



HAL
open science

Local and global properties of solutions of heat equation with superlinear absorption

Tai Nguyen Phuoc, Laurent Veron

► **To cite this version:**

Tai Nguyen Phuoc, Laurent Veron. Local and global properties of solutions of heat equation with superlinear absorption. 2010. hal-00459765v3

HAL Id: hal-00459765

<https://hal.science/hal-00459765v3>

Preprint submitted on 29 Mar 2010 (v3), last revised 23 Aug 2010 (v5)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Local and global properties of solutions of heat equation with superlinear absorption

Tai Nguyen Phuoc Laurent Véron

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We study the limit, when $k \rightarrow \infty$ of solutions of $u_t - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0, \infty)$ with initial data $k\delta$, when f is a positive increasing function. We prove that there exist essentially three types of possible behaviour according f^{-1} and $F^{-1/2}$ belong or not to $L^1(1, \infty)$, where $F(t) = \int_0^t f(s)ds$. We use these results for giving a general result on the existence of the initial trace and some non-uniqueness results for regular solutions with unbounded initial data.

1991 Mathematics Subject Classification. 35K58; 35K91; 35K15.

Key words. Heat equation; singularities; Borel measures; initial trace.

Contents

1	Introduction	2
2	Isolated singularities	8
3	About uniqueness	20
4	Initial trace	23
4.1	The regular part of the initial trace	23
4.2	The Keller-Osserman condition holds	25
4.3	The Keller-Osserman condition does not hold	34

1 Introduction

In this article we investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + f(u) = 0 \quad (1.1)$$

in $Q_\infty := \mathbb{R}^N \times (0, \infty)$ ($N \geq 2$) where $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, nondecreasing and positive on $(0, \infty)$, vanishes at 0 and tends to infinity at infinity. As a model equation we shall consider the following, with $\alpha > 0$,

$$\partial_t u - \Delta u + u \ln^\alpha(u + 1) = 0. \quad (1.2)$$

When $f(u) = |u|^\beta u$ with $\beta > 0$ much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x, t) = t^{-1/\beta} w(x\sqrt{t}). \quad (1.3)$$

The critical exponent $\beta_c = 2/N$ plays a fundamental role in the description of isolated singularities and the study of the initial trace, since if $0 < \beta < \beta_c$ there exists a self-similar solution with an isolated singularity at $(0, 0)$, while no such solution exists when $\beta \geq \beta_c$ and no solution with isolated singularities.

In the case of (1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any $k > 0$ there exists a unique $u := u_k \in C(\overline{Q_\infty} \setminus \{(0, 0)\}) \cap C^{2,1}(Q_\infty)$ solution of

$$\begin{cases} \partial_t u - \Delta u + u \ln^\alpha(u + 1) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \quad (1.4)$$

There are two critical values for α : $\alpha = 1$ and $\alpha = 2$, the explanation of which comes from the study of the two singular problems

$$\begin{cases} \phi' + \phi \ln^\alpha(\phi + 1) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty \end{cases} \quad (1.5)$$

and, for any $\epsilon > 0$,

$$\begin{cases} -\Delta \psi + \psi \ln^\alpha(\psi + 1) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty, \end{cases} \quad (1.6)$$

where $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$. When it exists, the solution ϕ_∞ of (1.5) is given implicitly by

$$\int_{\phi_\infty(t)}^{\infty} \frac{ds}{s \ln^\alpha(s+1)} = t \quad (1.7)$$

and such a formula is valid if and only if $\alpha > 1$. For problem (1.6) an explicit expression of the solution is not valid, but it exists if and only if the Keller-Osserman condition (1.12) holds $\alpha > 2$.

Having in mind this model we study (1.1) assuming the weak singularity condition on f :

$$\int_0^1 s^{N/2} f(s^{-N/2}) ds < \infty. \quad (1.8)$$

Proposition 1.1 *Assume (1.8) holds. Then for any $k > 0$, there exists a unique solution $u := u_k$ to*

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty \\ u(\cdot, 0) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \quad (1.9)$$

Another condition on f is

$$\int_1^\infty \frac{ds}{f(s)} < \infty. \quad (1.10)$$

Under assumption (1.10) there exists a solution $\phi := \phi_\infty$ to

$$\begin{cases} \phi' + f(\phi) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty. \end{cases} \quad (1.11)$$

The next important condition on f we shall encounter is the Keller-Osserman condition, i.e.

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad (1.12)$$

where

$$F(s) = \int_0^s f(\sigma) d\sigma \quad \forall s \in [1, \infty). \quad (1.13)$$

If (1.12) is satisfied, by [4, Theorem III] for any $\epsilon > 0$ there exists a maximal solution $\psi := \psi_\epsilon$ to

$$\begin{cases} -\Delta \psi + f(\psi) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty. \end{cases} \quad (1.14)$$

The first question we consider is the study of the limit of u_k when $k \rightarrow \infty$. This question is natural since $k \mapsto u_k$ is increasing. In order to treat it, we need some additional conditions.

(C1)- The function $s \mapsto \frac{f(s)}{s}$ is increasing on $(0, \infty)$ and satisfies

$$\lim_{s \rightarrow 0} \frac{f(s)}{s} = 0 \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{f(s)}{s} = \infty,$$

(C2)- The function f is convex on $(0, \infty)$,

(C3)- If $\liminf_{s \rightarrow \infty} f(s)/(s \ln^\alpha s) = 0, \forall \alpha > 2$, then there exists $\beta \in (1, 2]$ such that

$$\limsup_{s \rightarrow \infty} \frac{f(s)}{s \ln^\beta s} < \infty.$$

In the second section, we prove the following results.

Theorem 1.2 *Assume the conditions (C1) and (C3) hold. If f satisfies*

$$\int_1^\infty \frac{ds}{f(s)} = \infty \tag{1.15}$$

then the solutions u_k of (1.9) satisfy $\lim_{k \rightarrow \infty} u_k(x, t) = \infty$ for every $(x, t) \in Q_\infty$.

Theorem 1.3 *Assume the conditions (C1) – (C3) hold. If f satisfies (1.10) and*

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} = \infty \tag{1.16}$$

where F is defined in (1.13), then the solutions u_k of (1.9) satisfy $\lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t)$ for every $(x, t) \in Q_\infty$, where ϕ_∞ is the solution of (1.11).

We denote by \mathcal{U}_0 the set of positive solutions u of (1.1) in Q_∞ , which is continuous in $\overline{Q_\infty} \setminus \{(0, 0)\}$, vanishes on the set $\{(x, 0) : x \neq 0\}$ and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} u(x, t) dx = \infty \tag{1.17}$$

for any $\epsilon > 0$.

Theorem 1.4 *Assume f satisfies (1.8), (1.10) and (1.12). Then $\underline{U} := \lim_{k \rightarrow \infty} u_k$ is the minimal element of \mathcal{U}_0 .*

In the third section we study the set of positive and locally bounded solutions of (1.1) in Q_∞ . This set differs considerably according the assumption on f . This is due to the properties of the radial solutions of the associated stationary equation

$$-\Delta w + f(w) = 0 \quad \text{in } \mathbb{R}^N. \quad (1.18)$$

The next result bases upon the Picard-Lipschitz fixed point theorem and a result of Vazquez and Véron [11].

Proposition 1.5 *Assume (1.16) holds. For any $a > 0$, there exists a unique positive function $w := w_a \in C^2([0, \infty))$ to the problem*

$$\begin{cases} -w'' - \frac{N-1}{r}w' + f(w) = 0 & \text{in } \mathbb{R}_+ \\ w'(0) = 0 \\ w(0) = a. \end{cases} \quad (1.19)$$

A striking consequence of the existence of such solutions is the following non-uniqueness result.

Theorem 1.6 *Assume f satisfies (1.10) and (1.16). Then for any $u_0 \in C(\mathbb{R}^N)$ satisfying $w_a(x) \leq u_0(x) \leq w_b(x)$ for any $x \in \mathbb{R}^N$ and for some $b > a > 0$ there exist two solutions $\underline{u}, \bar{u} \in C(\overline{Q_\infty})$ of (1.1) with initial value u_0 . They satisfy respectively*

$$0 \leq \underline{u}(x, t) \leq \min\{w_b(x), \phi_\infty(t)\} \quad \forall (x, t) \in Q_\infty, \quad (1.20)$$

thus $\lim_{t \rightarrow \infty} \underline{u}(x, t) = 0$, uniformly with respect to $x \in \mathbb{R}^N$, and

$$w_a(x) \leq \bar{u}(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty, \quad (1.21)$$

thus $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$, uniformly with respect to $t \geq 0$.

The next theorem shows that if two solutions of (1.1) have the same initial data and the same asymptotic behaviour as $|x| \rightarrow \infty$ then they coincide.

Theorem 1.7 *Assume f satisfies (1.16) and (C1). Let $u, \tilde{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ be two positive solutions of (1.1) with initial data u_0 . If for any $\epsilon > 0$,*

$$u(x, t) - \tilde{u}(x, t) = o(w_\epsilon(|x|)) \quad \text{as } x \rightarrow \infty \quad (1.22)$$

locally uniformly with respect to $t \geq 0$, then $u = \tilde{u}$.

On the contrary, if the Keller-Osserman condition holds, a continuous solution is uniquely determined by a positive initial value $u_0 \in C(\mathbb{R}^N)$.

Theorem 1.8 *Assume f satisfies (1.12) and (C2). Then for any nonnegative function $u_0 \in C(\mathbb{R}^N)$ there exists a unique solution $u \in C(\overline{Q_\infty})$ of (1.1) in Q_∞ with initial value u_0 .*

In the last section we study the initial trace of locally bounded positive solutions of (1.1) in $Q_T^\Omega := \Omega \times (0, T)$ where Ω is a domain in \mathbb{R}^N with C^2 boundary and $T > 0$.

Proposition 1.9 *Assume f is a continuous nonnegative function defined on \mathbb{R}_+ . Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (1.1) in Q_T^Ω . The set $\mathcal{R}(u)$ of the points $z \in \Omega$ such that there exists an open ball $B_r(z)$ such that $u, f(u) \in L^1(Q_T^{B_r(z)})$ is an open subset of Ω . Furthermore there exists a positive Radon measure $\mu := \mu(u)$ on $\mathcal{R}(u)$ such that*

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(\mathcal{R}(u)). \quad (1.23)$$

Due to Proposition 1.9, we introduce the definition of the initial trace.

Definition 1.10 *The couple $(\mathcal{S}(u), \mu)$ where $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ is called the initial trace of u in Ω and will be denoted by $tr_\Omega(u)$. The set $\mathcal{R}(u)$ is called the regular set of the initial trace of u and the measure μ the regular part of the initial trace. The set $\mathcal{S}(u)$ is relatively closed in Ω and is called the singular part of the initial trace of u .*

The initial trace can also be represented by a positive, outer regular Borel measure, not necessary locally bounded. The space of these measures on Ω will be denoted by $\mathcal{B}_+^{\text{reg}}(\Omega)$. If for every open subset $A \subset \Omega$ we denote by $\mathfrak{M}_+(A)$ the space of positive Radon measures on A , there is a (1 – 1) correspondence between $\mathcal{B}_+^{\text{reg}}(\Omega)$ and the set of couples:

$$CM_+(\Omega) = \{(\mathcal{S}, \mu) : \mathcal{S} \subset \Omega \text{ relatively closed}, \mu \in \mathfrak{M}_+(\mathcal{R}) \text{ with } \mathcal{R} = \Omega \setminus \mathcal{S}\}. \quad (1.24)$$

The measure $\nu \in \mathcal{B}_+^{\text{reg}}(\Omega)$ corresponding to a couples $(\mathcal{S}, \mu) \in CM_+(\Omega)$ is given by

$$\nu(A) = \begin{cases} \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \mu(A) & \text{if } A \subseteq \mathcal{S}, \end{cases} \quad \forall A \subset \Omega, A \text{ Borel}. \quad (1.25)$$

If u is a solution of (1.1), we shall use the notation $tr_\Omega(u)$ (resp. $Tr_\Omega(u)$) for the trace considered as an element of $CM_+(\Omega)$ (resp. $\mathcal{B}_+^{\text{reg}}(\Omega)$).

We consider the case when the Keller-Osserman holds.

