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# Local and global properties of solutions of heat equation with superlinear absorption

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## Abstract

We study the limit, when  $k \rightarrow \infty$  of the solutions of  $\partial_t u - \Delta u + f(u) = 0$  in  $\mathbb{R}^N \times (0, \infty)$  with initial data  $k\delta$ , when  $f$  is a positive superlinear increasing function. We prove that there exist essentially three types of possible behaviour according  $f^{-1}$  and  $F^{-1/2}$  belong or not to  $L^1(1, \infty)$ , where  $F(t) = \int_0^t f(s)ds$ . We use these results for giving a general result on the existence of the initial trace and some uniqueness and non-uniqueness results for solutions with unbounded initial data.

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*Key words.* Heat equation; singularities; Borel measures; initial trace.

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# 1 Introduction

In this article we investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + f(u) = 0 \quad (1.1)$$

in  $Q_\infty := \mathbb{R}^N \times (0, \infty)$  ( $N \geq 2$ ) where  $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$  is continuous, nondecreasing and positive on  $(0, \infty)$ , vanishes at 0 and tends to infinity at infinity. As a model equation we shall consider the following, with  $\alpha > 0$ ,

$$\partial_t u - \Delta u + u \ln^\alpha(u + 1) = 0. \quad (1.2)$$

When  $f(u) = |u|^\beta u$  with  $\beta > 0$  much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x, t) = t^{-1/\beta} w(x\sqrt{t}). \quad (1.3)$$

The critical exponent  $\beta_c = 2/N$  plays a fundamental role in the description of isolated singularities and the study of the initial trace. This is due to the fact that, for  $0 < \beta < \beta_c$ , there exists a positive self-similar solution with an isolated singularity at  $(0, 0)$  and vanishing on  $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$ , while no such solution exists when  $\beta \geq \beta_c$  and more generally, no solution with isolated singularities.

In the case of (1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any  $k > 0$  there exists a unique  $u = u_k \in C(\overline{Q_\infty} \setminus \{(0, 0)\}) \cap C^{2,1}(Q_\infty)$  solution of

$$\begin{cases} \partial_t u - \Delta u + u \ln^\alpha(u + 1) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \quad (1.4)$$

There are two critical values for  $\alpha$ :  $\alpha = 1$  and  $\alpha = 2$ , the explanation of which comes from the study of the two singular problems

$$\begin{cases} \phi' + \phi \ln^\alpha(\phi + 1) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty, \end{cases} \quad (1.5)$$

and, for any  $\epsilon > 0$ ,

$$\begin{cases} -\Delta \psi + \psi \ln^\alpha(\psi + 1) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty, \end{cases} \quad (1.6)$$

where  $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$ . When it exists, the solution  $\phi_\infty$  of (1.5) is given implicitly by

$$\int_{\phi_\infty(t)}^{\infty} \frac{ds}{s \ln^\alpha(s+1)} = t \quad \forall t > 0, \quad (1.7)$$

and such a formula is valid if and only if  $\alpha > 1$ . For problem (1.6) an explicit expression of the solution is not valid, but this solution exists if and only if  $\alpha > 2$ ; in this case of the Keller-Osserman condition (see (1.12) below) holds.

Having in mind this model we study (1.1) assuming the weak singularity condition on  $f$ :

$$\int_0^1 s^{N/2} f(s^{-N/2}) ds < \infty. \quad (1.8)$$

**Proposition 1.1** *Assume (1.8) holds. Then for any  $k > 0$ , there exists a unique solution  $u := u_k$  to*

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \quad (1.9)$$

Another condition on  $f$  is

$$\int_1^\infty \frac{ds}{f(s)} < \infty. \quad (1.10)$$

Under assumption (1.10) there exists a solution  $\phi := \phi_\infty$  to

$$\begin{cases} \phi' + f(\phi) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty. \end{cases} \quad (1.11)$$

The next important condition on  $f$  we shall encounter is the Keller-Osserman condition, i.e.

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad (1.12)$$

where

$$F(s) = \int_0^s f(\sigma) d\sigma, \quad \forall s \in [1, \infty). \quad (1.13)$$

If (1.12) is satisfied, by [4, Th III] for any  $\epsilon > 0$  there exists a maximal solution  $\psi := \psi_\epsilon$  to

$$\begin{cases} -\Delta \psi + f(\psi) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty. \end{cases} \quad (1.14)$$

The first question we consider is the study of the limit of  $u_k$  when  $k \rightarrow \infty$ . This question is natural since  $k \mapsto u_k$  is increasing. In order to treat it, we need some additional conditions.

(C1)- The function  $s \mapsto \frac{f(s)}{s}$  is increasing on  $(0, \infty)$  and satisfies

$$\lim_{s \rightarrow 0} \frac{f(s)}{s} = 0 \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{f(s)}{s} = \infty.$$

(C2)- The function  $f$  is convex on  $(0, \infty)$ .

(C3)- If  $\liminf_{s \rightarrow \infty} f(s)/(s \ln^\alpha s) = 0, \forall \alpha > 2$ , then there exists  $\beta \in (1, 2]$  such that

$$\limsup_{s \rightarrow \infty} \frac{f(s)}{s \ln^\beta s} < \infty.$$

In the second section, we prove the following results.

**Theorem 1.2** *Assume the conditions (C1) and (C3) hold. If  $f$  satisfies*

$$\int_1^\infty \frac{ds}{f(s)} = \infty, \tag{1.15}$$

*then the solutions  $u_k$  of (1.9) satisfy  $\lim_{k \rightarrow \infty} u_k(x, t) = \infty$  for every  $(x, t) \in Q_\infty$ .*

**Theorem 1.3** *Assume the conditions (C1) – (C3) hold. If  $f$  satisfies (1.10) and*

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} = \infty \tag{1.16}$$

*where  $F$  is defined in (1.13), then the solutions  $u_k$  of (1.9) satisfy  $\lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t)$  for every  $(x, t) \in Q_\infty$ , where  $\phi_\infty$  is the solution of (1.11).*

We denote by  $\mathcal{U}_0$  the set of positive solutions  $u$  of (1.1) in  $Q_\infty$ , which is continuous in  $\overline{Q_\infty} \setminus \{(0, 0)\}$ , vanishes on the set  $\{(x, 0) : x \neq 0\}$  and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} u(x, t) dx = \infty \tag{1.17}$$

for any  $\epsilon > 0$ .

**Theorem 1.4** *Assume  $f$  satisfies (1.8) and (1.12). Then  $\underline{U} := \lim_{k \rightarrow \infty} u_k$  is the minimal element of  $\mathcal{U}_0$ .*

In the third section we study the set of positive and locally bounded solutions of (1.1) in  $Q_\infty$ . This set differs considerably according the assumption on  $f$ . This is due to the properties of the radial solutions of the associated stationary equation

$$-\Delta w + f(w) = 0 \quad \text{in } \mathbb{R}^N. \quad (1.18)$$

The next result is based upon the Picard-Lipschitz fixed point theorem and a result of Vazquez and Véron [11].

**Proposition 1.5** *Assume (1.16) holds. For any  $a > 0$ , there exists a unique positive function  $w := w_a \in C^2([0, \infty))$  to the problem*

$$\begin{cases} -w'' - \frac{N-1}{r}w' + f(w) = 0 & \text{in } \mathbb{R}_+ \\ w'(0) = 0 \\ w(0) = a. \end{cases} \quad (1.19)$$

A striking consequence of the existence of such solutions is the following non-uniqueness result.

**Theorem 1.6** *Assume  $f$  satisfies (1.10) and (1.16). Then for any  $u_0 \in C(\mathbb{R}^N)$  satisfying, for some  $b > a > 0$ ,  $w_a(x) \leq u_0(x) \leq w_b(x) \forall x \in \mathbb{R}^N$ , there exist two solutions  $\underline{u}, \bar{u} \in C(\overline{Q_\infty})$  of (1.1) with initial value  $u_0$ . They satisfy respectively*

$$0 \leq \underline{u}(x, t) \leq \min\{w_b(x), \phi_\infty(t)\} \quad \forall (x, t) \in Q_\infty, \quad (1.20)$$

thus  $\lim_{t \rightarrow \infty} \underline{u}(x, t) = 0$ , uniformly with respect to  $x \in \mathbb{R}^N$ , and

$$w_a(x) \leq \bar{u}(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty, \quad (1.21)$$

thus  $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$ , uniformly with respect to  $t \geq 0$ .

The next theorem shows that if two solutions of (1.1) have the same initial data and the same asymptotic behaviour as  $|x| \rightarrow \infty$  then they coincide.

**Theorem 1.7** *Assume  $f$  satisfies (C1) and (1.16). Let  $u, \tilde{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$  be two positive solutions of (1.1) with initial data  $u_0$ . If for any  $\epsilon > 0$ ,*

$$u(x, t) - \tilde{u}(x, t) = o(w_\epsilon(|x|)) \text{ as } x \rightarrow \infty \quad (1.22)$$

locally uniformly with respect to  $t \geq 0$ , then  $u = \tilde{u}$ .

On the contrary, if the Keller-Osserman condition holds, a continuous solution is uniquely determined by the positive initial value  $u_0 \in C(\mathbb{R}^N)$ , and uniqueness still holds if  $C(\mathbb{R}^N)$  is replaced by  $\mathfrak{M}_+(\mathbb{R}^N)$ .

**Theorem 1.8** *Assume  $f$  satisfies (1.12) and (C2). Then*

(i) *For any nonnegative function  $u_0 \in C(\mathbb{R}^N)$  there exists a unique nonnegative solution  $u \in C(\overline{Q_\infty})$  of (1.1) in  $Q_\infty$  with initial value  $u_0$ .*

(ii) *For any for any nonnegative measure  $\mu \in \mathfrak{M}(\mathbb{R}^N)$ , there exists at most one nonnegative solution  $u \in C(Q_\infty)$  of (1.1) in  $Q_\infty$  such that  $f(u) \in L^1_{loc}(\overline{Q_\infty})$  satisfying*

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(\mathbb{R}^N). \quad (1.23)$$

In the last section we study the initial trace of locally bounded positive solutions of (1.1) in  $Q_\infty$ .

**Proposition 1.9** *Let  $u \in C^{2,1}(Q_\infty)$  be a positive solution of (1.1) in  $Q_\infty$ . The set  $\mathcal{R}(u)$  of the points  $z \in \mathbb{R}^N$  such that there exists an open ball  $B_r(z)$  such that  $u, f(u) \in L^1(Q_T^{B_r(z)})$  is an open subset. Furthermore there exists a positive Radon measure  $\mu := \mu(u)$  on  $\mathcal{R}(u)$  such that*

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(\mathcal{R}(u)). \quad (1.24)$$

Due to Proposition 1.9, we introduce the definition of the initial trace.

**Definition 1.10** *The couple  $(\mathcal{S}(u), \mu)$  where  $\mathcal{S}(u) = \mathbb{R}^N \setminus \mathcal{R}(u)$  is called the initial trace of  $u$  in  $\Omega$  and will be denoted by  $tr_{\mathbb{R}^N}(u)$ . The set  $\mathcal{R}(u)$  is called the regular set of the initial trace of  $u$  and the measure  $\mu$  the regular part of the initial trace. The set  $\mathcal{S}(u)$  is closed and is called the singular part of the initial trace of  $u$ .*

The initial trace can also be represented by a positive, outer regular Borel measure, not necessary locally bounded. The space of these measures on  $\mathbb{R}^N$  will be denoted by  $\mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$ . If for every open subset  $A \subset \mathbb{R}^N$  we denote by  $\mathfrak{M}_+(A)$  the space of positive Radon measures on  $A$ , there is a one-to-one correspondence between  $\mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$  and the set of couples:

$$CM_+(\mathbb{R}^N) = \{(\mathcal{S}, \mu) : \mathcal{S} \subset \mathbb{R}^N \text{ closed}, \mu \in \mathfrak{M}_+(\mathcal{R}) \text{ with } \mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}\}. \quad (1.25)$$

The Borel measure  $\nu \in \mathcal{B}_+^{\text{reg}}(\mathbb{R}^N)$  corresponding to a couples  $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$  is given by

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \mu(A) & \text{if } A \subseteq \mathcal{S}, \end{cases} \quad \forall A \subset \mathbb{R}^N, A \text{ Borel}. \quad (1.26)$$

If  $u$  is a solution of (1.1), we shall use the notation  $tr_{\mathbb{R}^N}(u)$  (resp.  $Tr_{\mathbb{R}^N}(u)$ ) for the trace considered as an element of  $CM_+(\mathbb{R}^N)$  (resp.  $\mathcal{B}_+^{reg}(\mathbb{R}^N)$ ).

We consider the case when the Keller-Osserman holds.

**Theorem 1.11** *Assume  $f$  is nondecreasing and satisfies (1.12). If  $u \in C^{2,1}(Q_\infty)$  is a positive solution of (1.1), it possesses an initial trace  $\nu \in \mathcal{B}_+^{reg}(\mathbb{R}^N)$ .*

Furthermore, the following theorem deals with the existence of the maximal solution and the minimal solution of (1.1) with a given initial trace  $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$ .

