# Local and global properties of solutions of heat equation with superlinear absorption

Tai Nguyen Phuoc, Laurent Veron

# Local and global properties of solutions of heat equation with superlinear absorption

**Tai Nguyen Phuoc  Laurent Véron**

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

**Abstract**

We study the limit, when $k \to \infty$ of solutions of $u_t - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0, \infty)$ with initial data $k\delta$, when $f$ is a positive increasing function. We prove that there exist essentially three types of possible behaviour according $f^{-1}$ and $F^{-1/2}$ belong or not to $L^1(1, \infty)$, where $F(t) = \int_0^t f(s)ds$. We use these results for giving a general result on the existence of the initial trace and some non-uniqueness results for regular solutions with unbounded initial data.

*1991 Mathematics Subject Classification.* 35K58; 35K91; 35K15.

*Key words.* Heat equation; singularities; Borel measures; initial trace.

## Contents

# 1 Introduction

In this article we investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + f(u) = 0 \tag{1.1}$$

in $Q_\infty := \mathbb{R}^N \times (0,\infty)$ $(N \geq 2)$ where $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, nondecreasing and positive on $(0,\infty)$, vanishes at 0 and tends to infinity at infinity. As a model equation we shall consider the following, with $\alpha > 0$,

$$\partial_t u - \Delta u + u \ln^\alpha(u+1) = 0. \tag{1.2}$$

When $f(u) = |u|^\beta u$ with $\beta > 0$ much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x,t) = t^{-1/\beta} w(x\sqrt{t}). \tag{1.3}$$

The critical exponent $\beta_c = 2/N$ plays a fundamental role in the description of isolated singularities and the study of the initial trace, since if $0 < \beta < \beta_c$ there exists a self-similar solution with an isolated singularity at $(0,0)$, while no such solution exists when $\beta \geq \beta_c$ and no solution with isolated singularities.

In the case of (1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any $k > 0$ there exists a unique $u := u_k \in C(\overline{Q_\infty} \setminus \{(0,0)\}) \cap C^{2,1}(Q_\infty)$ solution of

$$\begin{cases} \partial_t u - \Delta u + u \ln^\alpha(u+1) = 0 & \text{in } Q_\infty \\ u(.,0) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \tag{1.4}$$

There are two critical values for $\alpha$: $\alpha = 1$ and $\alpha = 2$, the explanation of which comes from the study of the two singular problems

$$\begin{cases} \phi' + \phi \ln^\alpha(\phi+1) = 0 & \text{in } (0,\infty) \\ \phi(0) = \infty \end{cases} \tag{1.5}$$

and, for any $\epsilon > 0$,

$$\begin{cases} -\Delta\psi + \psi \ln^\alpha(\psi+1) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim\limits_{|x|\to\epsilon} \psi(x) = \infty, \end{cases} \tag{1.6}$$

where $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$. When it exists, the solution $\phi_\infty$ of (1.5) is given implicitely by

$$\int_{\phi_\infty(t)}^{\infty} \frac{ds}{s \ln^\alpha(s+1)} = t \tag{1.7}$$

and such a formula is valid if an only if $\alpha > 1$. For problem (1.6) an explicit expression of the solution is not valid, but it exists if and only if the Keller-Osserman condition (1.12) holds $\alpha > 2$.

Having in mind this model we study (1.1) assuming the weak singularity condition on $f$:

$$\int_0^1 s^{N/2} f(s^{-N/2}) ds < \infty. \tag{1.8}$$

**Proposition 1.1** *Assume* (1.8) *holds. Then for any $k > 0$, there exists a unique solution $u := u_k$ to*

$$\begin{cases} \partial_t u - \Delta u + f(u) &= 0 \qquad \text{in } Q_\infty \\ u(.,0) &= k\delta_0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \tag{1.9}$$

Another condition on $f$ is

$$\int_1^\infty \frac{ds}{f(s)} < \infty. \tag{1.10}$$

Under assumption (1.10) there exists a solution $\phi := \phi_\infty$ to

$$\begin{cases} \phi' + f(\phi) &= 0 \qquad \text{in } (0,\infty) \\ \phi(0) &= \infty. \end{cases} \tag{1.11}$$

The next important condition on $f$ we shall encounter is the Keller-Osserman condition, i.e.

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \tag{1.12}$$

where

$$F(s) = \int_0^s f(\sigma) d\sigma \qquad \forall s \in [1,\infty). \tag{1.13}$$

If (1.12) is satisfied, by [4, Theorem III] for any $\epsilon > 0$ there exists a maximal solution $\psi := \psi_\epsilon$ to

$$\begin{cases} -\Delta\psi + f(\psi) &= 0 \qquad \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim\limits_{|x|\to\epsilon} \psi(x) &= \infty. \end{cases} \tag{1.14}$$

The first question we consider is the study of the limit of $u_k$ when $k \to \infty$. This question is natural since $k \mapsto u_k$ is increasing. In order to treat it, we need some additional conditions.

(C1)- The function $s \mapsto \dfrac{f(s)}{s}$ is increasing on $(0,\infty)$ and satisfies

$$\lim_{s\to 0}\frac{f(s)}{s} = 0 \quad \text{and} \quad \lim_{s\to\infty}\frac{f(s)}{s} = \infty,$$

(C2)- The function $f$ is convex on $(0,\infty)$,

(C3)- If $\liminf_{s\to\infty} f(s)/(s\ln^{\alpha} s) = 0, \forall \alpha > 2$, then there exists $\beta \in (1,2]$ such that

$$\limsup_{s\to\infty}\frac{f(s)}{s\ln^{\beta} s} < \infty.$$

In the second section, we prove the following results.

**Theorem 1.2** *Assume the conditions* $(C1)$ *and* $(C3)$ *hold. If* $f$ *satisfies*

$$\int_1^{\infty}\frac{ds}{f(s)} = \infty \tag{1.15}$$

*then the solutions* $u_k$ *of* (1.9) *satisfy* $\lim_{k\to\infty} u_k(x,t) = \infty$ *for every* $(x,t) \in Q_{\infty}$.

**Theorem 1.3** *Assume the conditions* $(C1)-(C3)$ *hold. If* $f$ *satisfies* (1.10) *and*

$$\int_1^{\infty}\frac{ds}{\sqrt{F(s)}} = \infty \tag{1.16}$$

*where* $F$ *is defined in* (1.13)*, then the solutions* $u_k$ *of* (1.9) *satisfy* $\lim_{k\to\infty} u_k(x,t) = \phi_{\infty}(t)$ *for every* $(x,t) \in Q_{\infty}$*, where* $\phi_{\infty}$ *is the solution of* (1.11).

We denote by $\mathcal{U}_0$ the set of possitive solutions $u$ of (1.1) in $Q_{\infty}$, which is continuous in $\overline{Q_{\infty}} \setminus \{(0,0)\}$, vanishes on the set $\{(x,0) : x \neq 0\}$ and satisfies

$$\lim_{t\to 0}\int_{B_\epsilon} u(x,t)dx = \infty \tag{1.17}$$

for any $\epsilon > 0$.

**Theorem 1.4** *Assume* $f$ *satisfies* (1.8), (1.10) *and* (1.12). *Then* $\underline{U} := \lim_{k\to\infty} u_k$ *is the minimal element of* $\mathcal{U}_0$.

In the third section we study the set of positive and locally bounded solutions of (1.1) in $Q_\infty$. This set differs considerably according the assumption on $f$. This is due to the properties of the radial solutions of the associated stationnary equation

$$-\Delta w + f(w) = 0 \qquad \text{in } \mathbb{R}^N. \tag{1.18}$$

The next result bases upon the Picard-Lipschitz fixed point theorem and a result of Vazquez and Véron [11].

**Proposition 1.5** *Assume* (1.16) *holds. For any $a > 0$, there exists a unique positive function $w := w_a \in C^2([0,\infty))$ to the problem*

$$\begin{cases} -w'' - \dfrac{N-1}{r} w' + f(w) &= 0 \qquad \text{in } \mathbb{R}_+ \\ w'(0) &= 0 \\ w(0) &= a. \end{cases} \tag{1.19}$$

A striking consequence of the existence of such solutions is the following non-uniqueness result.

**Theorem 1.6** *Assume $f$ satisfies* (1.10) *and* (1.16). *Then for any $u_0 \in C(\mathbb{R}^N)$ satisfying $w_a(x) \le u_0(x) \le w_b(x)$ for any $x \in \mathbb{R}^N$ and for some $b > a > 0$ there exist two solutions $\underline{u}, \overline{u} \in C(\overline{Q_\infty})$ of* (1.1) *with initial value $u_0$. They satisfy respectively*

$$0 \le \underline{u}(x,t) \le \min\{w_b(x), \phi_\infty(t)\} \qquad \forall (x,t) \in Q_\infty, \tag{1.20}$$

*thus $\lim_{t\to\infty} \underline{u}(x,t) = 0$, uniformly with respect to $x \in \mathbb{R}^N$, and*

$$w_a(x) \le \overline{u}(x,t) \le w_b(x) \qquad \forall (x,t) \in Q_\infty, \tag{1.21}$$

*thus $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$, uniformly with respect to $t \ge 0$.*

The next theorem shows that if two solutions of (1.1) have the same initial data and the same asymptotic behaviour as $|x| \to \infty$ then they concide.

**Theorem 1.7** *Assume $f$ satisfies* (1.16) *and* $(C1)$. *Let $u, \tilde{u} \in C(\overline{Q}_\infty) \cap C^{2,1}(Q_\infty)$ be two positive solutions of* (1.1) *with initial data $u_0$. If for any $\epsilon > 0$,*

$$u(x,t) - \tilde{u}(x,t) = o(w_\epsilon(|x|)) \text{ as } x \to \infty \tag{1.22}$$

*locally uniformly with respect to $t \ge 0$, then $u = \tilde{u}$.*

On the contrary, if the Keller-Osserman condition holds, a continuous solution is uniquely determined by a positive initial value $u_0 \in C(\mathbb{R}^N)$.

**Theorem 1.8** *Assume $f$ satisfies* (1.12) *and* $(C2)$. *Then for any nonnegative function $u_0 \in C(\mathbb{R}^N)$ there exists a unique solution $u \in C(\overline{Q_\infty})$ of* (1.1) *in $Q_\infty$ with initial value $u_0$.*

In the last section we study the initial trace of locally bounded positive solutions of (1.1) in $Q_T^\Omega := \Omega \times (0,T)$ where $\Omega$ is a domain in $\mathbb{R}^N$ with $C^2$ boundary and $T > 0$.

**Proposition 1.9** *Assume $f$ is a continuous nonnegative function defined on $\mathbb{R}_+$. Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of* (1.1) *in $Q_T^\Omega$. The set $\mathcal{R}(u)$ of the points $z \in \Omega$ such that there exists an open ball $B_r(z)$ such that $u, f(u) \in L^1(Q_T^{B_r(z)})$ is an open subset of $\Omega$. Furthermore there exists a positive Radon measure $\mu := \mu(u)$ on $\mathcal{R}(u)$ such that*

$$\lim_{t\to 0} \int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(\mathcal{R}(u)). \tag{1.23}$$

Due to Proposition 1.9, we introduce the definition of the initial trace.

**Definition 1.10** *The couple $(\mathcal{S}(u), \mu)$ where $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ is called the initial trace of $u$ in $\Omega$ and will be denoted by $tr_\Omega(u)$. The set $\mathcal{R}(u)$ is called the regular set of the initial trace of $u$ and the measure $\mu$ the regular part of the initial trace. The set $\mathcal{S}(u)$ is relatively closed in $\Omega$ and is called the singular part of the initial trace of $u$.*

The initial trace can also be represented by a positive, outer regular Borel measure, not necessary locally bounded. The space of these measures on $\Omega$ will be denoted by $\mathcal{B}_+^{\rm reg}(\Omega)$. If for every open subset $A \subset \Omega$ we denote by $\mathfrak{M}_+(A)$ the space of positive Radon measures on $A$, there is a $(1-1)$ correspondence between $\mathcal{B}_+^{\rm reg}(\Omega)$ and the set of couples:

$$CM_+(\Omega) = \{(\mathcal{S}, \mu) : \mathcal{S} \subset \Omega \text{ relatively closed }, \mu \in \mathfrak{M}_+(\mathcal{R}) \text{ with } \mathcal{R} = \Omega \setminus \mathcal{S}\}. \tag{1.24}$$

The measure $\nu \in \mathcal{B}_+^{\rm reg}(\Omega)$ corresponding to a couples $(\mathcal{S}, \mu) \in CM_+(\Omega)$ is given by

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \mu(A) & \text{if } A \subseteq \mathcal{S}, \end{cases} \qquad \forall A \subset \Omega, A \text{ Borel.} \tag{1.25}$$

If $u$ is a solution of (1.1), we shall use the notation $tr_\Omega(u)$ (resp. $Tr_\Omega(u)$) for the trace considered as an element of $CM_+(\Omega)$ (resp. $\mathcal{B}_+^{\rm reg}(\Omega)$).

We consider the case when the Keller-Osserman holds.

**Theorem 1.11** *Assume $f$ is nondecreasing and satisfies* (1.12). *If $u \in C^{2,1}(Q_T^\Omega)$ is a positive solution of* (1.1), *it possesses an initial trace $\nu \in \mathcal{B}_+^{reg}(\Omega)$.*

On the contrary, the following theorem concerns with the existence of the maximal solution and the minimal solution of (1.1) with a given initial trace $(\mathcal{S}, \mu) \in CM_+(\Omega)$.