Theorem 1.11 *Assume f is nondecreasing and satisfies (1.12). If $u \in C^{2,1}(Q_T^\Omega)$ is a positive solution of (1.1), it possesses an initial trace $\nu \in \mathcal{B}_+^{reg}(\Omega)$.*

On the contrary, the following theorem concerns with the existence of the maximal solution and the minimal solution of (1.1) with a given initial trace $(\mathcal{S}, \mu) \in CM_+(\Omega)$.

Theorem 1.12 *Assume f is nondecreasing and satisfies (1.12), (1.8) and (C2). Then for any $(\mathcal{S}, \mu) \in CM_+(\Omega)$ there exist a maximal solution $\bar{u}_{\Omega, \mathcal{S}, \mu}$ and a minimal solution $\underline{u}_{\Omega, \mathcal{S}, \mu}$ of (1.1) in Q_T^Ω with the initial trace (\mathcal{S}, μ) , in the following sense:*

$$\underline{u}_{\Omega, \mathcal{S}, \mu} \leq v \leq \bar{u}_{\Omega, \mathcal{S}, \mu} \quad (1.26)$$

for every positive solution $v \in C^{2,1}(Q_T^\Omega)$ of (1.1) in Q_T^Ω such that $tr_\Omega(v) = (\mathcal{S}, \mu)$.

If the Keller-Osserman does not holds, we obtain the following results.

Theorem 1.13 *Assume (1.10), (1.16), (C1) and (C3) are verified. If $u \in C^{2,1}(Q_\infty)$ is a positive solution of (1.1) in Q_∞ , it possesses an initial trace which is either the Borel measure infinity ν_∞ which satisfies $\nu_\infty(\mathcal{O}) = \infty$ for any open subset $\mathcal{O} \subset \mathbb{R}^N$, or is a positive Radon measure μ on \mathbb{R}^N .*

A consequence of Theorem 1.13 which is worth mentioning is the following.

Proposition 1.14 *Under the assumptions of Theorem 1.13, for any $b > 0$ there exists a positive solution $u \in C(Q_\infty)$ of (1.1) in (1.1) satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x, t) \leq \phi_\infty(t) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (1.27)$$

Consequently there exist infinitely many positive solutions of (1.1) with initial trace ν_∞ . Furthermore ϕ_∞ is the smallest of all these solutions.

Theorem 1.15 *Assume f is satisfies (1.15),(1.16), (C1) and (C3). If u is a positive solution of (1.1) in Q_∞ , it possesses an initial trace which is a positive Radon measure μ on \mathbb{R}^N .*

The proofs are combination of methods developed in [8] for elliptic equations, stability results and Theorem 1.2 and Theorem 1.3.

2 Isolated singularities

In order to study (1.1), we start proving Proposition 1.1.

Proof of Proposition 1.1

We denote by $E(x, t) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ the fundamental solution of the heat equation in Q_∞ . By [6, Remark 2.1], if for $k > 0$,

$$I := \int_0^1 \int_{B_R} f(kE(x, t)) dx dt < \infty \quad (2.1)$$

for any $R > 0$, then there exists a unique solution $u := u_k$ to (1.1) satisfying initial condition

$$u_k(\cdot, 0) = k\delta_0$$

in $\mathcal{D}'(\mathbb{R}^N)$. Furthermore the mapping $k \mapsto u_k$ is increasing. This existence result and the next proposition lead to the conclusion of Proposition 1.1. \square

Proposition 2.1 *If f satisfies (1.8) and (C1) then (2.1) is fulfilled.*

Proof. We set

$$h(r) = \frac{f(r)}{r} \quad r \in (0, \infty). \quad (2.2)$$

I is rewritten as

$$I = kC^* \int_0^1 \int_{B_R} t^{-N/2} e^{-|x|^2/4t} h(kC^* t^{-N/2} e^{-|x|^2/4t}) dx dt$$

where $C^* = (4\pi)^{-N/2}$. Put $r = |x|$ then $dx = r^{N-1} dr$, and

$$I = kC^* \int_0^1 t^{-N/2} \int_0^R e^{-r^2/4t} h(kC^* t^{-N/2} e^{-r^2/4t}) r^{N-1} dr dt.$$

We next put $\rho = \frac{r}{\sqrt{t}}$, then $r^{N-1} dr = \rho^{N-1} t^{N/2} d\rho$, and

$$I = kC^* \int_0^1 \int_0^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

We set

$$I_1 := kC^* \int_0^1 \int_0^1 e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt,$$

$$I_2 := kC^* \int_0^1 \int_1^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

Since $e^{-\rho^2/4}\rho^{N-1}$ is bounded in $[0, \infty)$, then there exists a constant c_1 depending only on k such that

$$I_1 < c_1 \int_0^1 \int_0^1 h(kC^*t^{-N/2})d\rho dt = c_1 \int_0^1 h(kC^*t^{-N/2})dt < \infty.$$

We next show that under the condition (1.8), $I_2 < \infty$. In order to do that we change to a new variable τ such that $t^{-N/2}e^{-\rho^2/4} = \tau^{-N/2}$. Then $t = \tau e^{-\frac{\rho^2}{2N}}$ and $dt = e^{-\frac{\rho^2}{2N}}d\tau$. Therefore

$$I_2 \leq kC^* \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} \left(\int_0^{e^{\rho^2/2N}} h(kC^*\tau^{-N/2})d\tau \right) d\rho. \quad (2.3)$$

Since f satisfies (1.8), there exists $\epsilon > 0$ (depending only on k) such that

$$\int_0^\epsilon h(kC^*\tau^{-N/2})d\tau$$

take a finite value, denoted by c_2 . Hence

$$\int_0^{e^{\rho^2/2N}} h(kC^*\tau^{-N/2})d\tau \leq c_2 + h(kC^*\epsilon^{-N/2})(e^{\frac{\rho^2}{2N}} - \epsilon). \quad (2.4)$$

Inserting (2.4) into the right hand side of (2.3), we obtain

$$I_2 \leq c_3 \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} d\rho + c_4 \int_1^\infty e^{-\frac{\rho^2}{4}} \rho^{N-1} d\rho < \infty$$

where $c_3 = kC^*c_2$ and $c_4 = kC^*h(kC^*\epsilon^{-N/2})$. Thus $I = I_1 + I_2 < \infty$. \square

The functions which satisfy the following ODE are particular solutions of (1.1)

$$\phi_t + f(\phi) = 0 \quad \text{in } (0, \infty). \quad (2.5)$$

For $a > 0$, we denote by ϕ_a the solution of (2.5) with initial data $\phi(0) = a$. If (1.15) holds then $\lim_{a \rightarrow \infty} \phi_a(t) = \infty$ for any $t \in (0, \infty)$. While, if (1.10) holds there exists a maximal solution ϕ_∞ given explicitly by

$$t = \int_{\phi_\infty(t)}^\infty \frac{ds}{f(s)} < \infty.$$

Lemma 2.2 *If (1.15) holds then*

$$\liminf_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha r} = 0, \quad \forall \alpha > 1.$$

If (1.10) holds then

$$\limsup_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha r} = \infty, \quad \forall 0 < \alpha \leq 1.$$

Proof. Case 1. Assume (1.15) holds then

$$J := \int_e^\infty \frac{ds}{f(s)} < \infty. \quad (2.6)$$

We change the variable $s = e^{r^{-1}}$ and derive

$$J = \int_0^1 \frac{dr}{r^2 h(e^{r^{-1}})}$$

where h is defined in (2.2). Suppose that there exists $\alpha > 1$ such that

$$\liminf_{s \rightarrow \infty} \frac{f(s)}{s \ln^\alpha s} > 0,$$

equivalently,

$$\liminf_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) > 0,$$

then there exists $l > 0$ and $r_0 \in (0, 1)$ such that

$$h(e^{r^{-1}}) > l r^{-\alpha} \quad \forall r \in (0, r_0).$$

Hence we derive the following contradiction

$$J < \frac{1}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^1 \frac{dr}{r^2 h(e^{r^{-1}})} < \infty.$$

Case 2. Assume (1.10) holds then $J = \infty$. Suppose that there exists $\alpha \in (0, 1]$ such that

$$\limsup_{s \rightarrow \infty} \frac{f(s)}{s \ln^\alpha s} < \infty,$$

equivalently,

$$\limsup_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) < \infty,$$

then there exists $l > 0$ and $r_0 \in (0, 1)$ such that

$$h(e^{r^{-1}}) < l r^{-\alpha} \quad \forall r \in (0, r_0).$$

Hence

$$J > \frac{1}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^1 \frac{dr}{r^2 h(e^{r-1})} = \infty,$$

which is a contradiction. \square

Proof of Theorem 1.2.

Since (1.15) holds, by Lemma 2.2 and the definition (2.2) of h ,

$$\liminf_{r \rightarrow \infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 1.$$

Thus

$$\liminf_{r \rightarrow \infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 2.$$

By (C3), there exists $\beta \in (1, 2]$ such that $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty$. Hence there exist $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r \quad \forall r \in (r_0, \infty). \quad (2.7)$$

Step 1. Let $k > 0$, we claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau \quad \forall t \in (0, 1) \quad (2.8)$$

where $\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$ with $C^* = (4\pi)^{-N/2}$. Set $r = kC^* \tau^{-N/2}$ then (2.7) becomes

$$h(kC^* \tau^{-N/2}) < M [\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{2/N} r_0^{-2/N}$. We put $a_1 = \ln k$, $a_2 = \frac{N}{2} \ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1} (a_1^\beta + a_2^\beta)$$

in order to obtain (notice that $C^* = (4\pi)^{-N/2} < 1$)

$$\begin{aligned} h(kC^* \tau^{-N/2}) &< M [\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M [(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})] \quad \forall \tau \in (0, \tau_0). \end{aligned} \quad (2.9)$$

Integrating over $[0, t]$ yields (2.8).

Step 2. It follows from (2.9) that (1.8) is fulfilled, hence by Proposition 1.1 there exists the unique solution u_k of (1.1) in Q_∞ with initial data $k\delta_0$. By the maximum principle, $u_k(x, t) \leq kE(x, t)$ for every $(x, t) \in Q_\infty$, which implies $u_k(x, t) \leq kC^*t^{-N/2}$ for every $(x, t) \in Q_\infty$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \geq 0.$$

If we set $v_k(x, t) = e^{\theta_k(t)}u_k(x, t)$, we obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)}[\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})] \geq 0$$

and $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$. By the maximum principle, there holds

$$v_k(x, t) \geq kC^*t^{-N/2}e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^*t^{-N/2}e^{-\theta_k(t)-|x|^2/4t}. \quad (2.10)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t(\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.11)$$

where

$$c_1 = \exp\left(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau\right)$$

and $M_\beta = M2^{\beta-1}$. Inserting (2.11) into the right hand side of (2.10), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1 := \mathbb{R}^N \times (0, 1).$$

If $\lim_{k \rightarrow \infty} u_k(x, t) < \infty$ for all $(x, t) \in Q_\infty$, we put $\underline{U} := \lim_{k \rightarrow \infty} u_k$, then

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t(\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1, \quad \forall k > 0.$$

Let $\{t_n\} \subset (0, 1]$ be a sequence converging to 0. We choose $k_n = \exp\left(\left(2M_\beta t_n\right)^{\frac{1}{1-\beta}}\right)$ then $\ln k_n - M_\beta t_n(\ln k_n)^\beta = \frac{1}{2} \ln k_n$. Next we restrict x in order to have

$$\ln k_n - M_\beta t_n(\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}} t_n^{\frac{\beta-2}{2(\beta-1)}}.$$

Therefore, since $1 < \beta \leq 2$,

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-1/2}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} if $\beta = 2$.

We pick some point x_0 in \mathbb{R}^N (resp. B_{r_2}) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, by comparison principle, it follows that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \rightarrow \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of x and therefore it is a solution of (1.11). By (1.15), $\underline{U}(x, t) = \infty$ for any $(x, t) \in Q_\infty$, which is a contradiction and the conclusion follows. \square

Proposition 2.3 *Assume (1.8) and (1.10) are satisfied. For any $k > 0$, there holds*

$$u_k(x, t) \leq \phi_\infty(t) \quad \forall (x, t) \in Q_\infty.$$

Proof. For any small $\epsilon > 0$, we set $\phi_{\infty\epsilon}(t) = \phi_\infty(t - \epsilon)$, $t \in [\epsilon, \infty)$ then $\phi_{\infty\epsilon}$ is a solution of (1.1) in (ϵ, ∞) , which dominates u_k on $\mathbb{R}^N \times \{\epsilon\}$ for any $k > 0$. By comparison principle, $u_k(x, t) \leq \phi_{\infty\epsilon}(t)$ for every $(x, t) \in \mathbb{R}^N \times [\epsilon, \infty)$. Letting $\epsilon \rightarrow 0$ yields the claim. \square

A necessary and sufficient condition for the existence of a maximal solution to the stationary equation

$$-\Delta w + f(w) = 0$$

in a bounded domain Ω is the Keller-Osserman condition (1.12) ([4], [9]). If f is convex and (1.12) holds, then (1.10) is fulfilled. The Keller-Osserman condition can be replaced by another condition, which owes to the following result.

