \left( \widehat{P} - i\widehat{\chi} - z \right)^{-1} &= \underbrace{\widehat{\chi}}_{\text{bounded compact}} \underbrace{\widehat{E}}_{\text{compact}} - \underbrace{\widehat{\chi} \widehat{R}_2}_{\text{compact}} \underbrace{\left( \widehat{P} - i\chi - z \right)^{-1}}_{\text{bounded}}, \end{aligned}$$

and deduce that  $\widehat{\chi} \left( \widehat{P} - i\widehat{\chi} - z \right)^{-1}$  is a compact operator. □

With the following Lemma we finish the proof of Theorem 12.

**Lemma 19.** *From the facts that for every  $z \in \mathbb{C}$ ,  $\Im(z) > -(C_m - C)$ ,  $(\widehat{P} - i\widehat{\chi} - z)$  is invertible,  $\widehat{\chi}(\widehat{P} - i\widehat{\chi} - z)^{-1}$  is compact and that  $(\widehat{P} - z_0)$  is invertible for at least one point  $z_0 \in D$ , we deduce that  $\widehat{P}$  has **discrete spectrum with locally finite multiplicity** on  $\Im(z) > -(C_m - C)$ .*

*Proof.* Write for  $\Im(z) > -(C_m - C)$ :

$$\widehat{P} - z = \left(1 + i\widehat{\chi}(\widehat{P} - i\widehat{\chi} - z)^{-1}\right)(\widehat{P} - i\widehat{\chi} - z).$$

Here  $(\widehat{P} - i\widehat{\chi} - z) : \mathcal{D}(\widehat{P}) \rightarrow L^2(X)$  is bijective with bounded inverse and hence Fredholm of index 0. Similarly  $\left(1 + i\widehat{\chi}(\widehat{P} - i\widehat{\chi} - z)^{-1}\right) : L^2(X) \rightarrow L^2(X)$  is Fredholm of index 0 by Lemma 18. Thus

$$\widehat{P} - z : \mathcal{D}(\widehat{P}) \rightarrow L^2(X), \quad \Im(z) > C_m - C,$$

is a holomorphic family of Fredholm operators (of index 0) invertible for  $\Im(z) \gg 0$ . It then suffices to apply the analytic Fredholm theorem ([37, p.201, case (b)], see also [25, p.220 Appendix A]).  $\square$

#### 4.4 Proof of theorem 13 that the eigenvalues are intrinsic to the Anosov vector field $V$

Let  $m$  and  $G_m$  be as in Lemma 7. Let  $\widehat{m} = f(m)$  where  $f \in C^\infty(\mathbb{R})$ ,  $f(t) \geq \max(0, t)$ ,  $f' \geq 0$ ,  $f(t) = 0$  for  $t \leq u/2$  and  $f(t) = t$  for  $t \geq s/2$ .  $\widehat{H}$  viewed as a closed unbounded operator in  $H^{\widehat{m}}$  has no spectrum in the half plane  $\Im(z) \geq C_1$  for  $C_1 \gg 0$ . The same holds for  $\widehat{H} : L^2 \rightarrow L^2$ . Since  $\widehat{m} \geq 0$  we have  $H^{\widehat{m}} \subset L^2$  so if  $v \in H^{\widehat{m}}$  then  $R_{L^2}(z)v = R_{H^{\widehat{m}}}(z)v$  for  $\Im(z) \geq C_1$  where  $R_{L^2}$  denotes the resolvent of  $\widehat{H} : L^2 \rightarrow L^2$  and similarly for  $R_{H^{\widehat{m}}}$ .

Since  $\widehat{m} \geq m$  we also have  $H^{\widehat{m}} \subset H^m$  and hence  $R_{H^{\widehat{m}}}(z)v = R_{H^m}(z)v$  for  $\Im(z) \geq C_1$ ,  $v \in H^{\widehat{m}}$ . Especially when  $v \in C^\infty$ , we get  $R_{L^2}(z)v = R_{H^m}(z)v$ ,  $\Im(z) \geq C_1$ . Applying Theorem 12, we conclude that  $R_{L^2}(z)$ , viewed as an operator  $C^\infty \rightarrow \mathcal{D}'$  has a meromorphic extension  $R(z)$  from the half plane  $\Im(z) \geq C_1$  to the half plane  $\Im(z) > -(C_m - C)$  which coincide with  $R_{H^m}$  restricted to  $C^\infty$ .

If  $\gamma$  is a simple positively oriented closed curve in the half plane  $\Im(z) > -(C_m - C)$  which avoids the eigenvalues of  $\widehat{H} : H^m \rightarrow H^m$  then the spectral projection, associated to the spectrum of  $\widehat{H}$  inside  $\gamma$ , is given by

$$\pi_\gamma^{H^m} = \frac{1}{2\pi i} \int_\gamma R_{H^m}(z) dz.$$

For  $v \in C^\infty$ , we have

$$\pi_\gamma^{H^m} v = \pi_\gamma v := \frac{1}{2\pi i} \left( \int_\gamma R_{H^m}(z) dz \right) v.$$

Now  $C^\infty$  is dense in  $H^m$  and  $\pi_\gamma^{H^m}$  is of finite rank, hence its range  $\pi_\gamma^{H^m}(H^m)$  is equal to the image  $\pi_\gamma(C^\infty)$  of  $C^\infty$ . The latter space is independent of the choice of  $H^m$ . More precisely if  $\tilde{m}, \tilde{f}$  are as in Theorem 12 and we choose  $\gamma$  as above, now in the half plane  $\Im(z) > -\min((C_m - C), (C_{\tilde{m}} - C))$  and avoiding the spectrum of  $\widehat{H} : H^m \rightarrow H^m$  and  $\widehat{H} : H^{\tilde{m}} \rightarrow H^{\tilde{m}}$ , then the spectral projections  $\pi_\gamma^{H^m}$  and  $\pi_\gamma^{H^{\tilde{m}}}$  have the same range.

Since one can find order functions  $m$  which are arbitrarily large in every direction except  $E_u^*$ , see remark 3.3 page 17, we deduce that the eigen-distributions are smooth in every direction except  $E_u^*$ . The Theorem follows.

## 4.5 Proof of theorem 15 for the upper bound on the density of resonances

The asymptotic regime  $\Re(z) \gg 1$  which is considered in Theorem 15 is a semiclassical regime in the sense that it involves large values of  $H(\xi) = V(\xi) \gg 1$ , hence large values of  $|\xi|$  in the cotangent space  $T^*X$ .

For convenience, we will switch to  $h$ -semiclassical analysis. Let  $0 < h \ll 1$  be a small parameter (we will set  $\alpha = 1/h$  in Theorem 15). In  $h$ -semiclassical analysis the symbol  $\xi$  will be quantized into the operator  $\text{Op}_h(\xi) := hD_x = -ih\partial/\partial x$  whereas for ordinary PDO,  $\xi$  is quantized into  $\text{Op}(\xi) := D_x = -i\partial/\partial x$ . This is simply a rescaling of the cotangent space  $T^*X$  by a factor  $h$ , i.e.

$$h \text{Op}(\xi) = \text{Op}_h(\xi). \quad (56)$$

In this Section we first recall the definition of symbols in  $h$ -semiclassical analysis. In Lemma 22 we derive again the expression of the symbol of  $\widehat{P}$ . In Section 4.5.3 we give the main idea of the proof and the next Sections give the details of this proof.

### 4.5.1 $h$ -semiclassical class of symbols

We first define the symbol classes we will need in  $h$ -semiclassical analysis.

**Definition 20.** The **symbol class**  $(h^{-k}S_\rho^\mu)$  with  $1/2 < \rho \leq 1$ , order  $\mu \in \mathbb{R}$  and  $k \in \mathbb{R}$  consists of  $C^\infty$  functions  $p(x, \xi; h)$  on  $T^*X$ , indexed by  $0 < h \ll 1$  such that in every trivialization  $(x, \xi) : T^*X|_U \rightarrow \mathbb{R}^{2n}$ , for every compact  $K \subset U$

$$\forall \alpha, \beta, \quad |\partial_\xi^\alpha \partial_x^\beta p| \leq C_{K, \alpha, \beta} h^{-k + (\rho-1)(|\alpha| + |\beta|)} \langle \xi \rangle^{\mu - \rho|\alpha| + (1-\rho)|\beta|}. \quad (57)$$

For short we will write  $S_\rho^\mu$  instead of  $(h^{-k}S_\rho^\mu)$  when  $k = 0$ , and write  $S^\mu$  instead of  $S_\rho^\mu$  when  $\rho = 1$ .

For symbols of variable orders we have:

**Definition 21.** Let  $m(x, \xi) \in S^0$ ,  $\frac{1}{2} < \rho \leq 1$  and  $k \in \mathbb{R}$ . A family of functions  $p(x, \xi; h) \in C^\infty(T^*X)$  indexed by  $0 < h \ll 1$ , belongs to the **class**  $(h^{-k}S_\rho^{m(x, \xi)})$  **of variable order** if in every trivialization  $(x, \xi) : T^*X|_U \rightarrow \mathbb{R}^{2n}$ , for every compact  $K \subset U$  and all multi-indices  $\alpha, \beta \in \mathbb{N}^n$ , there is a constant  $C_{K, \alpha, \beta}$  such that

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{K, \alpha, \beta} h^{-k - (1-\rho)(|\alpha| + |\beta|)} \langle \xi \rangle^{m(x, \xi) - \rho|\alpha| + (1-\rho)|\beta|}, \quad (58)$$

for every  $(x, \xi) \in T^*X|_U$ .