**Theorem 1.12** *Assume  $f$  is nondecreasing and satisfies (1.12), (1.8) and (C2). Then for any  $(\mathcal{S}, \mu) \in CM_+(\mathbb{R}^N)$  there exist a maximal solution  $\bar{u}_{\mathcal{S}, \mu}$  and a minimal solution  $\underline{u}_{\mathcal{S}, \mu}$  of (1.1) in  $Q_\infty$ , with initial trace  $(\mathcal{S}, \mu)$ , in the following sense:*

$$\underline{u}_{\mathcal{S}, \mu} \leq v \leq \bar{u}_{\mathcal{S}, \mu} \quad (1.27)$$

for every positive solution  $v \in C^{2,1}(Q_\infty)$  of (1.1) in  $Q_\infty$  such that  $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$ .

If the Keller-Osserman does not holds, we obtain the following results.

**Theorem 1.13** *Assume (1.10), (1.16), (C1) and (C3) are verified. If  $u$  is a positive solution of (1.1) in  $Q_\infty$ , it possesses an initial trace which is either the Borel measure infinity  $\nu_\infty$  which satisfies  $\nu_\infty(\mathcal{O}) = \infty$  for any non-empty open subset  $\mathcal{O} \subset \mathbb{R}^N$ , or is a positive Radon measure  $\mu$  on  $\mathbb{R}^N$ .*

A consequence of Theorem 1.13 which is worth mentioning is the following.

**Proposition 1.14** *Under the assumptions of Theorem 1.13, for any  $b > 0$  there exists a positive solution  $u \in C(Q_\infty)$  of (1.1) in (1.1) satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x, t) \leq \phi_\infty(t) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (1.28)$$

Consequently there exist infinitely many positive solutions of (1.1) with initial trace  $\nu_\infty$ . Furthermore  $\phi_\infty$  is the smallest of all these solutions.

**Theorem 1.15** *Assume  $f$  satisfies (1.15),(1.16), (C1) and (C3). If  $u$  is a positive solution of (1.1) in  $Q_\infty$ , it possesses an initial trace which is a positive Radon measure  $\mu$  on  $\mathbb{R}^N$ .*

The proofs are combination of methods developed in [8] for elliptic equations, stability results and Theorem 1.2 and Theorem 1.3.



## 2 Isolated singularities

In order to study (1.1), we start proving Proposition 1.1.

### Proof of Proposition 1.1

We denote by  $E(x, t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$  the fundamental solution of the heat equation in  $Q_\infty$ . By [6, Remark 2.1], if for  $k > 0$ ,

$$I := \int_0^1 \int_{B_R} f(kE(x, t)) dx dt < \infty \quad (2.1)$$

for any  $R > 0$ , then there exists a unique solution  $u = u_k$  to (1.1) satisfying initial condition

$$u_k(\cdot, 0) = k\delta_0$$

in  $\mathcal{D}'(\mathbb{R}^N)$ . Furthermore the mapping  $k \mapsto u_k$  is increasing. This existence result and the next proposition lead to the conclusion of Proposition 1.1.  $\square$

**Proposition 2.1** *If  $f$  satisfies (1.8) and (C1) then (2.1) is fulfilled.*

*Proof.* We set

$$h(r) = \frac{f(r)}{r} \quad r \in (0, \infty). \quad (2.2)$$

$I$  is rewritten as

$$I = kC^* \int_0^1 \int_{B_R} t^{-N/2} e^{-|x|^2/4t} h(kC^* t^{-N/2} e^{-|x|^2/4t}) dx dt$$

where  $C^* = (4\pi)^{-N/2}$ . Put  $r = |x|$  then  $dx = r^{N-1} dr$ , and

$$I = kC^* \int_0^1 t^{-N/2} \int_0^R e^{-r^2/4t} h(kC^* t^{-N/2} e^{-r^2/4t}) r^{N-1} dr dt.$$

We put  $\rho = \frac{r}{\sqrt{t}}$ , then  $r^{N-1} dr = \rho^{N-1} t^{N/2} d\rho$ , and

$$I = kC^* \int_0^1 \int_0^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

We set

$$I_1 := kC^* \int_0^1 \int_0^1 e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt,$$

$$I_2 := kC^* \int_0^1 \int_1^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

Since  $e^{-\rho^2/4}\rho^{N-1}$  is bounded in  $[0, \infty)$ , then there exists a constant  $c_1$  depending only on  $k$  such that

$$I_1 < c_1 \int_0^1 \int_0^1 h(kC^*t^{-N/2})d\rho dt = c_1 \int_0^1 h(kC^*t^{-N/2})dt < \infty.$$

Next we show that under the condition (1.8),  $I_2 < \infty$ . In order to do that we introduce the variable  $\tau$  such that  $t^{-N/2}e^{-\rho^2/4} = \tau^{-N/2}$ . Then  $t = \tau e^{-\frac{\rho^2}{2N}}$  and  $dt = e^{-\frac{\rho^2}{2N}}d\tau$ . Therefore

$$I_2 \leq kC^* \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} \left( \int_0^{e^{\rho^2/2N}} h(kC^*\tau^{-N/2})d\tau \right) d\rho. \quad (2.3)$$

Since  $h$  satisfies (1.8), there exists  $\epsilon > 0$  (depending only on  $k$ ) such that

$$\int_0^\epsilon h(kC^*\tau^{-N/2})d\tau$$

take a finite value, denoted by  $c_2$ . Hence

$$\int_0^{e^{\rho^2/2N}} h(kC^*\tau^{-N/2})d\tau \leq c_2 + h(kC^*\epsilon^{-N/2})(e^{\frac{\rho^2}{2N}} - \epsilon). \quad (2.4)$$

Inserting (2.4) into the right-hand side of (2.3), we obtain

$$I_2 \leq c_3 \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1}d\rho + c_4 \int_1^\infty e^{-\frac{\rho^2}{4}} \rho^{N-1}d\rho < \infty$$

where  $c_3 = kC^*c_2$  and  $c_4 = kC^*h(kC^*\epsilon^{-N/2})$ . Thus  $I = I_1 + I_2 < \infty$ .  $\square$

The functions which satisfy the following ODE are particular solutions of (1.1)

$$\phi' + f(\phi) = 0 \quad \text{in } (0, \infty). \quad (2.5)$$

For  $a > 0$ , we denote by  $\phi_a$  the solution of (2.5) with initial data  $\phi(0) = a$ . If (1.15) holds then  $\lim_{a \rightarrow \infty} \phi_a(t) = \infty$  for any  $t \in (0, \infty)$ . While, if (1.10) holds there exists a maximal solution  $\phi_\infty$  given explicitly by

$$t = \int_{\phi_\infty(t)}^\infty \frac{ds}{f(s)} < \infty.$$

**Lemma 2.2** *If (1.15) holds then*

$$\liminf_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha r} = 0, \quad \forall \alpha > 1.$$

*If (1.10) holds then*

$$\limsup_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha r} = \infty, \quad \forall 0 < \alpha \leq 1.$$

*Proof. Case 1.* Assume (1.15) holds then

$$J := \int_e^\infty \frac{ds}{f(s)} < \infty. \quad (2.6)$$

We put  $s = e^{r^{-1}}$  and derive

$$J = \int_0^1 \frac{dr}{r^2 h(e^{r^{-1}})}$$

where  $h$  is defined in (2.2). Suppose that there exists  $\alpha > 1$  such that

$$\liminf_{s \rightarrow \infty} \frac{f(s)}{s \ln^\alpha s} > 0,$$

equivalently,

$$\liminf_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) > 0,$$

then there exists  $l > 0$  and  $r_0 \in (0, 1)$  such that

$$h(e^{r^{-1}}) > l r^{-\alpha} \quad \forall r \in (0, r_0).$$

Hence we derive the following contradiction

$$J < \frac{1}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^1 \frac{dr}{r^2 h(e^{r^{-1}})} < \infty.$$

*Case 2.* Assume (1.10) holds then  $J = \infty$ . Suppose that there exists  $\alpha \in (0, 1]$  such that

$$\limsup_{s \rightarrow \infty} \frac{f(s)}{s \ln^\alpha s} < \infty,$$

equivalently,

$$\limsup_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) < \infty,$$

then there exists  $l > 0$  and  $r_0 \in (0, 1)$  such that

$$h(e^{r^{-1}}) < l r^{-\alpha} \quad \forall r \in (0, r_0).$$

Hence

$$J > \frac{1}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^1 \frac{dr}{r^2 h(e^{r-1})} = \infty,$$

which is a contradiction.  $\square$

**Proof of Theorem 1.2.**

Since (1.15) holds, by Lemma 2.2 and the definition (2.2) of  $h$ ,

$$\liminf_{r \rightarrow \infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 1.$$

Thus

$$\liminf_{r \rightarrow \infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 2.$$

By (C3), there exists  $\beta \in (1, 2]$  such that

$$\limsup_{r \rightarrow \infty} \frac{h(r)}{\ln^\beta r} < \infty.$$

Hence there exist  $M > 0$  and  $r_0 > 0$  such that

$$h(r) < M \ln^\beta r \quad \forall r \in (r_0, \infty). \quad (2.7)$$

*Step 1.* Let  $k > 0$ , we claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau \quad \forall t \in (0, 1) \quad (2.8)$$

where  $\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$  with  $C^* = (4\pi)^{-N/2}$ . Set  $r = kC^* \tau^{-N/2}$  then (2.7) becomes

$$h(kC^* \tau^{-N/2}) < M [\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta \quad \forall \tau \in (0, \tau_0)$$

where  $\tau_0 = (kC^*)^{2/N} r_0^{-2/N}$ . We put  $a_1 = \ln k$ ,  $a_2 = \frac{N}{2} \ln(\tau^{-1})$ , and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1} (a_1^\beta + a_2^\beta)$$

in order to obtain

$$\begin{aligned} h(kC^* \tau^{-N/2}) &< M [\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M [(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})] \quad \forall \tau \in (0, \tau_0), \end{aligned} \quad (2.9)$$

(notice that  $C^* = (4\pi)^{-N/2} < 1$ ). Integrating over  $[0, t]$  yields to (2.8).

*Step 2.* It follows from (2.9) that (1.8) is fulfilled; hence by Proposition 1.1 there exists a unique solution  $u_k$  of (1.1) in  $Q_\infty$  with initial data  $k\delta_0$ . By the maximum principle,  $u_k(x, t) \leq kE(x, t)$  for every  $(x, t) \in Q_\infty$ , which implies  $u_k(x, t) \leq kC^*t^{-N/2}$  for every  $(x, t) \in Q_\infty$ . Therefore, since  $h$  is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \geq 0.$$

If we set  $v_k(x, t) = e^{\theta_k(t)}u_k(x, t)$ , we obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)}[\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})] \geq 0$$

and  $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$ . By the maximum principle, there holds

$$v_k(x, t) \geq kC^*t^{-N/2}e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^*t^{-N/2}e^{-\theta_k(t)-|x|^2/4t}. \quad (2.10)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t(\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.11)$$

where

$$c_1 = \exp\left(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau\right)$$

and  $M_\beta = M2^{\beta-1}$ . Inserting (2.11) into the right-hand side of (2.10), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1 := \mathbb{R}^N \times (0, 1).$$

If  $\lim_{k \rightarrow \infty} u_k(x, t) < \infty$  for all  $(x, t) \in Q_\infty$ , we put  $\underline{U} := \lim_{k \rightarrow \infty} u_k$ , then

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1, \quad \forall k > 0.$$

Let  $\{t_n\} \subset (0, 1]$  be a sequence converging to 0. We choose  $k_n = \exp\left((2M_\beta t_n)^{\frac{1}{1-\beta}}\right)$  then  $\ln k_n - M_\beta t_n (\ln k_n)^\beta = \frac{1}{2} \ln k_n$ . Next we restrict  $x$  in order to have

$$\ln k_n - M_\beta t_n (\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}} t_n^{\frac{\beta-2}{2(\beta-1)}}.$$

Therefore, since  $1 < \beta \leq 2$ ,

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty$$

uniformly on  $\mathbb{R}^N$  if  $1 \leq \beta < 2$ , or uniformly on the ball  $B_{r_2}$  where  $r_2 = (2M)^{-1/2}$  if  $\beta = 2$ . Since the sequence  $\{t_n\}$  is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on  $\mathbb{R}^N$  if  $1 \leq \beta < 2$ , or uniformly on the ball  $B_{r_2}$  if  $\beta = 2$ .