Lemma 2.4 *Assume f is convex on $(0, \infty)$. Set*

$$L := \int_1^\infty \frac{ds}{\sqrt{sf(s)}}.$$

Then (1.12) holds if and only if $L < \infty$.

Proof. In order to obtain the assertion, it is sufficient to show that

$$s f\left(\frac{s}{2}\right) \leq F(s) \leq s f(s) \quad \forall s \geq 1. \quad (2.12)$$

The right-hand side estimate in (2.12) follows from the monotone property of f . The assumption of convexity of f in $(0, \infty)$ implies

$$f(s) \geq f\left(\frac{s}{2}\right) + \frac{s}{2}f'\left(\frac{s}{2}\right) \quad \forall s > 0.$$

Define $\varphi(s) = \int_0^s f(\sigma)d\sigma - sf\left(\frac{s}{2}\right)$, then $\varphi'(s) = f(s) - f\left(\frac{s}{2}\right) - \frac{s}{2}f'\left(\frac{s}{2}\right) \geq 0$. Hence $\varphi(s) > \varphi(0) = 0$, which leads to the left-hand side estimate in (2.12). \square

By using the same argument as in the proof of the Lemma 2.2 and thank to the Lemma 2.4, we obtain the following lemma.

Lemma 2.5 *If (1.16) holds then*

$$\liminf_{r \rightarrow \infty} \frac{f(r)}{r \ln^\alpha(r)} = 0 \quad \forall \alpha > 2.$$

If (1.12) holds then

$$\limsup_{r \rightarrow 0} \frac{f(r)}{r \ln^\alpha(r)} = \infty \quad \forall 0 < \alpha \leq 2.$$

Proof of Theorem 1.3.

Since (1.16) holds, by Lemma 2.5 and the definition (2.2) of h ,

$$\liminf_{r \rightarrow \infty} \frac{h(r)}{\ln^\alpha r} = 0 \quad \forall \alpha > 2.$$

By (C3), there exists $\beta \in (1, 2]$ such that $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty$. Hence there exists $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r \quad \forall r \in (r_0, \infty). \quad (2.13)$$

Step 1. For any $k > 0$ we set

$$\theta_k(t) = \int_0^t h(kC^*\tau^{-N/2})d\tau$$

where $C^* = (4\pi)^{-N/2}$. We claim that

$$\theta_k(t) < 2^{\beta-1}Mt(\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau \quad \forall t \in (0, 1). \quad (2.14)$$

If we define τ by $r = kC^*\tau^{-N/2}$, (2.13) becomes

$$h(kC^*\tau^{-N/2}) < M[\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{2/N} r_0^{-2/N}$. We set $a_1 = \ln k$, $a_2 = \frac{N}{2} \ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1}(a_1^\beta + a_2^\beta)$$

in order to obtain (notice that $C^* < 1$)

$$\begin{aligned} h(kC^*\tau^{-N/2}) &< M[\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M[(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})]. \end{aligned} \quad (2.15)$$

Integrating over $[0, t]$, we obtain (2.14).

Step 2. It follows from (2.15) that (1.8) is fulfilled, hence by Proposition 1.1 there exists the unique solution of (1.1) in Q_∞ with initial trace $k\delta_0$. By maximum principle, $u_k(x, t) \leq kE(x, t)$ for every $(x, t) \in Q_\infty$, which implies that $u_k(x, t) \leq kC^*t^{-N/2}$ for every $(x, t) \in Q_\infty$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \geq 0.$$

We set $v_k(x, t) = e^{\theta_k(t)} u_k(x, t)$ and obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)} [\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})] \geq 0,$$

with $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$. By maximum principle, it follows

$$v_k(x, t) \geq kC^*t^{-N/2} e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^*t^{-N/2} e^{-\theta_k(t)-|x|^2/4t}. \quad (2.16)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t (\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.17)$$

where $c_1 = \exp\left(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau\right)$ and $M_\beta = M2^{\beta-1}$. Inserting (2.17) into the right-hand side of (2.16), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1 = \mathbb{R}^N \times (0, 1).$$

Since $k \mapsto u_k$ is increasing and by Proposition 2.3, there exists $\underline{U} := \lim_{k \rightarrow \infty} u_k$ and $\underline{U} \geq u_k$. Hence

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t} \quad \forall (x, t) \in Q_1, \forall k > 0.$$

Let $\{t_n\} \subset (0, 1]$ be a sequence converging to 0. We choose $k_n = \exp((2M_\beta t_n)^{\frac{1}{1-\beta}})$, equivalently $\ln k_n - M_\beta t_n (\ln k_n)^\beta = \frac{1}{2} \ln k_n$. Next we restrict $|x|$ in order

$$\ln k_n - M_\beta t_n (\ln k_n)^\beta - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq r_\beta t_n^{\frac{\beta-2}{2(\beta-1)}},$$

where $r_\beta = 2^{\frac{\beta-2}{2(\beta-1)}} M^{\frac{1}{2(1-\beta)}}$. Because $1 < \beta \leq 2$, it follows

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty,$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-\frac{1}{2}}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} if $\beta = 2$.

We pick some point x_0 in \mathbb{R}^N (resp. B_{r_2}) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, it follows, by comparison principle, that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \rightarrow \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t) \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of x and therefore it is a solution of (1.11). Since (1.10) holds, $\underline{U}(x, t) = \phi_\infty(t)$ for every $(x, t) \in Q_\infty$. \square

Proposition 2.6 *Assume (1.12) and (1.8) are satisfied. Then for any $k > 0$ there holds*

$$u_k(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty$$

where Φ is a solution to the problem

$$\begin{cases} -\Phi'' + f(\Phi) = 0 & \text{in } (0, \infty) \\ \lim_{s \rightarrow 0} \Phi(s) = \infty. \end{cases}$$

Proof. Step 1: Upper estimate. Since f satisfies (1.12), by [4] for any $R > 0$, there exists a solution w_R to the problem

$$\begin{cases} -\Delta w_R + f(w_R) = 0 & \text{in } B_R, \\ \lim_{|x| \rightarrow R} w_R(x) = \infty. \end{cases} \quad (2.18)$$

Let $x_0 \neq 0$ arbitrary in \mathbb{R}^N . Set $\mathbb{E} = \{\vec{e} : |\vec{e}| = 1\}$ and take $\vec{e} \in \mathbb{E}$. Put $x_{\vec{e}} = |x_0| \vec{e}$ and for $n > |x_0|$ put $a_n = n\vec{e}$. Denote by $\mathbb{H}_{\vec{e}}$ the open half-space generated by \vec{e}

and its orthogonal hyperplane at the origin, then $x_{\vec{e}}, a_n \in \mathbb{H}_{\vec{e}}$. Take R such that $n - |x_0| < R < n$. We set $W_{\vec{e},n,R}(x) = w_R(x - a_n)$, then $W_{\vec{e},n,R}$ is a solution of (1.1) in $B_R(a_n)$ and blows-up on the boundary $\lim_{|x-a_n| \rightarrow R} W_{\vec{e},n,R}(x) = \infty$. By the maximum principle,

$$u_k(x, t) \leq W_{\vec{e},n,R}(x) \quad \forall (x, t) \in B_R(a_n) \times (0, \infty). \quad (2.19)$$

The sequence $\{W_{\vec{e},n,R}\}$ is decreasing with respect to R and is bounded from below by u_k , then there exists $W_{\vec{e},n} := \lim_{R \rightarrow n} W_{\vec{e},n,R}$ satisfying

$$u_k(x, t) \leq W_{\vec{e},n}(x) \quad \forall (x, t) \in B_n(a_n) \times (0, \infty). \quad (2.20)$$

The sequence $\{W_{\vec{e},n}\}$ is also decreasing with respect to n and is bounded from below by u_k , then there exists $W_{\vec{e},\infty} := \lim_{n \rightarrow \infty} W_{\vec{e},n}$. Letting $n \rightarrow \infty$ in (2.20) yields to

$$u_k(x, t) \leq W_{\vec{e},\infty}(x) \quad \forall (x, t) \in \mathbb{H}_{\vec{e}} \times (0, \infty). \quad (2.21)$$

In particular,

$$u_k(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}). \quad (2.22)$$

Since u_k is radial, it follows that

$$u_k(x_0, t) = u_k(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}).$$

For any $r > 0$, $n > r$, $n - r < R < n$ and $\vec{e}, \vec{e}' \in \mathbb{E}$, since w_R is radial,

$$w_R(r\vec{e} - n\vec{e}) = w_R(r\vec{e}' - n\vec{e}').$$

Letting successively $R \rightarrow n$, $n \rightarrow \infty$ yields to

$$W_{\vec{e},\infty}(r\vec{e}) = W_{\vec{e}',\infty}(r\vec{e}').$$

Define $\tilde{\Phi}(r) := W_{\vec{e},\infty}(r\vec{e})$, $\forall r \in (0, \infty)$ then it satisfies

$$\begin{cases} -\tilde{\Phi}'' - \frac{N-1}{r}\tilde{\Phi}' + f(\tilde{\Phi}) = 0 & \text{in } (0, \infty) \\ \lim_{r \rightarrow 0} \tilde{\Phi}(r) = \infty \end{cases} \quad (2.23)$$

and

$$u_k(x, t) \leq \tilde{\Phi}(|x|) \quad \forall (x, t) \in Q_{\infty}. \quad (2.24)$$

Step 2: End of the proof. We claim that

$$\tilde{\Phi}(r) \leq \Phi(r) \quad \forall r \in (0, \infty). \quad (2.25)$$

For any $\epsilon > 0$, we set $\Phi_\epsilon(r) = \Phi(r - \epsilon)$, $r > \epsilon$ then Φ_ϵ is a solution of

$$-\Phi_\epsilon'' + f(\Phi_\epsilon) = 0 \quad \text{in } (\epsilon, \infty) \quad (2.26)$$

verifying $\lim_{r \rightarrow \epsilon} \Phi_\epsilon(r) = \infty$. Since $\Phi_\epsilon' \leq 0$, Φ_ϵ is a supersolution of the equation in (2.23) in (ϵ, ∞) , which dominates $\tilde{\Phi}$ at $r = \epsilon$. By the maximum principle, $\tilde{\Phi} \leq \Phi_\epsilon$ in (ϵ, ∞) . Letting $\epsilon \rightarrow 0$ yields (2.25). Combining (2.24) and (2.25) leads to the conclusion. \square

Remark. Combining Proposition 2.3 and Proposition 2.6 yields

$$u_k(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \quad \forall (x, t) \in Q_\infty, \forall k > 0. \quad (2.27)$$

Proof of Theorem 1.4.

The sequence $\{u_k\}$ increase with respect to k and is bounded from above by (2.27) then there exists $\underline{U} := \lim_{k \rightarrow \infty} u_k$ satisfying

$$\underline{U}(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \quad \forall (x, t) \in Q_\infty, \forall k > 0. \quad (2.28)$$

Moreover, the function $\underline{U} \in \mathcal{U}_0$ because it has the following properties:

- (i) It is positive in Q_∞ , belongs to $C(\overline{Q_\infty} \setminus \{(0, 0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$.
- (ii) It satisfies (1.1) and

$$\lim_{t \rightarrow 0} \int_{B_\sigma} \underline{U}(x, t) dx = \infty \quad \forall \sigma > 0. \quad (2.29)$$

In the sense of initial trace in Definition 4.3, \underline{U} has initial trace $tr_{\mathbb{R}^N}(\underline{U}) = (\{0\}, 0)$ (here $\{0\}$ is the singular part and the Radon measure on $\mathbb{R}^N \setminus \{0\}$ is the zero measure) and the conclusion follows due to Lemma 4.5. \square

By a simple adaptation of the proof of Proposition 2.3 and Proposition 2.6 it is possible to extend (2.28) to any positive solution vanishing on $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$.

