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Local and global properties of solutions of heat equation with superlinear absorption

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Abstract

We study the limit, when $k \rightarrow \infty$ of solutions of $u_t - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0, \infty)$ with initial data $k\delta$, when f is a positive increasing function. We prove that there exist essentially three types of possible behaviour according f^{-1} and $F^{-1/2}$ belong or not to $L^1(1, \infty)$, where $F(t) = \int_0^t f(s)ds$. We use these results for giving a general result on the existence of the initial trace and some non-uniqueness results for regular solutions with unbounded initial data.

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Key words. Heat equation; singularities; Borel measures; initial trace.

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1 Introduction

In this article we investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + uh(u) = 0 \quad (1.1)$$

in $Q_\infty := \mathbb{R}^N \times (0, \infty)$ ($N \geq 2$) where $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, nondecreasing and positive on $(0, \infty)$, vanishes at 0 and tends to infinity at infinity. As a model equation we shall consider the following, with $\alpha > 0$,

$$\partial_t u - \Delta u + u \ln^\alpha(u + 1) = 0 \quad (1.2)$$

When $h(u) = |u|^\beta$ with $\beta > 0$ much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x, t) = t^{-1/\beta} w(x\sqrt{t}) \quad (1.3)$$

The critical exponent $\beta_c = 2/N$ plays a fundamental role in the description of isolated singularities and the study of the initial trace, since if $0 < \beta < \beta_c$ there exists a self-similar solution with an isolated singularity at $(0, 0)$, while no such solution exists when $\beta \geq \beta_c$ and no solution with isolated singularities.

In the case of (1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any $k > 0$ there exists a unique $u = u_k \in C(\overline{Q_\infty} \setminus \{0\}) \cap C^{2,1}(Q_\infty)$ solution of

$$\begin{cases} u_t - \Delta u + u \ln^\alpha(u + 1) = 0 & \text{in } Q_\infty \\ u(0, \cdot) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \quad (1.4)$$

There are two critical values for α : $\alpha = 1$ and $\alpha = 2$, the explanation of which comes from the study of the two singular problems

$$\begin{cases} \phi' + \phi \ln^\alpha(\phi + 1) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty \end{cases} \quad (1.5)$$

and, for any $\epsilon > 0$,

$$\begin{cases} -\Delta \psi + \psi \ln^\alpha(\psi + 1) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty, \end{cases} \quad (1.6)$$

where $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$. When it exists, the solution ϕ_∞ of (1.5) is given implicitly by

$$\int_{\phi_\infty(t)}^{\infty} \frac{ds}{s \ln^\alpha(s + 1)} = t. \quad (1.7)$$

and such a formula is valid if and only if $\alpha > 1$. For problem (1.6) an explicit expression of the solution is not valid, but it exists if and only if the Keller-Osserman condition holds $\alpha > 2$.

Having in mind this model we study (1.1) assuming the weak singularity condition on h :

$$\int_0^1 h(t^{-N/2}) dt < \infty. \quad (1.8)$$

If (1.8) holds, for any $k > 0$, there exists a unique solution u_k of (1.1) satisfying $u_k(\cdot, 0) = k\delta_0$. Another condition on h is

$$\int_1^\infty \frac{dt}{th(t)} < \infty. \quad (1.9)$$

Under this assumption there exists a solution to

$$\begin{cases} \phi' + \phi h(\phi) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty \end{cases} \quad (1.10)$$

The last important condition on h we shall encounter is

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad (1.11)$$

where

$$F(t) = 2 \int_0^t sh(s) ds.$$

If (1.11) is satisfied, for any $\epsilon > 0$ there exists a maximal solution $\psi := \psi_\epsilon$ to

$$\begin{cases} -\Delta\psi + \psi h(\psi) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty, \end{cases} \quad (1.12)$$

The first question we consider is the study of the limit of u_k when $k \rightarrow \infty$. This question is natural since $k \mapsto u_k$ is increasing. We prove the following results.

Theorem A. *Assume h satisfies (1.8) and*

$$\int_1^\infty \frac{dt}{th(t)} = \infty. \quad (1.13)$$

Then $\lim_{k \rightarrow \infty} u_k(x, t) = \infty$.

Theorem B. *Assume h satisfies (1.8), (1.9) and*

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} = \infty, \quad (1.14)$$

Then $\lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t)$.

In the proof of the above two results we have to assume the following technical assumption, called **condition C**

C- If $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \forall \alpha > 2$, then there exists $\beta \in (1, 2]$ such that

$$\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty.$$

Theorem C. Assume h satisfies (1.8), (1.9) and (1.11). Then $\lim_{k \rightarrow \infty} u_k = u_\infty$ where u_∞ is the minimal solution of (1.1) in Q_∞ which is continuous in $\overline{Q_\infty} \setminus \{0, 0\}$, vanishes on the set $\{(x, 0) : x \neq 0\}$ and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} u(x, t) dx = \infty \quad (1.15)$$

for any $\epsilon > 0$.

In the second section we study the set \mathcal{U} of positive and locally bounded solutions of (1.1) in Q_∞ . This set differs considerably according the assumption on h . This is due to the properties of the radial solutions of the associated stationary equation

$$-\Delta w + wh(w) = 0 \quad \text{in } \mathbb{R}^N. \quad (1.16)$$

For any $a > 0$ the problem

$$\begin{cases} -w'' - \frac{N-1}{r}w' + wh(w) = 0 & \text{for } r > 0 \\ w'(0) = 0 \\ w(0) = a \end{cases} \quad (1.17)$$

admits a maximal solution w_a defined on $[0, r_a)$. The function w_a is increasing and $r_a < \infty$ if and only if h satisfies (1.11). If $r_a = \infty$ the mapping $a \mapsto w_a$ is increasing and $\lim_{r \rightarrow \infty} w_a(r) = \infty$. A striking consequence of the existence of such solutions is the following non-uniqueness result

Theorem D. Assume h satisfies (1.9) and (1.14). Then for any $u_0 \in C(\mathbb{R}^N)$ satisfying $w_a(x) \leq u_0(x) \leq w_b(x)$ for some $b > a > 0$ there exist at least two solutions $\underline{u}, \bar{u} \in C(\overline{Q_\infty})$ of (1.1) with initial value u_0 . They satisfy respectively

$$0 \leq \underline{u}(x, t) \leq \min\{w_b(x), \phi_\infty(t)\} \quad \forall (x, t) \in Q_\infty, \quad (1.18)$$

thus $\lim_{t \rightarrow \infty} \underline{u}(x, t) = 0$, uniformly with respect to x , and

$$w_a(x) \leq \bar{u}(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty, \quad (1.19)$$

thus $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$, uniformly with respect to $t \geq 0$.

We prove that, under the conditions of Theorem D, two solutions u_i ($i=1,2$) with the same initial data u_0 coincide if for any $\epsilon > 0$ there holds

$$u_1(x, t) - u_2(x, t) = o(w_\epsilon(|x|)) \quad \text{as } |x| \rightarrow \infty \quad (1.20)$$

locally uniformly with respect to $t \geq 0$.

On the contrary, if h satisfies (1.11) and $r \mapsto rh(r)$ is convex, a continuous solution is uniquely determined by the initial value u_0 .

In the last section we study the initial trace of locally bounded positive solutions of (1.1). We prove:

Theorem E. *If h satisfies condition C, (1.8) and (1.13) the initial trace is a Radon measure.*

Theorem F. *If h satisfies condition C, (1.8), (1.9) and (1.14), either the initial trace is a Radon measure, or $u \geq \phi_\infty$.*

Theorem G. *If h satisfies (1.8), (1.9) and (1.11) the initial trace is a positive outer regular Borel measure.*

The proofs are combination of methods developed in [8] for elliptic equations, stability results and Theorems A and B above. Among the consequences of the previous results we can mention.

