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ABSTRACT. Let $\mathcal{G}$ be the (open) set of $\dot{H}^{\frac{1}{2}}$ divergence free vector fields generating a global smooth solution to the three dimensional incompressible Navier-Stokes equations. We prove that any element of $\mathcal{G}$ can be perturbed by an arbitrarily large, smooth divergence free vector field which varies slowly in one direction, and the resulting vector field (which remains arbitrarily large) is an element of $\mathcal{G}$ if the variation is slow enough. This result implies that through any point in $\mathcal{G}$ passes an uncountable number of arbitrarily long segments included in $\mathcal{G}$.

1. INTRODUCTION

1.1. **Setting of the problem and statement of the result.** Let us first recall the classical Navier-Stokes system for incompressible fluids in three space dimensions:

$$(NS) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0 \end{cases}$$

where $u(t,x)$ denotes the fluid velocity and $p(t,x)$ the pressure. In this paper the space variable x is chosen in $\mathbb{R}^3$.

To start with, let us recall the history of results concerning small data. The first one states that if the initial data u_0 is such that $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$ is small enough, a global regular solution exists; this was proved by J. Leray in his seminal paper [12]. Then, starting with the paper by H. Fujita and T. Kato (see [5]), the following approach was developped: let us denote by $\mathbb{B}$ the bilinear operator defined by

$$\begin{cases} \partial_t \mathbb{B}(v,w) - \Delta \mathbb{B}(v,w) = \dfrac{1}{2}\mathbb{P}\operatorname{div}(v \otimes w + w \otimes v) \\ \mathbb{B}(v,w)_{|t=0} = 0 \end{cases}$$

where $\mathbb{P}$ denotes the Leray projection onto divergence free vector fields. Then, it is easily checked that u is a solution of (NS) if and only if

$$u = e^{t\Delta}u_0 + \mathbb{B}(u,u)$$

which is something like Duhamel's formula. Then the theory of small initial data reduces to finding a Banach space X of time-dependent divergence free vector fields on $\mathbb{R}^+ \times \mathbb{R}^3$ such that $\mathbb{B}$ is a bilinear map from $X \times X$ to X. An elementary abstract fixed point theorem claims that if X is a Banach space of time-dependent divergence free vector fields on $\mathbb{R}^+ \times \mathbb{R}^3$ such that

$$\|\mathbb{B}(v,w)\|_X \leq C\|v\|_X\|w\|_X$$

(X will be called from now on an adapted space), a solution of (NS) exists in X and is global as soon as

$$\|e^{t\Delta}u_0\|_X \leq (4C)^{-1}.$$

The search of the largest possible adapted space X is a long story. It started in 1964 with the paper [5] where the space X is defined by the norm

$$\|u\|_X \stackrel{\text{def}}{=} \sup_{t\geq 0} t^{\frac{1}{4}} \|\nabla u(t)\|_{L^2}.$$

After a number of important steps (see in particular [6], [9], [17] and [1]), the problem of finding the largest adapted space was achieved by H. Koch and D. Tataru. They proved in [10] that the space of time-dependent divergence free vector fields on $\mathbb{R}^+ \times \mathbb{R}^3$ such that

$$\|u\|_{X_{KT}} \stackrel{\text{def}}{=} \sup_{t\geq 0} t^{\frac{1}{2}} \|u(t)\|_{L^\infty} + \sup_{\substack{x\in\mathbb{R}^3 \\ R>0}} \frac{1}{R^{\frac{3}{2}}} \left(\int_{P(x,R)} |u(t,y)|^2 dydt \right)^{\frac{1}{2}} < \infty$$

where $P(x,R)$ is the parabolic ball $[0,R^2] \times B(x,R)$, is an adapted space.

Now let us observe that the incompressible Navier-Stokes system is translation and scaling invariant: if u is a solution of (NS) on $[0,T] \times \mathbb{R}^3$ then, for any positive λ and for any x_0 in $\mathbb{R}^3$, the vector field u_{λ,x_0} defined by

$$u_{\lambda,x_0}(t,x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda(x-x_0))$$

is also a solution of (NS) on $[0,\lambda^{-2}T] \times \mathbb{R}^3$. Thus, an adapted space must be translation and scaling invariant in the following sense: a constant C exists such that, for any positive λ and for any x_0 in $\mathbb{R}^3$,

$$C^{-1}\|u\|_X \leq \|u_{\lambda,x_0}\|_X \leq C\|u\|_X.$$

The second term appearing in the norm $\|\cdot\|_{X_{KT}}$ above comes from the fact that the solution of (NS) should be locally in L^2 in order to be able to define the product as a locally L^1 function. The relevant norm on the initial data is $\|e^{t\Delta}u_0\|_X$. In the case of the Koch and Tataru theorem, this norm turns out to be equivalent to the norm of the space ∂BMO of derivatives of BMO functions. Of course, the space of initial data which measures the size of the initial data must be translation and scaling invariant. A remark due to Y. Meyer (see [14]) is that the norm in such a space is always greater than the norm in the Besov space $\dot{B}^{-1}_{\infty,\infty}$ defined by

$$\|u\|_{\dot{B}^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t\geq 0} t^{\frac{1}{2}} \|u(t)\|_{L^\infty}.$$

This leads to the following definition of a large initial data for the incompressible Navier-Stokes equation.