**Theorem 1.12** *Assume $f$ is nondecreasing and satisfies* (1.12), (1.8) *and* $(C2)$. *Then for any $(\mathcal{S}, \mu) \in CM_+(\Omega)$ there exist a maximal solution $\overline{u}_{\Omega,\mathcal{S},\mu}$ and a minimal solution $\underline{u}_{\Omega,\mathcal{S},\mu}$ of* (1.1) *in $Q_T^\Omega$ with the initial trace $(\mathcal{S}, \mu)$, in the following sense:*

$$\underline{u}_{\Omega,\mathcal{S},\mu} \leq v \leq \overline{u}_{\Omega,\mathcal{S},\mu} \tag{1.26}$$

*for every positive solution $v \in C^{2,1}(Q_T^\Omega)$ of* (1.1) *in $Q_T^\Omega$ such that $tr_\Omega(v) = (\mathcal{S}, \mu)$.*

If the Keller-Osserman does not holds, we obtain the following results.

**Theorem 1.13** *Assume* (1.10), (1.16), $(C1)$ *and* $(C3)$ *are verified. If $u \in C^{2,1}(Q_\infty)$ is a positive solution of* (1.1) *in $Q_\infty$, it possesses an initial trace which is either the Borel measure infinity $\nu_\infty$ which satisfies $\nu_\infty(\mathcal{O}) = \infty$ for any open subset $\mathcal{O} \subset \mathbb{R}^N$, or is a positive Radon measure $\mu$ on $\mathbb{R}^N$.*

A consequence of Theorem 1.13 which is worth mentioning is the following.

**Proposition 1.14** *Under the assumptions of Theorem 1.13, for any $b > 0$ there exists a positive solution $u \in C(Q_\infty)$ of* (1.1) *in* (1.1) *satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x,t) \leq \phi_\infty(t) + w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{1.27}$$

*Consequently there exist infinitely many positive solutions of* (1.1) *with initial trace $\nu_\infty$. Furthermore $\phi_\infty$ is the smallest of all these solutions.*

**Theorem 1.15** *Assume $f$ is satisfies* (1.15),(1.16), $(C1)$ *and* $(C3)$. *If $u$ is a positive solution of* (1.1) *in $Q_\infty$, it possesses an initial trace which is a positive Radon measure $\mu$ on $\mathbb{R}^N$.*

The proofs are combination of methods developed in [8] for elliptic equations, stability results and Theorem 1.2 and Theorem 1.3.

## 2 Isolated singularities

In order to study (1.1), we start proving Proposition 1.1.

**Proof of Proposition 1.1**

We denote by $E(x,t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ the fundamental solution of the heat equation in $Q_\infty$. By [6, Remark 2.1], if for $k > 0$,

$$I := \int_0^1 \int_{B_R} f(kE(x,t))dx\,dt < \infty \tag{2.1}$$

for any $R > 0$, then there exists a unique solution $u := u_k$ to (1.1) satisfying initial condition

$$u_k(.,0) = k\delta_0$$

in $\mathcal{D}'(\mathbb{R}^N)$. Furthermore the mapping $k \mapsto u_k$ is increasing. This existence result and the next proposition lead to the conclusion of Proposition 1.1. □

**Proposition 2.1** *If $f$ satisfies* (1.8) *and* $(C1)$ *then* (2.1) *is fulfilled.*

*Proof.* We set

$$h(r) = \frac{f(r)}{r} \qquad r \in (0,\infty). \tag{2.2}$$

$I$ is rewritten as

$$I = kC^* \int_0^1 \int_{B_R} t^{-N/2} e^{-|x|^2/4t} h(kC^* t^{-N/2} e^{-|x|^2/4t}) dx\,dt$$

where $C^* = (4\pi)^{-N/2}$. Put $r = |x|$ then $dx = r^{N-1}dr$, and

$$I = kC^* \int_0^1 t^{-N/2} \int_0^R e^{-r^2/4t} h(kC^* t^{-N/2} e^{-r^2/4t}) r^{N-1} dr\,dt.$$

We next put $\rho = \frac{r}{\sqrt{t}}$, then $r^{N-1}dr = \rho^{N-1} t^{N/2} d\rho$, and

$$I = kC^* \int_0^1 \int_0^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho\,dt.$$

We set

$$I_1 := kC^* \int_0^1 \int_0^1 e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho\,dt,$$

$$I_2 := kC^* \int_0^1 \int_1^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho\,dt.$$

Since $e^{-\rho^2/4}\rho^{N-1}$ is bounded in $[0,\infty)$, then there exists a constant $c_1$ depending only on $k$ such that

$$I_1 < c_1 \int_0^1 \int_0^1 h(kC^* t^{-N/2}) d\rho\, dt = c_1 \int_0^1 h(kC^* t^{-N/2}) dt < \infty.$$

We next show that under the condition (1.8), $I_2 < \infty$. In order to do that we change to a new variable $\tau$ such that $t^{-N/2}e^{-\rho^2/4} = \tau^{-N/2}$. Then $t = \tau e^{-\frac{\rho^2}{2N}}$ and $dt = e^{-\frac{\rho^2}{2N}} d\tau$. Therefore

$$I_2 \leq kC^* \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} \left( \int_0^{e^{\rho^2/2N}} h(kC^* \tau^{-N/2}) d\tau \right) d\rho. \tag{2.3}$$

Since $f$ satisfies (1.8), there exists $\epsilon > 0$ (depending only on $k$) such that

$$\int_0^\epsilon h(kC^* \tau^{-N/2}) d\tau$$

take a finite value, denoted by $c_2$. Hence

$$\int_0^{e^{\rho^2/2N}} h(kC^* \tau^{-N/2}) d\tau \leq c_2 + h(kC^* \epsilon^{-N/2})(e^{\frac{\rho^2}{2N}} - \epsilon). \tag{2.4}$$

Inserting (2.4) into the right hand side of (2.3), we obtain

$$I_2 \leq c_3 \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} d\rho + c_4 \int_1^\infty e^{-\frac{\rho^2}{4}} \rho^{N-1} d\rho < \infty$$

where $c_3 = kC^* c_2$ and $c_4 = kC^* h(kC^* \epsilon^{-N/2})$. Thus $I = I_1 + I_2 < \infty$. □

The functions which satisfy the following ODE are particular solutions of (1.1)

$$\phi_t + f(\phi) = 0 \qquad \text{in } (0,\infty). \tag{2.5}$$

For $a > 0$, we denote by $\phi_a$ the solution of (2.5) with initial data $\phi(0) = a$. If (1.15) holds then $\lim_{a\to\infty} \phi_a(t) = \infty$ for any $t \in (0,\infty)$. While, if (1.10) holds there exists a maximal solution $\phi_\infty$ given explicitely by

$$t = \int_{\phi_\infty(t)}^\infty \frac{ds}{f(s)} < \infty.$$

**Lemma 2.2** *If* (1.15) *holds then*

$$\liminf_{r\to\infty}\frac{f(r)}{r\ln^{\alpha} r}=0,\quad \forall\alpha>1.$$

*If* (1.10) *holds then*

$$\limsup_{r\to\infty}\frac{f(r)}{r\ln^{\alpha} r}=\infty,\quad \forall 0<\alpha\leq 1.$$

*Proof. Case 1.* Assume (1.15) holds then

$$J:=\int_e^{\infty}\frac{ds}{f(s)}<\infty. \tag{2.6}$$

We change the variable $s=e^{r^{-1}}$ and derive

$$J=\int_0^1\frac{dr}{r^2h(e^{r^{-1}})}$$

where $h$ is defined in (2.2). Suppose that there exists $\alpha>1$ such that

$$\liminf_{s\to\infty}\frac{f(s)}{s\ln^{\alpha} s}>0,$$

equivalently,

$$\liminf_{r\to 0} r^{\alpha}h(e^{r^{-1}})>0,$$

then there exists $l>0$ and $r_0\in(0,1)$ such that

$$h(e^{r^{-1}})>lr^{-\alpha}\qquad \forall r\in(0,r_0).$$

Hence we derive the following contradiction

$$J<\frac{1}{l}\int_0^{r_0}r^{\alpha-2}dr+\int_{r_0}^1\frac{dr}{r^2h(e^{r^{-1}})}<\infty.$$

*Case 2.* Assume (1.10) holds then $J=\infty$. Suppose that there exists $\alpha\in(0,1]$ such that

$$\limsup_{s\to\infty}\frac{f(s)}{s\ln^{\alpha} s}<\infty,$$

equivalently,

$$\limsup_{r\to 0} r^{\alpha}h(e^{r^{-1}})<\infty,$$

then there exists $l>0$ and $r_0\in(0,1)$ such that

$$h(e^{r^{-1}})<lr^{-\alpha}\qquad \forall r\in(0,r_0).$$

Hence

$$J > \frac{1}{l}\int_0^{r_0} r^{\alpha-2}dr + \int_{r_0}^1 \frac{dr}{r^2 h(e^{r^{-1}})} = \infty,$$

which is a contradiction. □

**Proof of Theorem 1.2.**

Since (1.15) holds, by Lemma 2.2 and the definition (2.2) of $h$,

$$\liminf_{r\to\infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 1.$$

Thus

$$\liminf_{r\to\infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 2.$$

By (C3), there exists $\beta \in (1,2]$ such that $\limsup_{r\to\infty} h(r)/\ln^\beta r < \infty$. Hence there exist $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r \qquad \forall r \in (r_0, \infty). \tag{2.7}$$

*Step 1.* Let $k > 0$, we claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2}\int_0^1 (\ln(\tau^{-1}))^\beta d\tau \qquad \forall t \in (0,1) \tag{2.8}$$

where $\theta_k(t) = \int_0^t h(kC^*\tau^{-N/2})d\tau$ with $C^* = (4\pi)^{-N/2}$. Set $r = kC^*\tau^{-N/2}$ then (2.7) becomes

$$h(kC^*\tau^{-N/2}) < M[\ln(kC^*) + \frac{N}{2}\ln(\tau^{-1})]^\beta \qquad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{2/N} r_0^{-2/N}$. We put $a_1 = \ln k$, $a_2 = \frac{N}{2}\ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \le 2^{\beta-1}(a_1^\beta + a_2^\beta)$$

in order to obtain (notice that $C^* = (4\pi)^{-N/2} < 1$)

$$\begin{aligned} h(kC^*\tau^{-N/2}) &< M[\ln(k) + \tfrac{N}{2}\ln(\tau^{-1})]^\beta \\ &\le 2^{\beta-1} M[(\ln k)^\beta + (\tfrac{N}{2})^\beta \ln^\beta(\tau^{-1})] \qquad \forall \tau \in (0,\tau_0). \end{aligned} \tag{2.9}$$

Integrating over $[0,t]$ yields (2.8).

*Step 2.* It follows from (2.9) that (1.8) is fulfilled, hence by Proposition 1.1 there exists the unique solution $u_k$ of (1.1) in $Q_\infty$ with initial data $k\delta_0$. By the maximum principle, $u_k(x,t) \le kE(x,t)$ for every $(x,t) \in Q_\infty$, which implies $u_k(x,t) \le kC^*t^{-N/2}$ for every $(x,t) \in Q_\infty$. Therefore, since $h$ is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \ge 0.$$

If we set $v_k(x,t) = e^{\theta_k(t)}u_k(x,t)$, we obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)}[\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})] \ge 0$$

and $v_k(.,0) = u_k(.,0) = k\delta_0$. By the maximum principle, there holds

$$v_k(x,t) \ge kC^*t^{-N/2}e^{-|x|^2/4t} \Longleftrightarrow u_k(x,t) \ge kC^*t^{-N/2}e^{-\theta_k(t)-|x|^2/4t}. \tag{2.10}$$

By step 1,

$$e^{-\theta_k(t)} \ge c_1 e^{-M_\beta t(\ln k)^\beta} \qquad \forall t \in (0,1) \tag{2.11}$$

where

$$c_1 = \exp\Big(-\frac{M(N)^\beta}{2}\int_0^1 (\ln(\tau^{-1}))^\beta d\tau\Big)$$

and $M_\beta = M2^{\beta-1}$. Inserting (2.11) into the right hand side of (2.10), we get

$$u_k(x,t) \ge c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \qquad \forall (x,t) \in Q_1 := \mathbb{R}^N \times (0,1).$$

If $\lim_{k\to\infty} u_k(x,t) < \infty$ for all $(x,t) \in Q_\infty$, we put $\underline{U} := \lim_{k\to\infty} u_k$, then

$$\underline{U}(x,t) \ge c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x,t) \in Q_1, \quad \forall k > 0.$$

Let $\{t_n\} \subset (0,1]$ be a sequence converging to 0. We choose $k_n = \exp\Big((2M_\beta t_n)^{\frac{1}{1-\beta}}\Big)$ then $\ln k_n - M_\beta t_n (\ln k_n)^\beta = \frac{1}{2}\ln k_n$. Next we restrict $x$ in order to have

$$\ln k_n - M_\beta t_n (\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2}\ln k_n - \frac{|x|^2}{4t_n} \ge 0 \Longleftrightarrow |x| \le 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}} t_n^{\frac{\beta-2}{2(\beta-1)}}.$$

Therefore, since $1 < \beta \le 2$,

$$\lim_{n\to\infty} \underline{U}(x,t_n) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \le \beta < 2$, or uniformly on the ball $B_{r_2}$ where $r_2 = (2M)^{-1/2}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t\to 0} \underline{U}(x,t) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \le \beta < 2$, or uniformly on the ball $B_{r_2}$ if $\beta = 2$.

