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Tai Nguyen Phuoc, Laurent Veron. Local and global properties of solutions of heat equation with superlinear absorption. 2010. hal-00459765v1

HAL Id: hal-00459765

<https://hal.science/hal-00459765v1>

Preprint submitted on 25 Feb 2010 (v1), last revised 23 Aug 2010 (v5)

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Local and global properties of solutions of heat equation with superlinear absorption

Tai Nguyen Phuoc Laurent Véron

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We study the limit, when $k \rightarrow \infty$ of solutions of $u_t - \Delta u + f(u) = 0$ in $\mathbb{R}^N \times (0, \infty)$ with initial data $k\delta$, when f is a positive increasing function. We prove that there exist essentially three types of possible behaviour according f^{-1} and $F^{-1/2}$ belong or not to $L^1(1, \infty)$, where $F(t) = \int_0^t f(s)ds$. We use these results for giving a general result on the existence of the initial trace and some non-uniqueness results for regular solutions with unbounded initial data.

2010 Mathematics Subject Classification. 35K58; 35K91; 35K15.

Key words. Heat equation; singularities; Borel measures; initial trace.

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1 Introduction

In this article we investigate some local and global properties of solutions of a class of semilinear heat equations

$$\partial_t u - \Delta u + uh(u) = 0 \quad (1.1)$$

in $Q_\infty := \mathbb{R}^N \times (0, \infty)$ ($N \geq 2$) where $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, nondecreasing and positive on $(0, \infty)$, vanishes at 0 and tends to infinity at infinity. As a model equation we shall consider the following, with $\alpha > 0$,

$$\partial_t u - \Delta u + u \ln^\alpha(u + 1) = 0 \quad (1.2)$$

When $h(u) = |u|^\beta$ with $\beta > 0$ much is known about the structure of the set of solutions. The local and asymptotic behaviour of solutions is strongly linked to the existence of a self-similar solutions under the form

$$u(x, t) = t^{-1/\beta} w(x\sqrt{t}) \quad (1.3)$$

The critical exponent $\beta_c = 2/N$ plays a fundamental role in the description of isolated singularities and the study of the initial trace, since if $0 < \beta < \beta_c$ there exists a self-similar solution with an isolated singularity at $(0, 0)$, while no such solution exists when $\beta \geq \beta_c$ and no solution with isolated singularities.

In the case of (1.2), no self-similar structure exists. There is no critical exponent corresponding to isolated singularities since there always exist such singular solutions. Actually, for any $k > 0$ there exists a unique $u = u_k \in C(\overline{Q_\infty} \setminus \{0\}) \cap C^{2,1}(Q_\infty)$ solution of

$$\begin{cases} u_t - \Delta u + u \ln^\alpha(u + 1) = 0 & \text{in } Q_\infty \\ u(0, \cdot) = k\delta_0 & \text{in } \mathcal{D}'(\mathbb{R}^N). \end{cases} \quad (1.4)$$

There are two critical values for α : $\alpha = 1$ and $\alpha = 2$, the explanation of which comes from the study of the two singular problems

$$\begin{cases} \phi' + \phi \ln^\alpha(\phi + 1) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty \end{cases} \quad (1.5)$$

and, for any $\epsilon > 0$,

$$\begin{cases} -\Delta \psi + \psi \ln^\alpha(\psi + 1) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty, \end{cases} \quad (1.6)$$

where $B_\epsilon := \{x \in \mathbb{R}^N : |x| < \epsilon\}$. When it exists, the solution ϕ_∞ of (1.5) is given implicitly by

$$\int_{\phi_\infty(t)}^{\infty} \frac{ds}{s \ln^\alpha(s + 1)} = t. \quad (1.7)$$

and such a formula is valid if and only if $\alpha > 1$. For problem (1.6) an explicit expression of the solution is not valid, but it exists if and only if the Keller-Osserman condition holds $\alpha > 2$.

Having in mind this model we study (1.1) assuming the weak singularity condition on h :

$$\int_0^1 h(t^{-N/2}) dt < \infty. \quad (1.8)$$

If (1.8) holds, for any $k > 0$, there exists a unique solution u_k of (1.1) satisfying $u_k(\cdot, 0) = k\delta_0$. Another condition on h is

$$\int_1^\infty \frac{dt}{th(t)} < \infty. \quad (1.9)$$

Under this assumption there exists a solution to

$$\begin{cases} \phi' + \phi h(\phi) = 0 & \text{in } (0, \infty) \\ \phi(0) = \infty \end{cases} \quad (1.10)$$

The last important condition on h we shall encounter is

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad (1.11)$$

where

$$F(t) = 2 \int_0^t sh(s) ds.$$

If (1.11) is satisfied, for any $\epsilon > 0$ there exists a maximal solution $\psi := \psi_\epsilon$ to

$$\begin{cases} -\Delta\psi + \psi h(\psi) = 0 & \text{in } \mathbb{R}^N \setminus B_\epsilon \\ \lim_{|x| \rightarrow \epsilon} \psi(x) = \infty, \end{cases} \quad (1.12)$$

The first question we consider is the study of the limit of u_k when $k \rightarrow \infty$. This question is natural since $k \mapsto u_k$ is increasing. We prove the following results.

Theorem A. *Assume h satisfies*

$$\int_1^\infty \frac{dt}{th(t)} = \infty. \quad (1.13)$$

Then $\lim_{k \rightarrow \infty} u_k(x, t) = \infty$.

Theorem B. *Assume h satisfies (1.9) and*

$$\int_1^\infty \frac{dt}{\sqrt{F(t)}} = \infty, \quad (1.14)$$

Then $\lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t)$.

Theorem C. Assume h satisfies (1.8) and (1.9). Then $\lim_{k \rightarrow \infty} u_k = u_\infty$ where u_∞ is the minimal solution of (1.1) in Q_∞ which is continuous in $\overline{Q_\infty} \setminus \{0, 0\}$, vanishes on the set $\{(x, 0) : x \neq 0\}$ and satisfies

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} u(x, t) dx = \infty \quad (1.15)$$

for any $\epsilon > 0$.

In the second section we study the set \mathcal{U} of positive and locally bounded solutions of (1.1) in Q_∞ . This set differs considerably according the assumption on h . This is due to the properties of the radial solutions of the associated stationary equation

$$-\Delta w + wh(w) = 0 \quad \text{in } \mathbb{R}^N. \quad (1.16)$$

For any $a > 0$ the problem

$$\begin{cases} -w'' - \frac{N-1}{r}w' + wh(w) = 0 & \text{for } r > 0 \\ w'(0) = 0 \\ w(0) = a \end{cases} \quad (1.17)$$

admits a maximal solution w_a defined on $[0, r_a)$. The function w_a is increasing and $r_a < \infty$ if and only if h satisfies (1.11). If $r_a = \infty$ the mapping $a \mapsto w_a$ is increasing and $\lim_{r \rightarrow \infty} w_a(r) = \infty$. A striking consequence of the existence of such solutions is the following non-uniqueness result

Theorem D. Assume h satisfies (1.8) and (1.14). Then for any $u_0 \in C(\mathbb{R}^N)$ satisfying $w_a(x) \leq u_0(x) \leq w_b(x)$ for some $b > a > 0$ there exist two solutions $\underline{u}, \bar{u} \in C(\overline{Q_\infty})$ of (1.1) with initial value u_0 . They satisfy respectively

$$0 \leq \underline{u}(x, t) \leq \min\{w_b(x), \phi_\infty(t)\} \quad \forall (x, t) \in Q_\infty, \quad (1.18)$$

thus $\lim_{t \rightarrow \infty} \underline{u}(x, t) = 0$, uniformly with respect to x , and

$$w_a(x) \leq \bar{u}(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty, \quad (1.19)$$

thus $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$, uniformly with respect to $t \geq 0$.

We prove that, under the conditions of Theorem D, two solutions u_i ($i=1,2$) with the same initial data u_0 coincide if for any $\epsilon > 0$ there holds

$$u_1(x, t) - u_2(x, t) = o(w_\epsilon(|x|)) \quad \text{as } |x| \rightarrow \infty \quad (1.20)$$

locally uniformly with respect to $t \geq 0$.

On the contrary, if h satisfies (1.8) and (1.9), a continuous solution is uniquely determined by the initial value u_0 .

In the last section we study the initial trace of locally bounded positive solutions of (1.1). We prove:

Theorem E. *If h satisfies (1.14) the initial trace is a Radon measure.*

Theorem F. *If h satisfies (1.8) and (1.14), either the initial trace is a Radon measure, or $u \geq \phi_\infty$.*

Theorem G. *If h satisfies (1.8) and (1.9) the initial trace is a positive Borel measure.*

The proofs are combination of methods developed in [8] for elliptic equations, stability results and Theorems A and B above. Among the consequences of the previous results we can mention.