We pick some point  $x_0$  in  $\mathbb{R}^N$  (resp.  $B_{r_2}$ ) if  $1 < \beta < 2$  (resp.  $\beta = 2$ ). Since for any  $k > 0$ , the solution  $u_{k\delta_{x_0}}$  of (1.1) with initial data  $k\delta_{x_0}$  can be approximated by solutions with bounded initial data and support in  $B_\sigma(x_0)$  where  $0 < \sigma < r_2 - |x_0|$ , it follows

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t),$$

by comparison principle. Letting  $k \rightarrow \infty$  yields to  $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$ . Inverting the role of 0 and  $x_0$  yields to  $\underline{U}(x, t) = \underline{U}(x - x_0, t)$ . If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that  $\underline{U}(x, t)$  is independent of  $x$  and therefore it is a solution of (1.11). By (1.15),  $\underline{U}(x, t) = \infty$  for any  $(x, t) \in Q_\infty$ , which is a contradiction and the conclusion follows.  $\square$

**Proposition 2.3** *Assume (1.10) is satisfied. For any  $k > 0$ , there holds*

$$u_k(x, t) \leq \phi_\infty(t) \quad \forall (x, t) \in Q_\infty.$$

*Proof.* For any small  $\epsilon > 0$ , we set  $\phi_{\infty\epsilon}(t) = \phi_\infty(t - \epsilon), t \in [\epsilon, \infty)$  then  $\phi_{\infty\epsilon}$  is a solution of (1.1) in  $(\epsilon, \infty)$ , which dominates  $u_k$  on  $\mathbb{R}^N \times \{\epsilon\}$  for any  $k > 0$ . By comparison principle,  $u_k(x, t) \leq \phi_{\infty\epsilon}(t)$  for every  $(x, t) \in \mathbb{R}^N \times (\epsilon, \infty)$ . Letting  $\epsilon \rightarrow 0$  yields the claim.  $\square$

A necessary and sufficient condition for the existence of a maximal solution to the stationary equation

$$-\Delta w + f(w) = 0$$

in a bounded domain  $\Omega$  is the Keller-Osserman condition (1.12) ([4], [9]). If  $f$  is convex and (1.12) holds, then (1.10) is fulfilled. The Keller-Osserman condition can be replaced by another condition, which owes to the following result.

**Lemma 2.4** *Assume  $f$  is convex on  $(0, \infty)$ . Set*

$$L := \int_1^\infty \frac{ds}{\sqrt{sf(s)}}.$$

*Then (1.12) holds if and only if  $L < \infty$ .*

*Proof.* In order to obtain the assertion, it is sufficient to show that

$$s f\left(\frac{s}{2}\right) \leq F(s) \leq s f(s) \quad \forall s \geq 1. \quad (2.12)$$

The right-hand side estimate in (2.12) follows from the monotone property of  $f$ . The assumption of convexity of  $f$  in  $(0, \infty)$  implies

$$f(s) \geq f\left(\frac{s}{2}\right) + \frac{s}{2}f'\left(\frac{s}{2}\right) \quad \forall s > 0.$$

Define  $\varphi(s) = \int_0^s f(\sigma)d\sigma - sf\left(\frac{s}{2}\right)$ , then  $\varphi'(s) = f(s) - f\left(\frac{s}{2}\right) - \frac{s}{2}f'\left(\frac{s}{2}\right) \geq 0$ . Hence  $\varphi(s) > \varphi(0) = 0$ , which leads to the left-hand side estimate in (2.12).  $\square$

By using the same argument as in the proof of the Lemma 2.2 and thank to the Lemma 2.4, we obtain the following lemma.

**Lemma 2.5** *If (1.16) holds then*

$$\liminf_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha(r)} = 0 \quad \forall \alpha > 2.$$

*If (1.12) holds then*

$$\limsup_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha(r)} = \infty \quad \forall 0 < \alpha \leq 2.$$

**Proof of Theorem 1.3.**

Since (1.16) holds, by Lemma 2.5 and the definition (2.2) of  $h$ ,

$$\liminf_{r \rightarrow \infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 2.$$

By (C3), there exists  $\beta \in (1, 2]$  such that  $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty$ . Hence there exists  $M > 0$  and  $r_0 > 0$  such that

$$h(r) < M \ln^\beta r \quad \forall r \in (r_0, \infty). \quad (2.13)$$

*Step 1.* For any  $k > 0$  we set

$$\theta_k(t) = \int_0^t h(kC^*\tau^{-N/2})d\tau$$

where  $C^* = (4\pi)^{-N/2}$ . We claim that

$$\theta_k(t) < 2^{\beta-1}Mt(\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau \quad \forall t \in (0, 1). \quad (2.14)$$

If we define  $\tau$  by  $r = kC^*\tau^{-N/2}$ , (2.13) becomes

$$h(kC^*\tau^{-N/2}) < M[\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta \quad \forall \tau \in (0, \tau_0)$$

where  $\tau_0 = (kC^*)^{2/N} r_0^{-2/N}$ . We set  $a_1 = \ln k$ ,  $a_2 = \frac{N}{2} \ln(\tau^{-1})$ , and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1}(a_1^\beta + a_2^\beta)$$

in order to obtain (notice that  $C^* < 1$ )

$$\begin{aligned} h(kC^*\tau^{-N/2}) &< M[\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M[(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})]. \end{aligned} \quad (2.15)$$

Integrating over  $[0, t]$ , we obtain (2.14).

*Step 2.* It follows from (2.15) that (1.8) is fulfilled; hence by Proposition 1.1 there exists a unique solution of (1.1) in  $Q_\infty$  with initial trace  $k\delta_0$ . By maximum principle,  $u_k(x, t) \leq kE(x, t)$  for every  $(x, t) \in Q_\infty$ , which implies that  $u_k(x, t) \leq kC^*t^{-N/2}$  for every  $(x, t) \in Q_\infty$ . Therefore, since  $h$  is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \geq 0.$$

We set  $v_k(x, t) = e^{\theta_k(t)} u_k(x, t)$  and obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)} [\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})] \geq 0,$$

with  $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$ . By maximum principle, it follows

$$v_k(x, t) \geq kC^*t^{-N/2} e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^*t^{-N/2} e^{-\theta_k(t)-|x|^2/4t}. \quad (2.16)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t (\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.17)$$

where  $c_1 = \exp\left(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau\right)$  and  $M_\beta = M2^{\beta-1}$ . Inserting (2.17) into the right-hand side of (2.16), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1 = \mathbb{R}^N \times (0, 1).$$

Since  $k \mapsto u_k$  is increasing, by Proposition 2.3 there exists  $\underline{U} := \lim_{k \rightarrow \infty} u_k$  and  $\underline{U} \geq u_k$ . Hence

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1, \forall k > 0.$$

Let  $\{t_n\} \subset (0, 1]$  be a sequence converging to 0. We choose  $k_n = \exp((2M_\beta t_n)^{\frac{1}{1-\beta}})$ , equivalently  $\ln k_n - M_\beta t_n (\ln k_n)^\beta = \frac{1}{2} \ln k_n$ . Next we restrict  $|x|$  in order

$$\ln k_n - M_\beta t_n (\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq r_\beta t_n^{\frac{\beta-2}{2(\beta-1)}},$$



where  $r_\beta = 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}}$ . Because  $1 < \beta \leq 2$ , it follows

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty,$$

uniformly on  $\mathbb{R}^N$  if  $1 \leq \beta < 2$ , or uniformly on the ball  $B_{r_2}$  where  $r_2 = (2M)^{-\frac{1}{2}}$  if  $\beta = 2$ . Since the sequence  $\{t_n\}$  is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on  $\mathbb{R}^N$  if  $1 \leq \beta < 2$ , or uniformly on the ball  $B_{r_2}$  if  $\beta = 2$ .

We pick some point  $x_0$  in  $\mathbb{R}^N$  (resp.  $B_{r_2}$ ) if  $1 < \beta < 2$  (resp.  $\beta = 2$ ). Since for any  $k > 0$ , the solution  $u_{k\delta_{x_0}}$  of (1.1) with initial data  $k\delta_{x_0}$  can be approximated by solutions with bounded initial data and support in  $B_\sigma(x_0)$  where  $0 < \sigma < r_2 - |x_0|$ , it follows

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t),$$

by comparison principle. Letting  $k \rightarrow \infty$  yields to  $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$ . Reversing the role of 0 and  $x_0$  yields to  $\underline{U}(x, t) = \underline{U}(x - x_0, t)$ . If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that  $\underline{U}(x, t)$  is independent of  $x$  and therefore it is a solution of (1.11) Since (1.10) holds,  $\underline{U}(x, t) = \phi_\infty(t)$  for every  $(x, t) \in Q_\infty$ .  $\square$

**Proposition 2.6** *Assume (1.12) and (1.8) are satisfied. Then for any  $k > 0$  there holds*

$$u_k(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty$$

where  $\Phi$  is a solution to the problem

$$\begin{cases} -\Phi'' + f(\Phi) = 0 & \text{in } (0, \infty) \\ \lim_{s \rightarrow 0} \Phi(s) = \infty. \end{cases}$$

*Proof. Step 1: Upper estimate.* Since  $f$  satisfies (1.12), by [4] for any  $R > 0$ , there exists a solution  $w_R$  to the problem

$$\begin{cases} -\Delta w_R + f(w_R) = 0 & \text{in } B_R, \\ \lim_{|x| \rightarrow R} w_R(x) = \infty, \end{cases} \quad (2.18)$$

and  $w_R$  is nonnegative since  $f(0) = 0$ . Notice also that  $R \mapsto w_R$  is decreasing, since  $f$  is nondecreasing; moreover  $\lim_{R \rightarrow \infty} w_R = 0$ , since  $f(0) = 0$  and  $f$  is positive on  $(0, \infty)$ . Let  $x_0 \neq 0$  arbitrary in  $\mathbb{R}^N$ . Set  $\mathbb{E} = \{\vec{e} : |\vec{e}| = 1\}$  and take  $\vec{e} \in \mathbb{E}$ .

Put  $x_{\vec{e}} = |x_0| \vec{e}$  and for  $n > |x_0|$  put  $a_n = n\vec{e}$ . Denote by  $\mathbb{H}_{\vec{e}}$  the open half-space generated by  $\vec{e}$  and its orthogonal hyperplane at the origin, then  $x_{\vec{e}}, a_n \in \mathbb{H}_{\vec{e}}$ . Take  $R$  such that  $n - |x_0| < R < n$ . We set  $W_{\vec{e},n,R}(x) = w_R(x - a_n)$ , then  $W_{\vec{e},n,R}$  is a solution of (1.1) in  $B_R(a_n)$  and blows-up on the boundary  $\lim_{|x-a_n| \rightarrow R} W_{\vec{e},n,R}(x) = \infty$ .

By the maximum principle,

$$u_k(x, t) \leq W_{\vec{e},n,R}(x) \quad \forall (x, t) \in B_R(a_n) \times (0, \infty). \quad (2.19)$$

The sequence  $\{W_{\vec{e},n,R}\}$  is decreasing with respect to  $R$  and is bounded from below by  $u_k$ , then there exists  $W_{\vec{e},n} := \lim_{R \rightarrow n} W_{\vec{e},n,R}$  satisfying

$$u_k(x, t) \leq W_{\vec{e},n}(x) \quad \forall (x, t) \in B_n(a_n) \times (0, \infty). \quad (2.20)$$

The sequence  $\{W_{\vec{e},n}\}$  is also decreasing with respect to  $n$  and is bounded from below by  $u_k$ , then there exists  $W_{\vec{e},\infty} := \lim_{n \rightarrow \infty} W_{\vec{e},n}$ . Letting  $n \rightarrow \infty$  in (2.20) yields to

$$u_k(x, t) \leq W_{\vec{e},\infty}(x) \quad \forall (x, t) \in \mathbb{H}_{\vec{e}} \times (0, \infty). \quad (2.21)$$

In particular,

$$u_k(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}). \quad (2.22)$$

Since  $u_k$  is radial, it follows that

$$u_k(x_0, t) = u_k(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}).$$

For any  $r > 0$ ,  $n > r$ ,  $n - r < R < n$  and  $\vec{e}, \vec{e}' \in \mathbb{E}$ , since  $w_R$  is radial,

$$w_R(r\vec{e} - n\vec{e}) = w_R(r\vec{e}' - n\vec{e}').$$

Letting successively  $R \rightarrow n$ ,  $n \rightarrow \infty$  yields to

$$W_{\vec{e},\infty}(r\vec{e}) = W_{\vec{e}',\infty}(r\vec{e}').$$

Define  $\tilde{\Phi}(r) := W_{\vec{e},\infty}(r\vec{e})$ ,  $\forall r \in (0, \infty)$  then it satisfies

$$\begin{cases} -\tilde{\Phi}'' - \frac{N-1}{r}\tilde{\Phi}' + f(\tilde{\Phi}) = 0 & \text{in } (0, \infty) \\ \lim_{r \rightarrow 0} \tilde{\Phi}(r) = \infty, \end{cases} \quad (2.23)$$

and

$$u_k(x, t) \leq \tilde{\Phi}(|x|) \quad \forall (x, t) \in Q_{\infty}. \quad (2.24)$$

*Step 2: End of the proof.* We claim that

$$\tilde{\Phi}(r) \leq \Phi(r) \quad \forall r \in (0, \infty). \quad (2.25)$$

For any  $\epsilon > 0$ , we set  $\Phi_\epsilon(r) = \Phi(r - \epsilon)$ ,  $r > \epsilon$  then  $\Phi_\epsilon$  is a solution of

$$-\Phi_\epsilon'' + f(\Phi_\epsilon) = 0 \quad \text{in } (\epsilon, \infty) \quad (2.26)$$

verifying  $\lim_{r \rightarrow \epsilon} \Phi_\epsilon(r) = \infty$ . Since  $\Phi_\epsilon' \leq 0$ ,  $\Phi_\epsilon$  is a supersolution of the equation in (2.23) in  $(\epsilon, \infty)$ , which dominates  $\tilde{\Phi}$  at  $r = \epsilon$ . By the maximum principle,  $\tilde{\Phi} \leq \Phi_\epsilon$  in  $(\epsilon, \infty)$ . Letting  $\epsilon \rightarrow 0$  yields (2.25). Combining (2.24) and (2.25) leads to the conclusion.  $\square$

*Remark.* Combining Proposition 2.3 and Proposition 2.6 yields to

$$u_k(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \quad \forall (x, t) \in Q_\infty, \forall k > 0. \quad (2.27)$$

#### Proof of Theorem 1.4.

The sequence  $\{u_k\}$  is increasing with respect to  $k$  and is bounded from above by (2.27) then there exists  $\underline{U} := \lim_{k \rightarrow \infty} u_k$  satisfying

$$\underline{U}(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \quad \forall (x, t) \in Q_\infty, \forall k > 0. \quad (2.28)$$

Moreover,  $\underline{U} \in \mathcal{U}_0$  because  $\underline{U}$  has the following properties:

- (i) It is positive in  $Q_\infty$ , belongs to  $C(\overline{Q_\infty} \setminus \{(0, 0)\})$  and vanishes on the set  $\{(x, 0) : x \neq 0\}$ .
- (ii) It satisfies (1.1) and

$$\lim_{t \rightarrow 0} \int_{B_\sigma} \underline{U}(x, t) dx = \infty, \quad \forall \sigma > 0. \quad (2.29)$$

In the sense of initial trace in Definition 4.3,  $\underline{U}$  has initial trace  $tr_{\mathbb{R}^N}(\underline{U}) = (\{0\}, 0)$  (here  $\{0\}$  is the singular part and the Radon measure on  $\mathbb{R}^N \setminus \{0\}$  is the zero measure) and the conclusion follows from Proposition 4.5.  $\square$

By a simple adaptation of the proof of Proposition 2.3 and Proposition 2.6 it is possible to extend (2.28) to any positive solution vanishing on  $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$ .