Theorem H. *If h verifies (1.9) and (1.14) there exist infinitely many solutions u of (1.1). They all satisfy $u \geq \phi_\infty$ and ϕ_∞ is the minimal one.*

Theorem I. *If h satisfies (1.8), (1.9) and (1.11), then for any outer regular Borel measure ν on \mathbb{R}^N , there exists a positive solution u of (1.1) with initial trace ν .*

2 Isolated singularities

In order to study (1.1), we shall assume the following conditions on h

C1- h is a positive, increasing function and $h(0) = 0$.

C2- the mapping $r \mapsto f(r) = rh(r)$ is convex.

C3- If $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \forall \alpha > 2$, then there exists $\beta \in (1, 2]$ such that

$$\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty.$$

It is well-known (e.g. [14]) that, if for $k > 0$,

$$I = \int_0^1 \int_{B_R} kE(x, t)h(kE(x, t))dxdt < \infty \quad (2.1)$$

for any $R > 0$, then there exists a unique solution $u = u_k$ to (1.1) satisfying initial condition

$$u_k(\cdot, 0) = k\delta_0$$

in the sense of distributions in \mathbb{R}^N . Furthermore the mapping $k \mapsto u_k$ is increasing. The condition (2.1) can be reduced to a weaker form as follows

$$\int_0^1 h(t^{-N/2})dt < \infty. \quad (2.2)$$

Proposition 2.1 *If (2.2) is satisfied then (2.1) is fulfilled.*

Proof. We have

$$I = kC^* \int_0^1 \int_{B_R} t^{-N/2} e^{-|x|^2/4t} h(kC^* t^{-N/2} e^{-|x|^2/4t}) dx dt$$

where $C^* = (4\pi)^{-N/2}$. Put $r = |x|$ then $dx = r^{N-1} dr$, and

$$I = kC^* \int_0^1 t^{-N/2} \int_0^R e^{-r^2/4t} h(kC^* t^{-N/2} e^{-r^2/4t}) r^{N-1} dr dt.$$

We next put $\rho = \frac{r}{\sqrt{t}}$, then $r^{N-1} dr = \rho^{N-1} t^{N/2} d\rho$, and

$$I = kC^* \int_0^1 \int_0^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

We set

$$I_1 = kC^* \int_0^1 \int_0^1 e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt,$$

$$I_2 = kC^* \int_0^1 \int_1^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

Since $e^{-\rho^2/4} \rho^{N-1}$ is bounded in $[0, \infty)$, then there exists a constant c_1 depending only on k such that

$$I_1 < c_1 \int_0^1 \int_0^1 h(kC^* t^{-N/2}) d\rho dt = c_1 \int_0^1 h(kC^* t^{-N/2}) dt < \infty$$

We next show that under the condition (2.2), $I_2 < \infty$. In order to do that we change to a new variable τ such that $t^{-N/2} e^{-\rho^2/4} = \tau^{-N/2}$. Then $t = \tau e^{-\frac{\rho^2}{2N}}$ and $dt = e^{-\frac{\rho^2}{2N}} d\tau$. Therefore

$$I_2 \leq kC^* \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} \left(\int_0^{e^{\rho^2/2N}} h(kC^* \tau^{-N/2}) d\tau \right) d\rho \quad (2.3)$$

Since h satisfies (2.2), there exists $\epsilon > 0$ (depending only on k) such that

$$\int_0^\epsilon h(kC^*\tau^{-N/2})d\tau$$

take a finite value, denoted by c_2 . Hence

$$\int_0^{e^{\rho^2/2N}} h(kC^*\tau^{-N/2})d\tau \leq c_2 + h(kC^*\epsilon^{-N/2})(e^{\frac{\rho^2}{2N}} - \epsilon) \quad (2.4)$$

Inserting (2.4) into the right hand side of (2.3), we obtain

$$I_2 \leq c_3 \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} d\rho + c_4 \int_1^\infty e^{-\frac{\rho^2}{4}} \rho^{N-1} d\rho < \infty$$

where $c_3 = kC^*c_2$ and $c_4 = kC^*h(kC^*\epsilon^{-N/2})$. Thus, $I = I_1 + I_2 < \infty$. \square

The functions which satisfy the following ODE are particular solutions of (1.1)

$$\phi_t + \phi h(\phi) = 0 \quad \text{in } (0, \infty) \quad (2.5)$$

For $a > 0$, we denote by ϕ_a the solution of (2.5) with initial data $\phi(0) = a$. This solution is given by the formula

$$t = \int_{\phi(t)}^a \frac{ds}{sh(s)} < \infty. \quad (2.6)$$

If

$$J = \int_a^\infty \frac{ds}{sh(s)} = \infty. \quad (2.7)$$

then $\lim_{a \rightarrow \infty} \phi_a(t) = \infty$. While, if

$$J = \int_a^\infty \frac{ds}{sh(s)} < \infty. \quad (2.8)$$

there exists a maximal solution ϕ_∞ given explicitly by

$$t = \int_{\phi_\infty(t)}^\infty \frac{ds}{sh(s)} < \infty.$$

Lemma 2.2 *If $J = \infty$ then $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0$, $\forall \alpha > 1$.
If $J < \infty$ then $\limsup_{r \rightarrow \infty} h(r)/\ln^\alpha r = \infty$, $\forall 0 < \alpha \leq 1$.*

Proof. Set $s = e^{r^{-1}}$ then

$$J = \int_0^b \frac{dr}{r^{\alpha+1}h(e^{r^{-1}})}$$

where $b = (\ln a)^{-1}$.

Case 1: $J = \infty$. Suppose that there exist $\alpha > 1$ such that

$$\liminf_{s \rightarrow \infty} \frac{h(s)}{\ln^\alpha s} > 0,$$

equivalently,

$$\liminf_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) > 0$$

then there exists $l > 0$ and $r_0 > 0$ such that

$$h(e^{r^{-1}}) > lr^{-\alpha}, \quad \forall r \in (0, r_0).$$

Hence we derive the following contradiction

$$J < \frac{\alpha}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^b \frac{dr}{r^{\alpha+1}h(e^{r^{-1}})} < \infty.$$

Case 2: $J < \infty$. Suppose that there exist $\alpha \in (0, 1]$ such that

$$\limsup_{s \rightarrow \infty} \frac{h(s)}{\ln^\alpha s} < \infty,$$

equivalently,

$$\limsup_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) < \infty$$

then there exists $l > 0$ and $r_0 > 0$ such that

$$h(e^{r^{-1}}) < lr^{-\alpha}, \quad \forall r \in (0, r_0).$$

Hence

$$J > \frac{\alpha}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^b \frac{dr}{r^{\alpha+1}h(e^{r^{-1}})} = \infty,$$

contradiction. □

Theorem 2.3 *Assume (2.7) is satisfied. Then*

$$\lim_{k \rightarrow \infty} u_k(x, t) = \infty \quad \forall (x, t) \in Q_\infty.$$

Proof. Since $J = \infty$, by Lemma 2.2, $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0$ for all $\alpha > 1$. Thus

$$\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \quad \forall \alpha > 2.$$

By (C3), there exists $\beta \in (1, 2]$ such that $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r = 0 < \infty$. Hence there exist $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r, \quad \forall r \in (r_0, \infty). \quad (2.9)$$

Step 1. Let $k > 0$, we claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau, \quad t \in (0, 1) \quad (2.10)$$

where $\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$ and $C^* = (4\pi)^{-N/2}$. Set $r = kC^* \tau^{-N/2}$ then (2.9) becomes

$$h(kC^* \tau^{-N/2}) < M [\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta, \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{\frac{2}{N}} e^{-\frac{2}{N} r_0^{-1}}$. We put $a_1 = \ln k$, $a_2 = \frac{N}{2} \ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1} (a_1^\beta + a_2^\beta)$$

in order to obtain

$$\begin{aligned} h(kC^* \tau^{-N/2}) &< M [\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M [(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})], \quad \forall \tau \in (0, \tau_0) \end{aligned} \quad (2.11)$$

Integrating over $[0, t]$ yields to (2.10).