Theorem G. *If h satisfies (1.8) and (1.13) there exist infinitely many solutions of (1.1) $u \geq \phi_\infty$, and ϕ_∞ is the minimal one.*

Theorem I. *If h satisfies (1.8) and (1.9), then for any outer regular Borel measure ν on \mathbb{R}^N , there exists a positive solution u of (1.1) with initial trace ν .*

2 Isolated singularities

In order to study (1.1), we shall assume the following conditions on h

C1- h is a positive, increasing function and $h(0) = 0$.

C2- the mapping $r \mapsto f(r) = rh(r)$ is convex.

C3- If $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \forall \alpha > 2$, then there exists $\beta \in (1, 2]$ such that

$$\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty.$$

It is well-known (e.g. [14]) that, if for $k > 0$,

$$I = \int_0^1 \int_{B_R} kE(x, t)h(kE(x, t))dxdt < \infty \quad (2.1)$$

for any $R > 0$, then there exists a unique solution $u = u_k$ to (1.1) satisfying initial condition

$$u_k(\cdot, 0) = k\delta_0$$

in the sense of distributions in \mathbb{R}^N . Furthermore the mapping $k \mapsto u_k$ is increasing. The condition (2.1) can be reduced to a weaker form as follows

$$\int_0^1 h(t^{-N/2})dt < \infty. \quad (2.2)$$

Proposition 2.1 *If (2.2) is satisfied then (2.1) is fulfilled.*

Proof. We have

$$I = kC^* \int_0^1 \int_{B_R} t^{-N/2} e^{-|x|^2/4t} h(kC^* t^{-N/2} e^{-|x|^2/4t}) dx dt$$

where $C^* = (4\pi)^{-N/2}$. Put $r = |x|$ then $dx = r^{N-1} dr$, and

$$I = kC^* \int_0^1 t^{-N/2} \int_0^R e^{-r^2/4t} h(kC^* t^{-N/2} e^{-r^2/4t}) r^{N-1} dr dt.$$

We next put $\rho = \frac{r}{\sqrt{t}}$, then $r^{N-1} dr = \rho^{N-1} t^{N/2} d\rho$, and

$$I = kC^* \int_0^1 \int_0^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

We set

$$I_1 = kC^* \int_0^1 \int_0^1 e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt,$$

$$I_2 = kC^* \int_0^1 \int_1^{R/\sqrt{t}} e^{-\rho^2/4} h(kC^* t^{-N/2} e^{-\rho^2/4}) \rho^{N-1} d\rho dt.$$

Since $e^{-\rho^2/4} \rho^{N-1}$ is bounded in $[0, \infty)$, then there exists a constant c_1 depending only on k such that

$$I_1 < c_1 \int_0^1 \int_0^1 h(kC^* t^{-N/2}) d\rho dt = c_1 \int_0^1 h(kC^* t^{-N/2}) dt < \infty$$

We next show that under the condition (2.2), $I_2 < \infty$. In order to do that we change to a new variable τ such that $t^{-N/2} e^{-\rho^2/4} = \tau^{-N/2}$. Then $t = \tau e^{-\frac{\rho^2}{2N}}$ and $dt = e^{-\frac{\rho^2}{2N}} d\tau$. Therefore

$$I_2 \leq kC^* \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} \left(\int_0^{e^{\rho^2/2N}} h(kC^* \tau^{-N/2}) d\tau \right) d\rho \quad (2.3)$$

Since h satisfies (2.2), there exists $\epsilon > 0$ (depending only on k) such that

$$\int_0^\epsilon h(kC^*\tau^{-N/2})d\tau$$

take a finite value, denoted by c_2 . Hence

$$\int_0^{e^{\rho^2/2N}} h(kC^*\tau^{-N/2})d\tau \leq c_2 + h(kC^*\epsilon^{-N/2})(e^{\frac{\rho^2}{2N}} - \epsilon) \quad (2.4)$$

Inserting (2.4) into the right hand side of (2.3), we obtain

$$I_2 \leq c_3 \int_1^\infty e^{-\frac{(N+2)\rho^2}{4N}} \rho^{N-1} d\rho + c_4 \int_1^\infty e^{-\frac{\rho^2}{4}} \rho^{N-1} d\rho < \infty$$

where $c_3 = kC^*c_2$ and $c_4 = kC^*h(kC^*\epsilon^{-N/2})$. Thus, $I = I_1 + I_2 < \infty$. \square

The functions which satisfy the following ODE are particular solutions of (1.1)

$$\phi_t + \phi h(\phi) = 0 \quad \text{in } (0, \infty) \quad (2.5)$$

For $a > 0$, we denote by ϕ_a the solution of (2.5) with initial data $\phi(0) = a$. This solution is given by the formula

$$t = \int_{\phi(t)}^a \frac{ds}{sh(s)} < \infty. \quad (2.6)$$

If

$$J = \int_a^\infty \frac{ds}{sh(s)} = \infty. \quad (2.7)$$

then $\lim_{a \rightarrow \infty} \phi_a(t) = \infty$. While, if

$$J = \int_a^\infty \frac{ds}{sh(s)} < \infty. \quad (2.8)$$

there exists a maximal solution ϕ_∞ given explicitly by

$$t = \int_{\phi_\infty(t)}^\infty \frac{ds}{sh(s)} < \infty.$$

Lemma 2.2 *If $J = \infty$ then $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0$, $\forall \alpha > 1$.
If $J < \infty$ then $\limsup_{r \rightarrow \infty} h(r)/\ln^\alpha r = \infty$, $\forall 0 < \alpha \leq 1$.*

Proof. Set $s = e^{r^{-1}}$ then

$$J = \int_0^b \frac{dr}{r^{\alpha+1}h(e^{r^{-1}})}$$

where $b = (\ln a)^{-1}$.

Case 1: $J = \infty$. Suppose that there exist $\alpha > 1$ such that

$$\liminf_{s \rightarrow \infty} \frac{h(s)}{\ln^\alpha s} > 0,$$

equivalently,

$$\liminf_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) > 0$$

then there exists $l > 0$ and $r_0 > 0$ such that

$$h(e^{r^{-1}}) > lr^{-\alpha}, \quad \forall r \in (0, r_0).$$

Hence we derive the following contradiction

$$J < \frac{\alpha}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^b \frac{dr}{r^{\alpha+1}h(e^{r^{-1}})} < \infty.$$

Case 2: $J < \infty$. Suppose that there exist $\alpha \in (0, 1]$ such that

$$\limsup_{s \rightarrow \infty} \frac{h(s)}{\ln^\alpha s} < \infty,$$

equivalently,

$$\limsup_{r \rightarrow 0} r^\alpha h(e^{r^{-1}}) < \infty$$

then there exists $l > 0$ and $r_0 > 0$ such that

$$h(e^{r^{-1}}) < lr^{-\alpha}, \quad \forall r \in (0, r_0).$$

Hence

$$J > \frac{\alpha}{l} \int_0^{r_0} r^{\alpha-2} dr + \int_{r_0}^b \frac{dr}{r^{\alpha+1}h(e^{r^{-1}})} = \infty,$$

contradiction. □

Theorem 2.3 *Assume (2.7) is satisfied. Then*

$$\lim_{k \rightarrow \infty} u_k(x, t) = \infty \quad \forall (x, t) \in Q_\infty.$$

Proof. Since $J = \infty$, by Lemma 2.2, $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0$ for all $\alpha > 1$. Thus

$$\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \quad \forall \alpha > 2.$$

By (C3), there exists $\beta \in (1, 2]$ such that $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r = 0 < \infty$. Hence there exist $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r, \quad \forall r \in (r_0, \infty). \quad (2.9)$$

Step 1. Let $k > 0$, we claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau, \quad t \in (0, 1) \quad (2.10)$$

where $\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$ and $C^* = (4\pi)^{-N/2}$. Set $r = kC^* \tau^{-N/2}$ then (2.9) becomes

$$h(kC^* \tau^{-N/2}) < M [\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta, \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{\frac{2}{N}} e^{-\frac{2}{N} r_0^{-1}}$. We put $a_1 = \ln k$, $a_2 = \frac{N}{2} \ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1} (a_1^\beta + a_2^\beta)$$

in order to obtain

$$\begin{aligned} h(kC^* \tau^{-N/2}) &< M [\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M [(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})], \quad \forall \tau \in (0, \tau_0) \end{aligned} \quad (2.11)$$

Integrating over $[0, t]$ yields to (2.10).