Definition 1.1. *A divergence free vector field u_0 is a large initial data for the incompressible Navier-Stokes system if its $\dot{B}^{-1}_{\infty,\infty}$ norm is large.*

Let us point out that this approach using Duhamel's formula does not use the very special structure of the incompressible Navier-Stokes system. A family of results does use the special structure of (NS): in those cases some geometrical invariance on the initial data is preserved by the flow of (NS) and this leads to some unexpected conservation of quantities, which makes the problem subcritical and thus prevents blow up. We refer for instance to [11], [13], [15], or [16], where special symmetries (like helicoidal, or axisymmetric without swirl) allow to prove global wellposedness for any data.

Some years ago, the first two authors investigated the possible existence of large initial data (in the sense of Definition 1.1) which have no preserved geometrical invariance and which nevertheless generate global regular solution. The first result in this direction was proved in [2] where such a family of large initial data was constructed, with strong oscillations in one direction. The main point of the proof is that for any element of this family, the first iterate $\mathbb{B}(e^{t\Delta}u_0, e^{t\Delta}u_0)$ is exponentially small with respect to the large initial data u_0 in

some appropriate norm. Let us notice that this result does use the fine structure of the non linear term of (NS): M. Paicu and the second author proved in [8] that for a modified incompressible Navier-Stokes system, this family of initial data generates solutions that blow up at finite time.

In [3], the first two authors constructed another class of examples, in which the initial data has slow variations in one direction. The proof of global regularity uses the fact that the 2D Navier-Stokes equations are globally wellposed. The initial data presented in the next theorem will be referred to in the following as "quasi-2D" (see [3]).

Theorem 1. *Let $v_0^h = (v_0^1, v_0^2)$ be a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ (i.e. v_0^h is in $L^2(\mathbb{R}^3)$ as well as all its derivatives), belonging, as well as all its derivatives, to $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$; let w_0 be a smooth divergence free vector field on $\mathbb{R}^3$. Then there exists a positive ε_0 such that if $\varepsilon \leq \varepsilon_0$, the initial data*

$$u_{0,\varepsilon}(x) \stackrel{def}{=} (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3)$$

generates a unique, global solution u^ε of (NS).

Remark 1.2. It is clear from the proof of [3] that the dependence of the parameter ε_0 on the profiles v_0^h and w_0 is only through their norms.

Note that such an initial data may be arbitrarily large in the sense of Definition 1.1 (see [3]). We recall for the convenience of the reader the result proved in [3].

Proposition 1.3. *Let (f, g) be in $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R})$ and define $h_\varepsilon(x_h, x_3) \stackrel{def}{=} f(x_h)g(\varepsilon x_3)$. We have, if ε is small enough,*

$$\|h_\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3)} \geq \frac{1}{4}\|f\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})}.$$

In this paper we consider the global wellposedness of the Navier-Stokes equations with data which is the sum of an initial data (which may be large) giving rise to a global solution, and a quasi-2D initial data as presented above (which may also be large). The theorem is the following.

Theorem 2. *Let u_0, v_0^h and w_0 be three smooth divergence free vector fields defined on $\mathbb{R}^3$, satisfying*

- *u_0 belongs to $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and generates a unique global solution to the Navier-Stokes equations;*
- *$v_0^h = (v_0^1, v_0^2)$ is a horizontal, smooth divergence free vector field on $\mathbb{R}^3$ belonging, as well as all its derivatives, to the space $L^2(\mathbb{R}_{x_3}; \dot{H}^{-1}(\mathbb{R}^2))$;*
- *$v_0^h(x_1, x_2, 0) = w_0^3(x_1, x_2, 0) = 0$ for all $(x_1, x_2) \in \mathbb{R}^2$.*

Then there exists a positive number ε_0 depending on u_0 and on norms of v_0^h and w_0 such that for any $\varepsilon \in (0, \varepsilon_0]$, there is a unique, global solution to the Navier-Stokes equations with initial data

$$u_{0,\varepsilon}(x) \stackrel{def}{=} u_0(x) + (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3).$$

Remark 1.4. Let u_0 be any element of the (open) set $\mathcal{G}$ of $\dot{H}^{\frac{1}{2}}$ divergence free vector fields generating a global smooth solution to (NS), and let N be an arbitrarily large number. Then for any smooth divergence free vector field f^h (over $\mathbb{R}^2$) and g (over $\mathbb{R}$) satisfying $\|f^h\|_{\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^2)}\|g\|_{L^\infty(\mathbb{R})} \geq 4N$, and such that $g(0) = 0$, Theorem 2 implies that there is ε_N depending on u_0 and on norms of f^h and g such that $u_0 + (f^h \otimes g, 0)(x_1, x_2, \varepsilon_N x_3)$ belongs to $\mathcal{G}$. Since ε_N only depends on norms of f^h and g, that implies that for any $\lambda \in [-1, 1]$, the initial data $u_0 + \lambda(f^h \otimes g, 0)(x_1, x_2, \varepsilon_N x_3)$ also belongs to $\mathcal{G}$. Using Proposition 1.3 one

concludes that through u_0 passes an uncountable number of segments of length N included in $\mathcal{G}$.