Step 2. It follows from (2.11) that (2.2) is fulfilled, hence there exists the unique solution of (1.1) in Q_∞ with initial data $k\delta_0$. By the maximum principle, $u_k(x, t) \leq kE(x, t)$ for any $(x, t) \in Q_\infty$, which implies $u_k(x, t) \leq kC^* t^{-N/2}$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^* t^{-N/2}) \geq 0.$$

If we set $v_k(x, t) = e^{\theta_k(t)} u_k(x, t)$, we obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)} (\partial_t u_k - \Delta u_k + u_k h(kC^* t^{-N/2})) \geq 0,$$

and $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$. By the maximum principle, there holds

$$v_k(x, t) \geq kC^* t^{-N/2} e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^* t^{-N/2} e^{-\theta_k(t) - |x|^2/4t}. \quad (2.12)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t (\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.13)$$

where $c_1 = \exp(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau)$ and $M_\beta = M2^{\beta-1}$. Inserting (2.13) into the right hand side of (2.12), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1.$$

Put $\underline{U} := \lim_{k \rightarrow \infty} u_k$, then

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1, \forall k > 0.$$

Let $\{t_n\} \subset (0, 1]$ be a sequence converging to 0. We choose $k = k_n = \exp\left(\left(2M_\beta t_n\right)^{\frac{1}{1-\beta}}\right)$, then $\ln k_n - M_\beta t_n \ln^\gamma k_n = \frac{1}{2} \ln t_n$. Next we restrict x in order to have

$$\ln k_n - M_\beta t_n \ln^\gamma k_n - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}} t_n^{\frac{\beta-2}{2(\beta-1)}}.$$

Therefore, since $1 < \beta \leq 2$,

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-1/2}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} .

We pick some point x_0 in \mathbb{R}^N (resp. B_{r_2}) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, by comparison principle, it follows that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \rightarrow \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t), \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of x and therefore it is a solution of

$$\begin{cases} \phi' + \phi h(\phi) = 0 & \text{on } (0, \infty) \\ \lim_{t \rightarrow 0} \phi(t) = \infty. \end{cases}$$

By (2.7), $\underline{U}(x, t) = \infty$ for all $(x, t) \in Q_\infty$, which ends the proof. \square

Proposition 2.4 *Assume (2.8) is satisfied. For any $k > 0$, there holds*

$$u_k(x, t) \leq \phi_\infty(t), \quad \forall (x, t) \in Q_\infty.$$

Proof. For any small $\epsilon > 0$, put $\phi_{\infty\epsilon}(t) = \phi_\infty(t - \epsilon)$, $t \in [\epsilon, \infty)$ then $\phi_{\infty\epsilon}$ is a solution of (1.1) in (ϵ, ∞) , which dominates u_k on $\mathbb{R}^N \times \epsilon$ for any $k > 0$. By comparison principle, $u_k(x, t) \leq \phi_{\infty\epsilon}(t)$ in $\mathbb{R}^N \times [\epsilon, \infty)$. Letting $\epsilon \rightarrow 0$ yields to the claim. \square

A necessary and sufficient condition for the existence of a maximal solution to the stationary equation

$$-\Delta w + wh(w) = 0$$

in a bounded domain Ω is the Keller-Osserman condition ([4], [9])

$$K = \int_a^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad (2.14)$$

where $F(s) = \int_0^s \tau h(\tau) d\tau$. If (2.14) holds, then (2.8) is fulfilled. The Keller-Osserman condition can be replaced by another condition, which owes to the following result

Lemma 2.5 *Set*

$$L = \int_a^\infty \frac{ds}{s\sqrt{h(s)}}.$$

Then $K < \infty$ if and only if $L < \infty$.

Proof. In order to obtain the assertion, it is sufficient to show that there exists two positive constants c_1 and c_2 such that

$$c_1 s^2 h\left(\frac{s}{2}\right) \leq F(s) \leq c_2 s^2 h(s) \quad \forall s \geq a. \quad (2.15)$$

The right-hand side estimate in (2.15) follows from the monotone property of h (with $c_2 = 1/2$). The assumption of convexity of $s \mapsto f(s) := sh(s)$ in $(0, \infty)$ implies

$$f(s) \geq f\left(\frac{s}{2}\right) + \frac{s}{2} f'\left(\frac{s}{2}\right) \quad \forall s > 0.$$

Define $\varphi(s) = \int_0^s f(\tau) d\tau - sf\left(\frac{s}{2}\right)$, then $\varphi'(s) = f(s) - f\left(\frac{s}{2}\right) - \frac{s}{2} f'\left(\frac{s}{2}\right) > 0$. Hence $\varphi(s) > \varphi(0) = 0$, which is the left-hand side estimate in (2.15) (with $c_1 = 1/2$). \square

By using the same argument as in the proof of the Lemma 2.2 and thank to the Lemma 2.5, we obtain the following lemma

Lemma 2.6 *If*

$$K = \infty \quad (2.16)$$

then $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha(r) = 0$ *for all* $\alpha > 2$.

If

$$K < \infty \quad (2.17)$$

then $\limsup_{r \rightarrow 0} h(r)/\ln^\alpha(r) = \infty$ *for all* $0 < \alpha \leq 2$.

For the case *time-type absorption*, we have the following result

Theorem 2.7 *Assume (2.8) and (2.16) are satisfied. Then*

$$\underline{U} = \lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t) \quad \forall (x, t) \in Q_\infty.$$

Proof. Since $K = \infty$, by Lemma 2.6, $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \forall \alpha > 2$. By (C3), there exists $\beta \in (1, 2]$ such that $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty$. Hence there exists $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r, \quad \forall r \in (r_0, \infty). \quad (2.18)$$

Step 1. For any $k > 0$ we set

$$\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$$

where $C^* = (4\pi)^{-\frac{N}{2}}$. We claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau, \quad t \in (0, 1). \quad (2.19)$$

If we define τ by $r = kC^* \tau^{-N/2}$, (2.18) becomes

$$h(kC^* \tau^{-\frac{N}{2}}) < M [\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta, \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{\frac{2}{N}} e^{-\frac{2}{N} r_0^{-1}}$. We put $a_1 = \ln k$, $a_2 = \frac{N}{2} \ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1} (a_1^\beta + a_2^\beta)$$

in order to obtain

$$\begin{aligned} h(kC^* \tau^{-N/2}) &< M [\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M [(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})] \end{aligned} \quad (2.20)$$

Integrating over $[0, t]$, we obtain (2.19).