Proposition 2.7 *Assume (1.8), (1.12) and (C2) are satisfied. Then any positive solution u of (1.1) which $C(\overline{Q_\infty} \setminus \{(0, 0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{(0, 0)\}$ satisfies*

$$u(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\} \quad \forall (x, t) \in Q_\infty. \quad (2.30)$$

Proof. Estimate (2.19) is valid with u_k replaced by u . The remaining of the proof of Proposition 2.6 is similar and yields to the first estimate

$$u(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty.$$

For the second estimate, since $f(0) = 0$ and due to the convexity of f , the following inequality holds

$$f(a+b) \geq f(a) + f(b) \quad \forall a, b > 0, \quad (2.31)$$

which implies that for any $R, \tau > 0$, $(x, t) \mapsto \phi_\infty(t - \tau) + w_R(x)$ is a supersolution of (1.1) in $B_R \times (\tau, \infty)$. This function dominates u on the parabolic boundary, thus in the domain itself by the comparison principle. Since $f(r) > 0$ if $r > 0$, $\lim_{R \rightarrow \infty} w_R = 0$ in \mathbb{R}^N . Therefore

$$u(x, t) \leq \phi_\infty(t) = \lim_{\tau \rightarrow 0} \lim_{R \rightarrow \infty} (\phi_\infty(t - \tau) + w_R(x)) \quad \forall (x, t) \in Q_\infty.$$

This implies (2.30). \square

It is also possible to construct a maximal element of \mathcal{U}_0 (\mathcal{U}_0 is defined in Theorem 1.4). For $\ell > 0$ and $\epsilon > 0$, let $u := U_{\epsilon, \ell}$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + f(u) = 0 & \text{in } Q_\infty \\ u(x, 0) = \ell \chi_{B_\epsilon} & \text{in } \mathbb{R}^N. \end{cases}$$

Lemma 2.8 *Assume (1.8), (1.12) and (C2) are satisfied. For any $\tau > 0$ and $\epsilon > 0$, there exist $\ell > 0$ and $m(\tau, \epsilon) > 0$ such that any positive solution u of (1.1) which verifies (i) in the proof of Theorem 1.4 satisfies*

$$u(x, t) \leq U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon) \quad \forall (x, t) \in Q_\infty, t \geq \tau. \quad (2.32)$$

Furthermore

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = 0 \quad \forall \epsilon > 0. \quad (2.33)$$

Finally

$$\bar{U}(x, t) = \lim_{\tau \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\ell \rightarrow \infty} (U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon)) \quad (2.34)$$

is the maximal element of \mathcal{U}_0 .

Proof. We set $\ell = \phi_\infty(\tau)$, then $u(x, \tau) \leq \ell$ for any $x \in \mathbb{R}^N$. Let $W := W_{\epsilon/2}$ be the solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t W - \Delta W + f(W) = 0 & \text{in } B_{\epsilon/2}^c \times (0, \infty) \\ W(x, 0) = 0 & \text{in } B_{\epsilon/2}^c \\ W(x, t) = \phi_\infty(t) & \text{in } \partial B_{\epsilon/2}^c \times (0, \infty) \end{cases} \quad (2.35)$$

and put $m(\tau, \epsilon) := \max\{W_{\epsilon/2}(x, \delta) : |x| > \epsilon, 0 < \delta \leq \tau\}$. It is clear to see that

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = W_{\epsilon/2}(x, 0) = 0. \quad (2.36)$$

From the fact that $u(x, 0) = 0$ in $B_{\epsilon/2}^c$, $u(x, t) \leq \phi_\infty(t)$ in $\partial B_{\epsilon/2}^c \times (0, \infty)$ and the maximum principle, it follows that $u(x, t) \leq W_{\epsilon/2}(x, t)$ in $B_{\epsilon/2}^c \times (0, \infty)$.

We next compare $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$ with u in $\mathbb{R}^N \times (\tau, \infty)$. The function $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$ is a supersolution of (1.1) in $\mathbb{R}^N \times (\tau, \infty)$. If $x \in B_\epsilon$, $U_{\epsilon, \ell}(x, 0) = \ell \geq u(x, \tau)$, which implies $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$. If $x \in B_\epsilon^c$, $m(\tau, \epsilon) \geq W_{\epsilon/2}(x, \tau) \geq u(x, \tau)$, hence $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$. So we always have $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$ for any $x \in \mathbb{R}^N$. Applying maximum principle yields $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon) \geq u$ in $\mathbb{R}^N \times (\tau, \infty)$. Finally, the function \bar{U} defined by (2.34) is the maximal solution because $U_{\epsilon, \ell}(x, t - \tau) \rightarrow U_{\epsilon, \ell}(x, t)$ as $\tau \rightarrow 0$ and $U_{\epsilon, \ell} \uparrow U_{\epsilon, \infty}$ when $\ell \rightarrow \infty$ and $U_{\epsilon, \infty} \downarrow \bar{U}$ when $\epsilon \rightarrow 0$. \square

3 About uniqueness

We first prove the existence of the global radial solutions of (1.18) under the Keller-Osserman condition.

Proof of Proposition 1.5.

A solution of (1.19) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N} \int_0^s t^{N-1} f(w) dt ds \quad (3.1)$$

Existence follows from Picard-Lipschitz fixed point theorem. The function is increasing and defined on a maximal interval $[0, r_a)$. By a result of Vazquez and Veron [11] $r_a = \infty$, thus the solution is global. Uniqueness on $[0, \infty)$ follows always from local uniqueness. The function $r \mapsto w(r)$ is increasing and

$$\begin{aligned} w'(r) &\geq \frac{ah(a)}{N}r, \\ w(r) &\geq a + \frac{ah(a)}{2N}r^2 \end{aligned}$$

for all $r > 0$. \square

Proposition 3.1 *Assume (1.16) holds. For any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.2)$$

for some $0 < a < b$, there exists a positive function $\bar{u} \in C(\bar{Q}_\infty) \cap C^{2,1}(Q_\infty)$ solution of (1.1) in Q_∞ and satisfying $\bar{u}(\cdot, 0) = u_0$ in \mathbb{R}^N . Furthermore

$$w_a(|x|) \leq u(x, t) \leq w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (3.3)$$

Proof. Clearly w_a and w_b are ordered solutions of (1.1). We denote by u_n the solution of the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) = 0 & \text{in } Q_n = B_n \times (0, \infty) \\ u_n(x, t) = (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0, \infty) \\ u_n(x, 0) = u_0(x) & \text{in } B_n. \end{cases} \quad (3.4)$$

By the maximum principle, u_n satisfies (3.3) in Q_n . Using locally parabolic equations regularity theory, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q_\infty}$. Using a diagonal sequence, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q_\infty}$ to some weak solution $\overline{u} \in C(\overline{Q_\infty})$ which satisfies $\overline{u}(\cdot, 0) = u_0$ in \mathbb{R}^N . By standard method, \overline{u} is a strong solution (at least $C^{2,1}(Q_\infty)$). \square

Proposition 3.2 *Assume (1.10) and (1.16) hold. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.5)$$

for some $0 < a < b$, there exists a positive function $\underline{u} \in C(\overline{Q_\infty})$ solution of (1.1) in Q_∞ satisfying $\underline{u}(\cdot, 0) = u_0$ in \mathbb{R}^N and

$$\underline{u}(x, t) \leq \min\{\phi_\infty(t), w_b(|x|)\} \quad \forall (x, t) \in Q_\infty. \quad (3.6)$$

Proof. For any $R > 0$, let u_R be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R + f(u_R) = 0 & \text{in } Q_\infty \\ u_R(x, 0) = u_0(x)\chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases} \quad (3.7)$$

The solution which is constructed is dominated by the solution of the heat equation with the same initial data. Thus

$$u_R(x, t) \leq (4\pi t)^{-N/2} \int_{B_R} e^{-|x-y|^2/4t} u_0(y) dy \quad \forall (x, t) \in Q_\infty. \quad (3.8)$$

and $\lim_{|x| \rightarrow \infty} u_R(x, t) = 0$ uniformly with respect to t . The functions ϕ_∞ and w_b are solutions of (1.1) in Q_∞ , which dominate u_R at $t = 0$. By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x, t) \quad \forall (x, t) \in Q_\infty. \quad (3.9)$$

The fact that the mapping $R \mapsto u_R$ is increasing and (3.9) imply that there exists $\underline{u} := \lim_{R \rightarrow \infty} u_R$ which satisfies $\underline{u}(\cdot, 0) = u_0$ in \mathbb{R}^N . Letting $R \rightarrow \infty$ in (3.9) yields (3.6). \square

Proof of Theorem 1.6.

Combining Proposition 3.1 and Proposition 3.2 we see that there exists two solutions \underline{u} and \bar{u} with the same initial data u_0 which are ordered and different since $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$ and $\lim_{|x| \rightarrow \infty} \underline{u}(x, t) \leq \phi_\infty(t) < \infty$ for all $t > 0$. \square

Proof of Theorem 1.7.

Step 1: There always holds

$$(ah(a) - bh(b))\text{sign}(a - b) \geq |a - b| h(|a - b|) \quad \forall a, b > 0 \quad (3.10)$$

where h is defined in (2.2) and

$$\text{sign}(z) = \begin{cases} 1 & \text{if } z > 0, \\ -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

In fact, since h is increasing and assuming $a > b$, we get

$$\begin{aligned} ah(a) - bh(b) &= (a - b)h(a) + b(h(a) - h(b)) \\ &\geq (a - b)h(a) \\ &\geq (a - b)h(a - b). \end{aligned}$$

Step 2: End of the proof. By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \leq [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})]\text{sign}(u - \tilde{u}),$$

therefore by step 1,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + |u - \tilde{u}| h(|u - \tilde{u}|) \leq 0. \quad (3.11)$$

Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any $R \geq R_\epsilon$,

$$0 \leq |u - \tilde{u}|(x, t) \leq w_\epsilon(|x|) \quad \forall (x, t) \in B_R^c \times [0, 1]. \quad (3.12)$$

Since w_ϵ is a positive solution of (1.1) which dominates $|u - \tilde{u}|$ on $\partial B_R \times [0, 1]$ and at $t = 0$, it follows that $|u - \tilde{u}| \leq w_\epsilon$ in $B_R \times [0, 1]$. Letting $R \rightarrow \infty$ yields to $|u - \tilde{u}| \leq w_\epsilon$ in $\mathbb{R}^N \times [0, 1]$. Letting $\epsilon \rightarrow 0$ and since $\lim_{\epsilon \rightarrow 0} w_\epsilon(|x|) = 0$ for any $x \in \mathbb{R}^N$, we derive $|u - \tilde{u}| = 0$, thus $u = \tilde{u}$ in $\mathbb{R}^N \times [0, 1]$. Iterating yields that equality holds in Q_∞ . \square

Remark. If we replace the condition (C1) by the condition (C2), the conclusion of Theorem 1.7 remains valid. Indeed, it follows by the convexity of f that

$$(f(a) - f(b))\text{sign}(a - b) \geq f(|a - b|) \quad \forall a, b > 0.$$

Then we proceed as in step 2 to get the desired conclusion.

Proof of Theorem 1.8.

Step 1: Construction of the minimal solution. The solution \underline{u} which is constructed in the Proposition 3.2 is a minimal solution of (1.1) in Q_∞ with the initial value u_0 . Indeed, if $u \in C^{2,1}(Q_\infty)$ is a solution of (1.1) in Q_∞ which satisfies $u(\cdot, 0) = u_0$ in \mathbb{R}^N then by maximum principle $u_R \leq u$ in Q_∞ where u_R is the solution of (3.7). Letting $R \rightarrow \infty$ yields $\underline{u} \leq u$ in Q_∞ .