#### 4.5.2 The symbol of the conjugated operator

Since the symbol  $H = V(\xi) \bmod S^0$  of  $\widehat{H}$ , given in (22), is *linear* in  $\xi$ , it follows from (56) that

$$h\widehat{H} = h\text{Op}(H) = \text{Op}_h(H).$$

Therefore we also rescale the spectral domain  $z \in \mathbb{C}$  by defining:

$$z_h := hz, \quad \widehat{H}_h := h\widehat{H}. \quad (59)$$

and get

$$\widehat{H}_h = \text{Op}_h(V(\xi) + \mathcal{O}(hS^0)) \in \text{Op}_h(S^1).$$

From now on we will work with these new variables and **we will often drop the indices  $h$  for short.**

We will take again the escape function to be  $G_m(x, \xi) := m(x, \xi) \log \langle \xi \rangle$  as in (32) but with the rescaled variable  $\xi$ , i.e. quantized by

$$\widehat{G}_m := \text{Op}_h(G_m).$$

Since the vector field  $\mathbf{X}$  is linear in the fibers of the bundle  $T^*X$  we get the same estimates (33) and (35). We can now proceed as in Section 4.1:  $G_m$  is a  $h$ -semiclassical symbol,  $G_m \in S^{+0}$  and quantization gives

$$\hat{A}_m := \text{Op}_h(\exp(G_m)),$$

which is a  $h$ -PDO with symbol  $A_m \in S_{1-0}^{m(x,\xi)}$  (the invertibility of  $\hat{A}_m$  is automatic if  $h$  is small enough). Notice that the Sobolev space defined now by  $H^m := \hat{A}_m^{-1}(L^2(X)) = (\text{Op}_h(A_m))^{-1}(L^2(X))$  is identical to (47) as a linear space. However the norm in  $H^m$  depends on  $h$ .

In the following Lemma, we will use again the notation  $\mathcal{O}_m(h^2S^{-1+0})$  which means that the term is a symbol in  $h^2S^{-1+0}$ . We add the index  $m$  to emphasize that it depends on the escape function  $m$  whereas  $\mathcal{O}(hS^0)$  means that the term is a symbol in  $hS^0$  which *does not* depend on  $m$ .

**Lemma 22.** *We define*

$$\hat{P} := \hat{A}_m \hat{H} \hat{A}_m^{-1},$$

*as in (49). Its symbol  $P \in S^1$  is*

$$P(x, \xi) = V(\xi) + ih\mathbf{X}(G_m)(x, \xi) + \mathcal{O}(hS^0) + \mathcal{O}_m(h^2S^{-1+0}). \quad (60)$$

*Proof.* Eq. (60) follows from Lemma 16. But for clarity we re-derive it. Let us define

$$\hat{A}_{m,t} := \text{Op}_h(e^{tG_m}) = \text{Op}_h(e^{G_{tm}}) = \hat{A}_{tm}, \quad 0 \leq t \leq 1$$

and

$$\hat{H}_{m,t} := \hat{A}_{m,t} \hat{H} \hat{A}_{m,t}^{-1}$$

which interpolates between  $\hat{H} = \hat{H}_{m,0}$  and  $\hat{P} = \hat{H}_{m,1}$ . We have<sup>10</sup>  $\hat{A}_{m,t} \in \text{Op}_h(S^{tm+0})$ ,  $\hat{A}_{m,t}^{-1} \in \text{Op}_h(S^{-tm+0})$ ,  $\hat{H} \in \text{Op}_h(S^1)$  therefore  $\hat{H}_{m,t} \in \text{Op}_h(S^{1+0})$ . Then

$$\left( \frac{d}{dt} \hat{A}_{m,t} \right) \hat{A}_{m,t}^{-1} = -\hat{A}_{m,t} \left( \frac{d}{dt} \hat{A}_{m,t}^{-1} \right) = \text{Op}(G_m + r_{m,t})$$

with  $r_{m,t} \in hS^{-1+0}$  and

$$\frac{d}{dt} \hat{H}_{m,t} = \left[ \text{Op}_h(G_m + r_{m,t}), \hat{H}_{m,t} \right].$$

---

<sup>10</sup>The Theorem of composition of  $h$ -semiclassical PDO says that if  $A \in S_\rho^{m_1}$  and  $B \in S_\rho^{m_2}$  then the symbol of  $\text{Op}_h(A) \text{Op}_h(B)$  is the product  $AB$  and belongs to  $S_\rho^{m_1+m_2}$  modulo  $hS_\rho^{m_1+m_2-(2\rho-1)}$ .

We deduce that  $\frac{d}{dt}\widehat{H}_{m,t} \in \text{Op}_h(hS^{+0})$  therefore  $\widehat{H}_{m,t} - \widehat{H} = \left(\int_0^t \frac{d}{ds}\widehat{H}_{m,s}ds\right) \in \text{Op}_h(hS^{+0})$  also and

$$\begin{aligned}\frac{d}{dt}\widehat{H}_{m,t} &= \left[\text{Op}_h(G_m), \widehat{H}\right] + \left[\text{Op}_h(r_{m,t}), \widehat{H}\right] + \left[\text{Op}_h(G_m + r_{m,t}), \widehat{H}_{m,t} - \widehat{H}\right] \\ &= \left[\text{Op}_h(G_m), \widehat{H}\right] + \mathcal{O}_m(\text{Op}_h(h^2S^{-1+0})).\end{aligned}$$

We deduce that

$$\widehat{P} = \widehat{H} + \left(\int_0^1 \frac{d}{dt}\widehat{H}_{m,t}dt\right) = \widehat{H} + \left[\text{Op}_h(G_m), \widehat{H}\right] + \mathcal{O}_m(\text{Op}_h(h^2S^{-1+0})).$$

Since<sup>11</sup>

$$\left[\text{Op}(G_m), \widehat{H}\right] = \text{Op}_h(ih(\mathbf{X}(G_m))(x, \xi) + \mathcal{O}_m(h^2S^{-1+0})),$$

we get

$$\widehat{P} = \widehat{H} + \text{Op}_h(ih(\mathbf{X}(G_m))(x, \xi) + \mathcal{O}_m(h^2S^{-1+0})).$$

Finally, since  $\widehat{H} = \text{Op}_h(V(\xi) + \mathcal{O}(hS^0))$  with a remainder in  $hS^0$  which depends on the quantization (see discussion in Section 3.2.1) but which is independent of the escape function  $m$ , we get (50).  $\square$

We recall the main properties of the different terms in (60). First  $V(\xi) \in S^1$  is real. In each fiber  $T_x^*X$ ,  $V(\xi)$  is linear in  $\xi$  and for every  $E \in \mathbb{R}$  the characteristic set  $\Sigma_E := \{(x, \xi), V(\xi) - E = 0\}$  is the energy shell defined in (28) and transverse to  $E_0^*$ .

The second term  $ih\mathbf{X}(G_m) \in hS^{+0}$  is purely imaginary and

$$\mathbf{X}(G_m)(x, \xi) \text{ is } \begin{cases} \leq 0 & \text{for } |\xi| \geq R \\ \leq \mathcal{O}_m(1) & \text{for } |\xi| < R \\ \leq -C_m, & C_m > 0, \text{ for } (x, \xi) \notin (D_R \cup \widetilde{N}_0) \end{cases} \quad (61)$$

where  $D_R = \{\xi, |\xi| \leq R\}$  and  $\widetilde{N}_0$  is the cone defined in Lemma 7. With a convenient choice of the order function  $m(x, \xi)$  we have independently:

$$\begin{cases} \widetilde{N}_0 & \text{with arbitrarily small aperture} \\ R & \text{arbitrarily small} \\ C_m > 0 & \text{arbitrarily large} \end{cases} \quad (62)$$

---

<sup>11</sup>If  $A \in S_\rho^{m_1}$  and  $B \in S_\rho^{m_2}$  then the symbol of  $[\text{Op}_h(A), \text{Op}_h(B)]$  is the Poisson bracket  $-ih\{A, B\} = ih\mathbf{X}_B(A)$  and belongs to  $hS_\rho^{m_1+m_2-(2\rho-1)}$  modulo  $h^2S_\rho^{m_1+m_2-2(2\rho-1)}$ . Here  $\mathbf{X}_B$  is the Hamiltonian vector field generated by  $B$ .