Step 2. It follows from (2.20) that (2.2) is fulfilled, hence there exists the unique solution of (1.1) in Q_∞ with initial trace $k\delta_0$. By maximum principle, $u_k(x, t) \leq kE(x, t)$ for any $(x, t) \in Q_\infty$, which implies that $u_k(x, t) \leq kC^*t^{-N/2}$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \geq 0.$$

We set $v_k(x, t) = e^{\theta_k(t)} u_k(x, t)$ and obtain

$$\partial_t v_k - \Delta v_k = e^{\Theta(t)} (\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})) \geq 0,$$

with $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$. By maximum principle, it follows

$$v_k(x, t) \geq kC^*t^{-N/2}e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^*t^{-N/2}e^{-\theta_k(t)-|x|^2/4t}. \quad (2.21)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t (\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.22)$$

where $c_1 = \exp(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau)$ and $M_\beta = M2^{\beta-1}$. Inserting (2.13) into the right-hand side of (2.12), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1.$$

Since $\underline{U} \geq u_k$, then

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1, \quad \forall k > 0.$$

Let $\{t_n\} \subset (0, 1]$ be a sequence converging to 0. We choose $k = k_n = \exp((2M_\beta t_n)^{\frac{1}{1-\beta}})$, equivalently $\ln k_n - M_\beta t_n \ln^\beta k_n = \frac{1}{2} \ln t_n$. Next we restrict $|x|$ in order

$$\ln k_n - M_\beta t_n \ln^\beta k_n - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln t_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq r_\beta t_n^{\frac{\beta-2}{2(\beta-1)}},$$

where $r_\beta = 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}}$. Because $1 < \beta \leq 2$, it follows

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty,$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-\frac{1}{2}}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} .

We pick some point x_0 in \mathbb{R}^N (resp. B_{r_2}) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, it follows, by comparison principle, that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \rightarrow \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t), \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of x and therefore it is a solution of

$$\begin{cases} \phi' + \phi h(\phi) = 0 \text{ on } (0, \infty) \\ \lim_{t \rightarrow 0} \phi(t) = \infty \end{cases}$$

Thus $\underline{U}(x, t) = \phi_\infty$, $\forall (x, t) \in Q_\infty$. \square

Proposition 2.8 *Assume (2.2) and the Keller-Osserman condition are satisfied. Then there holds*

$$\underline{U}(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty$$

where Φ is a solution to the problem

$$\begin{cases} -\Phi'' + \Phi h(\Phi) = 0 \text{ in } (0, \infty) \\ \lim_{s \rightarrow 0} \Phi(s) = \infty \end{cases}$$

Proof. Step 1: Upper estimate. Since $s \mapsto sh(s)$ satisfies (2.14) condition, for any $R > 0$, there exists a solution w_R to the problem

$$\begin{cases} -\Delta w_R + w_R h(w_R) = 0 \text{ in } B_R \\ \lim_{|x| \rightarrow R} w_R(x) = \infty \end{cases}$$

Let $x_0 \neq 0$ arbitrary be in \mathbb{R}^N . Set $\mathbb{E} = \{\vec{e} : |\vec{e}| = 1\}$ and take $\vec{e} \in \mathbb{E}$. Put $x_{\vec{e}} = |x_0| \vec{e}$ and for $n > |x_0|$ put $a_n = n\vec{e}$. Denote by $\mathbb{H}_{\vec{e}}$ the open half-space generated by \vec{e} , then $x_{\vec{e}}, a_n \in \mathbb{H}_{\vec{e}}$. Take R such that $n - |x_0| < R < n$. We put $W_{\vec{e}, n, R}(x) = w_R(x - a_n)$, then $W_{\vec{e}, n, R}$ is a solution of (1.1) in $B_R(a_n)$ and satisfies the blow-up condition on the boundary $\lim_{|x - a_n| \rightarrow R} W_{\vec{e}, n, R}(x) = \infty$. By the maximum principle,

$$u_k(x, t) \leq W_{\vec{e}, n, R}(x) \text{ in } B_R(a_n) \times (0, \infty). \quad (2.23)$$

The sequence $\{W_{\vec{e}, n, R}\}_R$ is decreasing and is bounded from below, then there exists $W_{\vec{e}, n} := \lim_{R \rightarrow n} W_{\vec{e}, n, R}$. Letting successively $R \rightarrow n$, $k \rightarrow \infty$ in (2.23) yields to

$$\underline{U}(x, t) \leq W_{\vec{e}, n}(x) \text{ in } B_n(a_n) \times (0, \infty). \quad (2.24)$$

The sequence $(W_{\vec{e},n})_n$ is also decreasing and is bounded from below, then there exists $W_{\vec{e},\infty} := \lim_{n \rightarrow \infty} W_{\vec{e},n}$. Letting $n \rightarrow \infty$ in (2.24) yields to

$$\underline{U}(x, t) \leq W_{\vec{e},\infty}(x) \text{ in } \mathbb{H}_{\vec{e}} \times (0, \infty). \quad (2.25)$$

In particular,

$$\underline{U}(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}). \quad (2.26)$$

Since \underline{U} is radial, it follows that

$$\underline{U}(x_0, t) = \underline{U}(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}).$$

Because \vec{e} is arbitrary in \mathbb{E} , we obtain that

$$\underline{U}(x_0, t) \leq W(x_0),$$

where $W(x) := \inf_{\vec{e} \in \mathbb{E}} W_{\vec{e},\infty}(|x|\vec{e})$. Note that W is radially symmetric, finite in \mathbb{R}_*^N and satisfies $\lim_{|x| \rightarrow 0} W(x) = \infty$.

Step 2: End of the proof. We claim that

$$W(x) = \Phi(|x|) \quad \forall x \neq 0. \quad (2.27)$$

Let $\vec{e} \in \mathbb{E}$, n be large enough and $R < n$. By the maximum principle, there holds $\Phi(|x|) \leq W_{\vec{e},n,R}(x)$ for all $x \in \mathbb{H}_{\vec{e}}$. Letting successively $R \rightarrow n$, $n \rightarrow \infty$ yields to $\Phi(|x|) \leq W_{\vec{e},\infty}(x)$, which implies that

$$\Phi(|x|) \leq W(x). \quad (2.28)$$

Next for any $\epsilon > 0$, we set $\Phi_\epsilon(s) = \Phi(s - \epsilon)$, $s > \epsilon$ and $\tilde{W}_\epsilon(x) = \Phi_\epsilon(|x|)$ in B_ϵ^c . Since $\Phi'_\epsilon \leq 0$, W_ϵ is a supersolution, which dominates W on ∂B_ϵ . By the maximum principle, $W \leq \tilde{W}_\epsilon$ in B_ϵ^c . Letting $\epsilon \rightarrow \infty$ yields to

$$W(x) \leq \Phi(|x|) \quad \forall x \neq 0. \quad (2.29)$$

Combining (2.28) and (2.29) leads to (2.27), which implies $\underline{U}(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty \setminus (\{0\} \times (0, \infty))$. Since $\lim_{s \rightarrow 0} \Phi(s) = \infty$, the conclusion follows. \square

Remark 1. In the proof, we have used the fact that $\Phi'_\epsilon \leq 0$. Indeed, if there exists $s_0 > 0$ such that $\Phi'(s_0) \geq 0$, then by the convexity property,

$$\Phi(s) \geq \Phi(s_0) + (s - s_0)\Phi'(s_0) \quad \forall s > s_0.$$

Hence $\lim_{s \rightarrow \infty} \Phi(s) = \infty$, which contradicts with (2.28).

Remark 2. Combining Proposition 2.4 and Proposition 2.8 yields to

$$\underline{U}(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\}, \quad \forall (x, t) \in Q_\infty. \quad (2.30)$$

As a consequence, the function \underline{U} has the following properties:

- (i) It is positive in Q_∞ , belongs to $C(\overline{Q} \setminus \{(0, 0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{0\}$.
- (ii) It satisfies (1.1) and

$$\lim_{t \rightarrow 0} \int_{B_\sigma} \underline{U}(x, t) dx = \infty, \quad \forall \sigma > 0. \quad (2.31)$$

In the sense of initial trace, \underline{U} has initial trace $\nu_0 := (\{0\}, 0)$ (here $\{0\}$ is the singular part and the Radon measure on $\mathbb{R}^N \setminus \{0\}$ is the zero measure).