Step 2. It follows from (2.11) that (2.2) is fulfilled, hence there exists the unique solution of (1.1) in Q_∞ with initial data $k\delta_0$. By the maximum principle, $u_k(x, t) \leq kE(x, t)$ for any $(x, t) \in Q_\infty$, which implies $u_k(x, t) \leq kC^* t^{-N/2}$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^* t^{-N/2}) \geq 0.$$

If we set $v_k(x, t) = e^{\theta_k(t)} u_k(x, t)$, we obtain

$$\partial_t v_k - \Delta v_k = e^{\theta_k(t)} (\partial_t u_k - \Delta u_k + u_k h(kC^* t^{-N/2})) \geq 0,$$

and $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$. By the maximum principle, there holds

$$v_k(x, t) \geq kC^* t^{-N/2} e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^* t^{-N/2} e^{-\theta_k(t) - |x|^2/4t}. \quad (2.12)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t (\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.13)$$

where $c_1 = \exp(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau)$ and $M_\beta = M2^{\beta-1}$. Inserting (2.13) into the right hand side of (2.12), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1.$$

Put $\underline{U} := \lim_{k \rightarrow \infty} u_k$, then

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1, \forall k > 0.$$

Let $\{t_n\} \subset (0, 1]$ be a sequence converging to 0. We choose $k = k_n = \exp\left(\left(2M_\beta t_n\right)^{\frac{1}{1-\beta}}\right)$, then $\ln k_n - M_\beta t_n \ln^\gamma k_n = \frac{1}{2} \ln t_n$. Next we restrict x in order to have

$$\ln k_n - M_\beta t_n \ln^\gamma k_n - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}} t_n^{\frac{\beta-2}{2(\beta-1)}}.$$

Therefore, since $1 < \beta \leq 2$,

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-1/2}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} .

We pick some point x_0 in \mathbb{R}^N (resp. B_{r_2}) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, by comparison principle, it follows that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \rightarrow \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t), \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of x and therefore it is a solution of

$$\begin{cases} \phi' + \phi h(\phi) = 0 & \text{on } (0, \infty) \\ \lim_{t \rightarrow 0} \phi(t) = \infty. \end{cases}$$

By (2.7), $\underline{U}(x, t) = \infty$ for all $(x, t) \in Q_\infty$, which ends the proof. \square

Proposition 2.4 *Assume (2.8) is satisfied. For any $k > 0$, there holds*

$$u_k(x, t) \leq \phi_\infty(t), \quad \forall (x, t) \in Q_\infty.$$

Proof. For any small $\epsilon > 0$, put $\phi_{\infty\epsilon}(t) = \phi_\infty(t - \epsilon)$, $t \in [\epsilon, \infty)$ then $\phi_{\infty\epsilon}$ is a solution of (1.1) in (ϵ, ∞) , which dominates u_k on $\mathbb{R}^N \times \epsilon$ for any $k > 0$. By comparison principle, $u_k(x, t) \leq \phi_{\infty\epsilon}(t)$ in $\mathbb{R}^N \times [\epsilon, \infty)$. Letting $\epsilon \rightarrow 0$ yields to the claim. \square

A necessary and sufficient condition for the existence of a maximal solution to the stationary equation

$$-\Delta w + wh(w) = 0$$

in a bounded domain Ω is the Keller-Osserman condition ([4], [9])

$$K = \int_a^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad (2.14)$$

where $F(s) = \int_0^s \tau h(\tau) d\tau$. If (2.14) holds, then (2.8) is fulfilled. The Keller-Osserman condition can be replaced by another condition, which owes to the following result

Lemma 2.5 *Set*

$$L = \int_a^\infty \frac{ds}{s\sqrt{h(s)}}.$$

Then $K < \infty$ if and only if $L < \infty$.

Proof. In order to obtain the assertion, it is sufficient to show that there exists two positive constants c_1 and c_2 such that

$$c_1 s^2 h\left(\frac{s}{2}\right) \leq F(s) \leq c_2 s^2 h(s) \quad \forall s \geq a. \quad (2.15)$$

The right-hand side estimate in (2.15) follows from the monotone property of h (with $c_2 = 1/2$). The assumption of convexity of $s \mapsto f(s) := sh(s)$ in $(0, \infty)$ implies

$$f(s) \geq f\left(\frac{s}{2}\right) + \frac{s}{2} f'\left(\frac{s}{2}\right) \quad \forall s > 0.$$

Define $\varphi(s) = \int_0^s f(\tau) d\tau - sf\left(\frac{s}{2}\right)$, then $\varphi'(s) = f(s) - f\left(\frac{s}{2}\right) - \frac{s}{2} f'\left(\frac{s}{2}\right) > 0$. Hence $\varphi(s) > \varphi(0) = 0$, which is the left-hand side estimate in (2.15) (with $c_1 = 1/2$). \square

By using the same argument as in the proof of the Lemma 2.2 and thank to the Lemma 2.5, we obtain the following lemma

Lemma 2.6 *If*

$$K = \infty \quad (2.16)$$

then $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha(r) = 0$ *for all* $\alpha > 2$.

If

$$K < \infty \quad (2.17)$$

then $\limsup_{r \rightarrow 0} h(r)/\ln^\alpha(r) = \infty$ *for all* $0 < \alpha \leq 2$.

For the case *time-type absorption*, we have the following result

Theorem 2.7 *Assume (2.8) and (2.16) are satisfied. Then*

$$\underline{U} = \lim_{k \rightarrow \infty} u_k(x, t) = \phi_\infty(t) \quad \forall (x, t) \in Q_\infty.$$

Proof. Since $K = \infty$, by Lemma 2.6, $\liminf_{r \rightarrow \infty} h(r)/\ln^\alpha r = 0, \forall \alpha > 2$. By (C3), there exists $\beta \in (1, 2]$ such that $\limsup_{r \rightarrow \infty} h(r)/\ln^\beta r < \infty$. Hence there exists $M > 0$ and $r_0 > 0$ such that

$$h(r) < M \ln^\beta r, \quad \forall r \in (r_0, \infty). \quad (2.18)$$

Step 1. For any $k > 0$ we set

$$\theta_k(t) = \int_0^t h(kC^* \tau^{-N/2}) d\tau$$

where $C^* = (4\pi)^{-\frac{N}{2}}$. We claim that

$$\theta_k(t) < 2^{\beta-1} M t (\ln k)^\beta + \frac{MN^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau, \quad t \in (0, 1). \quad (2.19)$$

If we define τ by $r = kC^* \tau^{-N/2}$, (2.18) becomes

$$h(kC^* \tau^{-\frac{N}{2}}) < M [\ln(kC^*) + \frac{N}{2} \ln(\tau^{-1})]^\beta, \quad \forall \tau \in (0, \tau_0)$$

where $\tau_0 = (kC^*)^{\frac{2}{N}} e^{-\frac{2}{N} r_0^{-1}}$. We put $a_1 = \ln k$, $a_2 = \frac{N}{2} \ln(\tau^{-1})$, and apply the following inequality

$$(a_1 + a_2)^\beta \leq 2^{\beta-1} (a_1^\beta + a_2^\beta)$$

in order to obtain

$$\begin{aligned} h(kC^* \tau^{-N/2}) &< M [\ln(k) + \frac{N}{2} \ln(\tau^{-1})]^\beta \\ &\leq 2^{\beta-1} M [(\ln k)^\beta + (\frac{N}{2})^\beta \ln^\beta(\tau^{-1})] \end{aligned} \quad (2.20)$$

Integrating over $[0, t]$, we obtain (2.19).

Step 2. It follows from (2.20) that (2.2) is fulfilled, hence there exists the unique solution of (1.1) in Q_∞ with initial trace $k\delta_0$. By maximum principle, $u_k(x, t) \leq kE(x, t)$ for any $(x, t) \in Q_\infty$, which implies that $u_k(x, t) \leq kC^*t^{-N/2}$. Therefore, since h is increasing,

$$\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2}) \geq 0.$$

We set $v_k(x, t) = e^{\theta_k(t)} u_k(x, t)$ and obtain

$$\partial_t v_k - \Delta v_k = e^{\Theta(t)} (\partial_t u_k - \Delta u_k + u_k h(kC^*t^{-N/2})) \geq 0,$$

with $v_k(\cdot, 0) = u_k(\cdot, 0) = k\delta_0$. By maximum principle, it follows

$$v_k(x, t) \geq kC^*t^{-N/2}e^{-|x|^2/4t} \iff u_k(x, t) \geq kC^*t^{-N/2}e^{-\theta_k(t)-|x|^2/4t}. \quad (2.21)$$

By step 1,

$$e^{-\theta_k(t)} \geq c_1 e^{-M_\beta t (\ln k)^\beta} \quad \forall t \in (0, 1) \quad (2.22)$$

where $c_1 = \exp(-\frac{M(N)^\beta}{2} \int_0^1 (\ln(\tau^{-1}))^\beta d\tau)$ and $M_\beta = M2^{\beta-1}$. Inserting (2.13) into the right-hand side of (2.12), we get

$$u_k(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1.$$

Since $\underline{U} \geq u_k$, then

$$\underline{U}(x, t) \geq c_1 C^* t^{-N/2} e^{\ln k - M_\beta t (\ln k)^\beta - |x|^2/4t}, \quad \forall (x, t) \in Q_1, \quad \forall k > 0.$$