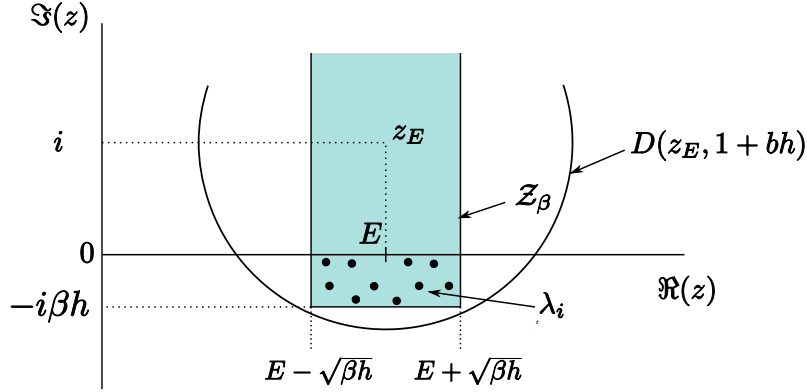


Figure 9: The objective is to bound from above the number of eigenvalues  $\lambda_i$  in the domain  $\mathcal{Z}_\beta$ . For that purpose, we will bound the number of resonances in the disk of radius  $1 + bh$  and center  $z_E = E + i$ .

#### 4.5.3 Main idea of the proof

Before giving the details of the proof we give here the main arguments that we will use in order to prove (48).

Let us consider the following complex valued function  $\tilde{p}(x, \xi) \in C^\infty(T^*X)$  made from the first two leading terms of the symbol (60):

$$\tilde{p}(x, \xi) := V(\xi) + ih\mathbf{X}(G_m). \quad (63)$$

Let  $E \in \mathbb{R} \setminus \{0\}$  and  $h \ll 1$ . We define the spectral domain  $\mathcal{Z} \subset \mathbb{C}$ :

$$\mathcal{Z} := \left\{ z \in \mathbb{C}, \quad |\Re(z) - E| \leq \sqrt{C_m h}, \quad \Im(z) \geq -C_m h \right\}.$$

See Figure 9. Let

$$\mathcal{V}_\mathcal{Z} := \{(x, \xi) \in T^*X, \quad \tilde{p}(x, \xi) \in \mathcal{Z}\}. \quad (64)$$

We have from (61)

$$(x, \xi) \in \mathcal{V}_\mathcal{Z} \Leftrightarrow \begin{cases} |V(\xi) - E|^2 \leq C_m h \\ h\mathbf{X}(G_m)(x, \xi) \geq -C_m h \end{cases} \Rightarrow \begin{cases} (x, \xi) \in \Sigma_{E \pm \sqrt{C_m h}} \\ (x, \xi) \in (D_R \cup \tilde{N}_0) \end{cases}, \quad (65)$$

where  $\Sigma_{E \pm \sqrt{C_m h}} := \left( \bigcup_{|E' - E| \leq \sqrt{C_m h}} \Sigma_{E'} \right)$  is a union of energy shells (28). We deduce that the symplectic volume of  $\mathcal{V}_\mathcal{Z}$  is

$$\text{Vol}(\mathcal{V}_\mathcal{Z}) \leq C \text{Vol}(X) \text{Vol}(\tilde{N}_0) \sqrt{h}, \quad (66)$$

with some constant  $C > 0$ . See Figure 10.

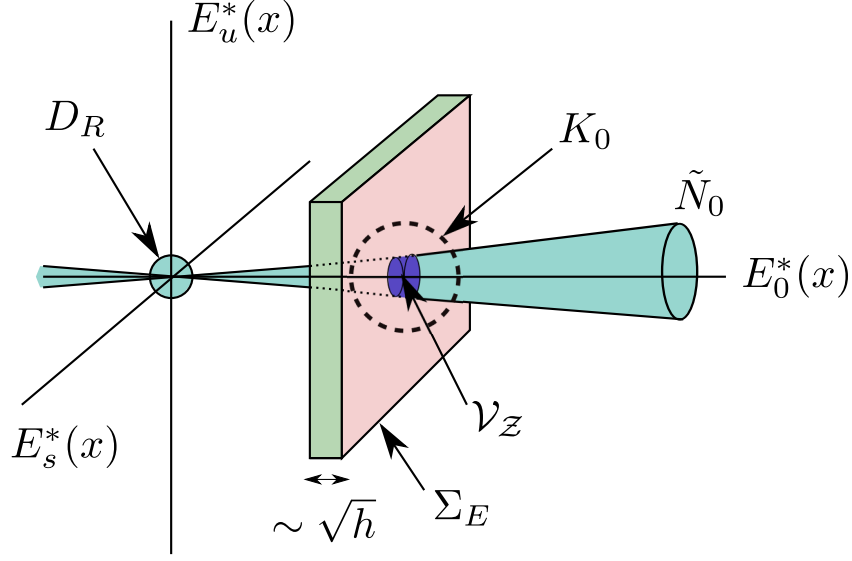


Figure 10: Picture in  $T_x^*X$  with  $x \in X$ , of the volume  $\mathcal{V}_Z$  which supports micro-locally the eigenvalues  $\lambda_i \in \mathcal{Z}$  of figure 9.

Using the “max-min formula” and “Weyl inequalities” we will obtain an upper bound for the number of eigenvalues (in a smaller domain  $\mathcal{Z}_\beta \subset \mathcal{Z}$ ) in terms of this upper bound:

$$\#\{\lambda_i \in \mathcal{Z}_\beta\} \leq \frac{C_m \text{Vol}(\mathcal{V}_Z)}{h^n} = C_m C \text{Vol}(X) \text{Vol}(\tilde{N}_0) h^{1/2-n}, \quad \beta < \frac{1}{4}C_m,$$

with

$$\mathcal{Z}_\beta := \left\{ \lambda \in \mathbb{C}, \quad |\Re(\lambda) - E| \leq \sqrt{\beta h}, \quad \Im(\lambda) \geq -\beta h \right\}. \quad (67)$$

Using  $C_m$  arbitrarily large and that  $\text{Vol}(\tilde{N}_0)$  is independently arbitrarily small, from Eq.(62), we deduce that

$$\#\{\lambda_i \in \mathcal{Z}_\beta\} \leq o(h^{1/2-n}),$$

which is precisely (48) with  $\alpha = 1/h$ .

The proof below follows these ideas but is not so simple because  $P(x, \xi)$  in Eq.(60) is a symbol and not simply a function (symbols belongs to a non commutative algebra of star product) and because the term  $h\mathbf{X}(G_m)$  is subprincipal. We will have to decompose the phase space  $T^*X$  in different parts in order to separate the different contributions as in (65). Another technical difficulty is that the width of the volume  $\mathcal{V}_Z$  is of order  $\sqrt{h}$ . We will use FBI quantization which is convenient for a sharp control on phase space at the scale  $\sqrt{h}$ .

#### 4.5.4 Proof of Theorem 15

We present in reverse order the main steps we will follow in the proof.



### Steps of the proof:

- Our purpose is to bound the cardinal of the spectrum  $\sigma(\hat{P})$  of the operator  $\hat{P}$  in the rectangular domain  $\mathcal{Z}_\beta$  given by (67). But as suggested by figure 9 and confirmed by Lemma 23 below, it suffices to bound the number of eigenvalues of  $\hat{P}$  in the disk

$$D(z_E, 1 + bh) := \{z \in \mathbb{C}, \quad |z - z_E| \leq (1 + bh)\}, \quad b > 0,$$

with radius  $(1 + bh)$  and center:

$$z_E := E + i \in \mathbb{C}.$$

**Lemma 23.** *If  $b > 2\beta$  and  $h$  small enough then*

$$\left(\sigma(\hat{P}) \cap \mathcal{Z}_\beta\right) \subset D(z_E, 1 + bh).$$

*Proof.* We know from a remark after Theorem 12 that  $z \in \sigma(\hat{P}) \Rightarrow \Im(z) \leq 0$ . Also, Pythagora's Theorem in the corner of  $\mathcal{Z}$  gives the condition  $(1 + bh)^2 > (1 + \beta h)^2 + (\sqrt{\beta h})^2$  which is fulfilled if  $b > 2\beta$  and  $h$  small enough.  $\square$

- In order to bound the number of eigenvalues of  $\hat{P}$  in the disk  $D(z_E, 1 + bh)$ , we will use Weyl inequalities in Corollary 28 page 47 and a bound for the number of small singular values of the operator  $(\hat{P} - z_E)$  (i.e. eigenvalues of  $(\hat{P} - z_E)^* (\hat{P} - z_E)$ ) obtained in Lemma 27.
- In order to get this bound on singular values, we will bound from below the expressions  $\left\| (\hat{P} - z_E) u \right\|^2 = \left( (\hat{P} - z_E)^* (\hat{P} - z_E) u | u \right)$ . From symbolic calculus (see footnote 10 page 36) we can compute the symbol of this operator and get:

$$\begin{aligned} (\hat{P} - z_E)^* (\hat{P} - z_E) &= \text{Op}(|V(\xi) - E|^2 + |1 - h\mathbf{X}(G_m)|^2 + \mathcal{O}(hS^1) + \mathcal{O}_m(h^2S^{+0})) \\ &= \text{Op}(|V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m) + \mathcal{O}(hS^1) + \mathcal{O}_m(h^2S^{+0})). \end{aligned} \tag{68}$$

However it is not possible to deduce directly estimates from this symbol because for large  $|\xi|$  the remainders  $\mathcal{O}(hS^1)$  and  $\mathcal{O}_m(h^2S^{+0})$  may dominate the important term  $2h\mathbf{X}(G_m) \in hS^{+0}$ . Therefore we first have to perform a partition of unity on phase space.