By a simple adaptation of the proof of Proposition 2.4 and Proposition 2.8 it is possible to extend (2.30) to any positive solution vanishing on $\mathbb{R}^N \times \{0\} \setminus \{0\}$.

Proposition 2.9 *Assume (2.2) and the Keller-Osserman condition are satisfied. Then any positive solution u of (1.1) which $C(\overline{Q} \setminus \{(0, 0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{0\}$ satisfies*

$$u(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\}, \quad \forall (x, t) \in Q_\infty. \quad (2.32)$$

Proof. Estimate (2.23) is valid with u_k replaced by u . The remaining of the proof of Proposition 2.8 is similar and yields to the first estimate

$$u(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty.$$

For the second estimate, due to the convexity, for any $R, \tau > 0$, $(x, t) \mapsto \phi_\infty(t - \tau) + w_R(x)$ is a supersolution of (1.1) in $B_R \times (\tau, \infty)$. It dominates u on the parabolic boundary, thus in the domain itself by the comparison principle. Since $h(r) > 0$ if $r > 0$, $\lim_{R \rightarrow \infty} w_R = 0$. Therefore

$$u(x, t) \leq \phi_\infty(t) = \lim_{\tau \rightarrow 0} \lim_{R \rightarrow \infty} (\phi_\infty(t - \tau) + w_R(x)) \quad \forall (x, t) \in Q_\infty.$$

This implies (2.32). □

It is also possible to construct a maximal solution of (1.1) in Q_∞ which satisfies (i). For $\ell > 0$ and $\epsilon > 0$, let $u := U_{\epsilon, \ell}$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + uh(u) = 0 & \text{in } Q_\infty \\ u(x, 0) = \ell \chi_{B_\epsilon} & \text{in } \mathbb{R}^N. \end{cases}$$

Lemma 2.10 *For any $\tau > 0$ and $\epsilon > 0$, there exist $\ell > 0$ and $m(\tau, \epsilon) > 0$ such that any positive solution u of (1.1) which verifies (i) satisfies*

$$u(x, t) \leq U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon) \quad \forall (x, t) \in Q_\infty, t \geq \tau. \quad (2.33)$$

Furthermore

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = 0 \quad \forall \epsilon > 0. \quad (2.34)$$

Finally

$$\bar{U}(x, t) = \lim_{\tau \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\ell \rightarrow \infty} (U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon)) \quad (2.35)$$

is the maximal solution.

Proof. We set $\ell = \phi_\infty(\tau)$, then $u(x, \tau) \leq \ell$ for any $x \in \mathbb{R}^N$. Let $W := W_{\epsilon/2}$ be the solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t W - \Delta W + Wh(W) = 0 & \text{in } B_{\epsilon/2}^c \times (0, \infty) \\ W(x, 0) = 0 & \text{in } B_{\epsilon/2}^c \\ W(x, t) = \phi_\infty(t) & \text{in } \partial B_{\epsilon/2}^c \times (0, \infty) \end{cases} \quad (2.36)$$

and put $m(\tau, \epsilon) := \max\{W_{\epsilon/2}(x, \delta) : |x| > \epsilon, 0 < \delta \leq \tau\}$. It is clear to see that

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = W_{\epsilon/2}(x, 0) = 0. \quad (2.37)$$

From the fact that $u(x, 0) = 0$ in $B_{\epsilon/2}^c$, $u(x, t) \leq \phi_\infty(t)$ in $\partial B_{\epsilon/2}^c \times (0, \infty)$ and the maximum principle, it follows that $u(x, t) \leq W_{\epsilon/2}(x, t)$ in $B_{\epsilon/2}^c \times (0, \infty)$.

Next we compare $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$ with u in $\mathbb{R}^N \times (\tau, \infty)$. The function $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$ is a supersolution of (1.1) in $\mathbb{R}^N \times (\tau, \infty)$. If $x \in B_\epsilon$, $U_{\epsilon, \ell}(x, 0) = \ell \geq u(x, \tau)$, which implies $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$. If $x \in B_\epsilon^c$, $m(\tau, \epsilon) \geq W_{\epsilon/2}(x, \tau) \geq u(x, \tau)$, hence $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$. So we always have $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$ for any $x \in \mathbb{R}^N$. Applying maximum principle yields to $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon) \geq u$ in $\mathbb{R}^N \times (\tau, \infty)$. Finally, the function \bar{U} defined by (2.35) is the maximal solution because $U_{\epsilon, \ell}(x, t - \tau) \rightarrow U_{\epsilon, \ell}(x, t)$ as $\tau \rightarrow 0$ and $U_{\epsilon, \ell} \uparrow U_{\epsilon, \infty}$ when $\ell \rightarrow \infty$ and $U_{\epsilon, \infty} \downarrow \bar{U}$ when $\epsilon \rightarrow 0$. \square

3 About uniqueness

Proposition 3.1 *Assume (2.16) holds. For any $a > 0$, there exists a unique positive function $w := w_a \in C^2([0, \infty))$ to the problem*

$$\begin{cases} -w'' - \frac{N-1}{r}w' + wh(w) = 0 & \text{in } \mathbb{R}_+ \\ w'(0) = 0 \\ w(0) = a \end{cases} \quad (3.1)$$

Proof. A solution of (3.1) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N} \int_0^s t^{N-1} w(t) h(w) dt ds \quad (3.2)$$

Existence follows from Picard-Lipschitz fixed point theorem. The function is increasing and defined on a maximal interval $[0, r_a)$. By a result of Vazquez and Veron [11] $r_a = \infty$, thus the solution is global. Uniqueness on $[0, r_a)$ follows always from local uniqueness. The function $r \mapsto w(r)$ is increasing and

$$\begin{aligned} w'(r) &\geq \frac{ah(a)}{N}r, \\ w(r) &\geq a + \frac{ah(a)}{2N}r^2 \end{aligned}$$

for all $r > 0$ □

Proposition 3.2 *Assume (2.16) holds. For any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.3)$$

for some $0 < a < b$, there exists a positive function $\bar{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ solution of (1.1) in Q_∞ and satisfying $\bar{u}(x, 0) = u_0$. Furthermore

$$w_a(|x|) \leq u(x, t) \leq w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (3.4)$$

Proof. Clearly w_a and w_b are ordered solutions of (1.1). We denote by u_n the solution of the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta u_n + u_n h(u_n) = 0 & \text{in } Q_n = B_n \times (0, \infty) \\ u_n(x, 0) = u_0(x) & \text{in } B_n \\ u_n(x, t) = (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0, \infty). \end{cases} \quad (3.5)$$

By the maximum principle, u_n satisfies (3.4) in Q_n . Using locally parabolic equations regularity theory, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q_\infty}$. Using a diagonal sequence, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q_\infty}$ to some weak solution $\bar{u} \in C(\overline{Q_\infty})$ which satisfies $\bar{u}(\cdot, 0) = u_0$. By standard method, \bar{u} is a strong solution (at least $C^{2,1}(Q_\infty)$). □

Proposition 3.3 *Assume (2.16) and (2.8) hold. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.6)$$

for some $0 < a < b$, there exists a positive function $\underline{u} \in C(\overline{Q_\infty})$ solution of (1.1) in Q_∞ satisfying $\underline{u}(x, 0) = u_0$ and

$$\underline{u}(x, t) \leq \min\{\phi_\infty(t), w_b(|x|)\} \quad \forall (x, t) \in Q_\infty. \quad (3.7)$$