We pick some point $x_0$ in $\mathbb{R}^N$ (resp. $B_{r_2}$) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, by comparison principle, it follows that

$$\underline{U}(x,t) \ge u_{k\delta_{x_0}}(x,t) = u_k(x - x_0, t).$$

Letting $k \to \infty$ yields to $\underline{U}(x,t) \ge \underline{U}(x - x_0, t)$. Reversing the role of 0 and $x_0$ yields to $\underline{U}(x,t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x,t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x,t)$ is independent of $x$ and therefore it is a solution of (1.11). By (1.15), $\underline{U}(x,t) = \infty$ for any $(x,t) \in Q_\infty$, which is a contradiction and the conclusion follows. □

**Proposition 2.3** *Assume* (1.8) *and* (1.10) *are satisfied. For any $k > 0$, there holds*

$$u_k(x,t) \le \phi_\infty(t) \qquad \forall (x,t) \in Q_\infty.$$

*Proof.* For any small $\epsilon > 0$, we set $\phi_{\infty\epsilon}(t) = \phi_\infty(t - \epsilon), t \in [\epsilon, \infty)$ then $\phi_{\infty\epsilon}$ is a solution of (1.1) in $(\epsilon, \infty)$, which dominates $u_k$ on $\mathbb{R}^N \times \{\epsilon\}$ for any $k > 0$. By comparison principle, $u_k(x,t) \le \phi_{\infty\epsilon}(t)$ for every $(x,t) \in \mathbb{R}^N \times [\epsilon, \infty)$. Letting $\epsilon \to 0$ yields the claim. □

A necessary and sufficient condition for the existence of a maximal solution to the stationary equation

$$-\Delta w + f(w) = 0$$

in a bounded domain $\Omega$ is the Keller-Osserman condition (1.12) ([4], [9]). If $f$ is convex and (1.12) holds, then (1.10) is fulfilled. The Keller-Osserman condition can be replaced by another condition, which owes to the following result.

**Lemma 2.4** *Assume $f$ is convex on $(0, \infty)$. Set*

$$L := \int_1^\infty \frac{ds}{\sqrt{sf(s)}}.$$

*Then* (1.12) *holds if and only if $L < \infty$.*

*Proof.* In order to obtain the assertion, it is sufficient to show that

$$s\, f(\frac{s}{2}) \le F(s) \le s\, f(s) \qquad \forall s \ge 1. \tag{2.12}$$

The right-hand side estimate in (2.12) follows from the monotone property of $f$. The assumption of convexity of $f$ in $(0,\infty)$ implies

$$f(s) \geq f(\frac{s}{2}) + \frac{s}{2} f'(\frac{s}{2}) \qquad \forall s > 0.$$

Define $\varphi(s) = \displaystyle\int_0^s f(\sigma)d\sigma - sf(\tfrac{s}{2})$, then $\varphi'(s) = f(s) - f(\tfrac{s}{2}) - \tfrac{s}{2} f'(\tfrac{s}{2}) \geq 0$. Hence $\varphi(s) > \varphi(0) = 0$, which leads to the left-hand side estimate in (2.12). □

By using the same argument as in the proof of the Lemma 2.2 and thank to the Lemma 2.4, we obtain the following lemma.

**Lemma 2.5** *If* (1.16) *holds then*

$$\liminf_{r\to\infty} \frac{f(r)}{r \ln^\alpha(r)} = 0 \quad \forall \alpha > 2.$$

*If* (1.12) *holds then*

$$\limsup_{r\to 0} \frac{f(r)}{r \ln^\alpha(r)} = \infty \quad \forall 0 < \alpha \leq 2.$$

**Proof of Theorem 1.3.**

Since (1.16) holds, by Lemma 2.5 and the definition (2.2) of $h$,

$$\liminf_{r\to\infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 2.$$

By (C3), there exists $\beta \in (1,2]$ such that $\limsup_{r\to\infty} h(r)/\ln^\beta r < \infty$. Hence there exists $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r \quad \forall r \in (r_0, \infty). \tag{2.13}$$

*Step 1.* For any $k > 0$ we set

$$\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$$

where $C^* = (4\pi)^{-N/2}$. We claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau \quad \forall t \in (0,1). \tag{2.14}$$

If we define $\tau$ by $r = kC^* \tau^{-N/2}$, (2.13) becomes

$$h(kC^* \tau^{-\frac{N}{2}}) < M[\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{2/N} r_0^{-2/N}$. We set $a_1 = \ln k$, $a_2 = \frac{N}{2}\ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \le 2^{\beta-1}(a_1^\beta + a_2^\beta)$$

in order to obtain (notice that $C^* < 1$)

$$\begin{aligned} h(kC^*\tau^{-N/2}) &< M[\ln(k) + \tfrac{N}{2}\ln(\tau^{-1})]^\beta \\ &\le 2^{\beta-1}M[(\ln k)^\beta + (\tfrac{N}{2})^\beta \ln^\beta(\tau^{-1})]. \end{aligned} \tag{2.15}$$

Integrating over $[0,t]$, we obtain (2.14).

*Step 2.* It follows from (2.15) that (1.8) is fulfilled, hence by Proposition 1.1 there exists the unique solution of (1.1) in $Q_\infty$ with initial trace $k\delta_0$. By maximum principle, $u_k(x,t) \le kE(x,t)$ for every $(x,t) \in Q_\infty$, which implies that $u_k(x,t) \le kC^*t^{-N/2}$ for every $(x,t) \in Q_\infty$. Therefore, since $h$ is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \ge 0.$$

We set $v_k(x,t) = e^{\theta_k(t)}u_k(x,t)$ and obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)}[\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})] \ge 0,$$

with $v_k(.,0) = u_k(.,0) = k\delta_0$. By maximum principle, it follows

$$v_k(x,t) \ge kC^*t^{-N/2}e^{-|x|^2/4t} \Longleftrightarrow u_k(x,t) \ge kC^*t^{-N/2}e^{-\theta_k(t)-|x|^2/4t}. \tag{2.16}$$

By step 1,

$$e^{-\theta_k(t)} \ge c_1 e^{-M_\beta t(\ln k)^\beta} \quad \forall t \in (0,1) \tag{2.17}$$

where $c_1 = \exp\Big(-\frac{M(N)^\beta}{2}\int_0^1 (\ln(\tau^{-1}))^\beta d\tau\Big)$ and $M_\beta = M2^{\beta-1}$. Inserting (2.17) into the right-hand side of (2.16), we get

$$u_k(x,t) \ge c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x,t) \in Q_1 = \mathbb{R}^N \times (0,1).$$

Since $k \mapsto u_k$ is increasing and by Proposition 2.3, there exists $\underline{U} := \lim\limits_{k\to\infty} u_k$ and $\underline{U} \ge u_k$. Hence

$$\underline{U}(x,t) \ge c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x,t) \in Q_1, \forall k > 0.$$

Let $\{t_n\} \subset (0,1]$ be a sequence converging to 0. We choose $k_n = \exp((2M_\beta t_n)^{\frac{1}{1-\beta}})$, equivalently $\ln k_n - M_\beta t_n (\ln k_n)^\beta = \frac{1}{2}\ln k_n$. Next we restrict $|x|$ in order

$$\ln k_n - M_\beta t_n (\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2}\ln k_n - \frac{|x|^2}{4t_n} \ge 0 \Longleftrightarrow |x| \le r_\beta\, t_n^{\frac{\beta-2}{2(\beta-1)}},$$

where $r_\beta = 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}}$. Because $1 < \beta \leq 2$, it follows

$$\lim_{n\to\infty} \underline{U}(x, t_n) = \infty,$$

uniformly on $\mathbb{R}^N$ if $1 \leq \beta < 2$, or uniformly on the ball $B_{r_2}$ where $r_2 = (2M)^{-\frac{1}{2}}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t\to 0} \underline{U}(x, t) = \infty$$

uniformly on $\mathbb{R}^N$ if $1 \leq \beta < 2$, or uniformly on the ball $B_{r_2}$ if $\beta = 2$.

We pick some point $x_0$ in $\mathbb{R}^N$ (resp. $B_{r_2}$) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, it follows, by comparison principle, that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \to \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and $x_0$ yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of $x$ and therefore it is a solution of (1.11) Since (1.10) holds, $\underline{U}(x, t) = \phi_\infty(t)$ for every $(x, t) \in Q_\infty$. □

**Proposition 2.6** *Assume* (1.12) *and* (1.8) *are satisfied. Then for any $k > 0$ there holds*

$$u_k(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty$$

*where $\Phi$ is a solution to the problem*

$$\begin{cases} -\Phi'' + f(\Phi) = 0 & \text{in } (0, \infty) \\ \lim\limits_{s\to 0} \Phi(s) = \infty. \end{cases}$$

*Proof. Step 1: Upper estimate.* Since $f$ satisfies (1.12), by [4] for any $R > 0$, there exists a solution $w_R$ to the problem

$$\begin{cases} -\Delta w_R + f(w_R) = 0 & \text{in } B_R, \\ \lim\limits_{|x|\to R} w_R(x) = \infty. \end{cases} \tag{2.18}$$

Let $x_0 \neq 0$ arbitrary in $\mathbb{R}^N$. Set $\mathbb{E} = \{\vec{e} : |\vec{e}| = 1\}$ and take $\vec{e} \in \mathbb{E}$. Put $x_{\vec{e}} = |x_0|\,\vec{e}$ and for $n > |x_0|$ put $a_n = n\vec{e}$. Denote by $\mathbb{H}_{\vec{e}}$ the open half-space generated by $\vec{e}$

and its orthogonal hyperplane at the origin, then $x_{\vec{e}}, a_n \in \mathbb{H}_{\vec{e}}$. Take $R$ such that $n - |x_0| < R < n$. We set $W_{\vec{e},n,R}(x) = w_R(x - a_n)$, then $W_{\vec{e},n,R}$ is a solution of (1.1) in $B_R(a_n)$ and blows-up on the boundary $\lim_{|x-a_n|\to R} W_{\vec{e},n,R}(x) = \infty$. By the maximum principle,

$$u_k(x,t) \le W_{\vec{e},n,R}(x) \quad \forall (x,t) \in B_R(a_n) \times (0,\infty). \tag{2.19}$$

The sequence $\{W_{\vec{e},n,R}\}$ is decreasing with respect to $R$ and is bounded from below by $u_k$, then there exists $W_{\vec{e},n} := \lim_{R\to n} W_{\vec{e},n,R}$ satisfying

$$u_k(x,t) \le W_{\vec{e},n}(x) \quad \forall (x,t) \in B_n(a_n) \times (0,\infty). \tag{2.20}$$

The sequence $\{W_{\vec{e},n}\}$ is also decreasing with respect to $n$ and is bounded from below by $u_k$, then there exists $W_{\vec{e},\infty} := \lim_{n\to\infty} W_{\vec{e},n}$. Letting $n \to \infty$ in (2.20) yields to

$$u_k(x,t) \le W_{\vec{e},\infty}(x) \quad \forall (x,t) \in \mathbb{H}_{\vec{e}} \times (0,\infty). \tag{2.21}$$

In particular,

$$u_k(x_{\vec{e}},t) \le W_{\vec{e},\infty}(x_{\vec{e}}). \tag{2.22}$$

Since $u_k$ is radial, it follows that

$$u_k(x_0,t) = u_k(x_{\vec{e}},t) \le W_{\vec{e},\infty}(x_{\vec{e}}).$$

For any $r > 0$, $n > r$, $n - r < R < n$ and $\vec{e}$, $\vec{e'} \in \mathbb{E}$, since $w_R$ is radial,

$$w_R(r\vec{e} - n\vec{e}) = w_R(r\vec{e'} - n\vec{e'}).$$

Letting successively $R \to n$, $n \to \infty$ yields to

$$W_{\vec{e},\infty}(r\vec{e}) = W_{\vec{e'},\infty}(r\vec{e'}).$$

Define $\tilde{\Phi}(r) := W_{\vec{e},\infty}(r\vec{e})$, $\forall r \in (0,\infty)$ then it satisfies

$$\begin{cases} -\tilde{\Phi}'' - \dfrac{N-1}{r}\tilde{\Phi}' + f(\tilde{\Phi}) = 0 & \text{in } (0,\infty) \\ \lim\limits_{r\to 0}\tilde{\Phi}(r) = \infty \end{cases} \tag{2.23}$$

and

$$u_k(x,t) \le \tilde{\Phi}(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{2.24}$$

*Step 2: End of the proof.* We claim that

$$\tilde{\Phi}(r) \le \Phi(r) \qquad \forall r \in (0,\infty). \tag{2.25}$$

For any $\epsilon > 0$, we set $\Phi_\epsilon(r) = \Phi(r-\epsilon)$, $r > \epsilon$ then $\Phi_\epsilon$ is a solution of

$$-\Phi_\epsilon'' + f(\Phi_\epsilon) = 0 \qquad \text{in } (\epsilon, \infty) \tag{2.26}$$

verifying $\lim_{r\to\epsilon}\Phi_\epsilon(r) = \infty$. Since $\Phi_\epsilon' \le 0$, $\Phi_\epsilon$ is a supersolution of the equation in (2.23) in $(\epsilon, \infty)$, which dominates $\tilde{\Phi}$ at $r = \epsilon$. By the maximum principle, $\tilde{\Phi} \le \Phi_\epsilon$ in $(\epsilon, \infty)$. Letting $\epsilon \to 0$ yields (2.25). Combining (2.24) and (2.25) leads to the conclusion. □

*Remark.* Combining Proposition 2.3 and Proposition 2.6 yields

$$u_k(x,t) \le min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty, \forall k > 0. \tag{2.27}$$

**Proof of Theorem 1.4.**

The sequence $\{u_k\}$ increase with respect to $k$ and is bounded from above by (2.27) then there exists $\underline{U} := \lim_{k\to\infty} u_k$ satisfying

$$\underline{U}(x,t) \le min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty, \forall k > 0. \tag{2.28}$$

Moreover, the function $\underline{U} \in \mathcal{U}_0$ because it has the following properties:
(i) It is positive in $Q_\infty$, belongs to $C(\overline{Q_\infty}\backslash\{(0,0)\})$ and vanishes on $\mathbb{R}^N \times \{0\}\backslash\{(0,0)\}$.
(ii) Is satisfies (1.1) and

$$\lim_{t\to 0}\int_{B_\sigma} \underline{U}(x,t)dx = \infty \qquad \forall \sigma > 0. \tag{2.29}$$

In the sense of initial trace in Definition 4.3, $\underline{U}$ has initial trace $tr_{\mathbb{R}^N}(\underline{U}) = (\{0\}, 0)$ (here $\{0\}$ is the singular part and the Radon measure on $\mathbb{R}^N\backslash\{0\}$ is the zero measure) and the conclusion follows due to Lemma 4.5. □

By a simple adaptation of the proof of Proposition 2.3 and Proposition 2.6 it is possible to extend (2.28) to any positive solution vanishing on $\mathbb{R}^N \times \{0\} \setminus \{(0,0)\}$.