**Partition of unity on phase space:** Let  $K_0 \subset T^*X$  be a compact subset (independent of  $h$ ) such that  $\mathcal{V}_Z \subset K_0$  with  $\mathcal{V}_Z$  defined in (64). See figure 10. Lemma 31 page 50 associates a “quadratic partition of unity of PDO” to the compact set  $K_0$ :

$$\boxed{\widehat{\chi}_0^2 + \widehat{\chi}_1^2 = 1 + \text{Op}(h^\infty S^{-\infty})} \quad (69)$$

with self-adjoint operators  $\widehat{\chi}_0, \widehat{\chi}_1$  with symbols  $\chi_0 \in S^{-\infty}$ ,  $\chi_1 \in S^0$ . On the compact set  $K_0$ ,  $\chi_0 = 1 + \mathcal{O}(h^\infty)$ ,  $\chi_1 = \mathcal{O}(h^\infty)$ .

Then from Lemma 32 page 51 called “IMS localization formula” we have: for every  $u \in L^2(X)$ ,

$$\boxed{\left\| \left( \widehat{P} - z_E \right) u \right\|^2 = \left\| \left( \widehat{P} - z_E \right) \widehat{\chi}_0 u \right\|^2 + \left\| \left( \widehat{P} - z_E \right) \widehat{\chi}_1 u \right\|^2 + \mathcal{O}(h^2) \|u\|^2.} \quad (70)$$

We will now study the different terms of (70) separately.

**Informal remark:** In order to show that the Lemma 24 below is expected, let us give an informal remark (non necessary for the proof). Using the function  $\widetilde{p}(x, \xi) := V(\xi) + ih\mathbf{X}(G_m)$ , as in (63), which is the dominant term of the symbol  $P(x, \xi)$ , we write:

$$|\widetilde{p}(x, \xi) - z_E|^2 = |V(\xi) - E|^2 + |1 - h\mathbf{X}(G_m)|^2 \quad (71)$$

$$= |V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m) + \mathcal{O}(h^2 S^{+0}). \quad (72)$$

If  $(x, \xi) \notin K_0$  there are two cases, according to (61):

1. Either  $\mathbf{X}(G_m)(x, \xi) \leq -C_m$ , therefore:

$$|\widetilde{p}(x, \xi) - z_E|^2 \geq 1 + 2hC_m.$$

2. or  $|V(\xi) - E|^2 \geq C_0 > 0$  and  $\mathbf{X}(G_m) \leq \mathcal{O}(1)$  from (61). Therefore

$$|\widetilde{p}(x, \xi) - z_E|^2 \geq 1 + C_0 + \mathcal{O}(h).$$

In both cases we have

$$(x, \xi) \notin K_0 \Rightarrow |\widetilde{p}(x, \xi) - z_E|^2 \geq 1 + 2hC_m. \quad (73)$$

Since  $\chi_1$  is negligible on  $K_0$ , the following Lemma 24 is not surprising in the light of property (73). It gives a lower bound for the second term in the right side of (70).

**Lemma 24.** For every  $u \in L^2(X)$ ,

$$\left\| \left( \widehat{P} - z_E \right) \widehat{\chi}_1 u \right\|^2 \geq (1 + 2h(C_m - C)) \|\widehat{\chi}_1 u\|^2 - \mathcal{O}(h^\infty) \|u\|^2. \quad (74)$$

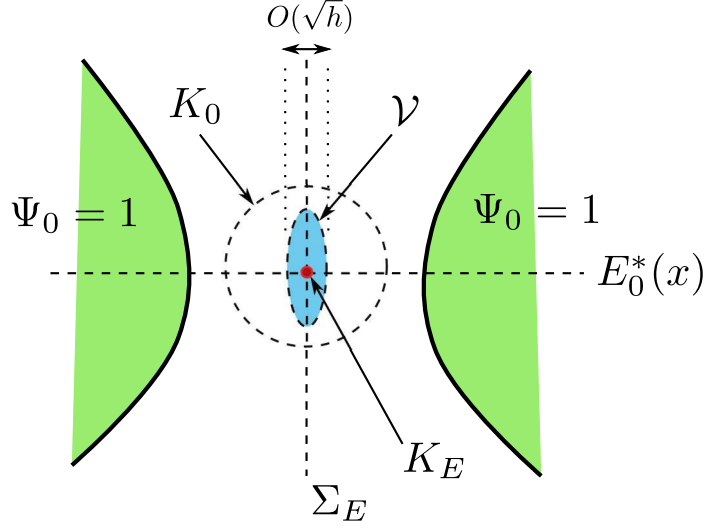


Figure 11: Picture in  $T_x^*X$  with  $x \in X$ , which shows the partition of unity of phase space used in the proof of Lemma 24. The support of  $\chi_1$  is outside the set  $K_0$ .

*Proof.* In order to prove (74) we have to consider a partition of unity in order to take into account two contributions as in the discussion after (71). Let  $\Psi_0 \in S^0$  which has its support inside the region where  $\chi_1 = 1$  and we set  $\Psi_0 = 1$  away from a conical neighborhood of the energy shell  $\Sigma_E$ , Eq.(28), which is the characteristic set  $V(\xi) - E = 0$ . See figure 11.

Since  $(V(\xi) - E)$  is the principal symbol of  $(P(x, \xi) - E) \in S^1$  and is non vanishing on the support of  $\Psi_0$ , there exists  $\widehat{Q} \in \text{Op}(S^{-1})$  such that

$$\widehat{Q}(\widehat{P} - E) = \widehat{\Psi}_0 + \widehat{R}, \quad \widehat{R} \in \text{Op}(h^\infty S^{-\infty}).$$

Since  $\widehat{Q}$  is continuous in  $L^2(X)$ , there exists  $C_0 > 0$  such that for every  $v \in L^2(X)$ ,  $\|v\|^2 \geq \frac{1}{C_0} \|\widehat{Q}v\|^2$ , hence for every  $u \in L^2(X)$

$$\begin{aligned} \left\| (\widehat{P} - E) \widehat{\chi}_1 u \right\|^2 &\geq \frac{1}{C_0} \left\| \widehat{Q}(\widehat{P} - E) \widehat{\chi}_1 u \right\|^2 = \frac{1}{C_0} \left\| (\widehat{\Psi}_0 + \widehat{R}) \widehat{\chi}_1 u \right\|^2 \\ &\geq \frac{1}{2C_0} \left\| \widehat{\Psi}_0 \widehat{\chi}_1 u \right\|^2 - \mathcal{O}(h^\infty) \|u\|^2. \end{aligned} \quad (75)$$

Writing  $\widehat{P} = \widehat{P}_1 + i\widehat{P}_2$  with  $\widehat{P}_i$  self-adjoint, we have

$$\begin{aligned} \left\| (\widehat{P} - z_E) \widehat{\chi}_1 u \right\|^2 &= \left( \left( (\widehat{P} - E)^* + i \right) \left( (\widehat{P} - E) - i \right) \widehat{\chi}_1 u \mid \widehat{\chi}_1 u \right) \\ &= \left\| (\widehat{P} - E) \widehat{\chi}_1 u \right\|^2 + \|\widehat{\chi}_1 u\|^2 - \left( 2\widehat{P}_2 \widehat{\chi}_1 u \mid \widehat{\chi}_1 u \right). \end{aligned} \quad (76)$$

Using (75) in (76) we get for every  $a > 0$ :

$$\begin{aligned}
\left\| \left( \hat{P} - z_E \right) \hat{\chi}_1 u \right\|^2 &= (1 - ah) \left\| \left( \hat{P} - E \right) \hat{\chi}_1 u \right\|^2 + ah \left\| \left( \hat{P} - E \right) \hat{\chi}_1 u \right\|^2 + \|\hat{\chi}_1 u\|^2 \quad (77) \\
&\quad - \left( 2\hat{P}_2 \hat{\chi}_1 u | \hat{\chi}_1 u \right) \\
&\geq (1 - ah) \left\| \left( \hat{P} - E \right) \hat{\chi}_1 u \right\|^2 + \|\hat{\chi}_1 u\|^2 - \left( 2\hat{P}_2 \hat{\chi}_1 u | \hat{\chi}_1 u \right) \\
&\quad + \frac{ah}{2C_0} \left\| \hat{\Psi}_0 \hat{\chi}_1 u \right\|^2 - \mathcal{O}(h^\infty) \|u\|^2 \\
&= (1 - ah) \left\| \left( \hat{P} - E \right) \hat{\chi}_1 u \right\|^2 + \|\hat{\chi}_1 u\|^2 \\
&\quad + \left( \left( -2\hat{P}_2 + \frac{ah}{2C_0} \hat{\Psi}_0^* \hat{\Psi}_0 \right) \hat{\chi}_1 u | \hat{\chi}_1 u \right) - \mathcal{O}(h^\infty) \|u\|^2.
\end{aligned}$$