Proof. For any $R > 0$, let u_R be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R + u_R h(u_R) = 0 & \text{in } Q_\infty \\ u_R(x, 0) = u_0(x) \chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The solution which is constructed is dominated by the solution of the heat equation with the same initial data. Thus

$$u_R(x, t) \leq (4\pi t)^{-N/2} \int_{B_R} e^{-|x-y|^2/4t} u_0(x) dy \quad \forall (x, t) \in Q_\infty. \quad (3.8)$$

and $\lim_{|x| \rightarrow \infty} u_R(x, t) = 0$ uniformly with respect to t . The function ϕ_∞ and w_b are solution of (1.1) in Q_∞ , which dominate u_R at $t = 0$. By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x, t) \quad \forall (x, t) \in Q_\infty. \quad (3.9)$$

The fact that the mapping $R \mapsto u_R$ is increasing and (3.9) imply that there exists $\underline{u} := \lim_{R \rightarrow \infty} u_R$ which satisfies $\underline{u}(x, 0) = u_0(x)$ in \mathbb{R}^N . Letting $R \rightarrow \infty$ in (3.9) yields to (3.7). \square

Corollary 3.4 *Assume (2.16) and (2.8) hold. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N, \quad (3.10)$$

there exists two solutions $u \in C(\overline{Q_\infty})$ of (1.1) satisfying $u(\cdot, 0) = u_0$.

Proof. Combining Proposition 3.2 and Proposition 3.3 we see that there exists two solutions \underline{u} and \bar{u} with the same initial data u_0 which are ordered and different since $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$ and $\lim_{|x| \rightarrow \infty} \underline{u}(x, t) \leq \phi_\infty(t) < \infty$ for all $t > 0$. \square

Theorem 3.5 *Assume (2.16) holds. Let $u, \tilde{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ be two positive solutions of (1.1) with initial data u_0 . If for any $\epsilon > 0$,*

$$u(x, t) - \tilde{u}(x, t) = o(w_\epsilon(|x|)) \text{ as } x \rightarrow \infty \quad (3.11)$$

locally uniformly with respect to $t \geq 0$, then $u = \tilde{u}$.

Proof. Step 1. There always holds

$$(ah(a) - bh(b)) \text{sign}(a - b) \geq |a - b| h(|a - b|) \quad \forall a, b > 0. \quad (3.12)$$

In fact, since h is increasing and assuming $a > b$,

$$\begin{aligned} ah(a) - bh(b) &= (a - b)h(a) + b(h(a) - h(b)) \\ &\geq (a - b)h(a) \\ &\geq (a - b)h(a - b) \end{aligned}$$

Step 2. End of the proof. By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \leq [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})] \text{sign}(u - \tilde{u}),$$

therefore,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + |u - \tilde{u}| h(|u - \tilde{u}|) \leq 0. \quad (3.13)$$

Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any $R \geq R_\epsilon$,

$$0 \leq |u - \tilde{u}|(x, t) \leq w_\epsilon(|x|) \quad \forall (x, t) \in B_R^c \times [0, 1]. \quad (3.14)$$

Since w_ϵ is a positive solution of (1.1) which dominates $|u - \tilde{u}|$ on $\partial B_R \times [0, 1]$ and at $t = 0$, it follows that $|u - \tilde{u}| \leq w_\epsilon$ in Q_R . Letting $R \rightarrow \infty$ yields to $|u - \tilde{u}| \leq w_\epsilon$ in Q_∞ . Letting $\epsilon \rightarrow 0$ and since $\lim_{\epsilon \rightarrow 0} w_\epsilon(|x|) = 0$ for any $x \in \mathbb{R}^N$, we derive $|u - \tilde{u}| = 0$, thus $u = u^*$ for $0 \leq t \leq 1$. Iterating yields that equality holds in Q_∞ . Similarly $\tilde{u} = u^*$ and the claim follows. \square

4 Initial trace

Let Ω is an arbitrary open domain in \mathbb{R}^N , we denote by $\mathfrak{M}(\Omega)$ (resp. $\mathfrak{M}^b(\Omega)$) the set of Radon measures in Ω (resp. bounded Radon measures), and by $\mathfrak{M}_+(\Omega)$ (resp. $\mathfrak{M}_+^b(\Omega)$) its positive cone. We set $Q_T^\Omega = \Omega \times (0, T)$.

4.1 The regular part of the initial trace

In this section we only assume that f is a continuous nonnegative function defined on \mathbb{R}_+ and that u is a $C^{2,1}$ positive solution of

$$u_t - \Delta u + f(u) = 0 \quad (4.1)$$

in Q_T^Ω .

Lemma 4.1 *Assume G is a bounded C^2 domain in \mathbb{R}^N , $Q_T^{\overline{G}} := \overline{G} \times (0, T]$ and let $u \in C^{2,1}(Q_T^{\overline{G}})$ be a positive solution of (4.1) in $Q_T^{\overline{G}}$ such that $f(u) \in L^1(Q_T^{\overline{G}})$. Then $u \in L^\infty(0, T; L^1(G'))$ for any domain $G' \subset \overline{G'} \subset G$ and there exists a positive Radon measure μ_G on G such that*

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G \zeta(x) d\mu_G(x) \quad \forall \zeta \in C_0(G). \quad (4.2)$$

Proof. Let $\phi := \phi_G$ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ with corresponding eigenvalue λ_G . We assume $\phi > 0$. Then

$$\frac{d}{dt} \int_G u \phi dx + \lambda_G \int_G u \phi dx + \int_G f(u) \phi dx + \int_{\partial G} u \phi_{\mathbf{n}} dS = 0$$

Since $\phi_{\mathbf{n}} < 0$, the function

$$t \mapsto e^{\lambda_G t} \int_G u(x, t) \phi(x) dx - \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

is increasing and

$$\int_G u(x, t) \phi(x) dx \leq e^{\lambda_G(T-t)} \int_G u(x, T) \phi(x) dx + e^{-\lambda_G t} \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

for $0 < t \leq T$. Thus $u \in L^\infty(0, T; L^1(G'))$ for any strict domain G' of G . If $\zeta \in C_0^2(G)$, there holds

$$\frac{d}{dt} \left(\int_G u(x, t) \zeta(x) dx - \int_t^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds \right) = 0. \quad (4.3)$$

Consequently

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G u(x, T) \zeta(x) dx + \int_0^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds. \quad (4.4)$$

This implies that $u(\cdot, t)$ admits a limit in $\mathcal{D}'(G)$ and this limit is a positive distribution, then a measure. Therefore (4.2) follows by density. \square

Corollary 4.2 *Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (4.1) in Q_T^Ω . The set $\mathcal{R}(u)$ of the points $z \in \Omega$ such that there exists an open ball $B_r(z)$ such that $f(u) \in L^1 Q_T^{B_r(z)}$ is an open subset of Ω . Furthermore there exists a positive Radon measure $\mu := \mu(u)$ on $\mathcal{R}(u)$ such that*

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x) \quad \forall \zeta \in C_0(\mathcal{R}(u)). \quad (4.5)$$

Proof. It is clear that $\mathcal{R}(u)$ is an open subset of Ω . If G is a strict subdomain of Ω , i.e. $\overline{G} \subset \Omega$, there exists a finite number of points z_j ($j = 1, \dots, k$) and $r'_j > r_j > 0$ such that $f(u) \in L^1(Q_T^{B_{r'_j}(z_j)})$ and $\overline{G} \subset \cup_j B_{r_j}(z_j)$. Let $\mu_j = \mu_{B_{r_j}(z_j)}$ the measure defined in Lemma 4.1. If $\zeta \in C_0(G)$ there exists a partition η_j of unity relative to the $B_{r_j}(z_j)$ such that $\eta_j \in C_0^\infty(B_{r_j}(z_j))$ and $\zeta = \sum_j \eta_j \zeta$. Since

$$\lim_{t \rightarrow 0} \int u(x, t) (\eta_j \zeta)(x) dx = \int (\eta_j \zeta)(x) d\mu_j(x) \quad \forall j = 1, \dots, k,$$

relation (4.5). Notice also that $u \in L^\infty(0, T; L^1(G))$ for any $G \subset \overline{G} \subset \mathcal{R}(u)$. \square

Definition 4.3 *The set $\mathcal{R}(u)$ is called the regular set of the initial trace of u and the measure $\mu(u)$ the regular part of the initial trace. The set $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ is the singular part of the initial trace of u . It is relatively closed in Ω .*

The main problem is to analyse what does happen on the singular set $\mathcal{S}(u)$.