**Proposition 2.7** *Assume (1.12) and (C2) are satisfied. Then any positive solution  $u \in C^{2,1}(Q_\infty)$  of (1.1) satisfies*

$$u(x, t) \leq \phi_\infty(t) \quad \forall (x, t) \in Q_\infty. \quad (2.30)$$

*If we assume moreover that  $u \in C(\overline{Q_\infty} \setminus \{(0, 0)\})$  vanishes on  $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$ , there holds*

$$u(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \quad \forall (x, t) \in Q_\infty. \quad (2.31)$$

*Proof.* Since  $f(0) = 0$  and due to the convexity of  $f$ , the following inequality holds

$$f(a+b) \geq f(a) + f(b) \quad \forall a, b > 0, \quad (2.32)$$

which implies that for any  $R, \tau > 0$ ,  $(x, t) \mapsto \phi_\infty(t - \tau) + w_R(x)$  is a supersolution of (1.1) in  $B_R \times (\tau, \infty)$ . This function dominates  $u$  on the parabolic boundary, thus in the domain itself by the comparison principle. Since  $f(r) > 0$  if  $r > 0$ ,  $\lim_{R \rightarrow \infty} w_R = 0$  in  $\mathbb{R}^N$ . Therefore

$$u(x, t) \leq \phi_\infty(t) = \lim_{\tau \rightarrow 0} \lim_{R \rightarrow \infty} (\phi_\infty(t - \tau) + w_R(x)) \quad \forall (x, t) \in Q_\infty.$$

For the second estimate we notice that (2.19) is valid with  $u_k$  replaced by  $u$  (and without assumption (1.8) since existence is assumed). The remaining of the proof of Proposition 2.6 is similar and yields to

$$u(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty.$$

This implies (2.31).  $\square$

It is also possible to construct a maximal element of  $\mathcal{U}_0$  ( $\mathcal{U}_0$  is defined in Theorem 1.4). For  $\ell > 0$  and  $\epsilon > 0$ , let  $u := U_{\epsilon, \ell}$  be the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty \\ u(x, 0) = \ell \chi_{B_\epsilon} & \text{in } \mathbb{R}^N. \end{cases}$$

**Lemma 2.8** *For any  $\tau > 0$  and  $\epsilon > 0$ , there exist  $\ell > 0$  and  $m(\tau, \epsilon) > 0$  such that any positive solution  $u$  of (1.1) which verifies (i) in the proof of Theorem 1.4 satisfies*

$$u(x, t) \leq U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon) \quad \forall (x, t) \in Q_\infty, t \geq \tau. \quad (2.33)$$

Furthermore

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = 0 \quad \forall \epsilon > 0. \quad (2.34)$$

Finally

$$\bar{U}(x, t) = \lim_{\tau \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\ell \rightarrow \infty} (U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon)) \quad (2.35)$$

is the maximal element of  $\mathcal{U}_0$ .

*Proof.* We set  $\ell = \phi_\infty(\tau)$ , then  $u(x, \tau) \leq \ell$  for any  $x \in \mathbb{R}^N$ . Let  $W := W_{\epsilon/2}$  be the solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t W - \Delta W + f(W) = 0 & \text{in } B_{\epsilon/2}^c \times (0, \infty) \\ W(x, 0) = 0 & \text{in } B_{\epsilon/2}^c \\ W(x, t) = \phi_\infty(t) & \text{in } \partial B_{\epsilon/2}^c \times (0, \infty) \end{cases} \quad (2.36)$$

and put  $m(\tau, \epsilon) := \max\{W_{\epsilon/2}(x, \delta) : |x| > \epsilon, 0 < \delta \leq \tau\}$ . It is clear to see that

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = W_{\epsilon/2}(x, 0) = 0. \quad (2.37)$$

From the fact that  $u(x, 0) = 0$  in  $B_{\epsilon/2}^c$ ,  $u(x, t) \leq \phi_\infty(t)$  in  $\partial B_{\epsilon/2}^c \times (0, \infty)$  and the maximum principle, it follows that  $u(x, t) \leq W_{\epsilon/2}(x, t)$  in  $B_{\epsilon/2}^c \times (0, \infty)$ .

Next, we compare  $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$  with  $u$  in  $\mathbb{R}^N \times (\tau, \infty)$ . The function  $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$  is a supersolution of (1.1) in  $\mathbb{R}^N \times (\tau, \infty)$ . If  $x \in B_\epsilon$ ,  $U_{\epsilon, \ell}(x, 0) = \ell \geq u(x, \tau)$ , which implies  $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$ . If  $x \in B_\epsilon^c$ ,  $m(\tau, \epsilon) \geq W_{\epsilon/2}(x, \tau) \geq u(x, \tau)$ , hence  $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$ . So we always have  $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$  for any  $x \in \mathbb{R}^N$ . Applying maximum principle yields to  $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon) \geq u$  in  $\mathbb{R}^N \times (\tau, \infty)$ . Finally, the function  $\bar{U}$  defined by (2.35) is the maximal solution because  $U_{\epsilon, \ell}(x, t - \tau) \rightarrow U_{\epsilon, \ell}(x, t)$  as  $\tau \rightarrow 0$  and  $U_{\epsilon, \ell} \uparrow U_{\epsilon, \infty}$  when  $\ell \rightarrow \infty$  and  $U_{\epsilon, \infty} \downarrow \bar{U}$  when  $\epsilon \rightarrow 0$ .  $\square$

### 3 About uniqueness

We prove first the existence of global radial solutions of (1.18) under the Keller-Osserman condition.

#### Proof of Proposition 1.5.

A solution of (1.19) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N} \int_0^s t^{N-1} f(w) dt ds \quad (3.1)$$

Existence follows from the Picard-Lipschitz fixed point theorem. The function is increasing and defined on a maximal interval  $[0, r_a)$ . By a result of Vazquez and Veron [11]  $r_a = \infty$ , thus the solution is global. Uniqueness on  $[0, \infty)$  follows always from local uniqueness. The function  $r \mapsto w(r)$  is increasing and

$$\begin{aligned} w'(r) &\geq \frac{ah(a)}{N}r, \\ w(r) &\geq a + \frac{ah(a)}{2N}r^2 \end{aligned}$$

for all  $r > 0$ .  $\square$

**Proposition 3.1** *Assume (1.16) holds. For any  $u_0 \in C(\mathbb{R}^N)$  which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.2)$$

*for some  $0 < a < b$ , there exists a positive function  $\bar{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$  solution of (1.1) in  $Q_\infty$  and satisfying  $\bar{u}(\cdot, 0) = u_0$  in  $\mathbb{R}^N$ . Furthermore*

$$w_a(|x|) \leq u(x, t) \leq w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (3.3)$$

*Proof.* Clearly  $w_a$  and  $w_b$  are ordered solutions of (1.1). We denote by  $u_n$  the solution of the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) = 0 & \text{in } Q_n = B_n \times (0, \infty) \\ u_n(x, t) = (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0, \infty) \\ u_n(x, 0) = u_0(x) & \text{in } B_n. \end{cases} \quad (3.4)$$

By the maximum principle,  $u_n$  satisfies (3.3) in  $Q_n$ . Using locally parabolic equations regularity theory, we derive that the set of functions  $\{u_n\}$  is eventually equicontinuous on any compact subset of  $\overline{Q_\infty}$ . Using a diagonal sequence, we conclude that there exists a subsequence  $\{u_{n_k}\}$  which converges locally uniformly in  $\overline{Q_\infty}$  to some weak solution  $\bar{u} \in C(\overline{Q_\infty})$  which satisfies  $\bar{u}(\cdot, 0) = u_0$  in  $\mathbb{R}^N$ . By standard method,  $\bar{u}$  is a strong solution (at least  $C^{2,1}(Q_\infty)$ ).  $\square$

**Proposition 3.2** *Assume (1.16) and (1.10) hold. Then for any  $u_0 \in C(\mathbb{R}^N)$  which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.5)$$

*for some  $0 < a < b$ , there exists a positive function  $\underline{u} \in C(\overline{Q_\infty})$  solution of (1.1) in  $Q_\infty$  satisfying  $\underline{u}(\cdot, 0) = u_0$  in  $\mathbb{R}^N$  and*

$$\underline{u}(x, t) \leq \min\{\phi_\infty(t), w_b(|x|)\} \quad \forall (x, t) \in Q_\infty. \quad (3.6)$$

*Proof.* For any  $R > 0$ , let  $u_R$  be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R + f(u_R) = 0 & \text{in } Q_\infty \\ u_R(x, 0) = u_0(x) \chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases} \quad (3.7)$$

The solution which is constructed is dominated by the solution of the heat equation with the same initial data. Thus

$$u_R(x, t) \leq (4\pi t)^{-N/2} \int_{B_R} e^{-|x-y|^2/4t} u_0(y) dy \quad \forall (x, t) \in Q_\infty. \quad (3.8)$$

and  $\lim_{|x| \rightarrow \infty} u_R(x, t) = 0$  uniformly with respect to  $t$ . The functions  $\phi_\infty$  and  $w_b$  are solutions of (1.1) in  $Q_\infty$ , which dominate  $u_R$  at  $t = 0$ . By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x, t) \quad \forall (x, t) \in Q_\infty. \quad (3.9)$$

The fact that the mapping  $R \mapsto u_R$  is increasing and (3.9) imply that there exists  $\underline{u} := \lim_{R \rightarrow \infty} u_R$  which satisfies  $\underline{u}(\cdot, 0) = u_0$  in  $\mathbb{R}^N$ . Letting  $R \rightarrow \infty$  in (3.9) yields (3.6).  $\square$

**Proof of Theorem 1.6.**

Combining Proposition 3.1 and Proposition 3.2 we see that there exists two solutions  $\underline{u}$  and  $\bar{u}$  with the same initial data  $u_0$  which are ordered and different since  $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$  and  $\lim_{|x| \rightarrow \infty} \underline{u}(x, t) \leq \phi_\infty(t) < \infty$  for all  $t > 0$ .  $\square$

**Proof of Theorem 1.7.**

*Step 1: There always holds*

$$(ah(a) - bh(b))\text{sign}(a - b) \geq |a - b|h(|a - b|) \quad \forall a, b > 0 \quad (3.10)$$

where  $h$  is defined in (2.2) and

$$\text{sign}(z) = \begin{cases} 1 & \text{if } z > 0, \\ -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

In fact, since  $h$  is increasing and assuming  $a > b$ , we get

$$\begin{aligned} ah(a) - bh(b) &= (a - b)h(a) + b(h(a) - h(b)) \\ &\geq (a - b)h(a) \\ &\geq (a - b)h(a - b). \end{aligned}$$

*Step 2: End of the proof.* By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \leq [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})]\text{sign}(u - \tilde{u}),$$

therefore by step 1,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + |u - \tilde{u}|h(|u - \tilde{u}|) \leq 0. \quad (3.11)$$

Let  $\epsilon > 0$ . There exists  $R_\epsilon > 0$  such that for any  $R \geq R_\epsilon$ ,

$$0 \leq |u - \tilde{u}|(x, t) \leq w_\epsilon(|x|) \quad \forall (x, t) \in B_R^c \times [0, 1]. \quad (3.12)$$

Since  $w_\epsilon$  is a positive solution of (1.1) which dominates  $|u - \tilde{u}|$  on  $\partial B_R \times [0, 1]$  and at  $t = 0$ , it follows that  $|u - \tilde{u}| \leq w_\epsilon$  in  $B_R \times [0, 1]$ . Letting  $R \rightarrow \infty$  yields to  $|u - \tilde{u}| \leq w_\epsilon$  in  $\mathbb{R}^N \times [0, 1]$ . Letting  $\epsilon \rightarrow 0$  and since  $\lim_{\epsilon \rightarrow 0} w_\epsilon(|x|) = 0$  for any  $x \in \mathbb{R}^N$ , we derive  $|u - \tilde{u}| = 0$ , thus  $u = \tilde{u}$  in  $\mathbb{R}^N \times [0, 1]$ . Iterating yields that equality holds in  $Q_\infty$ .  $\square$

*Remark.* If we replace the condition (C1) by the condition (C2), the conclusion of Theorem 1.7 remains valid. Indeed, it follows by the convexity of  $f$  that

$$(f(a) - f(b))\text{sign}(a - b) \geq f(|a - b|) \quad \forall a, b > 0.$$

Then we proceed as in step 2 to get the desired conclusion.