Let $\{t_n\} \subset (0, 1]$ be a sequence converging to 0. We choose $k = k_n = \exp((2M_\beta t_n)^{\frac{1}{1-\beta}})$, equivalently $\ln k_n - M_\beta t_n \ln^\gamma k_n = \frac{1}{2} \ln t_n$. Next we restrict $|x|$ in order

$$\ln k_n - M_\beta t_n \ln^\gamma k_n - \frac{|x|^2}{4t_n} = \frac{1}{2} \ln k_n - \frac{|x|^2}{4t_n} \geq 0 \iff |x| \leq r_\beta t_n^{\frac{\beta-2}{2(\beta-1)}},$$

where $r_\beta = 2^{\frac{\beta-2}{2(\beta-1)}} M_\beta^{\frac{1}{2(1-\beta)}}$. Because $1 < \beta \leq 2$, it follows

$$\lim_{n \rightarrow \infty} \underline{U}(x, t_n) = \infty,$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} where $r_2 = (2M)^{-\frac{1}{2}}$ if $\beta = 2$. Since the sequence $\{t_n\}$ is arbitrary,

$$\lim_{t \rightarrow 0} \underline{U}(x, t) = \infty$$

uniformly on \mathbb{R}^N if $1 \leq \beta < 2$, or uniformly on the ball B_{r_2} .

We pick some point x_0 in \mathbb{R}^N (resp. B_{r_2}) if $1 < \beta < 2$ (resp. $\beta = 2$). Since for any $k > 0$, the solution $u_{k\delta_{x_0}}$ of (1.1) with initial data $k\delta_{x_0}$ can be approximated by solutions with bounded initial data and support in $B_\sigma(x_0)$ where $0 < \sigma < r_2 - |x_0|$, it follows, by comparison principle, that

$$\underline{U}(x, t) \geq u_{k\delta_{x_0}}(x, t) = u_k(x - x_0, t).$$

Letting $k \rightarrow \infty$ yields to $\underline{U}(x, t) \geq \underline{U}(x - x_0, t)$. Reversing the role of 0 and x_0 yields to $\underline{U}(x, t) = \underline{U}(x - x_0, t)$. If we iterate this process we derive

$$\underline{U}(x, t) = \underline{U}(x - y, t), \quad \forall y \in \mathbb{R}^N.$$

This implies that $\underline{U}(x, t)$ is independent of x and therefore it is a solution of

$$\begin{cases} \phi' + \phi h(\phi) = 0 \text{ on } (0, \infty) \\ \lim_{t \rightarrow 0} \phi(t) = \infty \end{cases}$$

Thus $\underline{U}(x, t) = \phi_\infty$, $\forall (x, t) \in Q_\infty$. \square

Proposition 2.8 *Assume (2.2) and the Keller-Osserman condition are satisfied. Then there holds*

$$\underline{U}(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty$$

where Φ is a solution to the problem

$$\begin{cases} -\Phi'' + \Phi h(\Phi) = 0 \text{ in } (0, \infty) \\ \lim_{s \rightarrow 0} \Phi(s) = \infty \end{cases}$$

Proof. Step 1: Upper estimate. Since $s \mapsto sh(s)$ satisfies (2.14) condition, for any $R > 0$, there exists a solution w_R to the problem

$$\begin{cases} -\Delta w_R + w_R h(w_R) = 0 \text{ in } B_R \\ \lim_{|x| \rightarrow R} w_R(x) = \infty \end{cases}$$

Let $x_0 \neq 0$ arbitrary be in \mathbb{R}^N . Set $\mathbb{E} = \{\vec{e} : |\vec{e}| = 1\}$ and take $\vec{e} \in \mathbb{E}$. Put $x_{\vec{e}} = |x_0| \vec{e}$ and for $n > |x_0|$ put $a_n = n\vec{e}$. Denote by $\mathbb{H}_{\vec{e}}$ the open half-space generated by \vec{e} , then $x_{\vec{e}}, a_n \in \mathbb{H}_{\vec{e}}$. Take R such that $n - |x_0| < R < n$. We put $W_{\vec{e}, n, R}(x) = w_R(x - a_n)$, then $W_{\vec{e}, n, R}$ is a solution of (1.1) in $B_R(a_n)$ and satisfies the blow-up condition on the boundary $\lim_{|x - a_n| \rightarrow R} W_{\vec{e}, n, R}(x) = \infty$. By the maximum principle,

$$u_k(x, t) \leq W_{\vec{e}, n, R}(x) \text{ in } B_R(a_n) \times (0, \infty). \quad (2.23)$$

The sequence $\{W_{\vec{e}, n, R}\}_R$ is decreasing and is bounded from below, then there exists $W_{\vec{e}, n} := \lim_{R \rightarrow n} W_{\vec{e}, n, R}$. Letting successively $R \rightarrow n$, $k \rightarrow \infty$ in (2.23) yields to

$$\underline{U}(x, t) \leq W_{\vec{e}, n}(x) \text{ in } B_n(a_n) \times (0, \infty). \quad (2.24)$$

The sequence $(W_{\vec{e},n})_n$ is also decreasing and is bounded from below, then there exists $W_{\vec{e},\infty} := \lim_{n \rightarrow \infty} W_{\vec{e},n}$. Letting $n \rightarrow \infty$ in (2.24) yields to

$$\underline{U}(x, t) \leq W_{\vec{e},\infty}(x) \text{ in } \mathbb{H}_{\vec{e}} \times (0, \infty). \quad (2.25)$$

In particular,

$$\underline{U}(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}). \quad (2.26)$$

Since \underline{U} is radial, it follows that

$$\underline{U}(x_0, t) = \underline{U}(x_{\vec{e}}, t) \leq W_{\vec{e},\infty}(x_{\vec{e}}).$$

Because \vec{e} is arbitrary in \mathbb{E} , we obtain that

$$\underline{U}(x_0, t) \leq W(x_0),$$

where $W(x) := \inf_{\vec{e} \in \mathbb{E}} W_{\vec{e},\infty}(|x|\vec{e})$. Note that W is radially symmetric, finite in \mathbb{R}_*^N and satisfies $\lim_{|x| \rightarrow 0} W(x) = \infty$.

Step 2: End of the proof. We claim that

$$W(x) = \Phi(|x|) \quad \forall x \neq 0. \quad (2.27)$$

Let $\vec{e} \in \mathbb{E}$, n be large enough and $R < n$. By the maximum principle, there holds $\Phi(|x|) \leq W_{\vec{e},n,R}(x)$ for all $x \in \mathbb{H}_{\vec{e}}$. Letting successively $R \rightarrow n$, $n \rightarrow \infty$ yields to $\Phi(|x|) \leq W_{\vec{e},\infty}(x)$, which implies that

$$\Phi(|x|) \leq W(x). \quad (2.28)$$

Next for any $\epsilon > 0$, we set $\Phi_\epsilon(s) = \Phi(s - \epsilon)$, $s > \epsilon$ and $\tilde{W}_\epsilon(x) = \Phi_\epsilon(|x|)$ in B_ϵ^c . Since $\Phi'_\epsilon \leq 0$, W_ϵ is a supersolution, which dominates W on ∂B_ϵ . By the maximum principle, $W \leq \tilde{W}_\epsilon$ in B_ϵ^c . Letting $\epsilon \rightarrow \infty$ yields to

$$W(x) \leq \Phi(|x|) \quad \forall x \neq 0. \quad (2.29)$$

Combining (2.28) and (2.29) leads to (2.27), which implies $\underline{U}(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty \setminus (\{0\} \times (0, \infty))$. Since $\lim_{s \rightarrow 0} \Phi(s) = \infty$, the conclusion follows. \square

Remark 1. In the proof, we have used the fact that $\Phi'_\epsilon \leq 0$. Indeed, if there exists $s_0 > 0$ such that $\Phi'(s_0) \geq 0$, then by the convexity property,

$$\Phi(s) \geq \Phi(s_0) + (s - s_0)\Phi'(s_0) \quad \forall s > s_0.$$

Hence $\lim_{s \rightarrow \infty} \Phi(s) = \infty$, which contradicts with (2.28).