Remark 1.5. With the notation of Theorem 2, the data $u_0(x) + (v_0^h + \varepsilon w_0^h, w_0^3)(x_1, x_2, \varepsilon x_3)$ belongs to $\mathcal{G}$ as soon as ε is small enough, so one can add to that initial data any vector field of the type $(v_0^{h(1)} + \varepsilon_1 w_0^{h(1)}, w_0^{3(1)})(x_1, x_2, \varepsilon_1 x_3)$ and if ε_1 is small enough (depending on u_0, on ε, and on norms of v_0^h, w_0, $v_0^{h(1)}$ and $w_0^{(1)}$), then the resulting vector field belongs to $\mathcal{G}$. One thus immediately constructs by induction superpositions of the type

$$u_0(x) + \sum_{j=0}^{J} (v_0^{h(j)} + \varepsilon_j w_0^{h(j)}, w_0^{3(j)})(x_1, x_2, \varepsilon_j x_3)$$

which belong to $\mathcal{G}$ for small enough ε_j's, depending on u_0, on the norms of the profiles $v_0^{h(j)}$ and $w_0^{(j)}$, and on $(\varepsilon_k)_{k<j}$.

Finally notice that one can also require the slow variation on the profiles to hold on another coordinate than x_3, up to obvious modifications of the assumptions of the theorem.

Remark 1.6. In [4], an even larger initial data than the one of Theorem 1 is constructed (the size of the solution blows up when ε tends to 0); this is a strong obstacle to the use of a perturbative argument such as the one we use here.

1.2. **Scheme of the proof and organization of the paper.** Let us start by introducing some notation. In the following we shall denote, for any point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, its horizontal coordinates by $x_h \overset{\text{def}}{=} (x_1, x_2)$. Similarly the horizontal components of any vector field $u = (u^1, u^2, u^3)$ will be denoted by $u^h \overset{\text{def}}{=} (u^1, u^2)$ and the horizontal divergence will be defined by $\mathrm{div}_h u^h \overset{\text{def}}{=} \nabla^h \cdot u^h$, where $\nabla^h \overset{\text{def}}{=} (\partial_1, \partial_2)$. Finally we shall define the horizontal Laplacian by $\Delta_h \overset{\text{def}}{=} \partial_1^2 + \partial_2^2$. We shall often use the following shorthand notation for slowly varying functions: for any function f defined on $\mathbb{R}^3$, we write

$$[f]_\varepsilon(x_h, x_3) \overset{\text{def}}{=} f(x_h, \varepsilon x_3). \tag{1.1}$$

In order to prove Theorem 2, we look for the solution (which exists and is smooth for a short time depending on ε, due to classical existence theory) under the form

$$u_\varepsilon \overset{\text{def}}{=} u_\varepsilon^{app} + R_\varepsilon \tag{1.2}$$

where the approximate solution u_ε^{app} is defined by the sum of the global solution associated with u_0 and the quasi-2D approximation:

$$u_\varepsilon^{app} \overset{\text{def}}{=} u + [v_\varepsilon^{(2D)}]_\varepsilon \quad \text{with} \quad v_\varepsilon^{(2D)} \overset{\text{def}}{=} (v^h, 0) + (\varepsilon w_\varepsilon^h, w_\varepsilon^3) \tag{1.3}$$

while

- u is the global smooth solution of (NS) associated with the initial data u_0;
- v^h is the global smooth solution of the two dimensional Navier-Stokes equation (with parameter y_3 in $\mathbb{R}$) with pressure p_0 and data $v_0^h(\cdot, y_3)$

$$(NS2D_3) \qquad \begin{cases} \partial_t v^h + v^h \cdot \nabla^h v^h - \Delta_h v^h = -\nabla^h p_0 \\ \mathrm{div}_h\, v^h = 0 \\ v^h|_{t=0} = v_0^h(x_h, y_3); \end{cases}$$

- w_ε solves the linear equation with data w_0 (and pressure $p_{\varepsilon,1}$)

$$(T_v^\varepsilon) \begin{cases} \partial_t w_\varepsilon + v^h \cdot \nabla^h w_\varepsilon - \Delta_h w_\varepsilon - \varepsilon^2 \partial_3^2 w_\varepsilon = -(\nabla^h p_{\varepsilon,1}, \varepsilon^2 \partial_3 p_{\varepsilon,1}) \\ \mathrm{div}\, w_\varepsilon = 0 \\ w_\varepsilon|_{t=0} = w_0. \end{cases}$$

We will also define the approximate pressure

$$p_\varepsilon^{app} \stackrel{\text{def}}{=} p + [p_0 + \varepsilon p_{\varepsilon,1}]_\varepsilon. \tag{1.4}$$

The stability of this approximate solution is described by the following proposition, which is proved in [3], Lemma 2.1. As in the rest of this paper, we have used the following notation: if X (resp. Y) is a function space over $\mathbb{R}^2$ (resp. $\mathbb{R}$), then we write X_h for $X(\mathbb{R}^2)$ and Y_v for $Y(\mathbb{R})$. We also denote the space $Y(\mathbb{R}; X(\mathbb{R}^2))$ by $Y_v X_h$.