**Proof of Theorem 1.8.**

*Proof of statement (i).* The solution  $\underline{u}$  which is constructed in Proposition 3.2 is a minimal solution of (1.1) in  $Q_\infty$  with the initial value  $u_0$ . Indeed, if  $u \in C^{2,1}(Q_\infty)$  is a nonnegative solution of (1.1) in  $Q_\infty$  which satisfies  $u(\cdot, 0) = u_0$  in  $\mathbb{R}^N$  then, by maximum principle,  $u_R \leq u$  in  $Q_\infty$  where  $u_R$  is the solution of (3.7). Letting  $R \rightarrow \infty$  yields  $\underline{u} \leq u$  in  $Q_\infty$ . Next we construct the maximal solution. We recall that  $w_R$  is the solution of (2.18). Since  $f$  is convex and  $f(0) = 0$ , there holds  $f(w_R) + f(u_R) \leq f(w_R + u_R)$ . Consequently  $w_R + u_R$  is a supersolution of (1.1) in  $B_R \times (0, \infty)$ . If  $u \in C(\overline{Q_\infty})$  is a solution (1.1) in  $Q_\infty$  with initial data  $u_0$ , it is dominated by  $w_R + u_R$  on  $(\partial B_R \times (0, \infty)) \cup (B_R \times \{0\})$ . Thus, by maximum principle,  $u \leq w_R + u_R$  in  $B_R \times (0, \infty)$ . Since

$$u_R \leq u \leq w_R + u_R,$$

$w_R \rightarrow 0$  when  $R \rightarrow \infty$ , by Proposition 2.6-Step 1, and  $u_R \rightarrow \underline{u}$ , we derive that  $u = \underline{u}$ .

*Proof of statement (ii).*

*Step 1: Construction of a minimal solution.* Assume there exists at least one positive solution  $u$  of (1.1) satisfying (1.23) and  $f(u) \in L^1_{loc}(\overline{Q_\infty})$ , equivalently [7]

$$\int_0^\infty \int_{\mathbb{R}^N} (-u(\partial_t \eta + \Delta \eta) + f(u)\eta) dxdt = \int_{\mathbb{R}^N} \eta(x, 0) d\mu(x) \quad (3.13)$$

for all  $\eta \in C_c^{2,1}(\overline{Q_\infty})$ . We construct first a minimal solution in the following way: let  $n \in \mathbb{N}$  and  $R > 0$  and let  $v = v_{R,n}$  be the solution of

$$\begin{cases} \partial v - \Delta v + f(v) = 0 & \text{in } B_R \times (0, \infty) \\ v = 0 & \text{in } \partial B_R \times (0, \infty) \\ v(\cdot, 0) = u(\cdot, 2^{-n}) & \text{in } B_R. \end{cases} \quad (3.14)$$

By the maximum principle,  $v_{R,n}(\cdot, t) \leq u(\cdot, t + 2^{-n})$ . Furthermore,

$$v_{R,n}(x, 2^{-n}) \leq u(x, 2^{-n+1}) = v_{R,n-1}(x, 0),$$

therefore,

$$v_{R,n}(x, t + 2^{-n}) \leq v_{R,n-1}(x, t) \quad \text{in } B_R \times (0, \infty). \quad (3.15)$$

Using the formulation (3.13) with  $v_{R,n}$ , we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (-v_{R,n}(\partial_t \eta + \Delta \eta) + f(v_{R,n})\eta) dxdt = \int_{\mathbb{R}^N} \eta(x, 0) u(x, 2^{-n}) dx, \quad (3.16)$$



for any  $\eta \in C_c^{2,1}(\overline{Q_\infty^{B_R}})$ . The right-hand side of (3.16) converges to  $\int_{\mathbb{R}^N} \eta(x, 0) d\mu(x)$  as  $n \rightarrow \infty$ . Concerning the left-hand side, there holds  $f(v_{R,n}(x, t)) \leq f(u(x, t + 2^{-n}))$ . Since  $f(u) \in L_{loc}^1(\overline{Q_\infty})$ ,  $f(v_{R,n})$  is bounded in  $L_{loc}^1(\overline{Q_\infty^{B_R}})$ . By the  $L^1$  regularity theory for parabolic equations (see [6] and the references therein), the set of functions  $\{v_{R,n}\}$  is locally compact in  $L_{loc}^1(Q_\infty)$  and there exists a subsequence  $\{n_k\}$  and a function  $\underline{u}_R$  such that  $v_{R,n_k} \rightarrow \underline{u}_R$ , almost everywhere in  $Q_\infty^{B_R}$ , and  $\underline{u}_R \leq u$ . Noticing that the sets of functions  $\{f(u(\cdot, \cdot + 2^{-n}))\}$  and  $\{u(\cdot, \cdot + 2^{-n})\}$  are uniformly integrable, we obtain that the two sets  $\{f(v_{R,n})\}$  and  $\{v_{R,n}\}$  are also uniformly integrable in  $B_R \times (0, T)$ . It follows from Vitali's convergence theorem that, up to a subsequence still denoted by  $\{n_k\}$ ,  $v_{R,n_k} \rightarrow \underline{u}_R$  and  $f(v_{R,n_k}) \rightarrow f(\underline{u}_R)$  in  $L^1(B_R \times (0, T))$ . Letting  $n = n_k \rightarrow \infty$  in (3.16) we derive

$$\int_0^\infty \int_{\mathbb{R}^N} (-\underline{u}_R(\partial_t \eta + \Delta \eta) + f(\underline{u}_R)\eta) dx dt = \int_{\mathbb{R}^N} \eta(x, 0) d\mu(x). \quad (3.17)$$

This means that  $\underline{u}_R$  satisfies  $\underline{u}_R \leq u$  and

$$\begin{cases} \partial_t \underline{u}_R - \Delta \underline{u}_R + f(\underline{u}_R) = 0 & \text{in } B_R \times (0, \infty) \\ \underline{u}_R = 0 & \text{in } \partial B_R \times (0, \infty) \\ \underline{u}_R(\cdot, 0) = \chi_{B_R} \mu & \text{in } B_R. \end{cases} \quad (3.18)$$

If  $\tilde{u}$  is any other nonnegative solution of (1.1) in  $Q_\infty$  with initial data  $\mu$ , the same construction of  $\tilde{v}_{R,n}$  solution of (3.14) with initial data  $\tilde{u}(\cdot, 2^{-n})$  instead of  $u(\cdot, 2^{-n})$  converges, up to a subsequence to some  $\tilde{\underline{u}}_R$  which satisfies  $\tilde{\underline{u}}_R \leq \tilde{u}$  and is solution of problem (3.18). We know from [5], [6] that this problem admits at most one solution. Therefore  $\tilde{\underline{u}}_R = \underline{u}_R$ , which implies that  $\underline{u}_R \leq \tilde{u}$  in  $Q_\infty^{B_R}$ . Furthermore, in the above construction, we have only used the fact that  $\tilde{u}$  is defined in a domain larger than  $Q_\infty^{B_R}$  and is nonnegative. Consequently, the same comparison applies if we compare  $\underline{u}_R$  and  $\underline{u}_{R'}$  for  $R' > R$  and we obtain

$$\underline{u}_R \leq \underline{u}_{R'} \quad \text{in } Q_\infty^{B_R}.$$

Put  $\underline{u} = \lim_{R \rightarrow \infty} \underline{u}_R$ . Using the monotone convergence theorem and a test function  $\eta \in C_c^{2,1}(\overline{Q_\infty})$  with compact support in  $Q_\infty^{B_R}$ , we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} (-\underline{u}(\partial_t \eta + \Delta \eta) + f(\underline{u})\eta) dx dt = \int_{\mathbb{R}^N} \eta(x, 0) d\mu(x). \quad (3.19)$$

from (3.17). Thus  $\underline{u}$  satisfies (1.23) and  $f(\underline{u}) \in L_{loc}^1(\overline{Q_\infty})$ . By construction  $\underline{u}$  is smaller than any other nonnegative solution.

*Step 2: End of proof of statement (ii).* As in the proof of statement (i), we see that, for any  $n \in \mathbb{N}^*$ , there holds  $u \leq w_R + v_{R,n}$  in  $Q_\infty^{B_R}$  where  $w_R$  is the maximal solution of (2.18). Consequently  $u \leq w_R + \underline{u}_R$  and letting  $R \rightarrow \infty$ ,  $u \leq \underline{u}$ . Thus  $u = \underline{u}$ .  $\square$

## 4 Initial trace

If  $\Omega$  is an open domain in  $\mathbb{R}^N$ , we denote by  $\mathfrak{M}(\Omega)$  (resp.  $\mathfrak{M}^b(\Omega)$ ) the set of Radon measures in  $\Omega$  (resp. bounded Radon measures), and by  $\mathfrak{M}_+(\Omega)$  (resp.  $\mathfrak{M}_+^b(\Omega)$ ) its positive cone. For  $T > 0$ , we set  $Q_T^\Omega = \Omega \times (0, T)$ .

### 4.1 The regular part of the initial trace

In this section we only assume that  $f$  is a continuous nonnegative function defined on  $\mathbb{R}_+$  and that  $u$  is a  $C^{2,1}$  positive solution of (1.1) in  $Q_T$ .

**Lemma 4.1** *Assume  $G$  is a bounded  $C^2$  domain in  $\mathbb{R}^N$ ,  $Q_T^{\overline{G}} := \overline{G} \times (0, T]$  and let  $u \in C^{2,1}(Q_T^{\overline{G}})$  be a positive solution of (1.1) in  $Q_T^{\overline{G}}$  such that  $u, f(u) \in L^1(Q_T^{\overline{G}})$ . Then  $u \in L^\infty(0, T; L^1(G'))$  for any domain  $G' \subset \overline{G'} \subset G$  and there exists a positive Radon measure  $\mu_G$  on  $G$  such that*

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G \zeta(x) d\mu_G(x) \quad \forall \zeta \in C_c(G). \quad (4.1)$$

*Proof.* Let  $\phi := \phi_G$  be the first eigenfunction of  $-\Delta$  in  $W_0^{1,2}(G)$  with corresponding eigenvalue  $\lambda_G$ . We assume  $\phi > 0$  in  $G$ . Then

$$\frac{d}{dt} \int_G u \phi dx + \lambda_G \int_G u \phi dx + \int_G f(u) \phi dx + \int_{\partial G} u \phi_{\mathbf{n}} dS = 0$$

where  $\phi_{\mathbf{n}}$  denote the outward normal derivative of  $\phi$ . Since  $\phi_{\mathbf{n}} < 0$ , the function

$$t \mapsto e^{\lambda_G t} \int_G u(x, t) \phi(x) dx - \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

is increasing and

$$\int_G u(x, t) \phi(x) dx \leq e^{\lambda_G(T-t)} \int_G u(x, T) \phi(x) dx + e^{-\lambda_G t} \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

for  $0 < t \leq T$ . Thus  $u \in L^\infty(0, T; L^1(G'))$  for any strict domain  $G'$  of  $G$ . If  $\zeta \in C_c^2(G)$ , there holds

$$\frac{d}{dt} \left( \int_G u(x, t) \zeta(x) dx - \int_t^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds \right) = 0. \quad (4.2)$$

Consequently

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G u(x, T) \zeta(x) dx + \int_0^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds. \quad (4.3)$$

This implies that  $u(., t)$  admits a limit in  $\mathcal{D}'(G)$ , and this limit is a positive distribution. Therefore there exists a positive Radon measure  $\mu_G$  on  $G$  satisfies (4.1).  $\square$

### Proof of Proposition 1.9.

It is clear that  $\mathcal{R}(u)$  is an open subset. If  $G$  is a strict bounded subdomain of  $\mathcal{R}(u)$ , i.e.  $\overline{G} \subset \mathcal{R}(u)$ , there exists a finite number of points  $z_j$  ( $j = 1, \dots, k$ ) and  $r'_j > r_j > 0$  such that  $u, f(u) \in L^1(Q_T^{B_{r'_j}(z_j)})$  and  $\overline{G} \subset \cup_{j=1}^k B_{r_j}(z_j)$ . Let  $\mu_j = \mu_{B_{r_j}(z_j)}$  the measure defined in Lemma 4.1. If  $\zeta \in C_c(G)$  there exists a partition of unity  $\{\eta_j\}_{j=1}^k$  relative to the cover  $\{B_{r_j}(z_j)\}_{j=1}^k$  such that  $\eta_j \in C_0^\infty(G)$ ,  $\text{supp}(\eta_j) \subset B_{r_j}(z_j)$  and  $\zeta = \sum_{j=1}^k \eta_j \zeta$ . Since

$$\lim_{t \rightarrow 0} \int_{B_{r_j}(z_j)} u(x, t) (\eta_j \zeta)(x) dx = \int_{B_{r_j}(z_j)} (\eta_j \zeta)(x) d\mu_j(x) \quad \forall j = 1, \dots, k,$$

there exists a positive Radon measure  $\mu$  on  $\mathcal{R}(u)$  satisfying (1.24). Notice also that  $u \in L^\infty(0, T; L^1(G))$  for any  $G \subset \overline{G} \subset \mathcal{R}(u)$ .  $\square$

The main problem is to analyse the behaviour of  $u$  on the singular set  $\mathcal{S}(u)$ .