**Proposition 2.7** *Assume* (1.8), (1.12) *and* $(C2)$ *are satisfied. Then any positive solution* $u$ *of* (1.1) *which* $C(\overline{Q_\infty}\backslash\{(0,0)\})$ *and vanishes on* $\mathbb{R}^N \times \{0\}\backslash\{(0,0)\}$ *satisfies*

$$u(x,t) \le min\{\phi_\infty(t), \Phi(|x|)\} \qquad \forall (x,t) \in Q_\infty. \tag{2.30}$$

*Proof.* Estimate (2.19) is valid with $u_k$ replaced by $u$. The remaining of the proof of Proposition 2.6 is similar and yields to the first estimate

$$u(x,t) \le \Phi(|x|) \qquad \forall (x,t) \in Q_\infty.$$

For the second estimate, since $f(0) = 0$ and due to the convexity of $f$, the following inequality holds

$$f(a+b) \geq f(a) + f(b) \quad \forall a, b > 0, \tag{2.31}$$

which implies that for any $R, \tau > 0$, $(x,t) \mapsto \phi_\infty(t-\tau) + w_R(x)$ is a supersolution of (1.1) in $B_R \times (\tau, \infty)$. This function dominates $u$ on the parabolic boundary, thus in the domain itself by the comparison principle. Since $f(r) > 0$ if $r > 0$, $\lim\limits_{R\to\infty} w_R = 0$ in $\mathbb{R}^N$. Therefore

$$u(x,t) \leq \phi_\infty(t) = \lim_{\tau\to 0} \lim_{R\to\infty} (\phi_\infty(t-\tau) + w_R(x)) \qquad \forall (x,t) \in Q_\infty.$$

This implies (2.30). □

It is also possible to construct a maximal element of $\mathcal{U}_0$ ($\mathcal{U}_0$ is defined in Theorem 1.4). For $\ell > 0$ and $\epsilon > 0$, let $u := U_{\epsilon,\ell}$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) &= 0 & \text{in } Q_\infty \\ u(x,0) &= \ell \chi_{B_\epsilon} & \text{in } \mathbb{R}^N. \end{cases}$$

**Lemma 2.8** *Assume* (1.8), (1.12) *and* (C2) *are satisfied. For any $\tau > 0$ and $\epsilon > 0$, there exist $\ell > 0$ and $m(\tau,\epsilon) > 0$ such that any positive solution $u$ of* (1.1) *which verifies (i) in the proof of Theorem 1.4 satisfies*

$$u(x,t) \leq U_{\epsilon,\ell}(x, t-\tau) + m(\tau,\epsilon) \qquad \forall (x,t) \in Q_\infty,\ t \geq \tau. \tag{2.32}$$

*Furthermore*

$$\lim_{\tau\to 0} m(\tau,\epsilon) = 0 \qquad \forall \epsilon > 0. \tag{2.33}$$

*Finally*

$$\overline{U}(x,t) = \lim_{\tau\to 0} \lim_{\epsilon\to 0} \lim_{\ell\to\infty} (U_{\epsilon,\ell}(x,t-\tau) + m(\tau,\epsilon)) \tag{2.34}$$

*is the maximal element of $\mathcal{U}_0$.*

*Proof.* We set $\ell = \phi_\infty(\tau)$, then $u(x,\tau) \leq \ell$ for any $x \in \mathbb{R}^N$. Let $W := W_{\epsilon/2}$ be the solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t W - \Delta W + f(W) &= 0 & \text{in } B^c_{\epsilon/2} \times (0,\infty) \\ W(x,0) &= 0 & \text{in } B^c_{\epsilon/2} \\ W(x,t) &= \phi_\infty(t) & \text{in } \partial B^c_{\epsilon/2} \times (0,\infty) \end{cases} \tag{2.35}$$

and put $m(\tau,\epsilon) := max\{W_{\epsilon/2}(x,\delta) : |x| > \epsilon, 0 < \delta \leq \tau\}$. It is clear to see that

$$\lim_{\tau\to 0} m(\tau,\epsilon) = W_{\epsilon/2}(x,0) = 0. \tag{2.36}$$

From the fact that $u(x,0) = 0$ in $B^c_{\epsilon/2}$, $u(x,t) \le \phi_\infty(t)$ in $\partial B^c_{\epsilon/2} \times (0,\infty)$ and the maximum principle, it follows that $u(x,t) \le W_{\epsilon/2}(x,t)$ in $B^c_{\epsilon/2} \times (0,\infty)$.

We next compare $U_{\epsilon,\ell}(.,.-\tau) + m(\tau,\epsilon)$ with $u$ in $\mathbb{R}^N \times (\tau,\infty)$. The function $U_{\epsilon,\ell}(.,.-\tau) + m(\tau,\epsilon)$ is a supersolution of (1.1) in $\mathbb{R}^N \times (\tau,\infty)$. If $x \in B_\epsilon$, $U_{\epsilon,\ell}(x,0) = \ell \ge u(x,\tau)$, which implies $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$. If $x \in B^c_\epsilon$, $m(\tau,\epsilon) \ge W_{\epsilon/2}(x,\tau) \ge u(x,\tau)$, hence $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$. So we always have $U_{\epsilon,\ell}(x,0) + m(\tau,\epsilon) \ge u(x,\tau)$ for any $x \in \mathbb{R}^N$. Applying maximum principle yields $U_{\epsilon,\ell}(.,.-\tau) + m(\tau,\epsilon) \ge u$ in $\mathbb{R}^N \times (\tau,\infty)$. Finally, the function $\overline{U}$ defined by (2.34) is the maximal solution because $U_{\epsilon,\ell}(x,t-\tau) \to U_{\epsilon,\ell}(x,t)$ as $\tau \to 0$ and $U_{\epsilon,\ell} \uparrow U_{\epsilon,\infty}$ when $\ell \to \infty$ and $U_{\epsilon,\infty} \downarrow \overline{U}$ when $\epsilon \to 0$. □

# 3 About uniqueness

We first prove the existence of the global radial solutions of (1.18) under the Keller-Osserman condition.

**Proof of Proposition 1.5.**

A solution of (1.19) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N} \int_0^s t^{N-1} f(w) dt ds \tag{3.1}$$

Existence follows from Picard-Lipschitz fixed point theorem. The function is increasing and defined on a maximal interval $[0, r_a)$. By a result of Vazquez and Veron [11] $r_a = \infty$, thus the solution is global. Uniqueness on $[0,\infty)$ follows always from local uniqueness. The function $r \mapsto w(r)$ is increasing and

$$w'(r) \ge \frac{ah(a)}{N} r,$$

$$w(r) \ge a + \frac{ah(a)}{2N} r^2$$

for all $r > 0$. □

**Proposition 3.1** *Assume* (1.16) *holds. For any* $u_0 \in C(\mathbb{R}^N)$ *which satisfies*

$$w_a(|x|) \le u_0(x) \le w_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{3.2}$$

*for some* $0 < a < b$, *there exists a positive function* $\overline{u} \in C(\overline{Q}_\infty) \cap C^{2,1}(Q_\infty)$ *solution of* (1.1) *in* $Q_\infty$ *and satisfying* $\overline{u}(.,0) = u_0$ *in* $\mathbb{R}^N$. *Furthermore*

$$w_a(|x|) \le u(x,t) \le w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{3.3}$$

*Proof.* Clearly $w_a$ and $w_b$ are ordered solutions of (1.1). We denote by $u_n$ the solution of the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) &= 0 & \text{in } Q_n = B_n \times (0,\infty) \\ u_n(x,t) &= (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0,\infty) \\ u_n(x,0) &= u_0(x) & \text{in } B_n. \end{cases} \tag{3.4}$$

By the maximum principle, $u_n$ satisfies (3.3) in $Q_n$. Using locally parabolic equations regularity theory, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q}_\infty$. Using a diagonal sequence, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q}_\infty$ to some weak solution $\overline{u} \in C(\overline{Q}_\infty)$ which satisfies $\overline{u}(.,0) = u_0$ in $\mathbb{R}^N$. By standard method, $\overline{u}$ is a strong solution (at least $C^{2,1}(Q_\infty)$). □

**Proposition 3.2** *Assume* (1.10) *and* (1.16) *hold. Then for any* $u_0 \in C(\mathbb{R}^N)$ *which satisfies*

$$w_a(|x|) \le u_0(x) \le w_b(|x|) \qquad \forall x \in \mathbb{R}^N \tag{3.5}$$

*for some* $0 < a < b$*, there exists a positive function* $\underline{u} \in C(\overline{Q_\infty})$ *solution of* (1.1) *in* $Q_\infty$ *satisfying* $\underline{u}(.,0) = u_0$ *in* $\mathbb{R}^N$ *and*

$$\underline{u}(x,t) \le \min\{\phi_\infty(t), w_b(|x|)\} \qquad \forall (x,t) \in Q_\infty. \tag{3.6}$$

*Proof.* For any $R > 0$, let $u_R$ be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R + f(u_R) = 0 & \text{in } Q_\infty \\ u_R(x,0) = u_0(x)\chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.7}$$

The solution which is constructed is domainated by the solution of the heat equation with the same initial data. Thus

$$u_R(x,t) \le (4\pi t)^{-N/2} \int_{B_R} e^{-|x-y|^2/4t} u_0(x) dy \qquad \forall (x,t) \in Q_\infty. \tag{3.8}$$

and $\lim_{|x|\to\infty} u_R(x,t) = 0$ uniformly with respect to $t$. The functions $\phi_\infty$ and $w_b$ are solutions of (1.1) in $Q_\infty$, which dominate $u_R$ at $t = 0$. By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \ge u_R(x,t) \quad \forall (x,t) \in Q_\infty. \tag{3.9}$$

The fact that the mapping $R \mapsto u_R$ is increasing and (3.9) imply that there exists $\underline{u} := \lim_{R\to\infty} u_R$ which satisfies $\underline{u}(.,0) = u_0$ in $\mathbb{R}^N$. Letting $R \to \infty$ in (3.9) yields (3.6). □

**Proof of Theorem 1.6.**

Combining Proposition 3.1 and Proposition 3.2 we see that there exists two solutions $\underline{u}$ and $\overline{u}$ with the same initial data $u_0$ which are ordered and different since $\lim_{|x|\to\infty} \overline{u}(x,t) = \infty$ and $\lim_{|x|\to\infty} \underline{u}(x,t) \le \phi_\infty(t) < \infty$ for all $t > 0$. □

**Proof of Theorem 1.7.**

*Step 1: There always holds*

$$(ah(a) - bh(b))sign(a-b) \ge |a-b|\, h(|a-b|) \qquad \forall a, b > 0 \tag{3.10}$$

where $h$ is defined in (2.2) and

$$sign(z) = \begin{cases} 1 & \text{if } z > 0, \\ -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

In fact, since $h$ is increasing and assuming $a > b$, we get

$$\begin{aligned} ah(a) - bh(b) &= (a-b)h(a) + b(h(a) - h(b)) \\ &\ge (a-b)h(a) \\ &\ge (a-b)h(a-b). \end{aligned}$$

*Step 2: End of the proof..* By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \le [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})]sign(u - \tilde{u}),$$

therefore by step 1,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + |u - \tilde{u}|\, h(|u - \tilde{u}|) \le 0. \tag{3.11}$$

Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any $R \ge R_\epsilon$,

$$0 \le |u - \tilde{u}|\,(x,t) \le w_\epsilon(|x|) \quad \forall (x,t) \in B_R^c \times [0,1]. \tag{3.12}$$

Since $w_\epsilon$ is a positive solution of (1.1) which dominates $|u - \tilde{u}|$ on $\partial B_R \times [0,1]$ and at $t = 0$, it follows that $|u - \tilde{u}| \le w_\epsilon$ in $B_R \times [0,1]$. Letting $R \to \infty$ yields to $|u - \tilde{u}| \le w_\epsilon$ in $\mathbb{R}^N \times [0,1]$. Letting $\epsilon \to 0$ and since $\lim_{\epsilon\to 0} w_\epsilon(|x|) = 0$ for any $x \in \mathbb{R}^N$, we derive $|u - \tilde{u}| = 0$, thus $u = \tilde{u}$ in $\mathbb{R}^N \times [0,1]$. Iterating yields that equality holds in $Q_\infty$. □

*Remark.* If we replace the condition $(C1)$ by the condition $(C2)$, the conclusion of Theorem 1.7 remains valid. Indeed, it follows by the convexity of $f$ that

$$(f(a) - f(b))sign(a-b) \ge f(|a-b|) \quad \forall a, b > 0.$$

Then we proceed as in step 2 to get the desired conclusion.