Proposition 1.7 ([3])**.** *For any positive ε_0, the family $(u_\varepsilon^{app})_{\varepsilon\leq\varepsilon_0}$ of approximate solutions is uniformly bounded in $L^2(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$ and the family $(\nabla u_\varepsilon^{app})_{\varepsilon\leq\varepsilon_0}$ is uniformly bounded in $L^2(\mathbb{R}^+; L^\infty_v(L^2_h))$.*

The size of the error term E_ε (this denomination will become apparent in the next section) defined by

$$E_\varepsilon \stackrel{\text{def}}{=} (\partial_t - \Delta)u_\varepsilon^{app} + u_\varepsilon^{app}\cdot\nabla u_\varepsilon^{app} + (0, \partial_3[p_0]_\varepsilon) \tag{1.5}$$

can be estimated as follows.

Proposition 1.8. *The family $(E_\varepsilon)_{\varepsilon\leq\varepsilon_0}$ of error terms satisfies*

$$\lim_{\varepsilon\to 0} \|E_\varepsilon\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac{1}{2}})} = 0.$$

The structure of this article is the following:

- the second section is devoted the proof of Theorem 2 using the above two propositions;
- the third section consists in proving Proposition 1.8 using estimates on the product in anisotropic spaces.

2. Proof of Theorem 2

Assuming Proposition 1.8, the proof of Theorem 2 follows the sames lines as the proof of Theorem 3 of [3]; we recall it for the reader's convenience. Using the definition of the approximate solution $(u_\varepsilon^{app}, p_\varepsilon^{app})$ given in (1.3,1.4), and the error term E_ε given in (1.5), we find that the remainder R_ε satisfies the following modified three-dimensional Navier-Stokes equation

$$(MNS_\varepsilon) \qquad \left\{ \begin{array}{c} \partial_t R_\varepsilon + R_\varepsilon\cdot\nabla R_\varepsilon - \Delta R_\varepsilon + u_\varepsilon^{app}\cdot\nabla R_\varepsilon + R_\varepsilon\cdot\nabla u_\varepsilon^{app} = -E_\varepsilon - \nabla q_\varepsilon \\ \operatorname{div} R_\varepsilon = 0 \quad \text{and} \quad R_\varepsilon|_{t=0} = 0, \end{array} \right.$$

with $q_\varepsilon \stackrel{\text{def}}{=} p_\varepsilon - p_\varepsilon^{app}$. The proof of the theorem reduces to the proof that (MNS_ε) is globally wellposed. In order to do so, let us define

$$R_\varepsilon^\lambda(t) \stackrel{\text{def}}{=} R_\varepsilon(t)\exp\left(-\lambda\int_0^t V_\varepsilon(t')\,dt'\right) \quad \text{with} \quad V_\varepsilon(t) \stackrel{\text{def}}{=} \|u_\varepsilon^{app}(t)\|^2_{L^\infty} + \|\nabla u_\varepsilon^{app}(t)\|^2_{L^\infty_v L^2_h}.$$

Note that Proposition 1.7 implies that $\displaystyle\int_{\mathbb{R}^+} V_\varepsilon(t)\,dt$ is uniformly bounded, by a constant denoted by U in the following. Writing also $E_\varepsilon^\lambda(t) \stackrel{\text{def}}{=} E_\varepsilon(t)\exp\left(-\lambda\displaystyle\int_0^t V_\varepsilon(t')\,dt'\right)$, an $\dot{H}^{\frac{1}{2}}$ energy estimate on (MNS_ε) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|R_\varepsilon^\lambda(t)\|^2_{\dot{H}^{\frac{1}{2}}} + \|\nabla R_\varepsilon^\lambda(t)\|^2_{\dot{H}^{\frac{1}{2}}} &= -2\lambda V_\varepsilon(t)\|R_\varepsilon^\lambda(t)\|^2_{\dot{H}^{\frac{1}{2}}} - \big(E_\varepsilon^\lambda|R_\varepsilon^\lambda\big)_{\dot{H}^{\frac{1}{2}}}(t) \\ &\quad - \Big(\exp\Big(\lambda\int_0^t V_\varepsilon(t')dt'\Big)R_\varepsilon^\lambda\cdot\nabla R_\varepsilon^\lambda + u_\varepsilon^{app}\cdot\nabla R_\varepsilon^\lambda + R_\varepsilon^\lambda\cdot\nabla u_\varepsilon^{app}\big|R_\varepsilon^\lambda\Big)_{\dot{H}^{\frac{1}{2}}}(t).\end{aligned}$$

A law of product in Sobolev spaces and Proposition 1.7 imply that

$$\exp\Big(\lambda\int_0^t V_\varepsilon(t')dt'\Big)(R_\varepsilon^\lambda\cdot\nabla R_\varepsilon^\lambda|R_\varepsilon^\lambda)_{\dot H^{\frac12}}\leq Ce^{\lambda U}\|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}\|\nabla R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}.$$