4.2 The Keller-Osserman condition holds

If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (1.1) is based upon a dichotomy in the behaviour of those solutions near $t = 0$.

Lemma 4.4 *Assume u is a positive solution of (4.1) in Q_T^Ω and $z \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds*

(i) *There exists an open neighborhood G of z such that $u \in L^1(Q_T^G)$.*

(ii) *f is nondecreasing and (2.14) holds.*

Then for every open relative neighborhood G' of z

$$\lim_{t \rightarrow 0} \int_{G'} u(x, t) dx = \infty. \quad (4.6)$$

Proof. If z in $\mathcal{S}(u)$, then for every open relative neighborhood G' of z , there holds

$$\int_0^T \int_{G'} f(u) dx dt = \infty. \quad (4.7)$$

First, we assume (i) holds and let $\zeta \in C_0^2(G)$, $\zeta \geq 0$. Since there exists

$$\lim_{t \rightarrow 0} \int_t^T \int_{G'} u \Delta \zeta dx dt = L \in \mathbb{R},$$

it follows from (4.4) that

$$\int_{G'} u(x, t) \zeta(x) dx = \int_t^T \int_{G'} f(u) \zeta dx ds + O(1), \quad (4.8)$$

which implies (4.6).

Next we assume that (2.14) holds and $u \notin L^1(Q_T^G)$ for every relative neighborhood G of z . If there exists an open neighborhood $G \subset \Omega$ of z such that (4.6) does not hold, there exists a sequence $\{t_n\}$ decreasing to 0 and $0 \leq M < \infty$ such that

$$\sup_{t_n} \int_G u(x, t_n) dx = M. \quad (4.9)$$

Furthermore, we can always replace G by an open ball $B_R(z) \subset G$. Thus (4.9) holds with G replaced by $B_R(z)$. Let $w := w_R$ be the maximal solution of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } B_R(z) \\ \lim_{|x-z| \rightarrow R} w(x) = \infty. \end{cases} \quad (4.10)$$

Let $v := v_n$ be the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } B_R(z) \times (t_n, \infty) \\ v = 0 & \text{in } \partial B_R(z) \times (t_n, \infty) \\ v(\cdot, t_n) = u(\cdot, t_n) & \text{in } B_R(z). \end{cases} \quad (4.11)$$

Since $v_n \geq 0$, $f(w_R + v_n) \geq f(w_R)$, and $w_R + v_n$ is a supersolution. It dominates u on $\partial B_R(z) \times (t_n, T)$ and at $t = t_n$, thus $u \leq w_R + v_n$ in $B_R(z) \times (t_n, T)$. We can assume that $u(\cdot, t_n) \rightarrow \nu$ for some positive and bounded measure ν on $B_R(z)$. Therefore

$$u(x, t) \leq v(x, t) + w_R(x) \quad \text{in } Q_T^{B_R(z)} \quad (4.12)$$

where v is the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) = \nu & \text{in } \mathcal{D}'(B_R(z)). \end{cases} \quad (4.13)$$

Since $v \in L^1(Q_T^{B_R(z)})$ and w_R is uniformly bounded in any ball $B_{R'}(z)$ for $0 < R' < R$ we conclude that $u \in L^1(Q_T^{B_{R'}(z)})$, contradiction. \square

Definition 4.5 Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (4.1) in Q_T^Ω . We say that u possesses an initial trace with regular part $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ and singular part $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ if

(i) For any $\zeta \in C_0(\mathcal{R}(u))$,

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x). \quad (4.14)$$

(ii) For any open set $G \subset \Omega$ such that $G \cap \mathcal{S}(u) \neq \emptyset$

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \infty. \quad (4.15)$$

Remark. To the couple $(\mathcal{S}(u), \mu)$, where $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ we can associate a unique outer regular Borel measure ν (i.e. $\nu \in \mathcal{B}_+^{reg}(\Omega)$) which is the initial trace of u . We shall denote it by $Tr^\Omega(u)$.

Theorem 4.6 Assume f satisfies the Keller-Osserman condition and is nondecreasing. If $u \in C^{2,1}(Q_T^\Omega)$ is a positive solution of (4.1), it possesses an initial trace $\nu \in \mathcal{B}_+^{reg}(\Omega)$.

Proof. The set $\mathcal{R}(u)$ and the measure $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ are defined by Definition 4.3 thanks to Corollary 4.2. Because (2.14) holds, $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ inherits the property (ii) above because of Lemma 4.4 (ii). \square

As a counterpart of this result we have the following existence theorem.

Proposition 4.7 *Assume f satisfies the Keller-Osserman condition, (2.2) and is nondecreasing. Then for any outer regular positive Borel measure ν on \mathbb{R}^N there exists a positive solution $u \in C^{2,1}(Q_\infty)$ of (4.1) in Q_∞ , with initial trace ν .*

Proof. It is essentially similar to the one of [5, Th 3.4]-to be outlined. Notice that we can actually construct a maximal solution \bar{u}_ν and a minimal solution \underline{u}_ν with the same initial trace ν . \square

Remark. When $f(r) = |r|^{q-1}r$ with $1 < q < 1 + 2/N$, precise expansion of $u_{\infty\delta}(x, t)$, when $t \rightarrow 0$ allows to prove uniqueness. Even when $f(r) = r \ln^\alpha(r + 1)$ with $\alpha > 2$, uniqueness is not known.

4.3 The Keller-Osserman condition does not hold

In this section we assume that (2.14) does not hold but (2.1) is always satisfied. If Ω is a bounded domain with a C^2 boundary and $\mu \in \mathfrak{M}_+^b(\Omega)$, we denote by u_μ the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = \mu & \text{in } \mathcal{D}'\Omega. \end{cases} \quad (4.16)$$

We recall the following stability result proved in [6, Th 1.1].

Lemma 4.8 *Let Ω be a bounded domain with a C^2 boundary and $\{\mu_n\} \subset \mathfrak{M}_+^b(\Omega)$ a sequence weakly converging to some $\mu \in \mathfrak{M}_+^b(\Omega)$. Then $u_{\mu_n} \rightarrow u_\mu$ locally uniformly in $\bar{\Omega} \times (0, \infty)$ and in $L^1(Q_T^\Omega)$, and $f(u_{\mu_n}) \rightarrow f(u_\mu)$ in $L^1(Q_T^\Omega)$, for every $T > 0$.*

Remark. The result remains true if $\Omega = \mathbb{R}^N$ and μ_n have their support in a fixed compact set.