Step 2: Construction of the maximal solution. For any $R > 0$ and $k > 0$, let $v_{R,k}$ be the solution of the initial-boundary problem

$$\begin{cases} \partial_t v_{R,k} + \Delta v_{R,k} + f(v_{R,k}) = 0 & \text{in } B_R \times (0, \infty) \\ v_{R,k} = k & \text{on } \partial B_R \times (0, \infty) \\ v_{R,k}(\cdot, 0) = u_0 & \text{in } B_R. \end{cases} \quad (3.13)$$

Since f is convex, the solution $(x, t) \mapsto \phi_\infty(t) + w_R(x)$ (w_R is the solution of (2.18)) is a supersolution of (1.1) in $B_R \times (0, \infty)$, which dominates $v_{R,k}$ on $\partial B_R \times (0, \infty)$ and at $t = 0$. By comparison principle, $v_{R,k}(x, t) \leq \phi_\infty(t) + w_R(x)$ for every $(x, t) \in B_R \times (0, \infty)$. The sequence $\{v_{R,k}\}$ is increasing with respect to k and converges to a function $v_{R,\infty}$, which is a solution of (1.1) in $B_R \times (0, \infty)$ with initial data u_0 . The sequence $\{v_{R,\infty}\}$ is decreasing and converges to a solution v_∞ of (1.1) in Q_∞ , which satisfies $v_\infty(\cdot, 0) = u_0$ in \mathbb{R}^N . If $u \in C^{2,1}(Q_\infty)$ is a positive solution of (1.1) with initial data u_0 then for $R > 0$ and $k > 0$, by comparison principle, $u \leq v_{R,k} + w_R$ in $B_R \times (0, \infty)$. Since $\lim_{R \rightarrow \infty} w_R = 0$ in \mathbb{R}^N , letting successively $k \rightarrow \infty$ and $R \rightarrow \infty$ yields $u \leq v_\infty$ in Q_∞ .

Step 3: Uniqueness. Put $\tilde{v} = v_\infty - \underline{u}$ then $\tilde{v} \geq 0$ in Q_∞ . By convexity of f , \tilde{v} is a subsolution of (1.1) in Q_∞ , which satisfies $\tilde{v}(\cdot, 0) = 0$ in \mathbb{R}^N . It follows from the maximum principle that $\tilde{v} \leq 0$ in Q_∞ and hence $\tilde{v} = 0$ in Q_∞ , which leads to the uniqueness. \square .

4 Initial trace

Let Ω is an arbitrary open domain in \mathbb{R}^N with C^2 boundary, we denote by $\mathfrak{M}(\Omega)$ (resp. $\mathfrak{M}^b(\Omega)$) the set of Radon measures in Ω (resp. bounded Radon measures), and by $\mathfrak{M}_+(\Omega)$ (resp. $\mathfrak{M}_+^b(\Omega)$) its positive cone. For $0 < T \leq \infty$, we set $Q_T^\Omega = \Omega \times (0, T)$.

4.1 The regular part of the initial trace

In this section we only assume that f is a continuous nonnegative function defined on \mathbb{R}_+ and that u is a $C^{2,1}$ positive solution of (1.1) in Q_T^Ω .

Lemma 4.1 *Assume G is a bounded C^2 domain in \mathbb{R}^N , $Q_T^{\overline{G}} := \overline{G} \times (0, T]$ and let $u \in C^{2,1}(Q_T^{\overline{G}})$ be a positive solution of (1.1) in $Q_T^{\overline{G}}$ such that $u, f(u) \in L^1(Q_T^{\overline{G}})$. Then $u \in L^\infty(0, T; L^1(G'))$ for any domain $G' \subset \overline{G}' \subset G$ and there exists a positive Radon measure μ_G on G such that*

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G \zeta(x) d\mu_G(x) \quad \forall \zeta \in C_c(G). \quad (4.1)$$

Proof. Let $\phi := \phi_G$ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ with corresponding eigenvalue λ_G . We assume $\phi > 0$ in G . Then

$$\frac{d}{dt} \int_G u \phi dx + \lambda_G \int_G u \phi dx + \int_G f(u) \phi dx + \int_{\partial G} u \phi_{\mathbf{n}} dS = 0$$

where $\phi_{\mathbf{n}}$ denote the outward normal derivative of ϕ . Since $\phi_{\mathbf{n}} < 0$, the function

$$t \mapsto e^{\lambda_G t} \int_G u(x, t) \phi(x) dx - \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

is increasing and

$$\int_G u(x, t) \phi(x) dx \leq e^{\lambda_G(T-t)} \int_G u(x, T) \phi(x) dx + e^{-\lambda_G t} \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

for $0 < t \leq T$. Thus $u \in L^\infty(0, T; L^1(G'))$ for any strict domain G' of G . If $\zeta \in C_c(G)$, there holds

$$\frac{d}{dt} \left(\int_G u(x, t) \zeta(x) dx - \int_t^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds \right) = 0. \quad (4.2)$$

Consequently

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G u(x, T) \zeta(x) dx + \int_0^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds. \quad (4.3)$$

This implies that $u(\cdot, t)$ admits a limit in $\mathcal{D}'(G)$ and this limit is a positive distribution. Therefore there exists a positive Radon measure μ_G on G satisfies (4.1). \square

Proof of Proposition 1.9.

It is clear that $\mathcal{R}(u)$ is an open subset of Ω . If G is a strict bounded subdomain of $\mathcal{R}(u)$, i.e. $\overline{G} \subset \mathcal{R}(u)$, there exists a finite number of points z_j ($j = 1, \dots, k$) and $r'_j > r_j > 0$ such that $u, f(u) \in L^1(Q_T^{B_{r'_j}(z_j)})$ and $\overline{G} \subset \cup_{j=1}^k B_{r_j}(z_j)$. Let $\mu_j = \mu_{B_{r_j}(z_j)}$

the measure defined in Lemma 4.1. If $\zeta \in C_c(G)$ there exists a partition of unity $\{\eta_j\}_{j=1}^k$ relative to the cover $\{B_{r_j}(z_j)\}_{j=1}^k$ such that $\eta_j \in C_0^\infty(G)$, $\text{supp}(\eta_j) \subset B_{r_j}(z_j)$

and $\zeta = \sum_{j=1}^k \eta_j \zeta$. Since

$$\lim_{t \rightarrow 0} \int_{B_{r_j}(z_j)} u(x, t) (\eta_j \zeta)(x) dx = \int_{B_{r_j}(z_j)} (\eta_j \zeta)(x) d\mu_j(x) \quad \forall j = 1, \dots, k,$$

there exists a positive Radon measure μ on $\mathcal{R}(u)$ satisfying (1.23). Notice also that $u \in L^\infty(0, T; L^1(G))$ for any $G \subset \overline{G} \subset \mathcal{R}(u)$. \square

The main problem is to analyse what does happen on the singular set $\mathcal{S}(u)$.

4.2 The Keller-Osserman condition holds

If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (1.1) is based upon a dichotomy in the behaviour of those solutions near $t = 0$.

Lemma 4.2 *Assume u is a positive solution of (1.1) in Q_T^Ω and $z \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds*

- (i) *There exists an open neighborhood G of z such that $u \in L^1(Q_T^G)$.*
- (ii) *f is nondecreasing and (1.12) holds.*

Then for every open relative neighborhood G' of z

$$\lim_{t \rightarrow 0} \int_{G'} u(x, t) dx = \infty. \quad (4.4)$$

Proof. First, we assume (i) holds and let $\zeta \in C_c^2(G)$, $\zeta \geq 0$. Since $z \in \mathcal{S}(u)$, then for every open relative neighborhood G' of z , there holds

$$\int_0^T \int_{G'} f(u) dx dt = \infty. \quad (4.5)$$

Since there exists

$$\lim_{t \rightarrow 0} \int_t^T \int_{G'} u \Delta \zeta dx dt = L \in \mathbb{R},$$

it follows from (4.3) that

$$\int_{G'} u(x, t) \zeta(x) dx = \int_t^T \int_{G'} f(u) \zeta dx ds + O(1), \quad (4.6)$$

which implies (4.4).

Next we assume that (1.12) holds and $u \notin L^1(Q_T^G)$ for every relative neighborhood G of z . If there exists an open neighborhood $G \subset \Omega$ of z such that (4.4) does not hold, there exists a sequence $\{t_n\}$ decreasing to 0 and $0 \leq M < \infty$ such that

$$\sup_{t_n} \int_G u(x, t_n) dx = M. \quad (4.7)$$

Furthermore, we can always replace G by an open ball $B_R(z) \subset G$. Thus (4.7) holds with G replaced by $B_R(z)$. Let $w := w_R$ be the maximal solution of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } B_R(z) \\ \lim_{|x-z| \rightarrow R} w(x) = \infty. \end{cases} \quad (4.8)$$

Let $v := v_n$ be the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } B_R(z) \times (t_n, \infty) \\ v = 0 & \text{in } \partial B_R(z) \times (t_n, \infty) \\ v(\cdot, t_n) = u(\cdot, t_n) & \text{in } B_R(z). \end{cases} \quad (4.9)$$

Since $v_n \geq 0$, $f(w_R + v_n) \geq f(w_R)$, and $w_R + v_n$ is a supersolution of (1.1) in $B_R(z) \times (t_n, T)$. It dominates u on $\partial B_R(z) \times (t_n, T)$ and at $t = t_n$, thus $u \leq w_R + v_n$ in $B_R(z) \times (t_n, T)$. We can assume that $u(\cdot, t_n) \rightarrow \nu$ for some positive and bounded measure ν on $B_R(z)$. Therefore

$$u(x, t) \leq v(x, t) + w_R(x) \quad \text{in } Q_T^{B_R(z)} \quad (4.10)$$

where v is the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) = \nu & \text{in } \mathcal{D}'(B_R(z)). \end{cases} \quad (4.11)$$

Since $v \in L^1(Q_T^{B_R(z)})$ and w_R is uniformly bounded in any ball $B_{R'}(z)$ for $0 < R' < R$ we conclude that $u \in L^1(Q_T^{B_{R'}(z)})$, which is a contradiction. \square

Definition 4.3 Assume f is nondecreasing and satisfies (1.12). Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (1.1) in Q_T^Ω . We say that u possesses an initial trace with regular part $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ and singular part $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ if

(i) For any $\zeta \in C_c(\mathcal{R}(u))$,

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x). \quad (4.12)$$

(ii) For any open set $G \subset \Omega$ such that $G \cap \mathcal{S}(u) \neq \emptyset$

$$\lim_{t \rightarrow 0} \int_G u(x, t) dx = \infty. \quad (4.13)$$

Proof of Theorem 1.11

The set $\mathcal{R}(u)$ and the measure $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ are defined by Definition 1.10 thanks to Proposition 1.9. Because (1.12) holds, $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ inherits the property (ii) in Definition 4.3 because of Lemma 4.2 (ii). \square

If Ω is a bounded domain with a C^2 boundary and $\mu \in \mathfrak{M}_+^b(\Omega)$, we denote by u_μ the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = \mu & \text{in } \mathcal{D}'(\Omega). \end{cases} \quad (4.14)$$

We recall the following stability result proved in [6, Th 1.1].

Lemma 4.4 *Let Ω be a bounded domain with a C^2 boundary. Assume f is non-decreasing and satisfies (1.8). Then for any $\mu \in \mathfrak{M}^b(\Omega)$ problem (4.14) admits a unique solution u_μ . Moreover, if $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$ converges weakly to $\mu \in \mathfrak{M}^b(\Omega)$ then $u_{\mu_n} \rightarrow u_\mu$ locally uniformly in $\bar{\Omega} \times (0, \infty)$ and in $L^1(Q_T^\Omega)$, and $f(u_{\mu_n}) \rightarrow f(u_\mu)$ in $L^1(Q_T^\Omega)$, for every $T > 0$.*

Remark. The result remains true if $\Omega = \mathbb{R}^N$ and μ_n have their support in a fixed compact set.

If Ω is bounded, by Lemma 4.4, for every $y \in \Omega$ and $k > 0$, there exists a unique solution $v_{\Omega, y, k} := v$ to (4.14) with $\mu = k\delta_0$. By comparison principle [6, Proposition 1.2] $v_{\Omega, y, k}$ is positive and increases as k increases. By Proposition 2.7, $\{v_{\Omega, y, k}\}$ is bounded on compact subsets of Q_T^Ω and consequently

$$v_{\Omega, y, \infty} := \lim_{k \rightarrow \infty} v_{\Omega, y, k}$$

is a solution of (1.1) in Q_∞^Ω which vanishes on $\bar{\Omega} \times \{0\} \setminus \{(y, 0)\}$ and on $\partial\Omega \times (0, \infty)$. Note that the value of $v_{\Omega, y, \infty}(x, t)$ depends only on $|x - y|$ and t .