Lemma 2.3 of [3] claims that

$$(b\cdot\nabla a+a\cdot\nabla b|b)_{\dot H^{\frac12}}\leq C\big(\|a\|_{L^\infty}+\|\nabla a\|_{L^\infty_vL^2_h}\big)\|b\|_{\dot H^{\frac12}}\|\nabla b\|_{\dot H^{\frac12}},\tag{2.1}$$

so by definition of V_ε we get

$$(u_\varepsilon^{app}\cdot\nabla R_\varepsilon^\lambda+R_\varepsilon^\lambda\cdot\nabla u_\varepsilon^{app}|R_\varepsilon^\lambda)_{\dot H^{\frac12}}\leq\frac14\|\nabla R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}+CV_\varepsilon(t)\|R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}.$$

Let us choose $\lambda=C/2$. Then using the fact that

$$\big|(E_\varepsilon^\lambda|R_\varepsilon^\lambda)_{\dot H^{\frac12}}(t)\big|\leq\frac14\|\nabla R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}+C\|E_\varepsilon^\lambda(t)\|^2_{\dot H^{-\frac12}}$$

we obtain

$$\frac{d}{dt}\|R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}+\frac12\|\nabla R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}\leq C\|E_\varepsilon(t)\|^2_{\dot H^{-\frac12}}+Ce^{CU}\|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}\|\nabla R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}.$$

Since $R_{\varepsilon|t=0}=0$ and $\lim\limits_{\varepsilon\to0}\|E_\varepsilon\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12})}=0$ by Proposition 1.8, we deduce that as long as $\|R_\varepsilon^\lambda(t)\|_{\dot H^{\frac12}}$ is smaller than $1/4Ce^{-CU}$, then for any $\eta>0$ there is ε_0 such that

$$\forall\varepsilon\leq\varepsilon_0,\quad\|R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}+\frac14\int_0^t\|\nabla R_\varepsilon^\lambda(t')\|^2_{\dot H^{\frac12}}\,dt'\leq\eta,$$

which in turn implies that

$$\forall\varepsilon\leq\varepsilon_0,\quad\forall t\in\mathbb{R}^+,\quad\|R_\varepsilon^\lambda(t)\|^2_{\dot H^{\frac12}}+\frac14\int_0^t\|\nabla R_\varepsilon^\lambda(t')\|^2_{\dot H^{\frac12}}\,dt'\leq\eta.$$

That concludes the proof of the theorem. □

3. The estimate of the error term

In this section, we shall prove Proposition 1.8. Let us first remark that the error term E_ε can be decomposed as

$$E_\varepsilon=E_\varepsilon^1+E_\varepsilon^2\quad\text{with}\quad E_\varepsilon^2\overset{\text{def}}{=}u\cdot\nabla[v_\varepsilon^{(2D)}]_\varepsilon+[v_\varepsilon^{(2D)}]_\varepsilon\cdot\nabla u.$$

Thus the term E_ε^1 is exactly the error term which appears in [3], and Lemmas 4.1, 4.2 and 4.3 of [3] imply that

$$\|E_\varepsilon^1\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12})}\leq C_0\varepsilon^{\frac13}.\tag{3.2}$$

In order to estimate the term E_ε^2, let us first observe that Lemmas 3.1 and 3.2 of [3] imply the following proposition.

Proposition 3.1 ([3])**.** *For any s greater than -1, for any $\alpha\in\mathbb{N}^3$ and for any positive t, we have*

$$\|\partial^\alpha v_\varepsilon^{(2D)}(t)\|^2_{L^2_v\dot H^s_h}+\int_0^t\|\partial^\alpha\nabla^h v_\varepsilon^{(2D)}(t')\|^2_{L^2_v\dot H^s_h}dt'\leq C_0.$$

We shall also be using the following result, whose proof is postponed to the end of this paragraph.

Proposition 3.2. *The vector field $v_\varepsilon^{(2D)}$ satisfies*

$$\|v_\varepsilon^{(2D)}(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2_h)}+\|\nabla^h v_\varepsilon^{(2D)}(\cdot,0)\|_{L^2(\mathbb{R}^+;L^2_h)}\leq C\varepsilon^{\frac12}.$$

Assuming this result, let us prove Proposition 1.8. The stability theorem of [7] claims in particular that

$$\lim_{t\to\infty}\|u(t)\|_{\dot H^{\frac12}}=0.$$

As the set of smooth compactly supported divergence free vector field is dense in the space of $\dot H^{\frac12}$ divergence free vector fields, this allows to construct for any positive η, a family $(t_j)_{1\leq j\leq N}$ of positive real numbers and a family $(\phi_j)_{1\leq j\leq N}$ of smooth compactly supported divergence free vector fields such that (with $t_0=0$)

$$\|\underline{u}_\eta\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12})}\leq\eta\quad\text{with}\quad\underline{u}_\eta\overset{\text{def}}{=}u-\widetilde{u}_\eta\quad\text{and}\quad\widetilde{u}_\eta(t,x)\overset{\text{def}}{=}\sum_{j=1}^N\mathbf{1}_{[t_{j-1},t_j]}(t)\phi_j(x).\tag{3.3}$$

Then, for any positive η, let us decompose E^2_ε as

$$E^2_\varepsilon=\underline{E}_{\varepsilon,\eta}+\widetilde{E}_{\varepsilon,\eta}\quad\text{with}\quad\underline{E}_{\varepsilon,\eta}\overset{\text{def}}{=}\underline{u}_\eta\cdot\nabla[v^{(2D)}_\varepsilon]_\varepsilon+[v^{(2D)}_\varepsilon]_\varepsilon\cdot\nabla\underline{u}_\eta.\tag{3.4}$$

The term $\underline{E}_{\varepsilon,\eta}$ will be estimated thanks to the following lemma which is a generalization of (2.1).