Remark 2. Combining Proposition 2.4 and Proposition 2.8 yields to

$$\underline{U}(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\}, \quad \forall (x, t) \in Q_\infty. \quad (2.30)$$

As a consequence, the function \underline{U} has the following properties:

- (i) It is positive in Q_∞ , belongs to $C(\overline{Q} \setminus \{(0, 0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{0\}$.
- (ii) It satisfies (1.1) and

$$\lim_{t \rightarrow 0} \int_{B_\sigma} \underline{U}(x, t) dx = \infty, \quad \forall \sigma > 0. \quad (2.31)$$

In the sense of initial trace, \underline{U} has initial trace $\nu_0 := (\{0\}, 0)$ (here $\{0\}$ is the singular part and the Radon measure on $\mathbb{R}^N \setminus \{0\}$ is the zero measure).

By a simple adaptation of the proof of Proposition 2.4 and Proposition 2.8 it is possible to extend (2.30) to any positive solution vanishing on $\mathbb{R}^N \times \{0\} \setminus \{0\}$.

Proposition 2.9 *Assume (2.2) and the Keller-Osserman condition are satisfied. Then any positive solution u of (1.1) which $C(\overline{Q} \setminus \{(0, 0)\})$ and vanishes on $\mathbb{R}^N \times \{0\} \setminus \{0\}$ satisfies*

$$u(x, t) \leq \min\{\phi_\infty(t), \Phi(|x|)\}, \quad \forall (x, t) \in Q_\infty. \quad (2.32)$$

Proof. Estimate (2.23) is valid with u_k replaced by u . The remaining of the proof of Proposition 2.8 is similar and yields to the first estimate

$$u(x, t) \leq \Phi(|x|) \quad \forall (x, t) \in Q_\infty.$$

For the second estimate, due to the convexity, for any $R, \tau > 0$, $(x, t) \mapsto \phi_\infty(t - \tau) + w_R(x)$ is a supersolution of (1.1) in $B_R \times (\tau, \infty)$. It dominates u on the parabolic boundary, thus in the domain itself by the comparison principle. Since $h(r) > 0$ if $r > 0$, $\lim_{R \rightarrow \infty} w_R = 0$. Therefore

$$u(x, t) \leq \phi_\infty(t) = \lim_{\tau \rightarrow 0} \lim_{R \rightarrow \infty} (\phi_\infty(t - \tau) + w_R(x)) \quad \forall (x, t) \in Q_\infty.$$

This implies (2.32). □

It is also possible to construct a maximal solution of (1.1) in Q_∞ which satisfies (i). For $\ell > 0$ and $\epsilon > 0$, let $u := U_{\epsilon, \ell}$ be the solution of

$$\begin{cases} \partial_t u - \Delta u + uh(u) = 0 & \text{in } Q_\infty \\ u(x, 0) = \ell \chi_{B_\epsilon} & \text{in } \mathbb{R}^N. \end{cases}$$

Lemma 2.10 *For any $\tau > 0$ and $\epsilon > 0$, there exist $\ell > 0$ and $m(\tau, \epsilon) > 0$ such that any positive solution u of (1.1) which verifies (i) satisfies*

$$u(x, t) \leq U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon) \quad \forall (x, t) \in Q_\infty, t \geq \tau. \quad (2.33)$$

Furthermore

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = 0 \quad \forall \epsilon > 0. \quad (2.34)$$

Finally

$$\bar{U}(x, t) = \lim_{\tau \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{\ell \rightarrow \infty} (U_{\epsilon, \ell}(x, t - \tau) + m(\tau, \epsilon)) \quad (2.35)$$

is the maximal solution.

Proof. We set $\ell = \phi_\infty(\tau)$, then $u(x, \tau) \leq \ell$ for any $x \in \mathbb{R}^N$. Let $W := W_{\epsilon/2}$ be the solution of the following Cauchy-Dirichlet problem

$$\begin{cases} \partial_t W - \Delta W + Wh(W) = 0 & \text{in } B_{\epsilon/2}^c \times (0, \infty) \\ W(x, 0) = 0 & \text{in } B_{\epsilon/2}^c \\ W(x, t) = \phi_\infty(t) & \text{in } \partial B_{\epsilon/2}^c \times (0, \infty) \end{cases} \quad (2.36)$$

and put $m(\tau, \epsilon) := \max\{W_{\epsilon/2}(x, \delta) : |x| > \epsilon, 0 < \delta \leq \tau\}$. It is clear to see that

$$\lim_{\tau \rightarrow 0} m(\tau, \epsilon) = W_{\epsilon/2}(x, 0) = 0. \quad (2.37)$$

From the fact that $u(x, 0) = 0$ in $B_{\epsilon/2}^c$, $u(x, t) \leq \phi_\infty(t)$ in $\partial B_{\epsilon/2}^c \times (0, \infty)$ and the maximum principle, it follows that $u(x, t) \leq W_{\epsilon/2}(x, t)$ in $B_{\epsilon/2}^c \times (0, \infty)$.

Next we compare $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$ with u in $\mathbb{R}^N \times (\tau, \infty)$. The function $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon)$ is a supersolution of (1.1) in $\mathbb{R}^N \times (\tau, \infty)$. If $x \in B_\epsilon$, $U_{\epsilon, \ell}(x, 0) = \ell \geq u(x, \tau)$, which implies $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$. If $x \in B_\epsilon^c$, $m(\tau, \epsilon) \geq W_{\epsilon/2}(x, \tau) \geq u(x, \tau)$, hence $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$. So we always have $U_{\epsilon, \ell}(x, 0) + m(\tau, \epsilon) \geq u(x, \tau)$ for any $x \in \mathbb{R}^N$. Applying maximum principle yields to $U_{\epsilon, \ell}(\cdot, \cdot - \tau) + m(\tau, \epsilon) \geq u$ in $\mathbb{R}^N \times (\tau, \infty)$. Finally, the function \bar{U} defined by (2.35) is the maximal solution because $U_{\epsilon, \ell}(x, t - \tau) \rightarrow U_{\epsilon, \ell}(x, t)$ as $\tau \rightarrow 0$ and $U_{\epsilon, \ell} \uparrow U_{\epsilon, \infty}$ when $\ell \rightarrow \infty$ and $U_{\epsilon, \infty} \downarrow \bar{U}$ when $\epsilon \rightarrow 0$. \square

3 About uniqueness

Proposition 3.1 *Assume (2.16) holds. For any $a > 0$, there exists a unique positive function $w := w_a \in C^2([0, \infty))$ to the problem*

$$\begin{cases} -w'' - \frac{N-1}{r}w' + wh(w) = 0 & \text{in } \mathbb{R}_+ \\ w'(0) = 0 \\ w(0) = a \end{cases} \quad (3.1)$$

Proof. A solution of (3.1) is locally given by the formula

$$w(r) = a + \int_0^r s^{1-N} \int_0^s t^{N-1} w(t) h(w) dt ds \quad (3.2)$$

Existence follows from Picard-Lipschitz fixed point theorem. The function is increasing and defined on a maximal interval $[0, r_a)$. By a result of Vazquez and Veron [11] $r_a = \infty$, thus the solution is global. Uniqueness on $[0, r_a)$ follows always from local uniqueness. The function $r \mapsto w(r)$ is increasing and

$$\begin{aligned} w'(r) &\geq \frac{ah(a)}{N}r, \\ w(r) &\geq a + \frac{ah(a)}{2N}r^2 \end{aligned}$$

for all $r > 0$ □

Proposition 3.2 *Assume (2.16) holds. For any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.3)$$

for some $0 < a < b$, there exists a positive function $\bar{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ solution of (1.1) in Q_∞ and satisfying $\bar{u}(x, 0) = u_0$. Furthermore

$$w_a(|x|) \leq u(x, t) \leq w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (3.4)$$

Proof. Clearly w_a and w_b are ordered solutions of (1.1). We denote by u_n the solution of the initial-boundary problem

$$\begin{cases} \partial_t u_n - \Delta u_n + u_n h(u_n) = 0 & \text{in } Q_n = B_n \times (0, \infty) \\ u_n(x, 0) = u_0(x) & \text{in } B_n \\ u_n(x, t) = (w_a(|x|) + w_b(|x|))/2 & \text{in } \partial B_n \times (0, \infty). \end{cases} \quad (3.5)$$

By the maximum principle, u_n satisfies (3.4) in Q_n . Using locally parabolic equations regularity theory, we derive that the set of functions $\{u_n\}$ is eventually equicontinuous on any compact subset of $\overline{Q_\infty}$. Using a diagonal sequence, we conclude that there exists a subsequence $\{u_{n_k}\}$ which converges locally uniformly in $\overline{Q_\infty}$ to some weak solution $\bar{u} \in C(\overline{Q_\infty})$ which satisfies $\bar{u}(\cdot, 0) = u_0$. By standard method, \bar{u} is a strong solution (at least $C^{2,1}(Q_\infty)$). □