**Proof of Theorem 1.8.**

*Step 1: Construction of the minimal solution.* The solution $\underline{u}$ which is constructed in the Proposition 3.2 is a minimal solution of (1.1) in $Q_\infty$ with the initial value $u_0$. Indeed, if $u \in C^{2,1}(Q_\infty)$ is a solution of (1.1) in $Q_\infty$ which satisfies $u(.,0) = u_0$ in $\mathbb{R}^N$ then by maximum principle $u_R \leq u$ in $Q_\infty$ where $u_R$ is the solution of (3.7). Letting $R \to \infty$ yields $\underline{u} \leq u$ in $Q_\infty$.

*Step 2: Construction of the maximal solution.* For any $R > 0$ and $k > 0$, let $v_{R,k}$ be the solution of the initial-boundary problem

$$\begin{cases} \partial_t v_{R,k} + \Delta v_{R,k} + f(v_{R,k}) &= 0 \qquad \text{in } B_R \times (0,\infty) \\ v_{R,k} &= k \qquad \text{on } \partial B_R \times (0,\infty) \\ v_{R,k}(.,0) &= u_0 \qquad \text{in } B_R. \end{cases} \tag{3.13}$$

Since $f$ is convex, the solution $(x,t) \mapsto \phi_\infty(t) + w_R(x)$ ($w_R$ is the solution of (2.18)) is a supersolution of (1.1) in $B_R \times (0,\infty)$, which dominates $v_{R,k}$ on $\partial B_R \times (0,\infty)$ and at $t = 0$. By comparison principle, $v_{R,k}(x,t) \leq \phi_\infty(t) + w_R(x)$ for every $(x,t) \in B_R \times (0,\infty)$. The sequence $\{v_{R,k}\}$ is increasing with respect to $k$ and converges to a function $v_{R,\infty}$, which is a solution of (1.1) in $B_R \times (0,\infty)$ with initial data $u_0$. The sequence $\{v_{R,\infty}\}$ is decreasing and converges to a solution $v_\infty$ of (1.1) in $Q_\infty$, which satisfies $v_\infty(.,0) = u_0$ in $\mathbb{R}^N$. If $u \in C^{2,1}(Q_\infty)$ is a positive solution of (1.1) with initial data $u_0$ then for $R > 0$ and $k > 0$, by comparison principle, $u \leq v_{R,k} + w_R$ in $B_R \times (0,\infty)$. Since $\lim_{R\to\infty} w_R = 0$ in $\mathbb{R}^N$, letting successively $k \to \infty$ and $R \to \infty$ yields $u \leq v_\infty$ in $Q_\infty$.

*Step 3: Uniqueness.* Put $\tilde{v} = v_\infty - \underline{u}$ then $\tilde{v} \geq 0$ in $Q_\infty$. By convexity of $f$, $\tilde{v}$ is a subsolution of (1.1) in $Q_\infty$, which satisfies $\tilde{v}(.,0) = 0$ in $\mathbb{R}^N$. It follows from the maximum principle that $\tilde{v} \leq 0$ in $Q_\infty$ and hence $\tilde{v} = 0$ in $Q_\infty$, which leads to the uniquness. □.

# 4 Initial trace

Let $\Omega$ is an arbitrary open domain in $\mathbb{R}^N$ with $C^2$ boundary, we denote by $\mathfrak{M}(\Omega)$ (resp. $\mathfrak{M}^b(\Omega)$) the set of Radon measures in $\Omega$ (resp. bounded Radon measures), and by $\mathfrak{M}_+(\Omega)$ (resp. $\mathfrak{M}^b_+(\Omega)$) its positive cone. For $0 < T \leq \infty$, we set $Q^\Omega_T = \Omega \times (0,T)$.

## 4.1 The regular part of the initial trace

In this section we only assume that $f$ is a continuous nonnegative function defined on $\mathbb{R}_+$ and that $u$ is a $C^{2,1}$ positive solution of (1.1) in $Q^\Omega_T$.

**Lemma 4.1** *Assume $G$ is a bounded $C^2$ domain in $\mathbb{R}^N$, $Q_T^{\overline{G}} := \overline{G} \times (0,T]$ and let $u \in C^{2,1}(Q_T^{\overline{G}})$ be a positive solution of (1.1) in $Q_T^G$ such that $u, f(u) \in L^1(Q_T^G)$. Then $u \in L^\infty(0,T;L^1(G'))$ for any domain $G' \subset \overline{G'} \subset G$ and there exists a positive Radon measure $\mu_G$ on $G$ such that*

$$\lim_{t\to 0}\int_G u(x,t)\zeta(x)dx = \int_G \zeta(x)d\mu_G(x) \qquad \forall \zeta \in C_c(G). \tag{4.1}$$

*Proof.* Let $\phi := \phi_G$ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ with corresponding eigenvalue $\lambda_G$. We assume $\phi > 0$ in $G$. Then

$$\frac{d}{dt}\int_G u\phi dx + \lambda_G \int_G u\phi dx + \int_G f(u)\phi\, dx + \int_{\partial G} u\phi_{\mathbf{n}} dS = 0$$

where $\phi_{\mathbf{n}}$ denote the outward normal derivative of $\phi$. Since $\phi_{\mathbf{n}} < 0$, the function

$$t \mapsto e^{\lambda_G t}\int_G u(x,t)\phi(x)dx - \int_t^T \int_G e^{\lambda_G s} f(u)\phi dx\, ds$$

is increasing and

$$\int_G u(x,t)\phi(x)dx \le e^{\lambda_G(T-t)}\int_G u(x,T)\phi(x)dx + e^{-\lambda_G t}\int_t^T\int_G e^{\lambda_G s} f(u)\phi dx\, ds$$

for $0 < t \le T$. Thus $u \in L^\infty(0,T;L^1(G'))$ for any strict domain $G'$ of $G$. If $\zeta \in C_c(G)$, there holds

$$\frac{d}{dt}\left(\int_G u(x,t)\zeta(x)dx - \int_t^T\int_G (f(u)\zeta - u\Delta\zeta)\, dx\, ds\right) = 0. \tag{4.2}$$

Consequently

$$\lim_{t\to 0}\int_G u(x,t)\zeta(x)dx = \int_G u(x,T)\zeta(x)dx + \int_0^T\int_G (f(u)\zeta - u\Delta\zeta)\, dx\, ds. \tag{4.3}$$

This implies that $u(.,t)$ admits a limit in $\mathcal{D}'(G)$ and this limit is a positive distribution. Therefore there exists a positive Radon measure $\mu_G$ on $G$ satisfies (4.1). □

**Proof of Proposition 1.9.**

It is clear that $\mathcal{R}(u)$ is an open subset of $\Omega$. If $G$ is a strict bounded subdomain of $\mathcal{R}(u)$, i.e. $\overline{G} \subset \mathcal{R}(u)$, there exists a finite number of points $z_j$ $(j = 1, ..., k)$ and $r_j' > r_j > 0$ such that $u$, $f(u) \in L^1(Q_T^{B_{r_j'}(z_j)})$ and $\overline{G} \subset \cup_{j=1}^k B_{r_j}(z_j)$. Let $\mu_j = \mu_{B_{r_j}(z_j)}$

the measure defined in Lemma 4.1. If $\zeta \in C_c(G)$ there exists a partition of unity $\{\eta_j\}_{j=1}^k$ relative to the cover $\{B_{r_j}(z_j)\}_{j=1}^k$ such that $\eta_j \in C_0^\infty(G)$, $\mathrm{supp}(\eta_j) \subset B_{r_j}(z_j)$ and $\zeta = \sum_{j=1}^k \eta_j \zeta$. Since

$$\lim_{t\to 0} \int_{B_{r_j}(z_j)} u(x,t)(\eta_j\zeta)(x)dx = \int_{B_{r_j}(z_j)} (\eta_j\zeta)(x)d\mu_j(x) \qquad \forall j = 1, ..., k,$$

there exists a positive Radon measure $\mu$ on $\mathcal{R}(u)$ satisfying (1.23). Notice also that $u \in L^\infty(0,T;L^1(G))$ for any $G \subset \overline{G} \subset \mathcal{R}(u)$. □

The main problem is to analyse what does happen on the singular set $\mathcal{S}(u)$.

## 4.2 The Keller-Osserman condition holds

If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (1.1) is based upon a dichotomy in the behaviour of those solutions near $t = 0$.

**Lemma 4.2** *Assume $u$ is a positive solution of* (1.1) *in $Q_T^\Omega$ and $z \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds*

*(i) There exists an open neighborhood $G$ of $z$ such that $u \in L^1(Q_T^G)$.*

*(ii) $f$ is nondecreasing and* (1.12) *holds.*

*Then for every open relative neighborhood $G'$ of $z$*

$$\lim_{t\to 0} \int_{G'} u(x,t)dx = \infty. \tag{4.4}$$

*Proof.* First, we assume (i) holds and let $\zeta \in C_c^2(G)$, $\zeta \geq 0$. Since $z \in \mathcal{S}(u)$, then for every open relative neighborhood $G'$ of $z$, there holds

$$\int_0^T \int_{G'} f(u)dx\,dt = \infty. \tag{4.5}$$

Since there exists

$$\lim_{t\to 0} \int_t^T \int_{G'} u\Delta\zeta dx\,dt = L \in \mathbb{R},$$

it follows from (4.3) that

$$\int_{G'} u(x,t)\zeta(x)dx = \int_t^T \int_{G'} f(u)\zeta dxds + O(1), \tag{4.6}$$

which implies (4.4).

Next we assume that (1.12) holds and $u \notin L^1(Q_T^G)$ for every relative neighborhood $G$ of $z$. If there exists an open neighborhood $G \subset \Omega$ of $z$ such that (4.4) does not hold, there exists a sequence $\{t_n\}$ decreasing to 0 and $0 \le M < \infty$ such that

$$\sup_{t_n} \int_G u(x,t_n)dx = M. \tag{4.7}$$

Furthermore, we can always replace $G$ by an open ball $B_R(z) \subset G$. Thus (4.7) holds with $G$ replaced by $B_R(z)$. Let $w := w_R$ be the maximal solution of

$$\begin{cases} -\Delta w + f(w) &= 0 \qquad \text{in } B_R(z) \\ \lim\limits_{|x-z|\to R} w(x) &= \infty. \end{cases} \tag{4.8}$$

Let $v := v_n$ be the solution of

$$\begin{cases} v_t - \Delta v &= 0 & \text{in } B_R(z)\times(t_n,\infty) \\ v &= 0 & \text{in } \partial B_R(z)\times(t_n,\infty) \\ v(.,t_n) &= u(.,t_n) & \text{in } B_R(z). \end{cases} \tag{4.9}$$

Since $v_n \ge 0$, $f(w_R+v_n) \ge f(w_R)$, and $w_R+v_n$ is a supersolution of (1.1) in $B_R(z)\times(t_n,T)$. It dominates $u$ on $\partial B_R(z)\times(t_n,T)$ and at $t=t_n$, thus $u \le w_R+v_n$ in $B_R(z)\times(t_n,T)$. We can assume that $u(.,t_n)\to\nu$ for some positive and bounded measure $\nu$ on $B_R(z)$. Therefore

$$u(x,t) \le v(x,t) + w_R(x) \qquad \text{in } Q_T^{B_R(z)} \tag{4.10}$$

where $v$ is the solution of

$$\begin{cases} v_t - \Delta v &= 0 & \text{in } Q_\infty^{B_R(z)} \\ v &= 0 & \text{in } \partial B_R(z)\times(0,\infty) \\ v(.,0) &= \nu & \text{in } \mathcal{D}'(B_R(z)). \end{cases} \tag{4.11}$$

Since $v \in L^1(Q_T^{B_R(z)})$ and $w_R$ is uniformly bounded in any ball $B_{R'}(z)$ for $0<R'<R$ we conclude that $u \in L^1(Q_T^{B_{R'}(z)})$, which is a contradiction. □

**Definition 4.3** *Assume $f$ is nondecreasing and satisfies* (1.12). *Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of* (1.1) *in $Q_T^\Omega$. We say that $u$ possesses an initial trace with regular part $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ and singular part $\mathcal{S}(u) = \Omega\setminus\mathcal{R}(u)$ if*
*(i) For any $\zeta \in C_c(\mathcal{R}(u))$,*

$$\lim_{t\to 0}\int_{\mathcal{R}(u)} u(x,t)\zeta(x)dx = \int_{\mathcal{R}(u)} \zeta(x)d\mu(x). \tag{4.12}$$

*(ii) For any open set $G \subset \Omega$ such that $G \cap \mathcal{S}(u) \neq \emptyset$*

$$\lim_{t\to 0} \int_G u(x,t)dx = \infty. \tag{4.13}$$

**Proof of Theorem 1.11**

The set $\mathcal{R}(u)$ and the measure $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ are defined by Definition 1.10 thanks to Proposition 1.9. Because (1.12) holds, $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ inherits the property (ii) in Definition 4.3 because of Lemma 4.2 (ii). □

If $\Omega$ is a bounded domain with a $C^2$ boundary and $\mu \in \mathfrak{M}^b_+(\Omega)$, we denote by $u_\mu$ the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q^\Omega_\infty \\ u = 0 & \text{in } \partial\Omega \times (0,\infty) \\ u(.,0) = \mu & \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{4.14}$$

We recall the following stability result proved in [6, Th 1.1].