Lemma 3.3. *Let a and b be two smooth functions. We have*

$$\|ab\|_{\dot H^{\frac12}}\leq C\|a\|_{\dot H^{\frac12}}\big(\|\nabla^h b\|_{L^\infty_v(L^2_h)}+\|b\|_{L^\infty}+\|\partial_3 b\|_{L^2_v(\dot H^{\frac12}_h)}\big).$$

Proof. Let us first notice that, for any function f in $\dot H^{\frac12}(\mathbb{R}^3)$

$$\|f\|_{\dot H^{\frac12}}\leq\|f\|_{L^2_h\dot H^{\frac12}_v}+\|f\|_{L^2_v\dot H^{\frac12}_h}.\tag{3.5}$$

Now we observe that by two-dimensional product laws, one has for any x_3 in $\mathbb{R}$

$$\|a(\cdot,x_3)b(\cdot,x_3)\|_{\dot H^{\frac12}_h}\leq C\big(\|a(\cdot,x_3)\|_{\dot H^{\frac12}_h}\|\nabla^h b(\cdot,x_3)\|_{L^2_h}+\|a(\cdot,x_3)\|_{\dot H^{\frac12}_h}\|b(\cdot,x_3)\|_{L^\infty_h}\big).$$

One has of course

$$s\leq0\Longrightarrow\|a\|_{\dot H^s}\leq\|a\|_{L^2_v(\dot H^s_h)}\quad\text{and}\quad s\geq0\Longrightarrow\|a\|_{L^2_v(\dot H^s_h)}\leq\|a\|_{\dot H^s}\tag{3.6}$$

so taking $s=1/2$ gives

$$\begin{aligned}\|ab\|_{L^2_v\dot H^{\frac12}_h}&\leq C\|a\|_{L^2_v(\dot H^{\frac12}_h)}\big(\|\nabla^h b\|_{L^\infty_v(L^2_h)}+\|b\|_{L^\infty}\big)\\&\leq\|a\|_{\dot H^{\frac12}}\big(\|\nabla^h b\|_{L^\infty_v(L^2_h)}+\|b\|_{L^\infty}\big).\end{aligned}\tag{3.7}$$

Now let us estimate $\|ab\|_{L^2_h\dot H^{\frac12}_v}$. A law of product in the vertical variable implies that for any x_h in $\mathbb{R}^2$

$$\|a(x_h,\cdot)b(x_h,\cdot)\|_{\dot H^{\frac12}_v}\leq C\big(\|a(x_h,\cdot)\|_{\dot H^{\frac12}_v}\|b(x_h,\cdot)\|_{L^\infty_v}+\|a(x_h,\cdot)\|_{L^2_v}\|\partial_3 b(x_h,\cdot)\|_{L^2_v}\big).$$

Taking the L^2 norm in the horizontal variable gives

$$\|ab\|_{L^2_h\dot H^{\frac12}_v}\leq C\big(\|a\|_{L^2_h\dot H^{\frac12}_v}\|b\|_{L^\infty}+\|a\|_{L^4_hL^2_v}\|\partial_3 b\|_{L^4_hL^2_v}\big).$$

Using Minkowski's inequality, we get that

$$\|ab\|_{L^2_h\dot H^{\frac12}_v}\leq C\big(\|a\|_{L^2_h\dot H^{\frac12}_v}\|b\|_{L^\infty}+\|a\|_{L^2_vL^4_h}\|\partial_3 b\|_{L^2_vL^4_h}\big).$$

Then using the Sobolev embedding $L^4_h \hookrightarrow \dot{H}^{\frac{1}{2}}_h$ and again (3.6), we infer

$$\begin{aligned}\|ab\|_{L^2_h\dot{H}^{\frac12}_v} &\leq C\big(\|a\|_{L^2_h\dot{H}^{\frac12}_v}\|b\|_{L^\infty}+\|a\|_{L^2_v\dot{H}^{\frac12}_h}\|\partial_3 b\|_{L^2_v\dot{H}^{\frac12}_h}\big)\\ &\leq C\|a\|_{\dot{H}^{\frac12}}\big(\|b\|_{L^\infty}+\|\partial_3 b\|_{L^2_v\dot{H}^{\frac12}_h}\big).\end{aligned}$$

Together with (3.5) and (3.7), this proves Lemma 3.3. □

That lemma allows to obtain the required estimate for $\underline{E}_{\varepsilon,\eta}$. Using the divergence free condition, we indeed have that

$$\underline{E}_{\varepsilon,\eta} = \mathrm{div}(\underline{u}_\eta \otimes [v_\varepsilon^{(2D)}]_\varepsilon + [v_\varepsilon^{(2D)}]_\varepsilon \otimes \underline{u}_\eta).$$