Proposition 3.3 *Assume (2.16) and (2.8) hold. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N \quad (3.6)$$

for some $0 < a < b$, there exists a positive function $\underline{u} \in C(\overline{Q_\infty})$ solution of (1.1) in Q_∞ satisfying $\underline{u}(x, 0) = u_0$ and

$$\underline{u}(x, t) \leq \min\{\phi_\infty(t), w_b(|x|)\} \quad \forall (x, t) \in Q_\infty. \quad (3.7)$$

Proof. For any $R > 0$, let u_R be the solution of

$$\begin{cases} \partial_t u_R - \Delta u_R + u_R h(u_R) = 0 & \text{in } Q_\infty \\ u_R(x, 0) = u_0(x) \chi_{B_R}(x) & \text{in } \mathbb{R}^N. \end{cases}$$

The solution which is constructed is dominated by the solution of the heat equation with the same initial data. Thus

$$u_R(x, t) \leq (4\pi t)^{-N/2} \int_{B_R} e^{-|x-y|^2/4t} u_0(x) dy \quad \forall (x, t) \in Q_\infty. \quad (3.8)$$

and $\lim_{|x| \rightarrow \infty} u_R(x, t) = 0$ uniformly with respect to t . The function ϕ_∞ and w_b are solution of (1.1) in Q_∞ , which dominate u_R at $t = 0$. By the maximum principle,

$$\min\{\phi_\infty(t), w_b(|x|)\} \geq u_R(x, t) \quad \forall (x, t) \in Q_\infty. \quad (3.9)$$

The fact that the mapping $R \mapsto u_R$ is increasing and (3.9) imply that there exists $\underline{u} := \lim_{R \rightarrow \infty} u_R$ which satisfies $\underline{u}(x, 0) = u_0(x)$ in \mathbb{R}^N . Letting $R \rightarrow \infty$ in (3.9) yields to (3.7). \square

Corollary 3.4 *Assume (2.16) and (2.8) hold. Then for any $u_0 \in C(\mathbb{R}^N)$ which satisfies*

$$w_a(|x|) \leq u_0(x) \leq w_b(|x|) \quad \forall x \in \mathbb{R}^N, \quad (3.10)$$

there exists two solutions $u \in C(\overline{Q_\infty})$ of (1.1) satisfying $u(\cdot, 0) = u_0$.

Proof. Combining Proposition 3.2 and Proposition 3.3 we see that there exists two solutions \underline{u} and \bar{u} with the same initial data u_0 which are ordered and different since $\lim_{|x| \rightarrow \infty} \bar{u}(x, t) = \infty$ and $\lim_{|x| \rightarrow \infty} \underline{u}(x, t) \leq \phi_\infty(t) < \infty$ for all $t > 0$. \square

Theorem 3.5 *Assume (2.16) holds. Let $u, \tilde{u} \in C(\overline{Q_\infty}) \cap C^{2,1}(Q_\infty)$ be two positive solutions of (1.1) with initial data u_0 . If for any $\epsilon > 0$,*

$$u(x, t) - \tilde{u}(x, t) = o(w_\epsilon(|x|)) \text{ as } x \rightarrow \infty \quad (3.11)$$

locally uniformly with respect to $t \geq 0$, then $u = \tilde{u}$.

Proof. Step 1. There always holds

$$(ah(a) - bh(b)) \text{sign}(a - b) \geq |a - b| h(|a - b|) \quad \forall a, b > 0. \quad (3.12)$$

In fact, since h is increasing and assuming $a > b$,

$$\begin{aligned} ah(a) - bh(b) &= (a - b)h(a) + b(h(a) - h(b)) \\ &\geq (a - b)h(a) \\ &\geq (a - b)h(a - b) \end{aligned}$$

Step 2. End of the proof. By Kato's inequality,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| \leq [\partial_t(u - \tilde{u}) - \Delta(u - \tilde{u})] \text{sign}(u - \tilde{u}),$$

therefore,

$$\partial_t |u - \tilde{u}| - \Delta |u - \tilde{u}| + |u - \tilde{u}| h(|u - \tilde{u}|) \leq 0. \quad (3.13)$$

Let $\epsilon > 0$. There exists $R_\epsilon > 0$ such that for any $R \geq R_\epsilon$,

$$0 \leq |u - \tilde{u}|(x, t) \leq w_\epsilon(|x|) \quad \forall (x, t) \in B_R^c \times [0, 1]. \quad (3.14)$$

Since w_ϵ is a positive solution of (1.1) which dominates $|u - \tilde{u}|$ on $\partial B_R \times [0, 1]$ and at $t = 0$, it follows that $|u - \tilde{u}| \leq w_\epsilon$ in Q_R . Letting $R \rightarrow \infty$ yields to $|u - \tilde{u}| \leq w_\epsilon$ in Q_∞ . Letting $\epsilon \rightarrow 0$ and since $\lim_{\epsilon \rightarrow 0} w_\epsilon(|x|) = 0$ for any $x \in \mathbb{R}^N$, we derive $|u - \tilde{u}| = 0$, thus $u = u^*$ for $0 \leq t \leq 1$. Iterating yields that equality holds in Q_∞ . Similarly $\tilde{u} = u^*$ and the claim follows. \square

4 Initial trace

Let Ω is an arbitrary open domain in \mathbb{R}^N , we denote by $\mathfrak{M}(\Omega)$ (resp. $\mathfrak{M}^b(\Omega)$) the set of Radon measures in Ω (resp. bounded Radon measures), and by $\mathfrak{M}_+(\Omega)$ (resp. $\mathfrak{M}_+^b(\Omega)$) its positive cone. We set $Q_T^\Omega = \Omega \times (0, T)$.

4.1 The regular part of the initial trace

In this section we only assume that f is a continuous nonnegative function defined on \mathbb{R}_+ and that u is a $C^{2,1}$ positive solution of

$$u_t - \Delta u + f(u) = 0 \quad (4.1)$$

in Q_T^Ω .

Lemma 4.1 *Assume G is a bounded C^2 domain in \mathbb{R}^N , $Q_T^{\overline{G}} := \overline{G} \times (0, T]$ and let $u \in C^{2,1}(Q_T^{\overline{G}})$ be a positive solution of (4.1) in Q_T^G such that $f(u) \in L^1(Q_T^G)$. Then $u \in L^\infty(0, T; L^1(G'))$ for any domain $G' \subset \overline{G'} \subset G$ and there exists a positive Radon measure μ_G on G such that*

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G \zeta(x) d\mu_G(x) \quad \forall \zeta \in C_0(G). \quad (4.2)$$

Proof. Let $\phi := \phi_G$ be the first eigenfunction of $-\Delta$ in $W_0^{1,2}(G)$ with corresponding eigenvalue λ_G . We assume $\phi > 0$. Then

$$\frac{d}{dt} \int_G u \phi dx + \lambda_G \int_G u \phi dx + \int_G f(u) \phi dx + \int_{\partial G} u \phi_{\mathbf{n}} dS = 0$$

Since $\phi_{\mathbf{n}} < 0$, the function

$$t \mapsto e^{\lambda_G t} \int_G u(x, t) \phi(x) dx - \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

is increasing and

$$\int_G u(x, t) \phi(x) dx \leq e^{\lambda_G(T-t)} \int_G u(x, T) \phi(x) dx + e^{-\lambda_G t} \int_t^T \int_G e^{\lambda_G s} f(u) \phi dx ds$$

for $0 < t \leq T$. Thus $u \in L^\infty(0, T; L^1(G'))$ for any strict domain G' of G . If $\zeta \in C_0^2(G)$, there holds

$$\frac{d}{dt} \left(\int_G u(x, t) \zeta(x) dx - \int_t^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds \right) = 0. \quad (4.3)$$

Consequently

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \int_G u(x, T) \zeta(x) dx + \int_0^T \int_G (f(u) \zeta - u \Delta \zeta) dx ds. \quad (4.4)$$

This implies that $u(\cdot, t)$ admits a limit in $\mathcal{D}'(G)$ and this limit is a positive distribution, then a measure. Therefore (4.2) follows by density. \square

Corollary 4.2 *Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (4.1) in Q_T^Ω . The set $\mathcal{R}(u)$ of the points $z \in \Omega$ such that there exists an open ball $B_r(z)$ such that $f(u) \in L^1 Q_T^{B_r(z)}$ is an open subset of Ω . Furthermore there exists a positive Radon measure $\mu := \mu(u)$ on $\mathcal{R}(u)$ such that*

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x) \quad \forall \zeta \in C_0(\mathcal{R}(u)). \quad (4.5)$$

Proof. It is clear that $\mathcal{R}(u)$ is an open subset of Ω . If G is a strict subdomain of Ω , i.e. $\overline{G} \subset \Omega$, there exists a finite number of points z_j ($j = 1, \dots, k$) and $r'_j > r_j > 0$ such that $f(u) \in L^1(Q_T^{B_{r'_j}(z_j)})$ and $\overline{G} \subset \cup_j B_{r_j}(z_j)$. Let $\mu_j = \mu_{B_{r_j}(z_j)}$ the measure defined in Lemma 4.1. If $\zeta \in C_0(G)$ there exists a partition η_j of unity relative to the $B_{r_j}(z_j)$ such that $\eta_j \in C_0^\infty(B_{r_j}(z_j))$ and $\zeta = \sum_j \eta_j \zeta$. Since

$$\lim_{t \rightarrow 0} \int u(x, t) (\eta_j \zeta)(x) dx = \int (\eta_j \zeta)(x) d\mu_j(x) \quad \forall j = 1, \dots, k,$$

relation (4.5). Notice also that $u \in L^\infty(0, T; L^1(G))$ for any $G \subset \overline{G} \subset \mathcal{R}(u)$. \square

Definition 4.3 *The set $\mathcal{R}(u)$ is called the regular set of the initial trace of u and the measure $\mu(u)$ the regular part of the initial trace. The set $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ is the singular part of the initial trace of u . It is relatively closed in Ω .*

The main problem is to analyse what does happen on the singular set $\mathcal{S}(u)$.