Lemma 4.5 *Let Ω be a bounded domain with C^2 boundary. Assume f is nondecreasing and satisfies (1.8) and (1.12). Let $u \in C^{2,1}(Q_\infty^\Omega)$ is a positive solution of (1.1) in Q_∞^Ω with initial trace $(\mathcal{S}(u), \mu)$. Then for every $y \in \mathcal{S}(u)$,*

$$v_{\Omega, y, \infty} \leq u \quad (4.15)$$

in Q_∞^Ω .

Proof. By translation we may suppose that $y = 0$. Since $0 \in \mathcal{S}(u)$, for any $\eta > 0$ small enough

$$\lim_{t \rightarrow 0} \int_{B_\eta} u(x, t) dx = \infty.$$

For $\epsilon > 0$, denote $M_{\epsilon, \eta} = \int_{B_\eta} u(x, \epsilon) dx$. For any $m > m_\eta = \inf_{\sigma > 0} M_{\sigma, \eta}$ there exists $\epsilon = \epsilon(m, \eta)$ such that $m = M_{\epsilon, \eta}$ and $\lim_{\eta \rightarrow 0} \epsilon(m, \eta) = 0$. For $R > 0$ let v_η be the solution of the problem

$$\begin{cases} \partial_t v_\eta - \Delta v_\eta + f(v_\eta) = 0 & \text{in } \Omega \times (0, \infty) \\ v_\eta = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v_\eta(x, 0) = u(x, \epsilon) \chi_{B_\eta} & \text{in } \Omega \end{cases}$$

where $\eta < R$ and χ_{B_η} is the characteristic function of B_η . By the maximum principle $v_\eta \leq u$ in $\Omega \times (\epsilon, \infty)$. By Lemma 4.4, when η goes to zero v_η converges to $v_{\Omega, 0, m}$. Letting m goes to infinity yields (4.15). \square

The following convergence lemma is obtained by using the arguments as in [5, Lemma 3.2]

Lemma 4.6 *Assume f is nondecreasing and satisfies (1.8). Let $\{u_n\}$ be a sequence of positive solutions of (1.1) in Q_∞^Ω and let $A \subset \Omega$ is an open subset. Suppose that $Tr_A(u_n) = \nu_n \in \mathfrak{M}^b(A)$ and $u_n \rightarrow u$ locally uniformly in Q_∞^Ω . Thus u is a positive solution of (1.1) in Q_∞^Ω and we denote $tr_\Omega(u) = (\mathcal{S}, \mu)$. Under this assumption, if $A \subset \mathcal{S}$ and $\{\nu(A)\}$ is bounded then $\nu_n \rightarrow \mu|_A$ weakly where $\mu|_A := \chi_A \mu$.*

Proof. Let $\zeta \in C_c^2(A)$, $\xi \geq 0$ then

$$\lim_{n \rightarrow \infty} \int_0^T \int_\Omega u_n \Delta \zeta dx dt = \int_0^T \int_\Omega u \Delta \zeta dx dt. \quad (4.16)$$

Since

$$\int_A \zeta u_n(\cdot, T) dx + \int_0^T \int_A (-u_n \Delta \zeta + f(u_n) \zeta) dx dt = \int_A \zeta d\nu_n(x), \quad (4.17)$$

we derive by Fatou's lemma and (4.16) that

$$\int_A \zeta u(\cdot, T) dx + \int_0^T \int_A (-u \Delta \zeta + f(u) \zeta) dx dt \leq \liminf_{n \rightarrow \infty} \int_A \zeta d\nu_n(x). \quad (4.18)$$

Since $\{\nu_n(A)\}$ is bounded, we can extract a sunsequence (for simplification, we also denote by $\{\nu_n\}$ this subsequence) which converges weakly in $\mathfrak{M}(A)$ to a Borel

measure ν . We shall show that this limit does not depend on the subsequence and that in fact $\nu = \mu_{|A}$. Now letting $T \rightarrow 0$ in (4.18) yields

$$\int_A \zeta d\mu \leq \int_A \zeta d\nu, \quad (4.19)$$

which means

$$\mu_{|A} \leq \nu. \quad (4.20)$$

Now let A' be an open bounded subdomain of A with smooth boundary. Let $\{w_n\}$ (resp. w) be a solution of (1.1) in $Q_T^{A'}$ with initial data $\nu_n^* = \chi_{A'}\nu_n$ (resp. $\nu^* = \chi_{A'}\nu$) and boundary data zero. Clearly $\nu_n^* \rightarrow \nu^*$ weakly relative to functions in $C_c(\overline{A'})$. By Lemma 4.4, $w_n \rightarrow w$ locally uniformly in $\overline{A'} \times (0, \infty)$ and in $L^1(Q_T^{A'})$, and $f(w_n) \rightarrow f(w)$ in $L^1(Q_T^{A'})$ for every $T > 0$. Since $w_n \leq u_n$ in $Q_\infty^{A'}$, $w \leq u$ in $Q_\infty^{A'}$, and hence $\nu^* \leq \mu_{|A'}$. Since A' is arbitrary, it follows that $\nu \leq \mu_{|A}$ and we conclude that the equality holds by (4.20). \square

If A is an open subset of Ω and $\nu \in \mathfrak{M}^+(A)$, we define an extension $\underline{\nu}$ of ν to Ω by

$$\underline{\nu}(E) = \inf_{E \subseteq O} \nu(O \cap A) \quad (4.21)$$

for every Borel set $E \subset \Omega$, where the infimum is taken over the open subsets O ; $\underline{\nu}$ is an outer regular Borel measure on Ω and $\nu = \nu_{|A}$.

The next comparison result is obtained by using the argument as in [5, Lemma 2.7].

Lemma 4.7 *Let Ω be a bounded domain in C^2 boundary. Assume f is nondecreasing nonnegative on \mathbb{R}_+ . Let $(g_i, \mu_i) \in L^1(\partial\Omega \times (0, T)) \times \mathfrak{M}^b(\Omega)$, $i = 1, 2$ and denote by u_i , $i = 1, 2$ the solution to the problem*

$$\begin{cases} \partial_t u_i - \Delta u_i + f(u_i) = 0 & \text{in } Q_T^\Omega \\ u_i = g_i & \text{on } \partial\Omega \times (0, T) \\ u_i(\cdot, 0) = \mu_i & \text{in } \Omega. \end{cases}$$

If $g_1 \leq g_2$ in $\partial\Omega \times (0, T)$ and $\mu_1 \leq \mu_2$ then $u_1 \leq u_2$ in Q_T^Ω .

We next mimic the proof of M. Marcus and L. Véron in [5, Lemma 3.3] to show the existence of a minimal solution of (1.1) with a given initial trace in $\mathfrak{M}_+(A)$ for any open subset A in Ω .

Lemma 4.8 *Assume Ω is bounded, f is nondecreasing and satisfies (1.8), (1.12) and (C2).*

(i) Let A be an open subset of Ω and let $\nu \in \mathfrak{M}_+(A)$. Then there exists a positive solution of (1.1) in Q_∞^Ω denoted by \underline{u}_ν such that

$$\begin{cases} \underline{u}_\nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \text{Tr}_\Omega(\underline{u}_\nu) = \underline{\nu} & \text{and } \underline{u}_\nu \leq v, \end{cases} \quad (4.22)$$

for every positive solution v of (1.1) such that $\text{Tr}_A(v) \geq \nu$.

(ii) Let u_n be the solution of problem

$$\begin{cases} \partial_t u_n - \Delta u_n + f(u_n) = 0 & \text{in } Q_T^\Omega \\ u_n = n & \text{on } \partial\Omega \times (0, \infty) \\ u_n(\cdot, 0) = n dx & \text{in } \Omega. \end{cases} \quad (4.23)$$

Denote $U_{\infty, \Omega} := \lim_{n \rightarrow \infty} u_n$. Then $U_{\infty, \Omega}$ is the maximal solution of (1.1) in Q_∞^Ω in the sense that the following relation holds in Q_T^Ω for every positive solution v of (1.1)

$$U_{\infty, \Omega} \geq v. \quad (4.24)$$

Proof. Let D be an open subdomain of A such that $\overline{D} \subset A$ is compact and ∂D is smooth. Set u^D the solution of (1.1) in Q_∞^D with boundary data zero and initial data $\chi_D \nu \in \mathfrak{M}^+(D)$. If v is any positive solution of (1.1) in Q_∞^Ω with $\text{Tr}_A(v) \geq \nu$, then $\text{Tr}_A(v)|_D = \text{Tr}_D(v) \geq \chi_D \nu$ and $v|_{\partial D \times (0, \infty)} \geq 0$. Therefore by Lemma 4.7, $v \geq u^D$ in Q_∞^D . Now let $\{D_k\}$ be an increasing sequence of open and smooth subdomains of A such that $\overline{D}_k \subset A$ and $\cup_{k=1}^\infty D_k = A$. Then $\{u^{D_k}\}$ is monotone increasing and by Lemma 4.6 it admits a limit, which we denote by \underline{u}_ν , which satisfies (4.22).

To verify the statement (ii) we proceed as above with $\{D_k\}$ an increasing sequence of open smooth subdomains of Ω such that $\overline{D}_k \subset \Omega$ and $\cup_{k=1}^\infty D_k = \Omega$. For every k we construct U_{∞, D_k} in $Q_\infty^{D_k}$ as above and we have

$$U_{\infty, D_k}(x, t - \tau) \geq v(x, t) \quad \forall (x, t) \in D_k \times (\tau, \infty)$$

for any $\tau > 0$, if v is any positive solution of equation (1.1) in Q_∞^Ω . Letting $\tau \rightarrow 0$ and $k \rightarrow \infty$ yields $U_\infty(x, t) \geq v(x, t)$ for all $(x, t) \in Q_\infty^\Omega$ where U_∞ is the limit of the decreasing sequence $\{U_{\infty, D_k}\}$. In particular $U_\infty \geq U_{\infty, \Omega}$ in Q_T^Ω . In order to prove the reverse inequality, let $\omega_{\infty, \Omega}$ be the unique solution of the stationary problem

$$\begin{cases} -\Delta \omega_{\infty, \Omega} + f(\omega_{\infty, \Omega}) = 0 & \text{in } \Omega \\ \omega_{\infty, \Omega} = \infty & \text{on } \partial\Omega \end{cases} \quad (4.25)$$

Then $\omega_{\infty, \Omega}$ is the limit of the increasing sequence $\omega_{n, \Omega}$ of the same elliptic equation with boundary value n . Therefore

$$\max(\omega_{\infty, \Omega}(x), \phi_\infty(t)) \leq U_{\infty, \Omega}(x, t) \leq U_\infty(x, t) \leq \omega_{\infty, \Omega}(x) + \phi_\infty(t) \quad (4.26)$$

for every $(x, t) \in Q_T^\Omega$ since $(x, t) \mapsto \omega_{\infty, \Omega}(x) + \phi_\infty(t)$ is a supersolution of (1.1) in Q_∞^Ω due to the convexity of f . Consequently, for any $\epsilon > 0$ and $\tau > 0$, the function $(x, t) \mapsto (1+\epsilon)U_{\infty, \Omega}(x, t-\tau)$, which is a supersolution of (1.1) in $\Omega \times (\tau, \infty)$ dominates U_∞ on $\Omega \times \{\tau\} \cup \partial\Omega \times (\tau, \infty)$; hence it majorizes U_∞ in $\Omega \times (\tau, \infty)$. Letting ϵ and τ go to 0 implies $U_{\infty, \Omega} \geq U_\infty$ and hence $U_{\infty, \Omega} = U_\infty$ in Q_∞^Ω . Consequently, (4.24) holds. \square

As a counterpart of Theorem 1.11 we have the following existence theorem.

Proof of Theorem 1.12

The proof is essentially similar to the one of [5, Th 3.4]. We denote, for $\delta > 0$,

$$\begin{aligned} \mathcal{S}^\delta &:= \{x \in \Omega : \text{dist}(x, \mathcal{S}) \leq \delta\}, & \mathcal{R}^\delta &:= \Omega \setminus \mathcal{S}^\delta, \\ \mathcal{S}_\mu^{\partial\Omega} &:= \{y \in \partial\Omega : \mu(O \cap \mathcal{R}) = \infty, \text{ for every neighborhood } O \text{ of } y\}, \\ \hat{\mathcal{S}}_\mu &:= \overline{\mathcal{S}} \cup \mathcal{S}_\mu^{\partial\Omega}, \\ (\hat{\mathcal{S}}_\mu)^\delta &= \hat{\mathcal{S}}^\delta = \{x \in \Omega : \text{dist}(x, \hat{\mathcal{S}}_\mu) \leq \delta\}, & \hat{\mathcal{R}}^\delta &:= \Omega \setminus \hat{\mathcal{S}}^\delta, \end{aligned}$$

and let $\hat{\mu}_\delta$ be the measure given by

$$\hat{\mu}_\delta(E) := \mu(\hat{\mathcal{R}}_\delta \cap E) \quad \forall E \subset \Omega, E \text{ Borel.}$$

Step 1: Construction of the maximal solution in the case that Ω is bounded.