**Lemma 4.4** *Let $\Omega$ be a bounded domain with a $C^2$ boundary. Assume $f$ is non-decreasing and satisfies* (1.8)*. Then for any $\mu \in \mathfrak{M}^b(\Omega)$ problem* (4.14) *admits a unique solution $u_\mu$. Moreover, if $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$ converges weakly to $\mu \in \mathfrak{M}^b(\Omega)$ then $u_{\mu_n} \to u_\mu$ locally uniformly in $\overline{\Omega} \times (0,\infty)$ and in $L^1(Q^\Omega_T)$, and $f(u_{\mu_n}) \to f(u_\mu)$ in $L^1(Q^\Omega_T)$, for every $T > 0$.*

*Remark.* The result remains true if $\Omega = \mathbb{R}^N$ and $\mu_n$ have their support in a fixed compact set.

If $\Omega$ is bounded, by Lemma 4.4, for every $y \in \Omega$ and $k > 0$, there exists a unique solution $v_{\Omega,y,k} := v$ to (4.14) with $\mu = k\delta_0$. By comparison principle [6, Proposition 1.2] $v_{\Omega,y,k}$ is positive and increases as $k$ increases. By Proposition 2.7, $\{v_{\Omega,y,k}\}$ is bounded on compact subsets of $Q^\Omega_T$ and consequently

$$v_{\Omega,y,\infty} := \lim_{k\to\infty} v_{\Omega,y,k}$$

is a solution of (1.1) in $Q^\Omega_\infty$ which vanishes on $\overline{\Omega} \times \{0\} \setminus \{(y,0)\}$ and on $\partial\Omega \times (0,\infty)$. Note that the value of $v_{\Omega,y,\infty}(x,t)$ depends only on $|x-y|$ and $t$.

**Lemma 4.5** *Let $\Omega$ be a bounded domain with $C^2$ boundary. Assume $f$ is nondereasing and satisfies* (1.8) *and* (1.12)*. Let $u \in C^{2,1}(Q^\Omega_\infty)$ is a positive solution of* (1.1) *in $Q^\Omega_\infty$ with initial trace $(\mathcal{S}(u), \mu)$. Then for every $y \in \mathcal{S}(u)$,*

$$v_{\Omega,y,\infty} \leq u \tag{4.15}$$

*in $Q^\Omega_\infty$.*

*Proof.* By translation we may suppose that $y = 0$. Since $0 \in \mathcal{S}(u)$, for any $\eta > 0$ small enough

$$\lim_{t\to 0} \int_{B_\eta} u(x,t)dx = \infty.$$

For $\epsilon > 0$, denote $M_{\epsilon,\eta} = \int_{B_\eta} u(x,\epsilon)dx$. For any $m > m_\eta = \inf_{\sigma>0} M_{\sigma,\eta}$ there exists $\epsilon = \epsilon(m,\eta)$ such that $m = M_{\epsilon,\eta}$ and $\lim_{\eta\to 0} \epsilon(m,\eta) = 0$. For $R > 0$ let $v_\eta$ be the solution of the problem

$$\begin{cases} \partial_t v_\eta - \Delta v_\eta + f(v_\eta) &= 0 & \text{in } \Omega \times (0,\infty) \\ v_\eta &= 0 & \text{on } \partial\Omega \times (0,\infty) \\ v_\eta(x,0) &= u(x,\epsilon)\chi_{B_\eta} & \text{in } \Omega \end{cases}$$

where $\eta < R$ and $\chi_{B_\eta}$ is the characteristic function of $B_\eta$. By the maximum principle $v_\eta \le u$ in $\Omega \times (\epsilon,\infty)$. By Lemma 4.4, when $\eta$ goes to zero $v_\eta$ converges to $v_{\Omega,0,m}$. Letting $m$ goes to infinity yields (4.15). □

The following convergence lemma is obtained by using the arguments as in [5, Lemma 3.2]

**Lemma 4.6** *Assume $f$ is nondereasing and satisfies* (1.8). *Let $\{u_n\}$ be a sequence of positive solutions of* (1.1) *in $Q_\infty^\Omega$ and let $A \subset \Omega$ is an open subset. Suppose that $Tr_A(u_n) = \nu_n \in \mathfrak{M}^b(A)$ and $u_n \to u$ locally uniformly in $Q_\infty^\Omega$. Thus $u$ is a positive solution of* (1.1) *in $Q_\infty^\Omega$ and we denote $tr_\Omega(u) = (\mathcal{S},\mu)$. Under this assumption, if $A \subset \mathcal{S}$ and $\{\nu(A)\}$ is bounded then $\nu_n \to \mu_{|A}$ weakly where $\mu_{|A} := \chi_A\mu$.*

*Proof.* Let $\zeta \in C_c^2(A)$, $\xi \ge 0$ then

$$\lim_{n\to\infty} \int_0^T \int_\Omega u_n \Delta\zeta dx\, dt = \int_0^T \int_\Omega u\Delta\zeta dx\, dt. \tag{4.16}$$

Since

$$\int_A \zeta u_n(.,T)dx + \int_0^T \int_A (-u_n\Delta\zeta + f(u_n)\zeta)dx\, dt = \int_A \zeta d\nu_n(x), \tag{4.17}$$

we derive by Fatou's lemma and (4.16) that

$$\int_A \zeta u(.,T)dx + \int_0^T \int_A (-u\Delta\zeta + f(u)\zeta)dx\, dt \le \liminf_{n\to\infty} \int_A \zeta d\nu_n(x). \tag{4.18}$$

Since $\{\nu_n(A)\}$ is bounded, we can extract a sunsequence (for simplification, we also denote by $\{\nu_n\}$ this subsequence) which converges weakly in $\mathfrak{M}(A)$ to a Borel

measure $\nu$. We shall show that this limit does not depend on the subsequence and that in fact $\nu = \mu_{|A}$. Now letting $T \to 0$ in (4.18) yields

$$\int_A \zeta d\mu \leq \int_A \zeta d\nu, \tag{4.19}$$

which means

$$\mu_{|A} \leq \nu. \tag{4.20}$$

Now let $A'$ be an open bounded subdomain of $A$ with smooth boundary. Let $\{w_n\}$ (resp. $w$) be a solution of (1.1) in $Q_T^{A'}$ with initial data $\nu_n^* = \chi_{A'}\nu_n$ (resp. $\nu^* = \chi_{A'}\nu$) and boundary data zero. Cleary $\nu_n^* \to \nu^*$ weakly relative to functions in $C_c(\overline{A}')$. By Lemma 4.4, $w_n \to w$ locally uniformly in $\overline{A}' \times (0,\infty)$ and in $L^1(Q_T^{A'})$, and $f(w_n) \to f(w)$ in $L^1(Q_T^{A'})$ for every $T > 0$. Since $w_n \leq u_n$ in $Q_\infty^{A'}$, $w \leq u$ in $Q_\infty^{A'}$, and hence $\nu^* \leq \mu_{|A'}$. Since $A'$ is arbitrary, it follows that $\nu \leq \mu_{|A}$ and we conclude that the equality holds by (4.20). □

If $A$ is an open subset of $\Omega$ and $\nu \in \mathfrak{M}^+(A)$, we define an extension $\underline{\nu}$ of $\nu$ to $\Omega$ by

$$\underline{\nu}(E) = \inf_{E \subseteq O} \nu(O \cap A) \tag{4.21}$$

for every Borel set $E \subset \Omega$, where the infimum is taken over the open subsets $O$; $\underline{\nu}$ is an outer regular Borel measure on $\Omega$ and $\nu = \nu_{|A}$.

The next comparison result is obtained by using the argument as in [5, Lemma 2.7].

**Lemma 4.7** *Let $\Omega$ be a bounded domain in $C^2$ boundary.Assume $f$ is nondecreasing nonnegative on $\mathbb{R}_+$. Let $(g_i, \mu_i) \in L^1(\partial\Omega \times (0,T)) \times \mathfrak{M}^b(\Omega)$, $i = 1, 2$ and denote by $u_i$, $i = 1, 2$ the solution to the problem*

$$\begin{cases} \partial_t u_i - \Delta u_i + f(u_i) &= 0 & \text{in } Q_T^\Omega \\ u_i &= g_i & \text{on } \partial\Omega \times (0,T) \\ u_i(.,0) &= \mu_i & \text{in } \Omega. \end{cases}$$

*If $g_1 \leq g_2$ in $\partial\Omega \times (0,T)$ and $\mu_1 \leq \mu_2$ then $u_1 \leq u_2$ in $Q_T^\Omega$.*

We next mimic the proof of M. Marcus and L. Véron in [5, Lemma 3.3] to show the existence of a minimal solution of (1.1) with a given initial trace in $\mathfrak{M}_+(A)$ for any open subset $A$ in $\Omega$.

**Lemma 4.8** *Assume $\Omega$ is bounded, $f$ is nondereasing and satisfies* (1.8), (1.12) *and* $(C2)$.

*(i) Let $A$ be an open subset of $\Omega$ and let $\nu \in \mathfrak{M}_+(A)$. Then there exists a positive solution of* (1.1) *in $Q_\infty^\Omega$ denoted by $\underline{u}_\nu$ such that*

$$\begin{cases} \underline{u}_\nu = 0 & on \quad \partial\Omega \times (0,\infty), \\ Tr_\Omega(\underline{u}_\nu) = \underline{\nu} & and \quad \underline{u}_\nu \le v, \end{cases} \tag{4.22}$$

*for every positive solution $v$ of* (1.1) *such that $Tr_A(v) \ge \nu$.*

*(ii) Let $u_n$ be the solution of problem*

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) = 0 & in\ Q_T^\Omega \\ u_n = n & on\ \partial\Omega \times (0,\infty) \\ u_n(.,0) = ndx & in\ \Omega. \end{cases} \tag{4.23}$$

*Denote $U_{\infty,\Omega} := \lim_{n\to\infty} u_n$. Then $U_{\infty,\Omega}$ is the maximal solution of* (1.1) *in $Q_\infty^\Omega$ in the sense that the following relation holds in $Q_T^\Omega$ for every positive solution $v$ of* (1.1)

$$U_{\infty,\Omega} \ge v. \tag{4.24}$$

*Proof.* Let $D$ be an open subdomain of $A$ such that $\overline{D} \subset A$ is compact and $\partial D$ is smooth. Set $u^D$ the solution of (1.1) in $Q_\infty^D$ with boundary data zero and initial data $\chi_D\nu \in \mathfrak{M}^+(D)$. If $v$ is any positive solution of (1.1) in $Q_\infty^\Omega$ with $Tr_A(v) \ge \nu$, then $Tr_A(v)_{|D} = Tr_D(v) \ge \chi_D\nu$ and $v_{|\partial D\times(0,\infty)} \ge 0$. Therefore by Lemma 4.7, $v \ge u^D$ in $Q_\infty^D$. Now let $\{D_k\}$ be an increasing sequence of open and smooth subdomains of $A$ such that $\overline{D}_k \subset A$ and $\cup_{k=1}^\infty D_k = A$. Then $\{u^{D_k}\}$ is monotone increasing and by Lemma 4.6 it admits a limit, which we denote by $\underline{u}_\nu$, which satisfies (4.22).

To verify the statement (ii) we proceed as above with $\{D_k\}$ an increasing sequence of open smooth subdomains of $\Omega$ such that $\overline{D}_k \subset \Omega$ and $\cup_{k=1}^\infty D_k = \Omega$. For every $k$ we construct $U_{\infty,D_k}$ in $Q_\infty^{D_k}$ as above and we have

$$U_{\infty,D_k}(x,t-\tau) \ge v(x,t) \quad \forall (x,t) \in D_k \times (\tau,\infty)$$

for any $\tau > 0$, if $v$ is any positive solution of equation (1.1) in $Q_\infty^\Omega$. Letting $\tau \to 0$ and $k \to \infty$ yields $U_\infty(x,t) \ge v(x,t)$ for all $(x,t) \in Q_\infty^\Omega$ where $U_\infty$ is the limit of the decreasing sequence $\{U_{\infty,D_k}\}$. In particular $U_\infty \ge U_{\infty,\Omega}$ in $Q_T^\Omega$. In order to prove the reserve inequality, let $\omega_{\infty,\Omega}$ be the unique solution of the stationary problem

$$\begin{cases} -\Delta\omega_{\infty,\Omega} + f(\omega_{\infty,\Omega}) = 0 & \text{in } \Omega \\ \omega_{\infty,\Omega} = \infty & \text{on } \partial\Omega \end{cases} \tag{4.25}$$

Then $\omega_{\infty,\Omega}$ is the limit of the increasing sequence $\omega_{n,\Omega}$ of the same elliptic equation with boundary value $n$. Therefore

$$\max(\omega_{\infty,\Omega}(x), \phi_\infty(t)) \le U_{\infty,\Omega}(x,t) \le U_\infty(x,t) \le \omega_{\infty,\Omega}(x) + \phi_\infty(t) \tag{4.26}$$

for every $(x,t) \in Q_T^\Omega$ since $(x,t) \mapsto \omega_{\infty,\Omega}(x) + \phi_\infty(t)$ is a supersolution of (1.1) in $Q_\infty^\Omega$ due to the convexity of $f$. Consequently, for any $\epsilon > 0$ and $\tau > 0$, the function $(x,t) \mapsto (1+\epsilon)U_{\infty,\Omega}(x,t-\tau)$, which is a supersolution of (1.1) in $\Omega \times (\tau,\infty)$ dominates $U_\infty$ on $\Omega \times \{\tau\} \cup \partial\Omega \times (\tau,\infty)$; hence it majorizes $U_\infty$ in $\Omega \times (\tau,\infty)$. Letting $\epsilon$ and $\tau$ go to 0 implies $U_{\infty,\Omega} \geq U_\infty$ and hence $U_{\infty,\Omega} = U_\infty$ in $Q_\infty^\Omega$. Consequently, (4.24) holds. □

As a counterpart of Theorem 1.11 we have the following existence theorem.