So the above lemma implies that for any positive time t

$$\begin{aligned}\|\underline{E}_{\varepsilon,\eta}(t)\|_{\dot{H}^{-\frac12}} &\leq C\|\underline{u}_\eta \otimes [v_\varepsilon^{(2D)}]_\varepsilon + [v_\varepsilon^{(2D)}]_\varepsilon \otimes \underline{u}_\eta\|_{\dot{H}^{\frac12}}(t)\\ &\leq C\|\underline{u}_\eta\|_{\dot{H}^{\frac12}}\big(\|\nabla^h[v_\varepsilon^{(2D)}]_\varepsilon\|_{L^\infty_v L^2_h}+\|[v_\varepsilon^{(2D)}]_\varepsilon\|_{L^\infty}+\|\partial_3[v_\varepsilon^{(2D)}]_\varepsilon\|_{L^2_v\dot{H}^{\frac12}_h}\big)(t).\end{aligned}$$

By definition of $[\,\cdot\,]_\varepsilon$ and using (3.3), we get

$$\|\underline{E}_{\varepsilon,\eta}(t)\|_{\dot{H}^{-\frac12}} \leq C\eta\big(\|\nabla^h v_\varepsilon^{(2D)}\|_{L^\infty_v L^2_h}+\|v_\varepsilon^{(2D)}\|_{L^\infty}+\varepsilon^{\frac12}\|\partial_3 v_\varepsilon^{(2D)}\|_{L^2_v\dot{H}^{\frac12}_h}\big).$$

Proposition 3.1 gives finally

$$\|\underline{E}_{\varepsilon,\eta}\|_{L^2(\mathbb{R}^+;\dot{H}^{-\frac12})} \leq C_0\eta. \tag{3.8}$$

In order to estimate the term $\widetilde{E}_{\varepsilon,\eta}$ let us observe that thanks to the divergence free condition, we have

$$\widetilde{E}_{\varepsilon,\eta} = \widetilde{u}^h_\eta \cdot \nabla^h[v_\varepsilon^{(2D)}]_\varepsilon + \varepsilon\widetilde{u}^3_\eta[\partial_3 v_\varepsilon^{(2D)}]_\varepsilon + [v_\varepsilon^{(2D)}]_\varepsilon \cdot \nabla\widetilde{u}_\eta. \tag{3.9}$$

Using the 3D law of product gives

$$\|\varepsilon\widetilde{u}^3_\eta[\partial_3 v_\varepsilon^{(2D)}]_\varepsilon\|_{\dot{H}^{-\frac12}} \leq C\varepsilon\|\widetilde{u}_\eta\|_{\dot{H}^{\frac12}}\|[\partial_3 v_\varepsilon^{(2D)}]_\varepsilon\|_{\dot{H}^{\frac12}}$$

Using (3.6), this gives

$$\|\varepsilon\widetilde{u}^3_\eta[\partial_3 v_\varepsilon^{(2D)}]_\varepsilon\|_{\dot{H}^{-\frac12}} \leq \varepsilon^{\frac12}\|\widetilde{u}_\eta\|_{\dot{H}^{\frac12}}\|v_\varepsilon^{(2D)}\|_{\dot{H}^{\frac32}}. \tag{3.10}$$

The two other terms of (3.9) are estimated using the following lemma.

Lemma 3.4. *Let a and b be two smooth functions. We have*

$$\|ab\|_{\dot{H}^{-\frac12}} \leq C\|a\|_{L^2_v\dot{H}^{\frac12}_h}\|b(\cdot,0)\|_{L^2_h} + C\|x_3 a\|_{L^2}\|\partial_3 b\|_{L^\infty_v\dot{H}^{\frac12}_h}.$$

Proof. Let us decompose b in the following way:

$$b(x_h,x_3) = b(x_h,0) + \int_0^{x_3}\partial_3 b(x_h,y_3)dy_3. \tag{3.11}$$

Laws of product for Sobolev spaces on $\mathbb{R}^2$ together with Assertion (3.6) gives

$$\begin{aligned}\|a(b_{|x_3=0})\|_{\dot{H}^{-\frac12}} &\leq \|a(b_{|x_3=0})\|_{L^2_v\dot{H}^{-\frac12}_h}\\ &\leq \Big(\int_{\mathbb{R}}\|a(\cdot,x_3)b(\cdot,0)\|^2_{\dot{H}^{-\frac12}_h}dx_3\Big)^{\frac12}\\ &\leq C\|b(\cdot,0)\|_{L^2}\Big(\int_{\mathbb{R}}\|a(\cdot,x_3)\|^2_{\dot{H}^{\frac12}_h}dx_3\Big)^{\frac12}\\ &\leq C\|a\|_{L^2_v\dot{H}^{\frac12}_h}\|b(\cdot,0)\|_{L^2_h}. \qquad (3.12)\end{aligned}$$