Recall from (60) that

$$\hat{P}_2 = \text{Op} \left( h\mathbf{X} (G_m) + \mathcal{O}(hS^0) \right) \in \text{Op} (hS^{+0}).$$

Therefore

$$\left( -2\hat{P}_2 + \frac{ah}{2C_0} \hat{\Psi}_0^* \hat{\Psi}_0 \right) \in \text{Op} (hS^{+0}).$$

Assume  $a \geq 4C_0(C_m - C)$ . Then from (61) and the hypothesis on  $\Psi_0$ , for every  $(x, \xi) \in \text{supp}(\chi_1)$  we have

$$\left( -2P_2 + \frac{ah}{2C_0} \Psi_0^* \Psi_0 \right) (x, \xi) \geq \min \left( 2h(C_m - C), \frac{ah}{2C_0} \right) \geq 2h(C_m - C).$$

We can add a symbol  $\Psi_1 \in S^0$  positive, which vanishes on  $\text{supp}(\chi_1)$  so that  $\hat{\Psi}_1 \hat{\chi}_1 \in \text{Op}(h^\infty S^{-\infty})$  and such that for *every*  $(x, \xi) \in T^*X$  we have

$$\left( -2P_2 + \frac{ah}{2C_0} \Psi_0^* \Psi_0 + \Psi_1 \right) (x, \xi) \geq \min \left( 2h(C_m - C), \frac{ah}{2C_0} \right) \geq 2h(C_m - C).$$

The semiclassical sharp Gårding inequality implies that:

$$\begin{aligned}
\forall u \in L^2(X), \quad &\left( \left( -2\hat{P}_2 + \frac{ah}{2C_0} \hat{\Psi}_0^* \hat{\Psi}_0 \right) \hat{\chi}_1 u | \hat{\chi}_1 u \right) \geq (2h(C_m - C) - \mathcal{O}(h^2)) \|\hat{\chi}_1 u\|^2 \\
&\quad - \mathcal{O}(h^\infty) \|u\|^2,
\end{aligned}$$

where the remainder term  $\mathcal{O}(h^\infty) \|u\|^2$  comes from  $\left( \hat{\Psi}_1 \hat{\chi}_1 u | \hat{\chi}_1 u \right)$ . With (77) we get:

$$\begin{aligned}
\left\| \left( \hat{P} - z_E \right) \hat{\chi}_1 u \right\|^2 &\geq (1 - ah) \left\| \left( \hat{P} - E \right) \hat{\chi}_1 u \right\|^2 + \|\hat{\chi}_1 u\|^2 \\
&\quad + (2h(C_m - C) - \mathcal{O}(h^2)) \|\hat{\chi}_1 u\|^2 - \mathcal{O}(h^\infty) \|u\|^2 \\
&\geq (1 + 2h(C_m - C) - \mathcal{O}(h^2)) \|\hat{\chi}_1 u\|^2 - \mathcal{O}(h^\infty) \|u\|^2.
\end{aligned}$$

The term  $\mathcal{O}(h^2) \|\hat{\chi}_1 u\|^2$  can be absorbed in the constant  $C$ . □

**Lemma 25.** *There exists a family of trace class operators  $\widehat{B}_h$  (depending on  $h$ ) such that*

$$\left\| \widehat{B}_h \right\|_{\text{Tr}} \leq \mathcal{O}(1) C_m \text{Vol}(\widetilde{N}_0) h^{1/2-n}, \quad \widehat{B}_h \geq 0, \quad (78)$$

(where the constant  $\mathcal{O}(1)$  does not depend on the escape function  $m$ ) and for every  $u \in L^2(X)$ ,

$$\left\| \left( \widehat{P} - z_E \right) \widehat{\chi}_0 u \right\|^2 + \left( h \widehat{B}_h u | u \right) \geq (1 + 2h(C_m - \mathcal{O}(1))) \|\widehat{\chi}_0 u\|^2 - \mathcal{O}(h^\infty) \|u\|^2, \quad (79)$$

and

$$\left\| \left( \widehat{P} - z_E \right) \widehat{\chi}_0 u \right\|^2 \geq (1 - \mathcal{O}(h)) \|\widehat{\chi}_0 u\|^2 - \mathcal{O}(h^\infty) \|u\|^2, \quad (80)$$

where the term  $\mathcal{O}(h)$  does not depend on  $m$ .

**Remarks** Lemma 25 concerns the first term of the right hand side of (70). In order to obtain (79), which is similar to (74), it has been necessary to add a new term which involves a trace class operator  $\widehat{B}_h$ . Its role is to “hide” the domain  $\mathcal{V}_Z$  (65). Eq.(80) shows that without this term the lower bound is smaller.

*Proof.* The construction is based on ideas around Anti-Wick quantization, Berezin quantization, FBI transforms, Bargmann-Segal transforms, Gabor frames and Toeplitz operators, see e.g. [26]. We review some definitions in Appendix A.4 page 52. We will use the following two properties for an operator obtained by Toeplitz quantization of a symbol  $A(x, \xi; h)$ . Let

$$\widehat{A} := \text{Op}_T(A) := \int A(x, \xi; h) \widehat{\pi}_\alpha d\alpha.$$

Gårding’s inequality writes

$$A(x, \xi) \geq 0 \Rightarrow \left( \widehat{A} u | u \right) \geq 0 + \mathcal{O}(h^\infty) \|u\|^2, \quad (81)$$

also

$$\text{Tr}(\widehat{A}) = \frac{\mathcal{O}(1)}{h^n} \int A(x, \xi) dx d\xi + \mathcal{O}(h^\infty), \quad (82)$$

and

$$(\forall (x, \xi), \quad A(x, \xi) \geq 0) \Rightarrow \left\| \widehat{A} \right\|_{\text{Tr}} = \text{Tr}(\widehat{A}) + \mathcal{O}(h^\infty). \quad (83)$$

From (68) we have

$$\widehat{\chi}_0 \left( \widehat{P} - z_E \right)^* \left( \widehat{P} - z_E \right) \widehat{\chi}_0 = \widehat{\chi}_0 \widehat{S} \widehat{\chi}_0 + \widehat{R}, \quad (84)$$

with  $\widehat{R} \in \text{Op}(h^\infty S^{-\infty})$  and

$$\widehat{S} = \text{Op}_T(S),$$

with the Toeplitz symbol

$$S(x, \xi; h) = |V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m)(x, \xi) + \mathcal{O}(hS^{-\infty}) + \mathcal{O}_m(h^2S^{-\infty})$$

(the remainders are in  $S^{-\infty}$  since  $\chi_0$  has compact support in (84). Since  $\mathbf{X}(G_m) \leq 0$  from (61), we deduce (80) using Gårding's inequality (81).

In order to improve this lower bound and get (79), let  $B_h \in C_0^\infty(T^*X)$  such that  $\forall (x, \xi), B_h(x, \xi) \geq 0$  and

$$(x, \xi) \in \mathcal{V}_Z \Rightarrow B_h(x, \xi) \geq 2C_m. \quad (85)$$

Notice that from (66)  $B_h$  can be chosen such that

$$\int_{T^*X} B_h(x, \xi) dx d\xi \leq \mathcal{O}(1) C_m \text{Vol}(\mathcal{V}_Z) = \mathcal{O}(1) C_m \text{Vol}(X) \text{Vol}(\widetilde{N}_0) \sqrt{h}. \quad (86)$$

From (83), (82) and (86) we deduce (78). Recall that from (65) we have

$$(x, \xi) \notin \mathcal{V}_Z \Rightarrow |V(\xi) - E|^2 \geq hC_m \text{ or } -h\mathbf{X}(G_m) \geq hC_m.$$

Therefore in view of (85) for every  $(x, \xi) \in T^*X$  we have

$$\begin{aligned} S(x, \xi; h) + hB_h(x, \xi) &= |V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m) + hB_h(x, \xi) \\ &\quad + \mathcal{O}(hS^{-\infty}) + \mathcal{O}_m(h^2S^{-\infty}) \\ &\geq 1 + 2hC_m + \mathcal{O}(hS^{-\infty}) + \mathcal{O}_m(h^2S^{-\infty}). \end{aligned}$$

Let  $\widehat{B}_h := \text{Op}_T(B_h)$ . After multiplying both sides by  $\widehat{\chi}_0$ , using  $\widehat{\chi}_0 \widehat{B}_h \widehat{\chi}_0 = \widehat{B}_h + \text{Op}(h^\infty S^{-\infty})$  and Gårding's inequality we deduce that

$$\forall u \in L^2(X), \quad \left( \widehat{\chi}_0 \widehat{S} \widehat{\chi}_0 u | u \right) + \left( h \widehat{B}_h u | u \right) \geq ((\widehat{\chi}_0 (1 + 2h(C_m - \mathcal{O}(1)))) \widehat{\chi}_0) u | u) + \mathcal{O}(h^\infty) \|u\|^2.$$

Replacing the first term by (84) this gives (79).  $\square$

**Corollary 26.** *Eq. (70) with (74), (79), (69) gives:*

$$\forall u \in L^2(X), \quad \left\| \left( \widehat{P} - z_E \right) u \right\|^2 + \left( h \widehat{B}_h u | u \right) \geq (1 + 2(C_m - \mathcal{O}(1))h) \|u\|^2, \quad (87)$$

where  $\mathcal{O}(1)$  does not depend on  $m$ . Using (80) instead we get

$$\forall u \in L^2(X), \quad \left\| \left( \widehat{P} - z_E \right) u \right\|^2 \geq (1 - \mathcal{O}(h)) \|u\|^2. \quad (88)$$

Let us show that these last relations imply an upper bound for the number of eigenvalues of the operator  $\left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right)$  smaller than  $(1 + 2(C_m - \mathcal{O}(1))h)$ .