## 4.2 The Keller-Osserman condition holds

If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (1.1) is based upon a dichotomy in the behaviour of those solutions near  $t = 0$ .

**Lemma 4.2** *Assume  $u$  is a positive solution of (1.1) in  $Q_T$  and  $z \in \mathcal{S}(u)$ . Suppose that at least one of the following sets of conditions holds.*

- (i) *There exists an open neighborhood  $G$  of  $z$  such that  $u \in L^1(Q_T^G)$ .*
- (ii)  *$f$  is nondecreasing and (1.12) holds.*

*Then, for every open relative neighborhood  $G'$  of  $z$ ,*

$$\lim_{t \rightarrow 0} \int_{G'} u(x, t) dx = \infty. \quad (4.4)$$

*Proof.* First, we assume that (i) holds and let  $\zeta \in C_c^2(G)$ ,  $\zeta \geq 0$ . Since  $z \in \mathcal{S}(u)$ , then for every open relative neighborhood  $G'$  of  $z$ , there holds

$$\int_0^T \int_{G'} f(u) dx dt = \infty. \quad (4.5)$$

Since there exists

$$\lim_{t \rightarrow 0} \int_t^T \int_{G'} u \Delta \zeta dx dt = L \in \mathbb{R},$$

it follows from (4.3) that

$$\int_{G'} u(x, t) \zeta(x) dx = \int_t^T \int_{G'} f(u) \zeta dx ds + O(1), \quad (4.6)$$

which implies (4.4).

Next we assume that (1.12) holds and  $u \notin L^1(Q_T^G)$  for every relative neighborhood  $G$  of  $z$ . If there exists an open neighborhood  $G \subset \Omega$  of  $z$  such that (4.4) does not hold, there exists a sequence  $\{t_n\}$  decreasing to 0 and  $0 \leq M < \infty$  such that

$$\sup_{t_n} \int_G u(x, t_n) dx = M. \quad (4.7)$$

Furthermore, we can always replace  $G$  by an open ball  $B_R(z) \subset G$ . Thus (4.7) holds with  $G$  replaced by  $B_R(z)$ . Let  $w := w_R$  be the maximal solution of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } B_R(z) \\ \lim_{|x-z| \rightarrow R} w(x) = \infty. \end{cases} \quad (4.8)$$

Let  $v := v_n$  be the solution of

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } B_R(z) \times (t_n, \infty) \\ v = 0 & \text{in } \partial B_R(z) \times (t_n, \infty) \\ v(\cdot, t_n) = u(\cdot, t_n) & \text{in } B_R(z). \end{cases} \quad (4.9)$$

Since  $v_n \geq 0$ ,  $f(w_R + v_n) \geq f(w_R)$ , and  $w_R + v_n$  is a supersolution of (1.1) in  $B_R(z) \times (t_n, T)$ . It dominates  $u$  on  $\partial B_R(z) \times (t_n, T)$  and at  $t = t_n$ , thus  $u \leq w_R + v_n$  in  $B_R(z) \times (t_n, T)$ . We can assume that  $u(\cdot, t_n) \rightarrow \nu$  for some positive and bounded measure  $\nu$  on  $B_R(z)$ . Therefore

$$u(x, t) \leq v(x, t) + w_R(x) \quad \text{in } Q_T^{B_R(z)} \quad (4.10)$$

where  $v$  is the solution of

$$\begin{cases} \partial_t v - \Delta v = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) = \nu & \text{in } \mathcal{D}'(B_R(z)). \end{cases} \quad (4.11)$$

Since  $v \in L^1(Q_T^{B_R(z)})$  and  $w_R$  is uniformly bounded in any ball  $B_{R'}(z)$  for  $0 < R' < R$ , we conclude that  $u \in L^1(Q_T^{B_{R'}(z)})$ , which is a contradiction.  $\square$

**Definition 4.3** Assume  $f$  is nondecreasing and satisfies (1.12). Let  $u \in C^{2,1}(Q_T)$  be a positive solution of (1.1) in  $Q_T$ . We say that  $u$  possesses an initial trace with regular part  $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$  and singular part  $\mathcal{S}(u) = \mathbb{R}^N \setminus \mathcal{R}(u)$  if

(i) For any  $\zeta \in C_c(\mathcal{R}(u))$ ,

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x). \quad (4.12)$$

(ii) For any open set  $G \subset \mathbb{R}^N$  such that  $G \cap \mathcal{S}(u) \neq \emptyset$

$$\lim_{t \rightarrow 0} \int_G u(x, t) dx = \infty. \quad (4.13)$$

### Proof of Theorem 1.11

The set  $\mathcal{R}(u)$  and the measure  $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$  are defined by Definition 1.10 thanks to Proposition 1.9. Because (4.12) holds,  $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$  inherits the property (ii) in Definition 4.3 because of Lemma 4.2 (ii).  $\square$

If  $\Omega$  is a bounded domain with a  $C^2$  boundary and  $\mu \in \mathfrak{M}_+^b(\Omega)$ , we denote by  $u_\mu$  the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = \mu & \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (4.14)$$

We recall the following stability result proved in [6, Th 1.1].

**Lemma 4.4** Let  $\Omega$  be a bounded domain with a  $C^2$  boundary. Assume  $f$  is nondecreasing and satisfies (1.8). Then for any  $\mu \in \mathfrak{M}^b(\Omega)$  problem (4.14) admits a unique solution  $u_\mu$ . Moreover, if  $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$  converges weakly to  $\mu \in \mathfrak{M}^b(\Omega)$  then  $u_{\mu_n} \rightarrow u_\mu$  locally uniformly in  $\bar{\Omega} \times (0, \infty)$  and in  $L^1(Q_T^\Omega)$ , and  $f(u_{\mu_n}) \rightarrow f(u_\mu)$  in  $L^1(Q_T^\Omega)$ , for every  $T > 0$ .

*Remark.* The result remains true if  $\Omega$  is unbounded, with a  $C^2$  compact (possibly empty) boundary and the  $\mu_n$  have their support in a fixed compact set. In such a case  $u_{\mu_n}(x, t) \rightarrow 0$  when  $|x| \rightarrow \infty$ , uniformly with respect to  $n$  and  $t$  since

$$|u_{\mu_n}(x, t)| \leq \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-|x-y|^2/4t} d|\mu_n|(y) \quad \forall (x, t) \in Q_\infty. \quad (4.15)$$

By Lemma 4.4 and the remark hereafter, for every  $y \in \Omega$  and  $k > 0$ , there exists a unique solution  $v_{y,k,\Omega} := v$  to (4.14) with  $\mu = k\delta_0$ . By comparison principle (see

[6, Prop 1.2])  $v_{y,k,\Omega}$  is positive, increases as  $k$  increases and depends continuously on  $y$ . Note that if  $\Omega = \mathbb{R}^N$ ,  $v_{y,k,\mathbb{R}^N}(x,t) := v_{y,k}(x,t) = u_k(|x-y|,t)$ ; furthermore, if  $f$  satisfies (1.12), we recall that  $\underline{U} = \lim_{k \rightarrow \infty} u_k$  is the minimal solution of (1.1) in  $Q_\infty$  with initial trace  $(\{0\}, 0)$ .

**Proposition 4.5** *Assume  $f$  is nondecreasing and satisfies (1.8) and (1.12). Let  $u \in C^{2,1}(Q_\infty)$  is a positive solution of (1.1) in  $Q_\infty$  with initial trace  $(\mathcal{S}, \mu)$ . Then for every  $y \in \mathcal{S}$ ,*

$$\underline{U}_y(x,t) := \underline{U}(x-y,t) \leq u(x,t) \quad (4.16)$$

in  $Q_\infty$ .

*Proof.* By translation we may suppose that  $y = 0$ . Since  $0 \in \mathcal{S}(u)$ , for any  $\eta > 0$  small enough

$$\lim_{t \rightarrow 0} \int_{B_\eta} u(x,t) dx = \infty.$$

For  $\epsilon > 0$ , denote  $M_{\epsilon,\eta} = \int_{B_\eta} u(x,\epsilon) dx$ . For any  $m > m_\eta = \inf_{\sigma > 0} M_{\sigma,\eta}$  there exists  $\epsilon = \epsilon(m,\eta)$  such that  $m = M_{\epsilon,\eta}$  and  $\lim_{\eta \rightarrow 0} \epsilon(m,\eta) = 0$ . Let  $v_\eta$  be the solution of the problem

$$\begin{cases} \partial_t v_\eta - \Delta v_\eta + f(v_\eta) = 0 & \text{in } Q_\infty \\ v_\eta(x,0) = u(x,\epsilon)\chi_{B_\eta} & \text{in } \mathbb{R}^N \end{cases}$$

where  $\chi_{B_\eta}$  is the characteristic function of  $B_\eta$ . By the maximum principle  $v_\eta \leq u$  in  $\mathbb{R}^N \times (\epsilon, \infty)$ . By Lemma 4.4 and the remark after  $v_\eta$  converges to  $v_{0,m}$  when  $\eta$  goes to zero. Letting  $m$  go to infinity yields (4.16).  $\square$

**Corollary 4.6** *Under the assumption of Proposition 4.5, there exists a minimal positive solution  $\underline{U}_\mathcal{S}$  of (1.1) in  $Q_\infty$  with initial trace  $(\mathcal{S}, 0)$  in the sense that*

$$\underline{U}_\mathcal{S}(x,t) \leq u(x,t) \quad \forall (x,t) \in Q_\infty, \quad (4.17)$$

for all positive solution  $u \in C^{2,1}(Q_\infty)$  of (1.1) with initial trace  $(\mathcal{S}(u), \mu)$ .

*Proof.* If we set  $\tilde{\underline{U}}_\mathcal{S} = \sup\{U_y : y \in \mathcal{S}\}$ , then  $\tilde{\underline{U}}_\mathcal{S}$  is a subsolution of (1.1). If  $u$  is a positive solution of (1.1) with initial trace  $(\mathcal{S}, \mu)$ , then  $u \geq \tilde{\underline{U}}_\mathcal{S}$  by Proposition 4.5. Therefore  $u$  is larger than the smallest solution of (1.1) in  $Q_\infty$  which is above  $\tilde{\underline{U}}_\mathcal{S}$ . We denote this minimal solution by  $\underline{U}_\mathcal{S}$ .  $\square$

If  $\mathcal{S}$  contains some ball  $B_R$  we have a more precise result.

**Proposition 4.7** *Let  $u$  be a positive solution of (1.1) in  $Q_\infty$  with initial trace  $(\mathcal{S}, \mu)$ . We assume that  $\mathcal{S}$  has a non-empty interior, and for  $R > 0$ , we denote by  $\text{int}_R(\mathcal{S})$  the set of  $y \in \mathcal{S}$  such that  $\overline{B}_R(y) \subset \mathcal{S}$ . Then for any  $R' \in (0, R)$  there holds*

$$\lim_{t \rightarrow 0} \frac{u(x, t)}{\phi_\infty(t)} = 1 \quad (4.18)$$

uniformly for  $x \in \overline{B}_{R'}(y)$  and  $y \in \text{int}_R(\mathcal{S})$ .