4.2 The Keller-Osserman condition holds

If the Keller-Osserman condition holds, the existence of an initial trace of arbitrary positive solutions of (1.1) is based upon a dichotomy in the behaviour of those solutions near $t = 0$.

Lemma 4.4 *Assume u is a positive solution of (4.1) in Q_T^Ω and $z \in \mathcal{S}(u)$. Suppose that at least one of the following sets of conditions holds*

(i) *There exists an open neighborhood G of z such that $u \in L^1(Q_T^G)$.*

(ii) *f is nondecreasing and (2.14) holds.*

Then for every open relative neighborhood G' of z

$$\lim_{t \rightarrow 0} \int_{G'} u(x, t) dx = \infty. \quad (4.6)$$

Proof. If z in $\mathcal{S}(u)$, then for every open relative neighborhood G' of z , there holds

$$\int_0^T \int_{G'} f(u) dx dt = \infty. \quad (4.7)$$

First, we assume (i) holds and let $\zeta \in C_0^2(G)$, $\zeta \geq 0$. Since there exists

$$\lim_{t \rightarrow 0} \int_t^T \int_{G'} u \Delta \zeta dx dt = L \in \mathbb{R},$$

it follows from (4.4) that

$$\int_{G'} u(x, t) \zeta(x) dx = \int_t^T \int_{G'} f(u) \zeta dx ds + O(1), \quad (4.8)$$

which implies (4.6).

Next we assume that (2.14) holds and $u \notin L^1(Q_T^G)$ for every relative neighborhood G of z . If there exists an open neighborhood $G \subset \Omega$ of z such that (4.6) does not hold, there exists a sequence $\{t_n\}$ decreasing to 0 and $0 \leq M < \infty$ such that

$$\sup_{t_n} \int_G u(x, t_n) dx = M. \quad (4.9)$$

Furthermore, we can always replace G by an open ball $B_R(z) \subset G$. Thus (4.9) holds with G replaced by $B_R(z)$. Let $w := w_R$ be the maximal solution of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } B_R(z) \\ \lim_{|x-z| \rightarrow R} w(x) = \infty. \end{cases} \quad (4.10)$$

Let $v := v_n$ be the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } B_R(z) \times (t_n, \infty) \\ v = 0 & \text{in } \partial B_R(z) \times (t_n, \infty) \\ v(\cdot, t_n) = u(\cdot, t_n) & \text{in } B_R(z). \end{cases} \quad (4.11)$$

Since $v_n \geq 0$, $f(w_R + v_n) \geq f(w_R)$, and $w_R + v_n$ is a supersolution. It dominates u on $\partial B_R(z) \times (t_n, T)$ and at $t = t_n$, thus $u \leq w_R + v_n$ in $B_R(z) \times (t_n, T)$. We can assume that $u(\cdot, t_n) \rightarrow \nu$ for some positive and bounded measure ν on $B_R(z)$. Therefore

$$u(x, t) \leq v(x, t) + w_R(x) \quad \text{in } Q_T^{B_R(z)} \quad (4.12)$$

where v is the solution of

$$\begin{cases} v_t - \Delta v = 0 & \text{in } Q_\infty^{B_R(z)} \\ v = 0 & \text{in } \partial B_R(z) \times (0, \infty) \\ v(\cdot, 0) = \nu & \text{in } \mathcal{D}'(B_R(z)). \end{cases} \quad (4.13)$$

Since $v \in L^1(Q_T^{B_R(z)})$ and w_R is uniformly bounded in any ball $B_{R'}(z)$ for $0 < R' < R$ we conclude that $u \in L^1(Q_T^{B_{R'}(z)})$, contradiction. \square

Definition 4.5 Let $u \in C^{2,1}(Q_T^\Omega)$ be a positive solution of (4.1) in Q_T^Ω . We say that u possesses an initial trace with regular part $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ and singular part $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ if

(i) For any $\zeta \in C_0(\mathcal{R}(u))$,

$$\lim_{t \rightarrow 0} \int_{\mathcal{R}(u)} u(x, t) \zeta(x) dx = \int_{\mathcal{R}(u)} \zeta(x) d\mu(x). \quad (4.14)$$

(ii) For any open set $G \subset \Omega$ such that $G \cap \mathcal{S}(u) \neq \emptyset$

$$\lim_{t \rightarrow 0} \int_G u(x, t) \zeta(x) dx = \infty. \quad (4.15)$$

Remark. To the couple $(\mathcal{S}(u), \mu)$, where $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ we can associate a unique outer regular Borel measure ν (i.e. $\nu \in \mathcal{B}_+^{reg}(\Omega)$) which is the initial trace of u . We shall denote it by $Tr^\Omega(u)$.

Theorem 4.6 Assume f satisfies the Keller-Osserman condition and is nondecreasing. If $u \in C^{2,1}(Q_T^\Omega)$ is a positive solution of (4.1), it possesses an initial trace $\nu \in \mathcal{B}_+^{reg}(\Omega)$.

Proof. The set $\mathcal{R}(u)$ and the measure $\mu \in \mathfrak{M}_+(\mathcal{R}(u))$ are defined by Definition 4.3 thanks to Corollary 4.2. Because (2.14) holds, $\mathcal{S}(u) = \Omega \setminus \mathcal{R}(u)$ inherits the property (ii) above because of Lemma 4.4 (ii). \square

As a counterpart of this result we have the following existence theorem.

Proposition 4.7 *Assume f satisfies the Keller-Osserman condition, (2.2) and is nondecreasing. Then for any outer regular positive Borel measure ν on \mathbb{R}^N there exists a positive solution $u \in C^{2,1}(Q_\infty)$ of (4.1) in Q_∞ , with initial trace ν .*

Proof. It is essentially similar to the one of [5, Th 3.4]-to be outlined. Notice that we can actually construct a maximal solution \bar{u}_ν and a minimal solution \underline{u}_ν with the same initial trace ν . \square

Remark. When $f(r) = |r|^{q-1}r$ with $1 < q < 1 + 2/N$, precise expansion of $u_{\infty\delta}(x, t)$, when $t \rightarrow 0$ allows to prove uniqueness. Even when $f(r) = r \ln^\alpha(r + 1)$ with $\alpha > 2$, uniqueness is not known.

4.3 The Keller-Osserman condition does not hold

In this section we assume that (2.14) does not hold but (2.1) is always satisfied. If Ω is a bounded domain with a C^2 boundary and $\mu \in \mathfrak{M}_+^b(\Omega)$, we denote by u_μ the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = 0 & \text{in } \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = \mu & \text{in } \mathcal{D}'\Omega. \end{cases} \quad (4.16)$$

We recall the following stability result proved in [6, Th 1.1].

Lemma 4.8 *Let Ω be a bounded domain with a C^2 boundary and $\{\mu_n\} \subset \mathfrak{M}_+^b(\Omega)$ a sequence weakly converging to some $\mu \in \mathfrak{M}_+^b(\Omega)$. Then $u_{\mu_n} \rightarrow u_\mu$ locally uniformly in $\bar{\Omega} \times (0, \infty)$ and in $L^1(Q_T^\Omega)$, and $f(u_{\mu_n}) \rightarrow f(u_\mu)$ in $L^1(Q_T^\Omega)$, for every $T > 0$.*

Remark. The result remains true if $\Omega = \mathbb{R}^N$ and μ_n have their support in a fixed compact set.

Lemma 4.9 *Assume (2.14) does not hold and (2.8), and (C3) are satisfied. If u is a positive solution of (1.1) in Q_∞ which satisfies*

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx = \infty, \quad (4.17)$$

for some bounded open subset $G \subset \mathbb{R}^N$, then $u(x, t) \geq \phi_\infty(t)$.