Lemma 4.9 *Assume (2.14) does not hold and (2.8), and (C3) are satisfied. If u is a positive solution of (1.1) in Q_∞ which satisfies*

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx = \infty, \quad (4.17)$$

for some bounded open subset $G \subset \mathbb{R}^N$, then $u(x, t) \geq \phi_\infty(t)$.

Proof. By assumption, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n \rightarrow \infty} \int_G u(x, t_n) dx = \infty. \quad (4.18)$$

If (4.17), we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\text{diam}(G_k) = \epsilon_k \rightarrow 0$ when $k \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} \int_{G_k} u(x, t_n) dx = \infty \quad \forall k \in \mathbb{N}. \quad (4.19)$$

Furthermore there exists a unique $a \in \bigcap_k G_k$. We set

$$\int_{G_k} u(x, t_n) dx = M_{n,k}.$$

Since $\lim_{n \rightarrow \infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any k , there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x, t_{n(k)}) dx \geq m. \quad (4.20)$$

By induction, we define $n(1)$ as the smallest integer n such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer n larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any k , there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x, t_{n(k)}); \ell\} dx = m. \quad (4.21)$$

and we set

$$V_k(x) = \inf\{u(x, t_{n(k)}); \ell\} \chi_{G_k}(x).$$

Let $v_k = v$ be the unique bounded solution of

$$\begin{cases} v_t - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(\cdot, 0) = V_k & \text{in } \mathbb{R}^N \end{cases} \quad (4.22)$$

Since $v(x, 0) \leq u(x, t_{n(k)})$, it follows

$$u(x, t + t_{n(k)}) \geq v_k(x, t) \quad \forall (x, t) \in Q_\infty. \quad (4.23)$$

When $k \rightarrow \infty$, $V_k \rightarrow m\delta_a$, thus $v_k \rightarrow u_{m\delta_a}$ by Lemma 4.8. Therefore $u \geq u_{m\delta_a}$. Since m is arbitrary and $u_{m\delta_a} \rightarrow \phi_\infty$ when $m \rightarrow \infty$ by Theorem 2.7, it follows that $u \geq \phi_\infty$. \square

Lemma 4.10 *Assume (2.14) does not hold and (2.7), and (C3) are satisfied. There exists no u positive solution of (1.1) in Q_∞ which satisfies (4.17) for some bounded open subset $G \subset \mathbb{R}^N$.*

Proof. If we assume that such a u exists, we proceed as in the proof of the previous lemma. Since Lemma 4.8 holds, we derive that $u \geq u_{m\delta_a}$ for any m . Since $\lim_{m \rightarrow \infty} u_{m\delta_a}(x, t) = \infty$ for all $(x, t) \in Q_\infty$, we derive a contradiction. \square

Thanks to these results, we can characterize the initial trace of positive solutions of (4.1) when the Keller-Osserman condition does not hold.

Theorem 4.11 *Assume f is nondecreasing, (2.14) does not hold but (2.8), and (C3) are verified. If u is a positive solution of (4.1) in Q_∞ , it possesses an initial trace which is either the Borel measure infinity ν_∞ which satisfies $\nu_\infty(\mathcal{O}) = \infty$ for any open subset $\mathcal{O} \subset \mathbb{R}^N$, or is a positive Radon measure μ on \mathbb{R}^N .*

Proof. If there exists some open subset G of \mathbb{R}^N with the property (4.17), then $u \geq \phi_\infty$ and the initial trace of u is the Borel measure ν_∞ . Next we assume that for any bounded open subset G of \mathbb{R}^N there holds

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx < \infty, \quad (4.24)$$

If $\mathcal{S}(u) \neq \emptyset$, there exists $z \in \mathbb{R}^N$ and an bounded open neighborhood G of z such that

$$\int_0^T \int_G f(u) dx dx t = \infty$$

By (4.24), $u \in L^\infty(0, T; L^1(G)) \subset L^1(Q_T^G)$. Then, by Lemma 4.4, (4.6) holds, which contradict (4.24). Thus $\mathcal{S}(u) = \emptyset$ and $\mathcal{R}(u) = \mathbb{R}^N$. It follows from Corollary 4.2 that there exists a positive Radon measure μ such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) \quad \forall \zeta \in C_0(\mathbb{R}^N), \quad (4.25)$$

\square

Because of the lack of uniqueness from Corollary 3.4 it is difficult to give a complete characterization of admissible initial data for solutions of 4.1 under the assumptions of Theorem 4.11. However, we have the following result

Proposition 4.12 *Under the assumptions of Theorem 4.11, for any $b > 0$ there exists a positive solution $u \in C(Q_\infty)$ of (4.1) satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x, t) \leq \phi_\infty(t) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (4.26)$$

Consequently there exist infinitely many positive solutions of (4.1) with initial trace ν_∞ . Furthermore ϕ_∞ is the smallest of all these solutions.

Proof. We first notice that $\max\{\phi_\infty(t); w_b(|x|)\}$ is a subsolution of (4.1) which is dominated by the supersolution $\phi_\infty(t) + w_b(|x|)$. The process is standard: for $\tau > 0$ we set

$$\psi(x, \tau) = \frac{1}{2} (\max\{\phi_\infty(t); w_b(|x|)\} + \phi_\infty(t) + w_b(|x|))$$

There exists a function $u = u_\tau \in C(\overline{Q_\infty})$ solution of (4.1) in Q_∞ satisfying $u_\tau(\cdot, 0) = \psi(\cdot, \tau)$. Furthermore

$$\max\{\phi_\infty(t + \tau); w_b(|x|)\} \leq u_\tau(x, t) \leq \phi_\infty(t + \tau) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (4.27)$$

By parabolic equation regularity theory, the set $\{u_\tau\}$ is locally equicontinuous in Q_∞ . Thus there exists a subsequence $\{\tau_n\}$ and $u \in C(Q_\infty)$ such that $u_{\tau_n} \rightarrow u$ on any compact subset of Q_∞ . Clearly u is a weak, thus a strong solutions of (4.1) and it satisfies (4.26). Since any solution u with initial trace ν_∞ dominates ϕ_∞ by Lemma 4.9, it follows that ϕ_∞ is the minimal one. \square

Theorem 4.13 *Assume f is nondecreasing, (2.14) does not hold and (2.7), and (C3) are verified. If u is a positive solution of (4.1) in Q_∞ , it possesses an initial trace which is a positive Radon measure μ on \mathbb{R}^N .*

Proof. As in the proof of Theorem 4.11 and because of Lemma 4.10, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$ and the proof follows from Corollary 4.2. \square

Remark. Under the assumptions of Theorem 4.11, it is clear, from the proof of Proposition 3.2, that for any $0 < a < b$ and any initial data $u_0 \in C(\mathbb{R}^N)$ satisfying

$$w_a(x) \leq u_0(x) \leq w_b(x) \quad \forall x \in \mathbb{R}^N$$

there exists a solution $u \in C(\overline{Q_\infty})$ of (4.1) in Q_∞ satisfying $u(\cdot, 0) = u_0$ and

$$w_a(x) \leq u(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty.$$

We conjecture that for any positive measure μ on \mathbb{R}^N which satisfies, for some $b > 0$,

$$\int_{B_R} d\mu(x) \leq \int_{B_R} w_b(x) dx \quad \forall R > 0 \quad (4.28)$$

there exists a positive solution u of (4.1) in Q_∞ with initial trace μ . Another interesting open problem is to see if there exists local solutions in Q_T with an initial trace μ satisfying

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} d\mu(x)}{\int_{B_R} w_b(x) dx} = \infty \quad \forall b > 0. \quad (4.29)$$

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