**Lemma 27.** *Let  $s_1 \leq s_2 \leq \dots$  be the singular values  $\left(\widehat{P} - z_E\right)$  sorted from below. More precisely,  $s_1^2 \leq s_2^2 \leq \dots$  are the eigenvalues of the positive self-adjoint operator  $\widehat{A} := \left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right)$  below the infimum of the essential spectrum of  $\widehat{A}$ , possibly completed with an infinite repetition of that infimum if there are only finitely many such eigenvalues. Then the first eigenvalue is*

$$s_1 \geq 1 - \mathcal{O}(h) \quad (89)$$

and

$$\text{if } j > \mathcal{O}(1) C_m \text{Vol} \left( \widetilde{N}_0 \right) h^{\frac{1}{2}-n} \text{ then } s_j \geq 1 + (C_m - \mathcal{O}(1))h. \quad (90)$$

*In other words the number of singular values of  $\left(\widehat{P} - z_E\right)$  below  $1 + (C_m - \mathcal{O}(1))h$  is  $\mathcal{O}(1) C_m \text{Vol} \left( \widetilde{N}_0 \right) h^{\frac{1}{2}-n}$ .*

*Proof.* Eq.(89) is a direct consequence of (88). We use the “max-min formula” for self-adjoint operators [38, p.78] and Eq.(87). Put  $\lambda_m := C_m - \mathcal{O}(1)$ . We have for every  $j$

$$\begin{aligned} s_j^2 &= \max_{U \subseteq L^2(X), \text{codim}(U) \leq j-1} \min_{u \in U, \|u\|=1} \left( u, \widehat{A}u \right) \\ &\geq 1 + 2\lambda_m h + \max_{U \subseteq L^2(X), \text{codim}(U) \leq j-1} \min_{u \in U, \|u\|=1} \left( - \left( h \widehat{B}_h u | u \right) \right) \\ &= 1 + 2\lambda_m h - h \min_{U \subseteq L^2(X), \text{codim}(U) \leq j-1} \max_{u \in U, \|u\|=1} \left( \left( \widehat{B}_h u | u \right) \right) \\ &= 1 + 2\lambda_m h - h b_j, \end{aligned}$$

where  $U$  varies in the set of closed subspaces of  $L^2(X)$  and  $b_1 \geq b_2 \geq \dots$  denote the eigenvalues of  $\widehat{B}_h$  (possibly completed with an infinite repetition of 0 if there are only finitely many such eigenvalues). We have

$$\left\| \widehat{B}_h \right\|_{\text{Tr}} = \text{Tr} \left( \widehat{B}_h \right) = b_1 + b_2 + \dots$$

Eq.(78) implies that for every  $\varepsilon_0 > 0$ , if  $b_j \geq \varepsilon_0$  then  $j\varepsilon_0 \leq \text{Tr} \left( \widehat{B}_h \right) \leq \mathcal{O}(1) C_m \text{Vol} \left( \widetilde{N}_0 \right) h^{\frac{1}{2}-n}$  then  $j \leq \frac{1}{\varepsilon_0} \mathcal{O}(1) C_m \text{Vol} \left( \widetilde{N}_0 \right) h^{\frac{1}{2}-n}$ . Equivalently if  $j > \frac{1}{\varepsilon_0} \mathcal{O}(1) C_m \text{Vol} \left( \widetilde{N}_0 \right)$  then  $b_j < \varepsilon_0$  and  $s_j^2 \geq 1 + 2\lambda_m h - h b_j \geq 1 + 2(\lambda_m - \varepsilon_0)h$ . Taking the square root we get (90).  $\square$

We deduce now an upper bound for the number of eigenvalues of  $\widehat{P}$ .

**Corollary 28.** *We have*

$$\# \left\{ \sigma(\widehat{P}) \cap D \left( z_E, 1 + \frac{C_m}{2} h \right) \right\} \leq \mathcal{O}(1) C_m \text{Vol}(\widetilde{N}_0) h^{\frac{1}{2}-n}. \quad (91)$$

*Proof.* Let  $\lambda_1, \lambda_2, \lambda_3 \dots$  denote the eigenvalues of  $\widehat{P}$  sorted such that  $j \rightarrow |\lambda_j - z_E|$  is increasing. The Weyl inequalities (see [42, (a.8) p.38] for a proof) give

$$\prod_{j=1}^N s_j \leq \prod_{j=1}^N |\lambda_j - z_E|, \quad \forall N, \quad (92)$$

where  $(s_j)_j$  are the singular values defined in Lemma 27 above. Let

$$\widetilde{N} := \# \left\{ \lambda_j : |\lambda_j - z_E| \leq 1 + \frac{C_m}{2} h \right\} = \# \left\{ \sigma(P) \cap D \left( z_E, 1 + \frac{C_m}{2} h \right) \right\}$$

and let

$$\widetilde{M} := \mathcal{O}(1) C_m \text{Vol}(\widetilde{N}_0) h^{\frac{1}{2}-n}$$

be the factor which appears in (90). We want to show the bound:

$$\widetilde{N} \leq \left( 2 + \mathcal{O} \left( \frac{1}{C_m} \right) \right) \widetilde{M} \quad (93)$$

for  $C_m \gg 1$ .

If  $\widetilde{N} \leq \widetilde{M}$  then (93) is true. Conversely let us suppose that  $\widetilde{N} \geq \widetilde{M}$ . Using (92) we have

$$\left( \prod_{j=1}^{\widetilde{M}} s_j \right) \left( \prod_{j=\widetilde{M}}^{\widetilde{N}} s_j \right) \leq \left( 1 + \frac{C_m}{2} h \right)^{\widetilde{N}}.$$

Then using (89) and (90) we have

$$(1 - \mathcal{O}(h))^{\widetilde{M}} (1 + (C_m - \mathcal{O}(1)) h)^{\widetilde{N} - \widetilde{M}} \leq \left( 1 + \frac{C_m}{2} h \right)^{\widetilde{N}}.$$

We take the logarithm and since  $h \ll 1$  we get:

$$-\widetilde{M} \mathcal{O}(h) + (\widetilde{N} - \widetilde{M}) (C_m - \mathcal{O}(1)) h \leq \widetilde{N} \frac{C_m}{2} h$$



$$\Leftrightarrow \tilde{N} \left( \frac{C_m}{2} - \mathcal{O}(1) \right) \leq \widetilde{M} (C_m + \mathcal{O}(1)).$$

Now since  $C_m \gg 1$ ,

$$\Leftrightarrow \tilde{N} \leq \widetilde{M} \frac{\left(1 + \mathcal{O}\left(\frac{1}{C_m}\right)\right)}{\left(\frac{1}{2} + \mathcal{O}\left(\frac{1}{C_m}\right)\right)} = \widetilde{M} \left(2 + \mathcal{O}\left(\frac{1}{C_m}\right)\right),$$

so we have obtained (93). This implies (91).  $\square$

From Lemma 23 with  $b = \frac{C_m}{2}$  and  $\beta = \frac{C_m}{4}$  we deduce that the upper bound (91) implies an upper bound:

$$\# \left\{ \lambda_i \in \sigma(\widehat{P}), \quad |\Re(\lambda_i - E)| \leq \sqrt{\frac{C_m}{4}} h, \quad \Im(\lambda_i) \geq -\frac{C_m}{4} h \right\} = \mathcal{O}(1) C_m \text{Vol}(\tilde{N}_0) h^{\frac{1}{2}-n}.$$

We take  $\alpha = \frac{1}{h} \gg 1$  and return to the original spectral variable  $z = \frac{z_h}{h} = \alpha z_h$  after the scaling (59). From (62) we can choose the escape function  $m$  such that  $C_m \gg 1$  is arbitrarily large and  $\text{Vol}(\tilde{N}_0) < o\left(\frac{1}{C_m}\right)$  is arbitrarily small so that  $\mathcal{O}(1) C_m \text{Vol}(\tilde{N}_0) h^{\frac{1}{2}-n} = o\left(h^{\frac{1}{2}-n}\right)$ . Since the spectrum does not depend on the escape function  $m$ , we get (48). We have finished the proof of Theorem 15.