Let z_δ be the solution of (1.1) in Q_T^Ω with initial trace $\hat{\mu}_\delta$ and boundary data zero. For $0 < \tau < T/2$ let $g_{\delta, \tau}$ be defined by

$$g_{\delta, \tau}(x) := \begin{cases} z_\delta(x, \tau) & \text{if } x \in \hat{\mathcal{R}}^\delta, \\ U_{\infty, \Omega}(x, \tau) & \text{if } x \in \hat{\mathcal{S}}^\delta, \end{cases} \quad \forall x \in \Omega \quad (4.27)$$

and for $n \in \mathbb{N}$, let $u_{\delta, \tau, n}$ be the solution of (1.1) in Q_T^Ω with initial data $g_{\delta, \tau}$ and boundary data h_τ where

$$h_\tau(x, t) := \begin{cases} n & \text{if } (x, t) \in \partial\Omega \times (0, T - \tau), \\ 0 & \text{if } (x, t) \in \partial\Omega \times (T - \tau, T). \end{cases} \quad (4.28)$$

Since the sequence $\{u_{\delta, \tau, n}\}$ is increasing with respect to n and is bounded from above by $U_{\infty, \Omega}$, there exists $u_{\delta, \tau} := \lim_{n \rightarrow \infty} u_{\delta, \tau, n}$. By Lemma 4.8 (ii), the family $\{u_{\delta, \tau} : \tau \in (0, T/2]\}$ is locally uniformly bounded and therefore (by standard parabolic equations estimates) it is compact in $C^{2,1}(Q_T^\Omega)$. Consequently, there exists a sequence $\{\tau_m\}$ such that $\{u_{\delta, \tau_m}\}$ converges locally uniformly in Q_T^Ω to a solution \tilde{u}_δ of (1.1) in Q_T^Ω as $m \rightarrow \infty$, which blows-up on $\partial\Omega \times (0, T)$. By Lemma 4.6, $\text{tr}_\Omega(\tilde{u}_\delta) = (\hat{\mathcal{S}}^\delta, \mu|_{\hat{\mathcal{R}}^\delta})$.

Let v be a solution of (1.1) in Q_T^Ω and suppose that $tr_\Omega(v) = (\mathcal{S}', \mu')$ satisfies $\mathcal{S}' \subset (\hat{\mathcal{S}}^\delta)^{\text{int}}$ and $\mu' = \mu$ in $\hat{\mathcal{R}}^\delta$. Then

$$\lim_{\tau \rightarrow 0} (v(\cdot, \tau) - g_{\delta, \tau}(\cdot))_+ = 0 \quad (4.29)$$

in the sense of weak convergence of measures. In fact, by Lemma 4.8, $v(\cdot, \tau) \leq g_{\delta, \tau}$ in $\hat{\mathcal{S}}^\delta$ while $v(\cdot, \tau) \geq g_{\delta, \tau}$ in $\hat{\mathcal{R}}^\delta$. If $v_{\delta, \tau}$ is the solution of (1.1) in $\Omega \times (\tau, T)$ with boundary data zero and initial data $(v(\cdot, \tau) - g_{\delta, \tau}(\cdot))_+$, then by Lemma 4.4 $v_{\delta, \tau} \rightarrow 0$ locally uniformly in $\bar{\Omega} \times (0, T)$ as $\tau \rightarrow 0$. By the maximum principle, $v \leq u_{\delta, \tau} + v_{\delta, \tau}$ in $\Omega \times (\tau, T)$. Therefore

$$v \leq \tilde{u}_\delta. \quad (4.30)$$

In particular, it follows that $\{\tilde{u}_\delta\}$ is monotone increasing with respect to δ . Thus $\bar{u}_{\Omega, \mathcal{S}, \mu} := \lim_{\delta \rightarrow 0} \tilde{u}_\delta$ is a solution of (1.1), with initial data (\mathcal{S}, μ) , which blows-up on $\partial\Omega \times (0, T)$. If $tr_\Omega(v) = (\mathcal{S}, \mu)$ then (4.30) holds for every $\delta > 0$ and consequently $v \leq \bar{u}_{\Omega, \mathcal{S}, \mu}$ in Q_T^Ω .

Step 2: Construction of a minimal solution in the case that Ω is bounded.

First we construct a minimal solution under the additional assumption that \mathcal{S} is compact. Let $0 < \delta < \text{dist}(\mathcal{S}, \partial\Omega)$ and $0 < \tau < T/2$. Denote by $w_{\delta, \tau}$ the solution of (1.1) with boundary data 0 and initial data $h_{\delta, \tau}$ defined by

$$h_{\delta, \tau}(x) := \begin{cases} z_\delta(x, \tau) & \text{if } x \in \hat{\mathcal{R}}^\delta, \\ v_{\Omega, \hat{x}, \infty}(x, \tau) & \text{if } x \in \mathcal{S}^\delta, \\ 0 & \text{otherwise,} \end{cases} \quad \forall x \in \Omega \quad (4.31)$$

where $\hat{x} \in \mathcal{S}$ is a point such that $\text{dist}(x, \mathcal{S}) = |x - \hat{x}|$. If v is a positive solution of (1.1) in Q_T^Ω with

$$tr_\Omega(v) = (\mathcal{S}, \mu'), \mu' \geq \mu \quad (4.32)$$

then by Lemma 4.5 and maximum principle,

$$w_{\delta, \tau}(x, t) \leq v(x, t + \tau) \quad (4.33)$$

in $Q_{T-\tau}^\Omega$ for small positive τ . Let $\{\tau_n\}$ be a sequence converging to 0 such that $\{w_{\delta, \tau_n}\}$ converges in Q_T^Ω . Then by Lemma 4.4 and Lemma 4.6 the limit function $\underline{v}_{\mathcal{S}, \hat{\mu}_\delta}$ is a solution of (1.1) in Q_T^Ω with initial trace $(\mathcal{S}, \hat{\mu}_\delta)$ which minorizes v in Q_T^Ω . Therefore $\underline{v}_{\mathcal{S}, \mu} := \lim_{\delta \rightarrow 0} \underline{v}_{\mathcal{S}, \hat{\mu}_\delta}$ is a solution of (1.1) in Q_T^Ω with initial trace (\mathcal{S}, μ) and $\underline{v}_{\mathcal{S}, \mu} \leq v$ in Q_T^Ω .

Next we claim that if \mathcal{S}' is a compact subset of Ω ,

$$\mathcal{S} \subseteq \mathcal{S}', \mu|_{\mathcal{R}'} \leq \mu' \text{ and } \mu(\mathcal{S}' \setminus \mathcal{S}) = 0 \implies \underline{v}_{\mathcal{S}, \mu} \leq \underline{v}_{\mathcal{S}', \mu'}. \quad (4.34)$$

Suppose that, in addition to the conditions of (4.34), $\mu((\mathcal{S}' \setminus \mathcal{S})^\epsilon \cap \mathcal{R}) = 0$ for some $\epsilon > 0$. Then $\mathcal{S}' \subseteq \mathcal{S}$ and hence $\mathcal{S}' = \mathcal{S}$. From the previous construction we find that, for $0 < \delta < \epsilon$, $\underline{v}_{\mathcal{S}, \hat{\mu}_\delta} \leq \underline{v}_{\mathcal{S}, \hat{\mu}'_\delta}$. Letting $\delta \rightarrow 0$ yields (4.34). Next, assuming only the condition in (4.34), let μ^ϵ be defined by $\mu^\epsilon(E) := \mu(E \setminus [(\mathcal{S}' \setminus \mathcal{S})^\epsilon \cap \mathcal{R}])$, for every Borel subset $E \subseteq \mathcal{R}$. Then from the previous special case, we obtain $\underline{v}_{\mathcal{S}, \mu^\epsilon} \leq \underline{v}_{\mathcal{S}', \mu'^\epsilon}$. Letting $\delta \rightarrow 0$ leads to (4.34).

Using this fact we can construct the minimal solution for arbitrary couples (\mathcal{S}, μ) where \mathcal{S} is a relatively closed subset of Ω and $\mu \in \mathfrak{M}_+(\mathcal{R})$. For $\epsilon > 0$ put

$$\begin{aligned} \Omega_{(\epsilon)} &:= \{x \in \Omega : \text{dist}(x, \partial\Omega) > \epsilon\}, \\ \mathcal{S}_{(\epsilon)} &:= \{x \in \mathcal{S} : \text{dist}(x, \partial\Omega) \geq \epsilon\}, \quad \mathcal{R}_{(\epsilon)} := \Omega \setminus \mathcal{S}_{(\epsilon)}, \end{aligned}$$

and define the measure $\mu_{(\epsilon)}$ by $\mu_{(\epsilon)}(E) := \mu(E \cap \Omega_{(\epsilon)})$ for every Borel set $E \subseteq \mathcal{R}_{(\epsilon)}$. By (4.34), the sequence $\{\underline{v}_{\mathcal{S}_{(\epsilon)}, \mu_{(\epsilon)}}\}$ increases as ϵ decreases. Moreover this sequence is bounded from above by v . Thus $\underline{u}_{\Omega, \mathcal{S}, \mu} := \lim_{\epsilon \rightarrow 0} \underline{v}_{\mathcal{S}_{(\epsilon)}, \mu_{(\epsilon)}}$ is a solution of (1.1) in Q_T^Ω with initial trace (\mathcal{S}, μ) satisfying

$$\underline{u}_{\Omega, \mathcal{S}, \mu} \leq v \tag{4.35}$$

for every positive solution v of (1.1) in Q_T^Ω with $tr_\Omega(v) = (\mathcal{S}, \mu)$.

If $(\mathcal{S}, \mu), (\mathcal{S}', \mu') \in CM_+(\Omega)$ and $\mathcal{S} \subseteq \mathcal{S}'$ and $\mu|_{\mathcal{R}'} \leq \mu'$ where $\mathcal{R}' = \Omega \setminus \mathcal{S}'$ then from the previous construction, we get

$$\underline{v}_{\mathcal{S}_{(\epsilon)}, \mu_{(\epsilon)}} \leq \underline{v}_{\mathcal{S}'_{(\epsilon)}, \mu'_{(\epsilon)}}. \tag{4.36}$$

Letting $\epsilon \rightarrow 0$ yields

$$\underline{u}_{\Omega, \mathcal{S}, \mu} \leq \underline{u}_{\Omega, \mathcal{S}', \mu'}. \tag{4.37}$$

Step 3: Construction of the maximal solution when Ω is unbounded.

Let $\{D_k\}$ be an increasing sequence of open and bounded smooth subdomains of Ω such that $\overline{D_k} \subset \Omega$ and $\cup_{k=1}^\infty D_k = \Omega$. First we consider the case when $\mathcal{R} = \Omega \setminus \mathcal{S}$ is bounded and assume that $\mathcal{R} \subset D_k$ for every k . Proceeding as in step 1, we can construct a maximal solution $\overline{u}_{D_k, \mathcal{S}, \mu}$ of (1.1) in $Q_T^{D_k}$ with initial trace (\mathcal{S}_k, μ) where $\mathcal{S}_k = \mathcal{S} \cap D_k$, which blows-up everywhere on $\partial D_k \times (0, T)$. The sequence $\{\overline{u}_{D_k, \mathcal{S}, \mu}\}$ decreases with respect to k and is bounded from below. Therefore the limit function $\overline{u}_{\Omega, \mathcal{S}, \mu} := \lim_{k \rightarrow \infty} \overline{u}_{D_k, \mathcal{S}, \mu}$ is a maximal solution of (1.1) in Q_T^Ω in the sense of (1.26) with initial trace (\mathcal{S}, μ) . Indeed, if v is a solution of (1.1) in Q_T^Ω with initial trace (\mathcal{S}, μ) then by step 1, for any $k > 0$, $v|_{D_k \times (0, T)} \leq \overline{u}_{D_k, \mathcal{S}, \mu}$. Letting $k \rightarrow \infty$ yields $v \leq \overline{u}_{\Omega, \mathcal{S}, \mu}$ in Q_T^Ω .