Proof. By assumption, there exists a sequence $\{t_n\}$ decreasing to 0 such that

$$\lim_{n \rightarrow \infty} \int_G u(x, t_n) dx = \infty. \quad (4.18)$$

If (4.17), we can construct a decreasing sequence of open subsets $G_k \subset G$ such that $\overline{G_k} \subset G_{k-1}$, $\text{diam}(G_k) = \epsilon_k \rightarrow 0$ when $k \rightarrow \infty$, and

$$\lim_{n \rightarrow \infty} \int_{G_k} u(x, t_n) dx = \infty \quad \forall k \in \mathbb{N}. \quad (4.19)$$

Furthermore there exists a unique $a \in \bigcap_k G_k$. We set

$$\int_{G_k} u(x, t_n) dx = M_{n,k}.$$

Since $\lim_{n \rightarrow \infty} M_{n,k} = \infty$, we claim that for any $m > 0$ and any k , there exists $n = n(k) \in \mathbb{N}$ such that

$$\int_{G_k} u(x, t_{n(k)}) dx \geq m. \quad (4.20)$$

By induction, we define $n(1)$ as the smallest integer n such that $M_{n,1} \geq m$. This is always possible. Then we define $n(2)$ as the smallest integer larger than $n(1)$ such that $M_{n,2} \geq m$. By induction, $n(k)$ is the smallest integer n larger than $n(k-1)$ such that $M_{n,k} \geq m$. Next, for any k , there exists $\ell = \ell(k)$ such that

$$\int_{G_k} \inf\{u(x, t_{n(k)}); \ell\} dx = m. \quad (4.21)$$

and we set

$$V_k(x) = \inf\{u(x, t_{n(k)}); \ell\} \chi_{G_k}(x).$$

Let $v_k = v$ be the unique bounded solution of

$$\begin{cases} v_t - \Delta v + f(v) = 0 & \text{in } Q_\infty \\ v(\cdot, 0) = V_k & \text{in } \mathbb{R}^N \end{cases} \quad (4.22)$$

Since $v(x, 0) \leq u(x, t_{n(k)})$, it follows

$$u(x, t + t_{n(k)}) \geq v_k(x, t) \quad \forall (x, t) \in Q_\infty. \quad (4.23)$$

When $k \rightarrow \infty$, $V_k \rightarrow m\delta_a$, thus $v_k \rightarrow u_{m\delta_a}$ by Lemma 4.8. Therefore $u \geq u_{m\delta_a}$. Since m is arbitrary and $u_{m\delta_a} \rightarrow \phi_\infty$ when $m \rightarrow \infty$ by Theorem 2.7, it follows that $u \geq \phi_\infty$. \square

Lemma 4.10 *Assume (2.14) does not hold and (2.7), and (C3) are satisfied. There exists no u positive solution of (1.1) in Q_∞ which satisfies (4.17) for some bounded open subset $G \subset \mathbb{R}^N$.*

Proof. If we assume that such a u exists, we proceed as in the proof of the previous lemma. Since Lemma 4.8 holds, we derive that $u \geq u_{m\delta_a}$ for any m . Since $\lim_{m \rightarrow \infty} u_{m\delta_a}(x, t) = \infty$ for all $(x, t) \in Q_\infty$, we derive a contradiction. \square

Thanks to these results, we can characterize the initial trace of positive solutions of (4.1) when the Keller-Osserman condition does not hold.

Theorem 4.11 *Assume f is nondecreasing, (2.14) does not hold but (2.8), and (C3) are verified. If u is a positive solution of (4.1) in Q_∞ , it possesses an initial trace which is either the Borel measure infinity ν_∞ which satisfies $\nu_\infty(\mathcal{O}) = \infty$ for any open subset $\mathcal{O} \subset \mathbb{R}^N$, or is a positive Radon measure μ on \mathbb{R}^N .*

Proof. If there exists some open subset G of \mathbb{R}^N with the property (4.17), then $u \geq \phi_\infty$ and the initial trace of u is the Borel measure ν_∞ . Next we assume that for any bounded open subset G of \mathbb{R}^N there holds

$$\limsup_{t \rightarrow 0} \int_G u(x, t) dx < \infty, \quad (4.24)$$

If $\mathcal{S}(u) \neq \emptyset$, there exists $z \in \mathbb{R}^N$ and an bounded open neighborhood G of z such that

$$\int_0^T \int_G f(u) dx dx t = \infty$$

By (4.24), $u \in L^\infty(0, T; L^1(G)) \subset L^1(Q_T^G)$. Then, by Lemma 4.4, (4.6) holds, which contradict (4.24). Thus $\mathcal{S}(u) = \emptyset$ and $\mathcal{R}(u) = \mathbb{R}^N$. It follows from Corollary 4.2 that there exists a positive Radon measure μ such that

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^N} u(x, t) \zeta(x) dx = \int_{\mathbb{R}^N} \zeta(x) d\mu(x) \quad \forall \zeta \in C_0(\mathbb{R}^N), \quad (4.25)$$

\square

Because of the lack of uniqueness from Corollary 3.4 it is difficult to give a complete characterization of admissible initial data for solutions of 4.1 under the assumptions of Theorem 4.11. However, we have the following result

Proposition 4.12 *Under the assumptions of Theorem 4.11, for any $b > 0$ there exists a positive solution $u \in C(Q_\infty)$ of (4.1) satisfying*

$$\max\{\phi_\infty(t); w_b(|x|)\} \leq u(x, t) \leq \phi_\infty(t) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (4.26)$$

Consequently there exist infinitely many positive solutions of (4.1) with initial trace ν_∞ . Furthermore ϕ_∞ is the smallest of all these solutions.

Proof. We first notice that $\max\{\phi_\infty(t); w_b(|x|)\}$ is a subsolution of (4.1) which is dominated by the supersolution $\phi_\infty(t) + w_b(|x|)$. The process is standard: for $\tau > 0$ we set

$$\psi(x, \tau) = \frac{1}{2} (\max\{\phi_\infty(t); w_b(|x|)\} + \phi_\infty(t) + w_b(|x|))$$

There exists a function $u = u_\tau \in C(\overline{Q_\infty})$ solution of (4.1) in Q_∞ satisfying $u_\tau(\cdot, 0) = \psi(\cdot, \tau)$. Furthermore

$$\max\{\phi_\infty(t + \tau); w_b(|x|)\} \leq u_\tau(x, t) \leq \phi_\infty(t + \tau) + w_b(|x|) \quad \forall (x, t) \in Q_\infty. \quad (4.27)$$

By parabolic equation regularity theory, the set $\{u_\tau\}$ is locally equicontinuous in Q_∞ . Thus there exists a subsequence $\{\tau_n\}$ and $u \in C(Q_\infty)$ such that $u_{\tau_n} \rightarrow u$ on any compact subset of Q_∞ . Clearly u is a weak, thus a strong solutions of (4.1) and it satisfies (4.26). Since any solution u with initial trace ν_∞ dominates ϕ_∞ by Lemma 4.9, it follows that ϕ_∞ is the minimal one. \square

Theorem 4.13 *Assume f is nondecreasing, (2.14) does not hold and (2.7), and (C3) are verified. If u is a positive solution of (4.1) in Q_∞ , it possesses an initial trace which is a positive Radon measure μ on \mathbb{R}^N .*

Proof. As in the proof of Theorem 4.11 and because of Lemma 4.10, $\mathcal{S}(u) = \emptyset$. Therefore $\mathcal{R}(u) = \mathbb{R}^N$ and the proof follows from Corollary 4.2. \square

Remark. Under the assumptions of Theorem 4.11, it is clear, from the proof of Proposition 3.2, that for any $0 < a < b$ and any initial data $u_0 \in C(\mathbb{R}^N)$ satisfying

$$w_a(x) \leq u_0(x) \leq w_b(x) \quad \forall x \in \mathbb{R}^N$$

there exists a solution $u \in C(\overline{Q_\infty})$ of (4.1) in Q_∞ satisfying $u(\cdot, 0) = u_0$ and

$$w_a(x) \leq u(x, t) \leq w_b(x) \quad \forall (x, t) \in Q_\infty.$$

We conjecture that for any positive measure μ on \mathbb{R}^N which satisfies, for some $b > 0$,

$$\int_{B_R} d\mu(x) \leq \int_{B_R} w_b(x) dx \quad \forall R > 0 \quad (4.28)$$

there exists a positive solution u of (4.1) in Q_∞ with initial trace μ . Another interesting open problem is to see if there exists local solutions in Q_T with an initial trace μ satisfying

$$\lim_{R \rightarrow \infty} \frac{\int_{B_R} d\mu(x)}{\int_{B_R} w_b(x) dx} = \infty \quad \forall b > 0. \quad (4.29)$$

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