## A Some results in operator theory

### A.1 On minimal and maximal extensions

We show here that the pseudodifferential operator  $\widehat{P}$  defined in Eq.(49), has a unique closed extension on  $L^2(X)$ . This is a well known procedure for the case of elliptic PDO, we refer to [51, chap.13 p.125], and in general this is not true for PDO of order 2. The fact that  $P$  has order 1 (since it is defined from a vector field on  $X$ ) is therefore important.

The domain of the minimal closed extension  $\widehat{P}_{min}$  of the operator  $\widehat{P}$  with domain  $C^\infty(X)$  is

$$\mathcal{D}_{min} := \left\{ u \in L^2(X), \quad u_j \in C^\infty(X) \rightarrow u \text{ in } L^2(X) \text{ and } \widehat{P}u_j \rightarrow v \in L^2(X) \right\}. \quad (94)$$

The maximal closed extension  $\widehat{P}_{max}$  has domain

$$\mathcal{D}_{max} := \left\{ u \in L^2(X), \quad \widehat{P}u \in L^2(X) \right\}.$$

(Recall that  $\widehat{P}$  is defined a priori on  $C^\infty(X)$  and  $\mathcal{D}'(X)$ ).

**Lemma 29.** *For a PDO  $\widehat{P}$  of order 1 (i.e.  $\widehat{P} \in \text{Op}(S^1)$ ), the minimal and maximal extensions coincide:  $\mathcal{D}(\widehat{P}) := \mathcal{D}_{\min} = \mathcal{D}_{\max}$ , i.e. there is a unique closed extension of the operator  $\widehat{P}$  in  $L^2(X)$ .*

*Proof.*  $\mathcal{D}_{\min} \subset \mathcal{D}_{\max}$  is clear. Let us check that  $\mathcal{D}_{\max} \subset \mathcal{D}_{\min}$ . Let  $u \in \mathcal{D}_{\max}$ , i.e.  $u \in L^2(X)$ ,  $v := \widehat{P}u \in L^2(X)$ . We will construct a sequence  $u_h \in C^\infty(X)$  with  $h \rightarrow 0$ , such that  $u_h \rightarrow u$  in  $L^2(X)$  and show that  $\widehat{P}u_h \rightarrow v$  in  $L^2$ .

Let  $\chi : T^*X \rightarrow \mathbb{R}^+$  be a  $C^\infty$  function such that  $\chi(x, \xi) = 1$  for  $|\xi| \leq 1$ , and  $\chi(x, \xi) = 0$  for  $|\xi| \geq 2$ . For  $h > 0$ , let the function  $\chi_h$  on  $T^*X$  be defined by  $\chi_h(x, \xi) = \chi(x, h\xi)$ . Let the **truncation operator** be:

$$\widehat{\chi}_h := \text{Op}(\chi_h).$$

Notice that  $\widehat{\chi}_h$  is a smoothing operator which truncates large components in  $\xi$  (larger than  $1/h$ ),  $\widehat{\chi}_h$  is similar to a convolution in  $x$  coordinates.

Let

$$u_h := \widehat{\chi}_h u.$$

It is clear that  $u_h \rightarrow u$  in  $L^2(X)$  as  $h \rightarrow 0$ . We have

$$\widehat{P}u_h = \widehat{P}\widehat{\chi}_h u = \widehat{\chi}_h \widehat{P}u + [\widehat{P}, \widehat{\chi}_h] u.$$

The first term converges  $\widehat{\chi}_h \widehat{P}u \rightarrow v = \widehat{P}u$  as  $h \rightarrow 0$ . The principal symbol of the PDO  $[\widehat{P}, \widehat{\chi}_h]$  is

$$\frac{1}{i} \{P, \chi_h\} = \frac{1}{i} (\partial_\xi P \partial_x \chi_h - \partial_x P \partial_\xi \chi_h).$$

Now we use the fact that  $P \in S^1$  has order 1. In the first term,  $\partial_\xi P \in S^0$  is bounded (from (3)) and  $\partial_x \chi_h$  is non zero only on a large ring  $\frac{1}{h} \leq |\xi| \leq \frac{2}{h}$ . In the second term  $\partial_x P \in S^1$  has order 1 but  $\partial_\xi \chi_h = h \partial_\xi \chi(x, h\xi)$  is non zero on the same large ring and therefore of order  $(-1)$  (since  $h \simeq |\xi|^{-1}$  on the ring). Therefore the PDO  $[\widehat{P}, \widehat{\chi}_h]$  converges strongly to zero in  $L^2(X)$  as  $h \rightarrow 0$ . Hence  $[\widehat{P}, \widehat{\chi}_h] u \rightarrow 0$  as  $h \rightarrow 0$ . We deduce that  $\widehat{P}u_h \rightarrow v = \widehat{P}u$ , and that  $u \in \mathcal{D}_{\min}$ .  $\square$

## A.2 The sharp Gårding inequality

References: [23, p.52] or (100), [32, p.99], [52, p.1157] for a short proof using Toeplitz quantization.

**Proposition 30.** *If  $\widehat{P}$  is a PDO with symbol  $P \in S^\mu$ ,  $\mu \in \mathbb{R}$ ,  $\Re(P) \geq 0$  then there exists  $C \in \mathbb{R}$  such that*

$$\forall u \in C^\infty(X), \quad \Re(\widehat{P}u|u) \geq -C \|u\|_{H^{\frac{\mu-1}{2}}}^2 \quad (95)$$

where  $\|u\|_{H^\mu}^2 := \left( \langle \widehat{\xi} \rangle^\mu u | \langle \widehat{\xi} \rangle^\mu u \right)_{L^2(X)}$  denotes the norm in the Sobolev space  $H^\mu$ .

### A.3 Quadratic partition of unity on phase space

As usual in this paper, we denote  $\widehat{A} := \text{Op}_h(A)$  for a symbol  $A$ .

**Lemma 31.** *Let  $K_0 \subset T^*X$  compact. There exists symbols  $\chi_0 \in S^{-\infty}$  and  $\chi_1 \in S^0$  of self-adjoint operators  $\widehat{\chi}_0, \widehat{\chi}_1$  such that*

$$\widehat{\chi}_0^2 + \widehat{\chi}_1^2 = 1 + \widehat{R}.$$

The symbol  $R \in (h^\infty S^{-\infty})$  is negligible,  $\text{supp}(\chi_0)$  is compact and on  $K_0$ ,  $\chi_1(x, \xi) = \mathcal{O}(h^\infty)$ ,  $\chi_0(x, \xi) = 1 + \mathcal{O}(h^\infty)$ .

*Proof.* Let  $K_0 \subset T^*X$  be compact. We can find symbols  $0 \leq \chi_0 \in C_0^\infty(T^*X)$  (with compact support) and  $0 \leq \chi_1 \in C^\infty(T^*X)$  such that

$$\chi_1 = \begin{cases} 0 & \text{on } K_0 \\ 1 & \text{for } |\xi| \gg 1 \end{cases}$$

and

$$A := \chi_0^2 + \chi_1^2 \text{ is } \begin{cases} > 0 & \text{everywhere} \\ = 1 & \text{for } |\xi| \gg 1. \end{cases}$$

We replace  $\chi_0, \chi_1$  respectively by  $\chi_0 A^{-1/2}, \chi_1 A^{-1/2}$ . We obtain  $1 = \chi_0^2 + \chi_1^2$ .

Let  $\widehat{R} := \widehat{\chi}_0^2 + \widehat{\chi}_1^2 - 1$ . Then  $\widehat{R} \in \text{Op}(hS^{-\infty})$ . We write  $R = hr_0(x, \xi) + h^2 \dots$

We replace  $\widehat{\chi}_j$ ,  $j = 0, 1$  by

$$\widehat{\chi}_j' := (1 + h\widehat{r}_0)^{-1/4} \widehat{\chi}_j (1 + h\widehat{r}_0)^{-1/4}.$$

Which is also self-adjoint. We obtain

$$\begin{aligned} \widehat{\chi}_0'^2 + \widehat{\chi}_1'^2 &= (1 - h\widehat{r}_0) \widehat{\chi}_0^2 + (1 - h\widehat{r}_1) \widehat{\chi}_1^2 + \mathcal{O}(\text{Op}(h^2 S^{-\infty})) \\ &= 1 + \mathcal{O}(\text{Op}(h^2 S^{-\infty})). \end{aligned}$$

If we iterate this algorithm, we obtain the Lemma. □

### A.3.1 I.M.S. localization formula

The following Lemma is similar to the “I.M.S localization formula” given in [11, p.27]. It uses the quadratic partition of phase space obtained in Lemma 31 above.