If \mathcal{R} is unbounded, for $k > 0$, we put $\mathcal{R}_k := \mathcal{R} \cap D_k$, $\mathcal{S}_k := \Omega \setminus \mathcal{R}_k$ and $\mu_k(E) := \mu(E \cap \mathcal{R}_k)$ for every Borel subset $E \subset \mathcal{R}$. The sequence of maximal solutions $\{\overline{u}_{\Omega, \mathcal{S}_k, \mu_k}\}$ decreases with respect to k and its limit as $k \rightarrow \infty$ is the maximal solution of (1.1) in Q_T^Ω in the sense of (1.26) with initial trace (\mathcal{S}, μ) .

Step 4: Construction of the minimal solution when Ω is unbounded.

First assume that \mathcal{S} is bounded and that $\mu(E) = 0$ for every Borel subset $E \subset \mathcal{R} \setminus G$ for some compact subset $G \subset \Omega$. Let $\{D_k\}$ be a sequence as in step 3 and assume that $\mathcal{S}, G \subset D_k$ for every $k > 0$. We construct the minimal solution $\underline{u}_{D_k, \mathcal{S}, \mu_k}$ of (1.1) in $Q_T^{D_k}$ with initial data (\mathcal{S}, μ_k) where $\mu_k(E) = \mu(E)$ for every Borel subset $E \subset \mathcal{R}_k = D_k \setminus \mathcal{S}$ and boundary data 0. We observe that $\underline{u}_{D_k, \mathcal{S}, \mu_k}$ increases with respect to k and its limit as $k \rightarrow \infty$ is the minimal solution $\underline{u}_{\Omega, \mathcal{S}, \mu}$ of (1.1) in Q_T^Ω in the sense of (1.26) with initial data (\mathcal{S}, μ) . Indeed, (4.35) remains valid with $\underline{u}_{\Omega, \mathcal{S}, \mu}$ replaced by $\underline{u}_{D_k, \mathcal{S}, \mu_k}$; then letting $k \rightarrow \infty$ yields $\underline{u}_{\Omega, \mathcal{S}, \mu} \leq v$ in Q_T^Ω .

In the general case, we put $\mathcal{S}^{(k)} := \mathcal{S} \cap \overline{D}_k$ and $\mu^{(k)}(E) := \mu(E \cap D_k)$ for every Borel set $E \subseteq \Omega \setminus \mathcal{S}^{(k)}$. We denote by $\underline{u}_{\Omega, \mathcal{S}^{(k)}, \mu^{(k)}}$ the minimal solution of (1.1) in Q_T^Ω with initial trace $(\mathcal{S}^{(k)}, \mu^{(k)})$ and boundary data 0. By (4.37), the sequence of minimal solutions $\{\underline{u}_{\Omega, \mathcal{S}^{(k)}, \mu^{(k)}}\}$ increases with respect to k and its limit as $k \rightarrow \infty$ is the minimal solution $\underline{u}_{\Omega, \mathcal{S}, \mu}$ of (1.1) in Q_T^Ω in the sense of (1.26) with initial data (\mathcal{S}, μ) . \square

Remark. When $f(r) = |r|^{q-1}r$ with $1 < q < 1 + 2/N$, precise expansion of $u_{\infty\delta}(x, t)$, when $t \rightarrow 0$ allows to prove uniqueness. Even when $f(r) = r \ln^\alpha(r + 1)$ with $\alpha > 2$, uniqueness is not known.

4.3 The Keller-Osserman condition does not hold

In this section we assume that (1.16) holds and (1.8) is always satisfied.

Lemma 4.9 *Assume (1.10), (1.16), (C1) and (C3) are satisfied. If u is a positive solution of (1.1) in Q_∞ which satisfies*

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx = \infty, \quad (4.38)$$

for some bounded open subset $G \subset \mathbb{R}^N$, then $u(x, t) \geq \phi_\infty(t)$.

Proof. By assumption, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n \rightarrow \infty} \int_G u(x, t_n) dx = \infty. \quad (4.39)$$

If (4.38), we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\text{diam}(G_k) = \epsilon_k \rightarrow 0$ when $k \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} \int_{G_k} u(x, t_n) dx = \infty \quad \forall k \in \mathbb{N}. \quad (4.40)$$

Furthermore there exists a unique $a \in \bigcap_k G_k$. We set

$$\int_{G_k} u(x, t_n) dx = M_{n,k}.$$

Since $\lim_{n \rightarrow \infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any k , there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x, t_{n(k)}) dx \geq m. \quad (4.41)$$

By induction, we define $n(1)$ as the smallest integer n such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer n larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any k , there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x, t_{n(k)}); \ell\} dx = m \quad (4.42)$$

and we set

$$V_k(x) = \inf\{u(x, t_{n(k)}); \ell\} \chi_{G_k}(x).$$

Let $v_k = v$ be the unique bounded solution of

$$\begin{cases} v_t - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(\cdot, 0) = V_k & \text{in } \mathbb{R}^N. \end{cases} \quad (4.43)$$

Since $v(x, 0) \leq u(x, t_{n(k)})$, it follows

$$u(x, t + t_{n(k)}) \geq v_k(x, t) \quad \forall (x, t) \in Q_\infty. \quad (4.44)$$

When $k \rightarrow \infty$, $V_k \rightarrow m\delta_a$, thus $v_k \rightarrow u_{m\delta_a}$ by Lemma 4.4. Therefore $u \geq u_{m\delta_a}$. Since m is arbitrary and $u_{m\delta_a} \rightarrow \phi_\infty$ when $m \rightarrow \infty$ by Theorem 1.3, it follows that $u \geq \phi_\infty$. \square

Lemma 4.10 *Assume (1.15), (1.16), (C1) and (C3) are satisfied. There exists no positive solution u of (1.1) in Q_∞ which satisfies (4.38) for some bounded open subset $G \subset \mathbb{R}^N$.*

Proof. If we assume that such a u exists, we proceed as in the proof of the previous lemma. Since Lemma 4.4 holds, we derive that $u \geq u_{m\delta_a}$ for any m . Since $\lim_{m \rightarrow \infty} u_{m\delta_a}(x, t) = \infty$ for all $(x, t) \in Q_\infty$, we derive a contradiction. \square

Thanks to these results, we can characterize the initial trace of positive solutions of (1.1) when the Keller-Osserman condition does not hold.

Proof of Theorem 1.13

If there exists some open subset G of \mathbb{R}^N with the property (4.38), then $u \geq \phi_\infty$ and the initial trace of u is the Borel measure ν_∞ . Next we assume that for any bounded open subset G of \mathbb{R}^N there holds

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx < \infty. \quad (4.45)$$

If $\mathcal{S}(u) \neq \emptyset$, there exists $z \in \mathbb{R}^N$ and an bounded open neighborhood G of z such that

$$\int_0^T \int_G f(u) dx dx t = \infty.$$

By (4.45), $u \in L^\infty(0, T; L^1(G)) \subset L^1(Q_T^G)$. Then, by Lemma 4.2, (4.4) holds, which contradict (4.45). Thus $\mathcal{S}(u) = \emptyset$ and $\mathcal{R}(u) = \mathbb{R}^N$. It follows from Proposition 1.9 that there exists a positive Radon measure μ such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) \quad \forall \zeta \in C_c(\mathbb{R}^N). \quad (4.46)$$

\square

Because of the lack of uniqueness from Theorem 1.6 it is difficult to give a complete characterization of admissible initial data for solutions of (1.1) under the assumptions of Theorem 1.13. However, we have the result as in Proposition 1.14.

Proof of Proposition 1.14

We first notice that $\max\{\phi_\infty(t); w_b(|x|)\}$ is a subsolution of (1.1) which is dominated by the supersolution $\phi_\infty(t) + w_b(|x|)$. The process is standard: for $\tau > 0$ we set

$$\psi(x, \tau) = \frac{1}{2} (\max\{\phi_\infty(t); w_b(|x|)\} + \phi_\infty(t) + w_b(|x|)).$$

There exists a function $u = u_\tau \in C(\overline{Q_\infty})$ solution of (1.1) in Q_∞ satisfying $u_\tau(\cdot, 0) = \psi(\cdot, \tau)$. Furthermore

$$\max\{\phi_\infty(t + \tau); w_b(|x|)\} \leq u_\tau(x, t) \leq \phi_\infty(t + \tau) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (4.47)$$

By parabolic equation regularity theory, the set $\{u_\tau\}$ is locally equicontinuous in Q_∞ . Thus there exists a subsequence $\{\tau_n\}$ and $u \in C(Q_\infty)$ such that $u_{\tau_n} \rightarrow u$ on any compact subset of Q_∞ . Clearly u is a weak, thus a strong solutions of (1.1) and it satisfies (1.27). Since any solution u with initial trace ν_∞ dominates ϕ_∞ by Lemma 4.9, it follows that ϕ_∞ is the minimal one. \square

Proof of Theorem 1.15

As in the proof of Theorem 1.13 and because of Lemma 4.10, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$ and the proof follows from Proposition 1.9. \square

Remark. Under the assumptions of Theorem 1.13, it is clear, from the proof of Proposition 3.1, that for any $0 < a < b$ and any initial data $u_0 \in C(\mathbb{R}^N)$ satisfying

$$w_a(x) \leq u_0(x) \leq w_b(x) \quad \forall x \in \mathbb{R}^N$$

there exists a solution $u \in C(\overline{Q_\infty})$ of (1.1) in Q_∞ satisfying $u(\cdot, 0) = u_0$ and

$$w_a(x) \leq u(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty.$$

We conjecture that for any positive measure μ on \mathbb{R}^N which satisfies, for some $b > 0$,

$$\int_{B_R} d\mu(x) \leq \int_{B_R} w_b(x) dx \quad \forall R > 0 \quad (4.48)$$

there exists a positive solution u of (1.1) in Q_∞ with initial trace μ . Another interesting open problem is to see if there exists local solutions in Q_T with an initial trace μ satisfying

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} d\mu(x)}{\int_{B_R} w_b(x) dx} = \infty \quad \forall b > 0. \quad (4.49)$$

References

- [1] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions* CRC Press (1992)
- [2] L. C. Evans and B. F. Knerr, *Instantaneous shrinking of the support of non-negative solutions to certain nonlinear parabolic equations and variational inequalities*, **Illinois J. Math.** **23**, 153–166 (1979).
- [3] J. Fabbri and J. R. Licois, *Boundary behavior of solution of some weakly superlinear elliptic equations*, **Adv. Nonlinear Studies** **2**, 147-176 (2002).

- [4] Keller J.B., *On solutions of $\Delta u = f(u)$* , **Comm. Pure Appl. Math.** **10**, 503-510 (1957).
- [5] M. Marcus and L. Véron, *Initial trace of positive solutions of some nonlinear parabolic equations*, **Comm. Part. Diff. Equ.** **24**, 1445-1499 (1999).
- [6] M. Marcus and L. Véron, *Initial trace of positive solutions to semilinear parabolic inequalities*, **Adv. Nonlinear Studies** **2**, 395-436 (2002).
- [7] M. Marcus and L. Véron, *Boundary trace of positive solutions of nonlinear elliptic inequalities*, **Ann. Scu. Norm. Sup. Pisa** **5**, 481-533 (2004).
- [8] M. Marcus and L. Véron, *The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption*, **Comm. Pure Appl. Math.** **LVI**, 689-731 (2003).
- [9] Osserman R., *On the inequality $\Delta u \geq f(u)$* , **Pacific J. Math.** **7**, 1641-1647 (1957).
- [10] A. Shishkov and L. Véron, *The balance between diffusion and absorption in semilinear parabolic equations*, **Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei Mat. Appl.** **18**, 59-90 (2007).
- [11] J. L. Vazquez and L. Véron *Isolated singularities of some semilinear elliptic equations*, **J. Diff. Equ.** **60**, 301-321 (1985).
- [12] L. Véron, *Weak and strong singularities of nonlinear elliptic equations*, **Proc. Symp. Pure Math.** **45**, 477-495 (1986).
- [13] L. Véron, *Boundary trace of solutions of semilinear elliptic equalities and inequalities*, **Rend. Acad. Lincei: Mat. Appl. Ser. IV****15**, 301-314 (2004).
- [14] L. Véron, *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Research Notes in Mathematics Series **353**, pp 1-388 (1996).