**Proof of Theorem 1.12**

The proof is essentially similar to the one of [5, Th 3.4]. We denote, for $\delta > 0$,

$$\begin{array}{l} \mathcal{S}^\delta := \{x \in \Omega : dist(x,\mathcal{S}) \leq \delta\}, \quad \mathcal{R}^\delta := \Omega \setminus \mathcal{S}^\delta, \\ \mathcal{S}_\mu^{\partial\Omega} := \{y \in \partial\Omega : \mu(O \cap \mathcal{R}) = \infty, \text{ for every neighborhood O of y}\}, \\ \hat{\mathcal{S}}_\mu := \overline{\mathcal{S}} \cup \mathcal{S}_\mu^{\partial\Omega}, \\ (\hat{\mathcal{S}}_\mu)^\delta = \hat{\mathcal{S}}^\delta = \{x \in \Omega : dist(x,\hat{\mathcal{S}}_\mu) \leq \delta\}, \quad \hat{\mathcal{R}}^\delta := \Omega \setminus \hat{\mathcal{S}}^\delta, \end{array}$$

and let $\hat{\mu}_\delta$ be the measure given by

$$\hat{\mu}_\delta(E) := \mu(\hat{\mathcal{R}}_\delta \cap E) \quad \forall E \subset \Omega, E \text{ Borel}.$$

*Step 1: Construction of the maximal solution in the case that $\Omega$ is bounded.*

Let $z_\delta$ be the solution of (1.1) in $Q_T^\Omega$ with initial trace $\hat{\mu}_\delta$ and boundary data zero. For $0 < \tau < T/2$ let $g_{\delta,\tau}$ be defined by

$$g_{\delta,\tau}(x) := \begin{cases} z_\delta(x,\tau) & \text{if } x \in \hat{\mathcal{R}}^\delta, \\ U_{\infty,\Omega}(x,\tau) & \text{if } x \in \hat{\mathcal{S}}^\delta, \end{cases} \qquad \forall x \in \Omega \tag{4.27}$$

and for $n \in \mathbb{N}$, let $u_{\delta,\tau,n}$ be the solution of (1.1) in $Q_T^\Omega$ with initial data $g_{\delta,\tau}$ and boundary data $h_\tau$ where

$$h_\tau(x,t) := \begin{cases} n & \text{if } (x,t) \in \partial\Omega \times (0, T-\tau), \\ 0 & \text{if } (x,t) \in \partial\Omega \times (T-\tau, T). \end{cases} \tag{4.28}$$

Since the sequence $\{u_{\delta,\tau,n}\}$ is increasing with respect to $n$ and is bounded from above by $U_{\infty,\Omega}$, there exists $u_{\delta,\tau} := \lim_{n\to\infty} u_{\delta,\tau,n}$. By Lemma 4.8 (ii), the family $\{u_{\delta,\tau} : \tau \in (0,T/2]\}$ is locally uniformly bounded and therefore (by standard parabolic equations estimates) it is compact in $C^{2,1}(Q_T^\Omega)$. Consequently, there exists a sequence $\{\tau_m\}$ such that $\{u_{\delta,\tau_m}\}$ converges locally uniformly in $Q_T^\Omega$ to a solution $\tilde{u}_\delta$ of (1.1) in $Q_T^\Omega$ as $m \to \infty$, which blows-up on $\partial\Omega \times (0,T)$. By Lemma 4.6, $tr_\Omega(\tilde{u}_\delta) = (\hat{\mathcal{S}}^\delta, \mu_{|\hat{\mathcal{R}}^\delta})$.

Let $v$ be a solution of (1.1) in $Q_T^\Omega$ and suppose that $tr_\Omega(v) = (\mathcal{S}', \mu')$ satisfies $\mathcal{S}' \subset (\hat{\mathcal{S}}^\delta)^{\text{int}}$ and $\mu' = \mu$ in $\hat{\mathcal{R}}^\delta$. Then

$$\lim_{\tau\to 0}(v(.,\tau) - g_{\delta,\tau}(.))_+ = 0 \tag{4.29}$$

in the sense of weak convergence of measures. In fact, by Lemma 4.8, $v(.,\tau) \le g_{\delta,\tau}$ in $\hat{\mathcal{S}}^\delta$ while $v(.,\tau) \ge g_{\delta,\tau}$ in $\hat{\mathcal{R}}^\delta$. If $v_{\delta,\tau}$ is the solution of (1.1) in $\Omega \times (\tau, T)$ with boundary data zero and initial data $(v(.,\tau) - g_{\delta,\tau}(.))_+$, then by Lemma 4.4 $v_{\delta,\tau} \to 0$ locally uniformly in $\overline{\Omega} \times (0,T)$ as $\tau \to 0$. By the maximum principle, $v \le u_{\delta,\tau} + v_{\delta,\tau}$ in $\Omega \times (\tau, T)$. Therefore

$$v \le \tilde{u}_\delta. \tag{4.30}$$

In particular, it follows that $\{\tilde{u}_\delta\}$ is monotone increasing with respect to $\delta$. Thus $\overline{u}_{\Omega,\mathcal{S},\mu} := \lim_{\delta\to 0} \tilde{u}_\delta$ is a solution of (1.1), with initial data $(\mathcal{S}, \mu)$, which blows-up on $\partial\Omega \times (0,T)$. If $tr_\Omega(v) = (\mathcal{S}, \mu)$ then (4.30) holds for every $\delta > 0$ and consequently $v \le \overline{u}_{\Omega,\mathcal{S},\mu}$ in $Q_T^\Omega$.

*Step 2: Construction of a minimal solution in the case that $\Omega$ is bounded.*

First we construct a minimal solution under the additional assumption that $\mathcal{S}$ is compact. Let $0 < \delta < dist(\mathcal{S}, \partial\Omega)$ and $0 < \tau < T/2$. Denote by $w_{\delta,\tau}$ the solution of (1.1) with boundary data 0 and initial data $h_{\delta,\tau}$ defined by

$$h_{\delta,\tau}(x) := \begin{cases} z_\delta(x,\tau) & \text{if } x \in \hat{\mathcal{R}}^\delta, \\ v_{\Omega,\hat{x},\infty}(x,\tau) & \text{if } x \in \mathcal{S}^\delta, \\ 0 & \text{otherwise,} \end{cases} \qquad \forall x \in \Omega \tag{4.31}$$

where $\hat{x} \in \mathcal{S}$ is a point such that $dist(x, \mathcal{S}) = |x - \hat{x}|$. If $v$ is a positive solution of (1.1) in $Q_T^\Omega$ with

$$tr_\Omega(v) = (\mathcal{S}, \mu'), \mu' \ge \mu \tag{4.32}$$

then by Lemma 4.5 and maximum principle,

$$w_{\delta,\tau}(x,t) \le v(x, t+\tau) \tag{4.33}$$

in $Q_{T-\tau}^\Omega$ for small positive $\tau$. Let $\{\tau_n\}$ be a sequence converging to 0 such that $\{w_{\delta,\tau_n}\}$ converges in $Q_T^\Omega$. Then by Lemma 4.4 and Lemma 4.6 the limit function $\underline{v}_{\mathcal{S},\hat{\mu}_\delta}$ is a solution of (1.1) in $Q_T^\Omega$ with initial trace $(\mathcal{S}, \hat{\mu}_\delta)$ which minorizes $v$ in $Q_T^\Omega$. Therefore $\underline{v}_{\mathcal{S},\mu} := \lim_{\delta\to 0} \underline{v}_{\mathcal{S},\hat{\mu}_\delta}$ is a solution of (1.1) in $Q_T^\Omega$ with initial trace $(\mathcal{S}, \mu)$ and $\underline{v}_{\mathcal{S},\mu} \le v$ in $Q_T^\Omega$.

Next we claim that if $\mathcal{S}'$ is a compact subset of $\Omega$,

$$\mathcal{S} \subseteq \mathcal{S}', \mu_{|\mathcal{R}'} \le \mu' \text{ and } \mu(\mathcal{S}' \setminus \mathcal{S}) = 0 \Longrightarrow v_{\mathcal{S},\mu} \le v_{\mathcal{S}',\mu'}. \tag{4.34}$$

Suppose that, in addition to the conditions of (4.34), $\mu((\mathcal{S}' \setminus \mathcal{S})^\epsilon \cap \mathcal{R}) = 0$ for some $\epsilon > 0$. Then $\mathcal{S}' \subseteq \mathcal{S}$ and hence $\mathcal{S}' = \mathcal{S}$. From the previous contruction we find that, for $0 < \delta < \epsilon$, $\underline{v}_{\mathcal{S},\hat{\mu}_\delta} \leq \underline{v}_{\mathcal{S},\hat{\mu}'_\delta}$. Letting $\delta \to 0$ yields (4.34). Next, assuming only the condition in (4.34), let $\mu^\epsilon$ be defined by $\mu^\epsilon(E) := \mu(E \setminus [(\mathcal{S}' \setminus \mathcal{S})^\epsilon \cap \mathcal{R}])$, for every Borel subset $E \subseteq \mathcal{R}$. Then from the previous special case, we obtain $\underline{v}_{\mathcal{S},\mu^\epsilon} \leq \underline{v}_{\mathcal{S}',\mu'^\epsilon}$. Letting $\delta \to 0$ leads to (4.34).

Using this fact we can construct the minimal solution for arbitrary couples $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a relatively closed subset of $\Omega$ and $\mu \in \mathfrak{M}_+(\mathcal{R})$. For $\epsilon > 0$ put

$$\Omega_{(\epsilon)} := \{x \in \Omega : dist(x, \partial\Omega) > \epsilon\},$$
$$\mathcal{S}_{(\epsilon)} := \{x \in \mathcal{S} : dist(x, \partial\Omega) \geq \epsilon\}, \quad \mathcal{R}_{(\epsilon)} := \Omega \setminus \mathcal{S}_{(\epsilon)},$$

and define the measure $\mu_{(\epsilon)}$ by $\mu_{(\epsilon)}(E) := \mu(E \cap \Omega_{(\epsilon)})$ for every Borel set $E \subseteq \mathcal{R}_{(\epsilon)}$. By (4.34), the sequence $\{\underline{v}_{\mathcal{S}_{(\epsilon)},\mu_{(\epsilon)}}\}$ increases as $\epsilon$ decreases. Moreover this sequence is bounded from above by $v$. Thus $\underline{u}_{\Omega,\mathcal{S},\mu} := \lim_{\epsilon \to 0} \underline{v}_{\mathcal{S}_{(\epsilon)},\mu_{(\epsilon)}}$ is a solution of (1.1) in $Q_T^\Omega$ with initial trace $(\mathcal{S}, \mu)$ satisfying

$$\underline{u}_{\Omega,\mathcal{S},\mu} \leq v \tag{4.35}$$

for every positve solution $v$ of (1.1) in $Q_T^\Omega$ with $tr_\Omega(v) = (\mathcal{S}, \mu)$.

If $(\mathcal{S}, \mu)$, $(\mathcal{S}', \mu') \in CM_+(\Omega)$ and $\mathcal{S} \subseteq \mathcal{S}'$ and $\mu_{|\mathcal{R}'} \leq \mu'$ where $\mathcal{R}' = \Omega \setminus \mathcal{S}$ then from the previous construction, we get

$$\underline{v}_{\mathcal{S}_{(\epsilon)},\mu_{(\epsilon)}} \leq \underline{v}_{\mathcal{S}'_{(\epsilon)},\mu'_{(\epsilon)}}. \tag{4.36}$$

Letting $\epsilon \to 0$ yields

$$\underline{u}_{\Omega,\mathcal{S},\mu} \leq \underline{u}_{\Omega,\mathcal{S}',\mu'}. \tag{4.37}$$

*Step 3: Construction of the maximal solution when $\Omega$ is unbouded.*

Let $\{D_k\}$ be an increasing sequence of open and bounded smooth subdomains of $\Omega$ such that $\overline{D}_k \subset \Omega$ and $\cup_{k=1}^\infty D_k = \Omega$. First we consider the case when $\mathcal{R} = \Omega \setminus \mathcal{S}$ is bounded and assume that $\mathcal{R} \subset D_k$ for every $k$. Proceeding as in step 1, we can construct a maximal solution $\overline{u}_{D_k,\mathcal{S},\mu}$ of (1.1) in $Q_T^{D_k}$ with initial trace $(\mathcal{S}_k, \mu)$ where $\mathcal{S}_k = \mathcal{S} \cap D_k$, which blows-up everywhere on $\partial D_k \times (0, T)$. The sequence $\{\overline{u}_{D_k,\mathcal{S},\mu}\}$ decreases with respect to $k$ and is bounded from below. Therefore the limit function $\overline{u}_{\Omega,\mathcal{S},\mu} := \lim_{k \to \infty} \overline{u}_{D_k,\mathcal{S},\mu}$ is a maximal solution of (1.1) in $Q_T^\Omega$ in the sense of (1.26) with initial trace $(\mathcal{S}, \mu)$. Indeed, if $v$ is a solution of (1.1) in $Q_T^\Omega$ with initial trace $(\mathcal{S}, \mu)$ then by step 1, for any $k > 0$, $v_{|D_k \times (0,T)} \leq \overline{u}_{D_k,\mathcal{S},\mu}$. Letting $k \to \infty$ yields $v \leq \overline{u}_{\Omega,\mathcal{S},\mu}$ in $Q_T^\Omega$.