*Proof.* Let  $y \in \text{int}_R(\mathcal{S})$  and  $w(x, t) = u(x, t) + w_R(x - y)$  where  $w_R$  is the maximal solution of (2.18). Then  $w$  is a supersolution of (1.1) in  $Q_\infty^{B_R(y)}$  and  $\lim_{t \rightarrow 0} w(x, t) = \infty$ , uniformly with respect to  $x \in B_R(y)$ , by (4.16). Then, for any  $\epsilon > 0$ , there exists  $t_\epsilon > 0$  such that  $w(x, t) \geq \phi_\infty(\epsilon)$  in  $Q_{t_\epsilon}^{B_R(y)}$ . Since  $\phi_\infty(t + \epsilon)$  remains bounded on  $\partial B_R(y) \times (0, \infty)$ , it follows by the maximum principle that

$$w(x, t) \geq \phi_\infty(t + \epsilon) \quad \forall (x, t) \in Q_\infty^{B_R(y)}.$$

Letting  $\epsilon \rightarrow 0$  and using the fact that  $w_R(x - y)$  remains uniformly bounded when  $|x - y| \leq R'$ , we derive

$$u(x, t) \geq \phi_\infty(t) - K_{R'} \quad \forall (x, t) \in Q_\infty^{B_{R'}(y)}. \quad (4.19)$$

where  $K_{R'} = \max\{w_R(x - y) : |x - y| \leq R'\}$ . Combining this estimate with (2.30) yields to (4.18).  $\square$

**Proposition 4.8** *Assume  $f$  is nondecreasing and satisfies (1.8) and (1.12). Let  $\{u_n\}$  be a sequence of positive solutions of (1.1) in  $Q_\infty$  with initial trace  $(\mathcal{S}(u_n), \mu_n)$  such that  $u_n \rightarrow u$  locally uniformly in  $Q_\infty$  and let  $A$  be an open subset of  $\mathcal{R}(u_n) := \mathbb{R}^N \setminus \mathcal{S}(u_n)$ . Then  $u$  is a positive solution of (1.1) in  $Q_\infty$ , with initial trace denoted by  $\text{tr}_{\mathbb{R}^N}(u) = (\mathcal{S}, \mu)$ . Furthermore, if  $\mu_n(A)$  remains uniformly bounded, then  $A \subset \mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$  and  $\chi_A \mu_n \rightarrow \chi_A \mu$  weakly. Conversely, if  $A \subset \mathcal{R}(u)$ , then  $\mu_n(K)$  remains bounded independently of  $n$ , for every compact set  $K \subset A$ .*

*Proof.* The fact that  $u$  is a positive solution of (1.1) in  $Q_\infty$  is standard by the weak formulation of the equation. Assume now that  $A \cap \mathcal{S} \neq \emptyset$ . Let  $z \in A \cap \mathcal{S}$  and  $R > 0$  such that  $\overline{B}_R(z) \subset A$ . By convexity,  $u_n$  is bounded from above in  $Q_\infty^{B_R(z)}$  by  $v_{n,z} + w_R$ , where  $v_{n,z}$  satisfies

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) = \chi_{B_R(z)} \mu_n & \text{in } B_R(z), \end{cases} \quad (4.20)$$

and  $w_R$  is the maximal solution of (4.8). We can assume that, up to a subsequence,  $\chi_{B_R(z)}\mu_{n_k} \rightarrow \mu_z \in \mathfrak{M}_+^b(B_R(z))$  weakly, thus  $v_{n_k,z} \rightarrow v_z$  where  $v_z$  is the solution of

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) = \mu_z & \text{in } B_R(z), \end{cases} \quad (4.21)$$

Therefore

$$u \leq v_z + w_R \quad \text{in } Q_\infty^{B_R(z)}. \quad (4.22)$$

By Lemma 4.4, it implies that  $u \in L^1(Q_T^{B_{R'}(z)})$  for any  $0 < R' < R$ . Furthermore, if (1.8) is satisfied, then for any positive constant  $k$ ,  $s \mapsto s^{N/2}f(s^{-N/2} + k) \in L^1(0, 1)$ , thus if  $v$  is such that  $f(v) \in L^1(Q_T^{B_{R'}(z)})$ , there holds  $f(v + k) \in L^1(Q_T^{B_{R'}(z)})$ . In particular, since  $f(v_z) \in L^1(Q_T^{B_{R'}(z)})$ , and if we take  $k = \max\{w_R(x) : x \in B_{R'}(z)\}$ , we derive that  $f(u) \in L^1(Q_T^{B_{R'}(z)})$ , and therefore  $z \in \mathcal{R}$ , which is a contradiction; thus  $A \subset \mathcal{R}$ . Next, there exist a subsequence  $\{n_k\}$  and a bounded positive measure  $\tilde{\mu}$ , with support in  $A$  such that  $\chi_A \mu_{n_k} \rightarrow \tilde{\mu}$  weakly and suppose  $\overline{B_R(z)} \subset A$ . Since  $u_{n_k} \leq v_{n_k,z} + k$  and  $f(u_{n_k}) \leq f(v_{n_k,z} + k)$  in  $Q_T^{B_{R'}(z)}$  and  $v_{n_k,z} + k$  and  $f(v_{n_k,z} + k)$  are uniformly integrable in  $Q_T^{B_{R'}(z)}$ , it follows that  $u_{n_k}$  and  $f(u_{n_k,z})$  inherit this property. Therefore, if  $\zeta \in C_c^2(B_R(z))$  we can assume that it vanishes outside  $B_{R'}(z)$ . Because

$$\int_{B_R(z)} \zeta(x) d\mu_{n_k}(x) = \int_{B_R(z)} u_{n_k}(x, t) \zeta(x) dx + \int_0^t \int_{B_R(z)} (-u_{n_k} \Delta \zeta + f(u_{n_k}) \zeta) dx ds, \quad (4.23)$$

we derive from Vitali's convergence theorem

$$\int_{B_R(z)} \zeta(x) d\tilde{\mu}(x) = \int_{B_R(z)} u(x, t) \zeta(x) dx + \int_0^t \int_{B_R(z)} (-u \Delta \zeta + f(u) \zeta) dx ds. \quad (4.24)$$

This implies that  $\chi_{B_R(z)} \tilde{\mu} = \chi_{B_R(z)} \mu$  and, by a partition of unity, that  $\tilde{\mu} = \chi_A \mu$ .

Assume now that  $K \subset \mathcal{R}$  is compact. If  $\mu_n(K)$  is unbounded and up to a subsequence still denoted by  $\{n\}$ , there exists a point  $y \in K$  such that for any neighborhood  $\mathcal{O}$  of  $y$ ,  $\mathcal{O} \subset A$ ,  $\mu_n(\mathcal{O}) \rightarrow \infty$  as  $n \rightarrow \infty$ . We can take  $\mathcal{O} = B_r(y)$  and put  $M_{n,r} = \mu_n(B_r(y))$ . If  $m \in \mathbb{N}^*$ , there exists an integer  $n = n(m, r)$  such that  $m = M_{n,r}$ , and  $\lim_{r \rightarrow 0} n(m, r) = \infty$ . Let  $r_0 > r$  such that  $B_{r_0}(y) \subset A$ , and  $w_r$  be the solution of

$$\begin{cases} \partial_t w - \Delta w + f(w) = 0 & \text{in } Q_\infty^{B_{r_0}(y)} \\ w = 0 & \text{in } \partial B_{r_0}(y) \times (0, \infty) \\ w(\cdot, 0) = \chi_{B_r(y)} \mu_n & \text{in } B_{r_0}(y). \end{cases} \quad (4.25)$$



By the comparison principle,  $w_r \leq u_n$  in  $Q_\infty^{B_{r_0}(y)}$ . Since  $\chi_{B_r(y)}\mu_n \rightarrow m\delta_y$  as  $r \rightarrow 0$  and  $n \rightarrow \infty$ , we derive  $u_{y,m,B_{r_0}(y)} \leq u$  from Lemma 4.4 and the remark hereafter. Since  $m$  is arbitrary,  $u_{y,\infty,B_{r_0}(y)} \leq u$ . This implies that  $y \in \mathcal{S}$ , a contradiction.  $\square$

If  $A$  is an open subset of  $\Omega$  and  $\nu \in \mathfrak{M}_+(A)$ , we define an extension  $\underline{\nu}$  of  $\nu$  to  $\Omega$  by

$$\underline{\nu}(E) = \inf_{E \subseteq O} \nu(O \cap A) \quad (4.26)$$

for every Borel set  $E \subset \Omega$  where the infimum is taken over the open subsets  $O$ ;  $\underline{\nu}$  is an outer regular Borel measure on  $\Omega$  and  $\nu = \chi_A \underline{\nu}$ .

The following result which shows the existence of a minimal solution of (1.1) with a given initial trace in  $\mathfrak{M}_+(A)$  for any open subset  $A$  in  $\mathbb{R}^N$  is a straightforward adaptation of [5, Lemma 3.3].

**Proposition 4.9** *Assume  $f$  is nondecreasing and satisfies (1.8), (1.12) and (C2).*

(i) *Let  $A$  be an open subset of  $\mathbb{R}^N$  and let  $\nu \in \mathfrak{M}_+(A)$  with associated extension  $\underline{\nu}$ . Then there exists a positive solution of (1.1) in  $Q_\infty$  denoted by  $\underline{u}_\nu$  satisfying  $Tr_{\mathbb{R}^N}(\underline{u}_\nu) = \underline{\nu}$  and such that  $\underline{u}_\nu \leq v$  for every positive solution  $v$  of (1.1) in  $Q_\infty$  such that  $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$  and  $\chi_A \mu \geq \nu$ .*

(ii) *Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with a  $C^2$  boundary and  $u_n$  be the solution of problem*

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) = 0 & \text{in } Q_T^\Omega \\ u_n = n & \text{on } \partial\Omega \times (0, \infty) \\ u_n(\cdot, 0) = n & \text{in } \Omega. \end{cases} \quad (4.27)$$

*Denote  $U_{\infty,\Omega} := \lim_{n \rightarrow \infty} u_n$ . Then  $U_{\infty,\Omega}$  is the maximal solution of (1.1) in  $Q_\infty^\Omega$  in the sense that the following relation holds in  $Q_T^\Omega$  for every positive solution  $v$  of (1.1)*

$$U_{\infty,\Omega} \geq v. \quad (4.28)$$

Taking  $A = \mathcal{R} := \mathbb{R}^N \setminus \mathcal{S}$ , we obtain the existence of a minimal positive solution of (1.1) with a given positive Radon measure  $\mu \in \mathfrak{M}_+(\mathcal{R})$  as the regular part of the initial trace.

**Corollary 4.10** *Let  $\mathcal{S}$  be a closed subset of  $\mathbb{R}^N$ ,  $\mathcal{R} = \mathbb{R}^N \setminus \mathcal{S}$  and  $\mu \in \mathfrak{M}_+(\mathcal{R})$ . Then there exists a positive solution  $\underline{u}_\mu$  of (1.1) such that  $Tr_{\mathbb{R}^N}(\underline{u}_\mu) = \underline{\mu}$  and  $\underline{u}_\mu \leq v$  for every positive solution  $v$  of (1.1) in  $Q_\infty$  such that  $tr_{\mathbb{R}^N}(v) = (\mathcal{S}, \mu)$ .*

As a counterpart of Theorem 1.11 we have the following existence theorem.

**Proof of Theorem 1.12**

*Step 1: Construction of a minimal solution.* Let  $\underline{u}_{\mathcal{S}}$  and  $\underline{u}_{\mu}$  the minimal solution constructed in Corollary 4.6 and Corollary 4.10. Then  $\check{u}_{\mathcal{S},\mu} := \sup\{\underline{u}_{\mathcal{S}}, \underline{u}_{\mu}\}$  is a subsolution of (1.1) in  $Q_{\infty}$  while  $\hat{u}_{\mathcal{S},\mu} := \underline{u}_{\mathcal{S}} + \underline{u}_{\mu}$  is a supersolution. Furthermore  $\check{u}_{\mathcal{S},\mu} \leq \hat{u}_{\mathcal{S},\mu}$ . Therefore the set of solutions  $u$  in  $Q_{\infty}$  such that  $\check{u}_{\mathcal{S},\mu} \leq u \leq \hat{u}_{\mathcal{S},\mu}$  is not empty and we denote by  $\underline{u}_{\mathcal{S},\mu}$  the smallest solution larger than  $\check{u}_{\mathcal{S},\mu}$ ; it is a solution with initial trace  $(\mathcal{S}, \mu)$ . If  $u$  is any other positive solution with the same initial trace, it is larger than  $\underline{u}_{\mathcal{S}}$  and  $\underline{u}_{\mu}$  by Corollary 4.6 and Corollary 4.10. Therefore it is larger than  $\check{u}_{\mathcal{S},\mu}$  and consequently larger than  $\underline{u}_{\mathcal{S},\mu}$ .