**Lemma 32.** *Suppose that  $\widehat{P} \in \text{Op}(S^\mu)$  for some  $\mu \in \mathbb{R}$  and that  $(P - P^*) \in \text{Op}(hS^\mu)$ . Then for every  $u \in L^2(X)$ ,  $z \in \mathbb{C}$ ,*

$$\left\| (\widehat{P} - z) u \right\|^2 = \left\| (\widehat{P} - z) \widehat{\chi}_0 u \right\|^2 + \left\| (\widehat{P} - z) \widehat{\chi}_1 u \right\|^2 + \mathcal{O}(h^2) \|u\|^2. \quad (96)$$

*Proof.* For simplicity, we suppose  $z = i\beta$  with  $\beta \in \mathbb{R}$ , i.e.  $\Re(z) = 0$  (this is equivalent to replacing  $\widehat{P} - \Re(z)$  by some operator  $\widehat{P}'$ ). We use (69) and write

$$\left\| (\widehat{P} - i\beta) u \right\|^2 = \left( (\widehat{P} - i\beta)^* (\widehat{P} - i\beta) u | u \right) = \sum_{k=0,1} \left( (\widehat{P} - i\beta)^* \widehat{\chi}_k^2 (\widehat{P} - i\beta) u | u \right) \quad (97)$$

$$+ \mathcal{O}(h^\infty) \|u\|^2. \quad (98)$$

The aim is to move the operators  $\widehat{\chi}_k$  outside. One has for  $k = 0, 1$ :

$$\begin{aligned} & (\widehat{P} - i\beta)^* \widehat{\chi}_k^2 (\widehat{P} - i\beta) - \widehat{\chi}_k (\widehat{P} - i\beta)^* (\widehat{P} - i\beta) \widehat{\chi}_k = \\ &= \widehat{\chi}_k (\widehat{P} - i\beta)^* \widehat{\chi}_k (\widehat{P} - i\beta) + [\widehat{P}^*, \widehat{\chi}_k] \widehat{\chi}_k (\widehat{P} - i\beta) - \widehat{\chi}_k (\widehat{P} - i\beta)^* (\widehat{P} - i\beta) \widehat{\chi}_k \\ &= [\widehat{P}^*, \widehat{\chi}_k] \widehat{\chi}_k (\widehat{P} - i\beta) - \widehat{\chi}_k (\widehat{P} - i\beta)^* [\widehat{P}, \widehat{\chi}_k] \\ &= \left( \underbrace{[\widehat{P}^*, \widehat{\chi}_k] \widehat{\chi}_k \widehat{P} - \widehat{\chi}_k \widehat{P}^* [\widehat{P}, \widehat{\chi}_k]}_{\text{I}_k} \right) - i\beta \left( \underbrace{[\widehat{P}^*, \widehat{\chi}_k] \widehat{\chi}_k + \widehat{\chi}_k [\widehat{P}, \widehat{\chi}_k]}_{\text{II}_k} \right). \end{aligned} \quad (99)$$

First remark that for every PDO  $\widehat{A} \in \text{Op}(S^\mu)$  with some  $\mu \in \mathbb{R}$ , then

$$[\widehat{A}, \widehat{\chi}_k] \in \text{Op}(hS^{-\infty}).$$

This is obvious for  $k = 0$  since  $\widehat{\chi}_0 \in \text{Op}(S^{-\infty})$  and for  $k = 1$  this is because  $(\widehat{\chi}_1 - 1) \in \text{Op}(S^{-\infty})$  and  $[\widehat{A}, 1] = 0$ . We have assumed that

$$(\widehat{P}^* - \widehat{P}) \in \text{Op}(hS^\mu),$$

therefore

$$[\widehat{P}^* - \widehat{P}, \widehat{\chi}_k] \in \text{Op}(h^2 S^{-\infty}).$$

Also

$$\left[\widehat{P}, \widehat{\chi}_k\right] \in \text{Op}\left(hS^{-\infty}\right).$$

The first term of (99) is

$$\begin{aligned} \text{I}_k &= \left[\widehat{P}^*, \widehat{\chi}_k\right] \widehat{\chi}_k \widehat{P} - \widehat{\chi}_k \widehat{P}^* \left[\widehat{P}, \widehat{\chi}_k\right] \\ &= \left[\widehat{P}, \widehat{\chi}_k\right] \widehat{\chi}_k \widehat{P} - \widehat{\chi}_k \widehat{P} \left[\widehat{P}, \widehat{\chi}_k\right] + \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right) \\ &= \left[\left[\widehat{P}, \widehat{\chi}_k\right], \widehat{\chi}_k \widehat{P}\right] + \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right) \\ &= \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right). \end{aligned}$$

The second term of (99) is

$$\begin{aligned} \text{II}_k &= \left[\widehat{P}^*, \widehat{\chi}_k\right] \widehat{\chi}_k + \widehat{\chi}_k \left[\widehat{P}, \widehat{\chi}_k\right] \\ &= \left[\widehat{P}, \widehat{\chi}_k\right] \widehat{\chi}_k + \widehat{\chi}_k \left[\widehat{P}, \widehat{\chi}_k\right] + \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right) \\ &= \left[\widehat{P}, \widehat{\chi}_k^2\right] + \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right). \end{aligned}$$

Therefore using (69)

$$\begin{aligned} \text{II}_0 + \text{II}_1 &= \left[\widehat{P}, \widehat{\chi}_0^2 + \widehat{\chi}_1^2\right] + \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right) \\ &= \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right). \end{aligned}$$

We have shown that

$$\sum_{k=0,1} \left(\widehat{P} - i\beta\right)^* \widehat{\chi}_k^2 \left(\widehat{P} - i\beta\right) = \sum_{k=0,1} \widehat{\chi}_k \left(\widehat{P} - i\beta\right)^* \left(\widehat{P} - i\beta\right) \widehat{\chi}_k + \mathcal{O}\left(\text{Op}\left(h^2 S^{-\infty}\right)\right).$$

Coming back to (97) we get (96).  $\square$

## A.4 FBI transform and Toeplitz operators

References: [26],[32],[52], [43].

The manifold  $X$  is equipped with a smooth Riemannian metric so that we have a well-defined exponential map  $\exp_x : T_x X \rightarrow X$  which is a diffeomorphism from a neighborhood of  $0 \in T_x X$  onto a neighborhood of  $x \in X$ . Define the **coherent state** at point  $\alpha = (\alpha_x, \alpha_\xi) \in T^*X$  to be the function of  $y \in X$ :

$$e_\alpha(y) := \chi(\alpha_x, y) \exp\left(\frac{i}{h} \alpha_\xi \left(\exp_{\alpha_x}^{-1}(y)\right) - \frac{1}{2h} \langle \alpha_\xi \rangle \text{dist}(\alpha_x, y)^2\right), \quad \langle \alpha_\xi \rangle := (1 + \alpha_\xi^2)^{1/2},$$

where  $\chi \in C^\infty(X \times X)$  is a standard cutoff to a small neighborhood of the diagonal. In the Euclidean case  $X = \mathbb{R}^n$ , the cutoff is often superfluous and we get the complex Gaussian “wave packet”

$$e_\alpha(y) = \exp\left(\frac{i}{h} \alpha_\xi (y - \alpha_x) - \frac{1}{2h} \langle \alpha_\xi \rangle |y - \alpha_x|^2\right).$$

We can define the **FBI-transform** of  $u \in C^\infty(X)$  by

$$(Tu)(\alpha; h) := h^{-\frac{3n}{4}} (e_\alpha | u) = h^{-\frac{3n}{4}} \int_X \overline{e_\alpha(y)} u(y) dy,$$

which can be made asymptotically isometric after multiplication to the left by an elliptic symbol of order 0 and we can keep this point of view in mind. We have the following known facts [43, 26]:

- There exists  $a_0(\alpha; h) \in h^{-\frac{3n}{2}} S^{n/2}$  elliptic and  $a_0 > 0$  such that

$$u = \int_{T^*X} (\widehat{\pi}_\alpha u) d\alpha + \widehat{R}u, \quad \forall u \in L^2(X)$$

with

$$\widehat{\pi}_\alpha := a_0(\alpha; h) e_\alpha(e_\alpha | \cdot)$$

and  $\widehat{R} \in \text{Op}_h(h^\infty S^{-\infty})$  negligible.

- $\widehat{\pi}_\alpha \geq 0$  and

$$\|\widehat{\pi}_\alpha\|_{tr} = \text{Tr}(\widehat{\pi}_\alpha) = a_0(\alpha; h) \|e_\alpha\|^2 = \mathcal{O}(1) h^{-n}.$$

- If  $\widehat{B} \in \text{Op}_h(S^m)$  has the principal symbol  $b_0$  (modulo  $hS^{m-1}$ ), then

$$\widehat{B} = \int_{T^*X} b(\alpha; h) \widehat{\pi}_\alpha d\alpha + \widehat{R}$$

where  $\widehat{R}$  is negligible as above,  $b \in S^m$  and  $b = b_0 \bmod(hS^{m-1})$ .

- For a function  $A(x, \xi; h)$ , we define the **Toeplitz quantization** of  $A$  by

$$\text{Op}_T(A) := \int A(\alpha; h) \widehat{\pi}_\alpha d\alpha,$$

then the previous results imply a “Gårding’s inequality”:

$$A(x, \xi) \geq 0 \Rightarrow (\text{Op}_T(A) u | u) \geq 0 + \mathcal{O}(h^\infty) \|u\|^2 \quad (100)$$

and

$$\text{Tr}(\widehat{A}) = \frac{\mathcal{O}(1)}{h^n} \int A(x, \xi) dx d\xi + \mathcal{O}(h^\infty).$$

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