If $\mathcal{R}$ is unbounded, for $k > 0$, we put $\mathcal{R}_k := \mathcal{R} \cap D_k$, $\mathcal{S}_k := \Omega \setminus \mathcal{R}_k$ and $\mu_k(E) := \mu(E \cap \mathcal{R}_k)$ for every Borel subset $E \subset \mathcal{R}$. The sequence of maximal solutions $\{\overline{u}_{\Omega,\mathcal{S}_k,\mu_k}\}$ decreases with respect to $k$ and its limit as $k \to \infty$ is the maximal solution of (1.1) in $Q_T^\Omega$ in the sense of (1.26) with initial trace $(\mathcal{S}, \mu)$.

*Step 4: Construction of the minimal solution when $\Omega$ is unbounded.*

First assume that $\mathcal{S}$ is bounded and that $\mu(E) = 0$ for every Borel subset $E \subset \mathcal{R} \setminus G$ for some compact subset $G \subset \Omega$. Let $\{D_k\}$ be a sequence as in step 3 and assume that $\mathcal{S}$, $G \subset D_k$ for every $k > 0$. We construct the minimal solution $\underline{u}_{D_k,\mathcal{S},\mu_k}$ of (1.1) in $Q_T^{D_k}$ with initial data $(\mathcal{S}, \mu_k)$ where $\mu_k(E) = \mu(E)$ for every Borel subset $E \subset \mathcal{R}_k = D_k \setminus \mathcal{S}$ and boundary data 0. We observe that $\underline{u}_{D_k,\mathcal{S},\mu_k}$ increases with respect to $k$ and its limit as $k \to \infty$ is the minimal solution $\underline{u}_{\Omega,\mathcal{S},\mu}$ of (1.1) in $Q_T^\Omega$ in the sense of (1.26) with initial data $(\mathcal{S}, \mu)$. Indeed, (4.35) remains valid with $\underline{u}_{\Omega,\mathcal{S},\mu}$ replaced by $\underline{u}_{D_k,\mathcal{S},\mu_k}$; then letting $k \to \infty$ yields $\underline{u}_{\Omega,\mathcal{S},\mu} \le v$ in $Q_T^\Omega$.

In the general case, we put $\mathcal{S}^{(k)} := \mathcal{S} \cap \overline{D}_k$ and $\mu^{(k)}(E) := \mu(E \cap D_k)$ for every Borel set $E \subseteq \Omega \setminus \mathcal{S}^{(k)}$. We denote by $\underline{u}_{\Omega,\mathcal{S}^{(k)},\mu^{(k)}}$ the minimal solution of (1.1) in $Q_T^\Omega$ with initial trace $(\mathcal{S}^{(k)}, \mu^{(k)})$ and boundary data 0. By (4.37), the sequence of minimal solutions $\{\underline{u}_{\Omega,\mathcal{S}^{(k)},\mu^{(k)}}\}$ increases with respect to $k$ and its limit as $k \to \infty$ is the minimal solution $\underline{u}_{\Omega,\mathcal{S},\mu}$ of (1.1) in $Q_T^\Omega$ in the sense of (1.26) with initial data $(\mathcal{S}, \mu)$. □

*Remark.* When $f(r) = |r|^{q-1}r$ with $1 < q < 1 + 2/N$, precise expansion of $u_{\infty\delta}(x,t)$, when $t \to 0$ allows to prove uniqueness. Even when $f(r) = r\ln^\alpha(r+1)$ with $\alpha > 2$, uniqueness is not known.

### 4.3 The Keller-Osserman condition does not hold

In this section we assume that (1.16) holds and (1.8) is always satisfied.

**Lemma 4.9** *Assume* (1.10), (1.16), (*C*1) *and* (*C*3) *are satisfied. If $u$ is a positive solution of* (1.1) *in $Q_\infty$ which satisfies*

$$\limsup_{t\to 0} \int_G u(x,t)dx = \infty, \tag{4.38}$$

*for some bounded open subset $G \subset \mathbb{R}^N$, then $u(x,t) \ge \phi_\infty(t)$.*

*Proof.* By assumpion, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n\to\infty} \int_G u(x,t_n)dx = \infty. \tag{4.39}$$

If (4.38), we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\mathrm{diam}(G_k) = \epsilon_k \to 0$ when $k \to \infty$, and

$$\lim_{n\to\infty} \int_{G_k} u(x, t_n)dx = \infty \qquad \forall k \in \mathbb{N}. \tag{4.40}$$

Furthermore there exists a unique $a \in \cap_k G_k$. We set

$$\int_{G_k} u(x, t_n)dx = M_{n,k}.$$

Since $\lim_{n\to\infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any $k$, there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x, t_{n(k)})dx \geq m. \tag{4.41}$$

By induction, we define $n(1)$ as the smallest integer $n$ such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer $n$ larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any $k$, there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x, t_{n(k)}); \ell\}dx = m \tag{4.42}$$

and we set

$$V_k(x) = \inf\{u(x, t_{n(k)}); \ell\}\chi_{G_k}(x).$$

Let $v_k = v$ be the unique bounded solution of

$$\begin{cases} v_t - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(., 0) = V_k & \text{in } \mathbb{R}^N. \end{cases} \tag{4.43}$$

Since $v(x, 0) \leq u(x, t_{n(k)})$, it follows

$$u(x, t + t_{n(k)}) \geq v_k(x, t) \qquad \forall (x, t) \in Q_\infty. \tag{4.44}$$

When $k \to \infty$, $V_k \to m\delta_a$, thus $v_k \to u_{m\delta_a}$ by Lemma 4.4. Therefore $u \geq u_{m\delta_a}$. Since $m$ is arbitrary and $u_{m\delta_a} \to \phi_\infty$ when $m \to \infty$ by Theorem 1.3, it follows that $u \geq \phi_\infty$. □

**Lemma 4.10** *Assume* (1.15), (1.16), $(C1)$ *and* $(C3)$ *are satisfied. There exists no positive solution* $u$ *of* (1.1) *in* $Q_\infty$ *which satisfies* (4.38) *for some bounded open subset* $G \subset \mathbb{R}^N$.

*Proof.* If we assume that such a $u$ exists, we proceed as in the proof of the previous lemma. Since Lemma 4.4 holds, we derive that $u \geq u_{m\delta_a}$ for any $m$. Since $\lim_{m\to\infty} u_{m\delta_a}(x,t) = \infty$ for all $(x,t) \in Q_\infty$, we derive a contradiction. □

Thanks to these results, we can characterize the initial trace of positive solutions of (1.1) when the Keller-Osserman condition does not hold.

**Proof of Theorem 1.13**

If there exists some open subset $G$ of $\mathbb{R}^N$ with the property (4.38), then $u \geq \phi_\infty$ and the initial trace of $u$ is the Borel measure $\nu_\infty$. Next we assume that for any bounded open subset $G$ of $\mathbb{R}^N$ there holds

$$\limsup_{t\to 0} \int_G u(x,t)dx < \infty. \tag{4.45}$$

If $\mathcal{S}(u) \neq \emptyset$, there exists $z \in \mathbb{R}^N$ and an bounded open neighborhood $G$ of $z$ such that

$$\int_0^T \int_G f(u)dxdxt = \infty.$$

By (4.45), $u \in L^\infty(0,T;L^1(G)) \subset L^1(Q_T^G)$. Then, by Lemma 4.2, (4.4) holds, which contradict (4.45). Thus $\mathcal{S}(u) = \emptyset$ and $\mathcal{R}(u) = \mathbb{R}^N$. It follows from Proposition 1.9 that there exists a positive Radon measure $\mu$ such that

$$\lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\zeta(x)dx = \int_{\mathbb{R}^N} \zeta(x)d\mu(x) \qquad \forall \zeta \in C_c(\mathbb{R}^N). \tag{4.46}$$

□

Because of the lack of uniqueness from Theorem 1.6 it is difficult to give a complete characterization of admissible initial data for solutions of (1.1) under the asssumptions of Theorem 1.13. However, we have the result as in Proposition 1.14.

**Proof of Proposition 1.14**

We first notice that $\max\{\phi_\infty(t); w_b(|x|)\}$ is a subsolution of (1.1) which is dominated by the supersolution $\phi_\infty(t) + w_b(|x|)$. The process is standard: for $\tau > 0$ we set

$$\psi(x,\tau) = \frac{1}{2}\left(\max\{\phi_\infty(t); w_b(|x|)\} + \phi_\infty(t) + w_b(|x|)\right).$$

There exists a function $u = u_\tau \in C(\overline{Q_\infty})$ solution of (1.1) in $Q_\infty$ satisfying $u_\tau(.,0) = \psi(.,\tau)$. Furthermore

$$\max\{\phi_\infty(t+\tau); w_b(|x|)\} \leq u_\tau(x,t) \leq \phi_\infty(t+\tau) + w_b(|x|) \qquad \forall (x,t) \in Q_\infty. \tag{4.47}$$

By parabolic equation regularity theory, the set $\{u_\tau\}$ is locally equicontinuous in $Q_\infty$. Thus there exists a subsequence $\{\tau_n\}$ and $u \in C(Q_\infty)$ such that $u_{\tau_n} \to u$ on any compact subset of $Q_\infty$. Clearly $u$ is a weak, thus a strong solutions of (1.1) and it satisfies (1.27). Since any solution $u$ with initial trace $\nu_\infty$ dominates $\phi_\infty$ by Lemma 4.9, it follows that $\phi_\infty$ is the minimal one. □

**Proof of Theorem 1.15**

As in the proof of Theorem 1.13 and because of Lemma 4.10, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$ and the proof follows from Proposition 1.9. □

*Remark.* Under the assumptions of Theorem 1.13, it is clear, from the proof of Proposition 3.1, that for any $0 < a < b$ and any initial data $u_0 \in C(\mathbb{R}^N)$ satisfying

$$w_a(x) \le u_0(x) \le w_b(x) \qquad \forall x \in \mathbb{R}^N$$

there exists a solution $u \in C(\overline{Q_\infty})$ of (1.1) in $Q_\infty$ satisfying $u(.,0) = u_0$ and

$$w_a(x) \le u(x,t) \le w_b(x) \qquad \forall (x,t) \in Q_\infty.$$

We conjecture that for any positive measure $\mu$ on $\mathbb{R}^N$ which satisfies, for some $b > 0$,

$$\int_{B_R} d\mu(x) \le \int_{B_R} w_b(x)dx \qquad \forall R > 0 \tag{4.48}$$

there exists a positive solution $u$ of (1.1) in $Q_\infty$ with initial trace $\mu$. Another interesting open problem is to see if there exists local solutions in $Q_T$ with an initial trace $\mu$ satisfying

$$\lim_{R\to\infty} \frac{\int_{B_R} d\mu(x)}{\int_{B_R} w_b(x)dx} = \infty \qquad \forall b > 0. \tag{4.49}$$

# References


[1] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions* CRC Press (1992)

[2] L. C. Evans and B. F. Knerr, *Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities*, **Illinois J. Math. 23**, 153–166 (1979).

[3] J. Fabbri and J. R. Licois, *Boundary behavior of solution of some weakly superlinear elliptic equations*, **Adv. Nonlinear Studies 2**, 147-176 (2002).

[4] Keller J.B., *On solutions of* $\Delta u = f(u)$, **Comm. Pure Appl. Math. 10**, 503-510 (1957).

[5] M. Marcus and L. Véron, *Initial trace of positive solutions of some nonlinear parabolic equations*, **Comm. Part. Diff. Equ. 24**, 1445-1499 (1999).

[6] M. Marcus and L. Véron, *Initial trace of positive solutions to semilinear parabolic inequalities*, **Adv. Nonlinear Studies 2**, 395-436 (2002).

[7] M. Marcus and L. Véron, *Boundary trace of positive solutions of nonlinear elliptic inequalities*, **Ann. Scu. Norm. Sup. Pisa 5**, 481-533 (2004).

[8] M. Marcus and L. Véron, *The boundary trace and generalized boundary value problem for seminilear elliptic equations with coercive absorption*, **Comm. Pure Appl. Math. LVI**, 689-731 (2003).

[9] Osserman R., *On the inequality* $\Delta u \geq f(u)$, **Pacific J. Math. 7**, 1641-1647 (1957).

[10] A. Shishkov and L. Véron, *The balance between diffusion and absorption in semilinear parabolic equations*, **Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl. 18**, 59-90 (2007).

[11] J. L. Vazquez and L. Véron *Isolated singularities of some semilinear elliptic equations*, **J. Diff. Equ. 60**, 301-321 (1985).

[12] L. Véron, *Weak and strong singularities of nonlinear elliptic equations*, **Proc. Symp. Pure Math. 45**, 477-495 (1986).

[13] L. Véron, *Boundary trace of solutions of semilinear elliptic equalities and inequalities*, **Rend. Acad. Lincei: Mat. Appl. Ser. IV15**, 301-314 (2004).

[14] L. Véron, *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Research Notes in Mathematics Series **353**, pp 1-388 (1996).