*Step 2: Construction of the maximal solution.* The proof is somewhat similar to the one on [5, Th 3.4], but we give it for the sake of completeness. We denote, for  $\delta > 0$ ,

$$\mathcal{S}^{\delta} := \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{S}) \leq \delta\} \text{ and } \mathcal{R}^{\delta} := \mathbb{R}^N \setminus \mathcal{S}^{\delta}.$$

and let  $\mu_{\delta}$  be the measure given by

$$\mu_{\delta}(E) = \mu(\mathcal{R}_{\delta} \cap E) \quad \forall E \subset \mathbb{R}^N, E \text{ Borel.}$$

We denote by  $u_{\mathcal{S}^{\delta}}$  a solution of (1.1) in  $Q_{\infty}$  with initial trace  $(\mathcal{S}^{\delta}, 0)$ : a solution is easily constructed as the limit when  $R, k \rightarrow \infty$  of the solution  $v = v_{k,R}$  of

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_{\infty} \\ v(\cdot, 0) = k\chi_{(\overline{B}_R \cap \mathcal{S}^{\delta}) \cup (\overline{B}_R \cap \overline{B}_{R-\delta}^c)} \end{cases} \quad (4.29)$$

By Proposition 4.7, there holds, for any  $0 < \delta' < \delta$ ,

$$\lim_{t \rightarrow 0} \frac{u_{\mathcal{S}^{\delta}}(x, t)}{\phi_{\infty}(t)} = 1 \quad \text{uniformly on } \mathcal{S}_{\delta'} \quad (4.30)$$

Let  $u_{\mu_{\delta}}$  be the solution of (1.1) in  $Q_{\infty}$  with initial trace  $(\emptyset, \mu_{\delta})$ . This solution is constructed by approximation, as the limit, when  $R \rightarrow \infty$ , of the solution  $u = u_{\chi_{B_R} \mu_{\delta}}$  of

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_{\infty} \\ u(\cdot, 0) = \chi_{B_R} \mu_{\delta} & \text{in } \mathbb{R}^N. \end{cases} \quad (4.31)$$

For  $\tau > 0$ , let  $u_{\delta,\tau}$  be the solution of (1.1) in  $Q_{\infty}$  with initial data  $m_{\delta,\tau}$  defined by

$$m_{\delta,\tau}(x) = \begin{cases} \phi_{\infty}(\tau) & \text{if } x \in \mathcal{S}_{\delta} \\ u_{\mu_{\delta}}(x, \tau) & \text{if } x \in \mathcal{R}_{\delta}. \end{cases}$$

If  $u$  is a positive solution of (1.1) with initial trace  $(\mathcal{S}, \mu)$  then  $u(\cdot, \tau) \leq m_{\delta,\tau}$  in  $\mathcal{S}_{\delta}$  and  $u(\cdot, \tau) \geq m_{\delta,\tau}$  in  $\mathcal{R}_{\delta}$  by Proposition 4.9. Therefore

$$\lim_{\tau \rightarrow 0} (u(\cdot, \tau) - m_{\delta,\tau}(\cdot))_+ = 0$$

in the weak sense of measures. Furthermore, this convergence does not depend on  $u$ , but only on  $\mathcal{S}_\delta$  and  $\mu_\delta$ . The set of functions  $\{u_{\delta,\tau}\}_{\tau>0}$  is locally uniformly bounded in  $Q_\infty$ . By the regularity theory for parabolic equations, there exists a subsequence  $\{\tau_k\}$  and a positive solution  $u_\delta^*$  of (1.1) in  $Q_\infty$  such that  $u_{\delta,\tau_k} \rightarrow u_\delta^*$  locally uniformly in  $Q_\infty$ . By Proposition 4.8 and Proposition 4.9,  $tr_{\mathbb{R}^N}(u_\delta^*) = (\mathcal{S}^\delta, \mu_\delta)$ . Let  $\omega_{\delta,\tau}$  be the solution of (1.1) in  $Q_\infty$  with initial data  $(u(\cdot, \tau) - m_{\delta,\tau}(\cdot))_+$  (it is constructed in the same way as  $\underline{u}_\mu$  in Proposition 4.9 -(i)). By Theorem 1.8-(ii),  $\lim_{\tau \rightarrow 0} \omega_{\delta,\tau} = 0$ , locally uniformly. Since  $u \leq u_{\delta,\tau} + \omega_{\delta,\tau}$  in  $(\tau, \infty) \times \mathbb{R}^N$ , we obtain  $u \leq u_\delta^*$ . If  $0 < \delta' < \delta$ , we can compare similarly  $u_{\delta,\tau}$  with the solution  $u_{\delta',\tau}$  of (1.1) with initial data

$$m_{\delta',\tau}(x) = \begin{cases} \phi_\infty(\tau) & \text{if } x \in \mathcal{S}_{\delta'} \\ u_{\mu_{\delta'}}(x, \tau) & \text{if } x \in \mathcal{R}_{\delta'}. \end{cases}$$

If  $u_{\delta'}^*$  is the limit of any sequence  $\{u_{\delta',\tau_k}\}$ , it satisfies  $0 < u_{\delta'}^* \leq u_\delta^*$  and has initial trace  $(\mathcal{S}^{\delta'}, \mu_{\delta'})$ . If we take in particular  $\delta = \delta_n = 2^{-n}$ , we construct a decreasing sequence of positive solutions  $\{u_{2^{-n}}^*\}$  of (1.1) in  $Q_\infty$ , with  $tr_{\mathbb{R}^N}(u_{2^{-n}}^*) = (\mathcal{S}^{2^{-n}}, \mu_{2^{-n}})$ , satisfying

$$u \leq u_{2^{-n}}^* \quad \text{in } Q_\infty.$$

Clearly the limit  $\bar{u}_{\mathcal{S},\mu}$  of the sequence  $\{u_{2^{-n}}^*\}$  is a positive solution of (1.1) in  $Q_\infty$  with initial trace  $(\mathcal{S}, \mu)$  and is independent of  $u$ . It is the maximal solution of the equation with this initial trace.  $\square$

*Remark.* When  $f(r) = |r|^{q-1}r$  with  $1 < q < 1 + 2/N$ , precise expansion of  $u_{\infty\delta}(x, t)$ , when  $t \rightarrow 0$  allows to prove uniqueness. Even when  $f(r) = r \ln^\alpha(r+1)$  with  $\alpha > 2$ , uniqueness is not known. The first step would be to prove that uniqueness holds if  $tr_\Omega(u) = (\{a\}, 0)$  for some  $a \in \Omega$ . However, if  $\mathcal{S} = \emptyset$ , uniqueness holds from Theorem 1.8-(ii).

### 4.3 The Keller-Osserman condition does not hold

In this section we assume that (1.12) does not hold but (2.1) is always satisfied.

**Lemma 4.11** *Assume (1.10), (1.16), (C1) and (C3) are satisfied. If  $u$  is a positive solution of (1.1) in  $Q_\infty$  which satisfies*

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx = \infty, \quad (4.32)$$

for some bounded open subset  $G \subset \mathbb{R}^N$ , then  $u(x, t) \geq \phi_\infty(t)$ .

*Proof.* By assumption, there exists a sequence  $\{t_n\}$  decreasing to 0 such that

$$\lim_{n \rightarrow \infty} \int_G u(x, t_n) dx = \infty. \quad (4.33)$$

If (4.32), we can construct a decreasing sequence of open subsets  $G_k \subset G$  such that  $\overline{G_k} \subset G_{k-1}$ ,  $\text{diam}(G_k) = \epsilon_k \rightarrow 0$  when  $k \rightarrow \infty$ , and

$$\lim_{n \rightarrow \infty} \int_{G_k} u(x, t_n) dx = \infty \quad \forall k \in \mathbb{N}. \quad (4.34)$$

Furthermore there exists a unique  $a \in \bigcap_k G_k$ . We set

$$\int_{G_k} u(x, t_n) dx = M_{n,k}.$$

Since  $\lim_{n \rightarrow \infty} M_{n,k} = \infty$ , we claim that for any  $m > 0$  and any  $k$ , there exists  $n = n(k) \in \mathbb{N}$  such that

$$\int_{G_k} u(x, t_{n(k)}) dx \geq m. \quad (4.35)$$

By induction, we define  $n(1)$  as the smallest integer  $n$  such that  $M_{n,1} \geq m$ . This is always possible. Then we define  $n(2)$  as the smallest integer larger than  $n(1)$  such that  $M_{n,2} \geq m$ . By induction,  $n(k)$  is the smallest integer  $n$  larger than  $n(k-1)$  such that  $M_{n,k} \geq m$ . Next, for any  $k$ , there exists  $\ell = \ell(k)$  such that

$$\int_{G_k} \inf\{u(x, t_{n(k)}); \ell\} dx = m \quad (4.36)$$

and we set

$$V_k(x) = \inf\{u(x, t_{n(k)}); \ell\} \chi_{G_k}(x).$$

Let  $v_k = v$  be the unique bounded solution of

$$\begin{cases} \partial_t v - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(\cdot, 0) = V_k & \text{in } \mathbb{R}^N. \end{cases} \quad (4.37)$$

Since  $v(x, 0) \leq u(x, t_{n(k)})$ , we derive

$$u(x, t + t_{n(k)}) \geq v_k(x, t) \quad \forall (x, t) \in Q_\infty. \quad (4.38)$$

When  $k \rightarrow \infty$ ,  $V_k \rightarrow m\delta_a$ , thus  $v_k \rightarrow u_{m\delta_a}$  by Lemma 4.4. Therefore  $u \geq u_{m\delta_a}$ . Since  $m$  is arbitrary and  $u_{m\delta_a} \rightarrow \phi_\infty$  when  $m \rightarrow \infty$  by Theorem 1.3, it follows that  $u \geq \phi_\infty$ .  $\square$

**Lemma 4.12** *Assume (1.15), (C1) and (C3) are satisfied. There exists no positive solution  $u$  of (1.1) in  $Q_\infty$  which satisfies (4.32) for some bounded open subset  $G \subset \mathbb{R}^N$ .*

*Proof.* If we assume that such a  $u$  exists, we proceed as in the proof of the previous lemma. Since Lemma 4.4 holds, we derive that  $u \geq u_{m\delta_a}$  for any  $m$ . Since  $\lim_{m \rightarrow \infty} u_{m\delta_a}(x, t) = \infty$  for all  $(x, t) \in Q_\infty$ , we are led to a contradiction.  $\square$

Thanks to these results, we can characterize the initial trace of positive solutions of (1.1) when the Keller-Osserman condition does not hold.

### Proof of Theorem 1.13

If there exists some open subset  $G$  of  $\mathbb{R}^N$  with the property (4.32), then  $u \geq \phi_\infty$  and the initial trace of  $u$  is the Borel measure  $\nu_\infty$ . Next we assume that for any bounded open subset  $G$  of  $\mathbb{R}^N$  there holds

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx < \infty. \quad (4.39)$$

If  $\mathcal{S}(u) \neq \emptyset$ , there exist  $z \in \mathbb{R}^N$  and an bounded open neighborhood  $G$  of  $z$  such that

$$\int_0^T \int_G f(u) dx dx t = \infty.$$

By (4.39),  $u \in L^\infty(0, T; L^1(G)) \subset L^1(Q_T^G)$ . Then, by Lemma 4.2, (4.4) holds, which contradict (4.39). Thus  $\mathcal{S}(u) = \emptyset$  and  $\mathcal{R}(u) = \mathbb{R}^N$ . It follows from Proposition 1.9 that there exists a positive Radon measure  $\mu$  such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(\mathbb{R}^N). \quad (4.40)$$

$\square$

Because of the lack of uniqueness from Theorem 1.6 it is difficult to give a complete characterization of admissible initial data for solutions of (1.1) under the assumptions of Theorem 1.13. However, we have the result as in Proposition 1.14.

### Proof of Proposition 1.14

We first notice that  $\max\{\phi_\infty(t); w_b(|x|)\}$  is a subsolution of (1.1) which is dominated by the supersolution  $\phi_\infty(t) + w_b(|x|)$ . The construction is standard: for  $\tau > 0$  we set

$$\psi(x, \tau) = \frac{1}{2} (\max\{\phi_\infty(t); w_b(|x|)\} + \phi_\infty(t) + w_b(|x|)).$$

There exists a function  $u = u_\tau \in C(\overline{Q_\infty})$  solution of (1.1) in  $Q_\infty$  satisfying  $u_\tau(\cdot, 0) = \psi(\cdot, \tau)$ . Furthermore

$$\max\{\phi_\infty(t + \tau); w_b(|x|)\} \leq u_\tau(x, t) \leq \phi_\infty(t + \tau) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (4.41)$$

By the parabolic equation regularity theory, the set  $\{u_\tau\}_{\tau>0}$  is locally equicontinuous in  $Q_\infty$ . Thus there exist a subsequence  $\{\tau_n\}$  and  $u \in C(Q_\infty)$  such that  $u_{\tau_n} \rightarrow u$  on any compact subset of  $Q_\infty$ . Clearly  $u$  is a weak, thus a strong solutions of (1.1) and it satisfies (1.28). Since any solution  $u$  with initial trace  $\nu_\infty$  dominates  $\phi_\infty$  by Lemma 4.11, it follows that  $\phi_\infty$  is the minimal one.  $\square$

### Proof of Theorem 1.15

As in the proof of Theorem 1.13 and because of Lemma 4.12,  $\mathcal{S}(u) = \emptyset$ . Therefore  $\mathcal{R}(u) = \mathbb{R}^N$  and the proof follows from Proposition 1.9.  $\square$

*Remark.* Under the assumptions of Theorem 1.13, it is clear, from the proof of Proposition 3.1, that for any  $0 < a < b$  and any initial data  $u_0 \in C(\mathbb{R}^N)$  satisfying

$$w_a(x) \leq u_0(x) \leq w_b(x) \quad \forall x \in \mathbb{R}^N$$

there exists a solution  $u \in C(\overline{Q_\infty})$  of (1.1) in  $Q_\infty$  satisfying  $u(\cdot, 0) = u_0$  and

$$w_a(x) \leq u(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty.$$

We conjecture that for any positive measure  $\mu$  on  $\mathbb{R}^N$  which satisfies, for some  $b > 0$ ,

$$\int_{B_R} d\mu(x) \leq \int_{B_R} w_b(x) dx \quad \forall R > 0 \quad (4.42)$$

there exists a positive solution  $u$  of (1.1) in  $Q_\infty$  with initial trace  $\mu$ . Another interesting open problem is to see if there exist local solutions in  $Q_T$  with an initial trace  $\mu$  satisfying

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} d\mu(x)}{\int_{B_R} w_b(x) dx} = \infty \quad \forall b > 0. \quad (4.43)$$

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