In order to use (3.11), let us observe that for any x_3, two-dimensional product laws give

$$\begin{aligned}\left\|a(\cdot,x_3)\int_0^{x_3}\partial_3 b(\cdot,y_3)dy_3\right\|_{\dot H_h^{-\frac12}} &\leq C\|a(\cdot,x_3)\|_{L^2_h}\left|\int_0^{x_3}\|\partial_3 b(\cdot,y_3)\|_{\dot H_h^{\frac12}}dy_3\right|\\ &\leq C|x_3|\|a(\cdot,x_3)\|_{L^2_h}\|\partial_3 b\|_{L^\infty_v\dot H_h^{\frac12}}.\end{aligned}$$

The above estimate integrated in x_3 together with (3.11) and (3.12) gives the result. □

Now let us apply this lemma to estimate $\widetilde{u}^h_\eta\cdot\nabla^h[v^{(2D)}_\varepsilon]_\varepsilon$ and $[v^{(2D)}_\varepsilon]_\varepsilon\cdot\nabla\widetilde{u}_\eta$. We get

$$\begin{aligned}\|\widetilde{u}^h_\eta\cdot\nabla^h[v^{(2D)}_\varepsilon]_\varepsilon(t)\|_{\dot H^{-\frac12}}\leq C\|\widetilde{u}^h_\eta(t,\cdot)\|_{L^2_v(\dot H^{\frac12}_h)}\|\nabla^h v^{(2D)}_\varepsilon(t,\cdot,0)\|_{L^2_h}\\ +\varepsilon\|x_3\widetilde{u}^h_\eta(t)\|_{L^2}\|\partial_3\nabla^h v^{(2D)}_\varepsilon(t,\cdot)\|_{L^\infty_v(\dot H^{\frac12}_h)}\end{aligned}$$

and

$$\begin{aligned}\|[v^{(2D)}_\varepsilon]_\varepsilon\cdot\nabla\widetilde{u}_\eta(t)\|_{\dot H^{-\frac12}}\leq C\|\nabla\widetilde{u}_\eta(t,\cdot)\|_{L^2_v(\dot H^{\frac12}_h)}\|v^{(2D)}_\varepsilon(t,\cdot,0)\|_{L^2_h}\\ +\varepsilon\|x_3\nabla\widetilde{u}_\eta(t,\cdot)\|_{L^2}\|\partial_3 v^{(2D)}_\varepsilon(t,\cdot)\|_{L^\infty_v(\dot H^{\frac12}_h)}.\end{aligned}$$

By construction of $\widetilde{u}_\eta$ and by Proposition 3.1 and 3.2, together with (3.9) and (3.10), we infer that

$$\|\widetilde{E}_{\varepsilon,\eta}\|_{L^2(\mathbb{R}^+;\dot H^{-\frac12})}\leq C_\eta\varepsilon^{\frac12} \tag{3.13}$$

and putting (3.2), (3.8) and (3.13) together proves Proposition 1.8, up to the proof of Proposition 3.2.

So let us finally prove Proposition 3.2. We recall that $v^{(2D)}_\varepsilon=(v^h,0)+(\varepsilon w^h_\varepsilon,w^3_\varepsilon)$, and due to the form of $(NS2D_3)$ it is clear that $v^h(t,x_h,0)=0$ for any (t,x_h) in $\mathbb{R}^+\times\mathbb{R}^2$. So it remains to estimate $(\varepsilon w^h_\varepsilon,w^3_\varepsilon)$. We first notice that due to Proposition 3.1

$$\|\varepsilon w^h_\varepsilon(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2_h)}+\|\varepsilon\nabla^h w^h_\varepsilon(\cdot,0)\|_{L^2(\mathbb{R}^+;L^2_h)}\leq C\varepsilon,$$

so we are left with the computation of $w^3_\varepsilon(t,\cdot,0)$. By definition of w_ε we have

$$\left\{\begin{array}{c}\partial_t w^3_\varepsilon+v^h\cdot\nabla^h w^3_\varepsilon-\Delta_h w^3_\varepsilon=\varepsilon^2F_\varepsilon\\ w^3_{\varepsilon|t=0}=w^3_0\end{array}\right.\qquad\text{with}\quad F_\varepsilon\overset{\text{def}}{=}\partial_3^2w^3_\varepsilon-\partial_3 p_{\varepsilon,1}.$$

We shall start by writing an $\dot H^{\frac12}_h$ energy estimate (with y_3 seen as a parameter) which will imply that $w^3_\varepsilon(t,\cdot,0)$ is smaller than $C\varepsilon$ in $L^\infty(\mathbb{R}^+;\dot H^{\frac12}_h)\cap L^2(\mathbb{R}^+;\dot H^{\frac32}_h)$. The result in the space $L^\infty(\mathbb{R}^+;L^2_h)\cap L^2(\mathbb{R}^+;\dot H^1_h)$ will follow by interpolation with a bound in a negative order Sobolev space, given by Proposition 3.1.

Let us start by the $\dot H^{\frac12}_h$ energy estimate. We claim that there is a constant C_0 such that for any $\varepsilon\leq\varepsilon_0$,

$$\varepsilon\|F_\varepsilon\|_{L^2(\mathbb{R}^+;L^\infty_v\dot H_h^{-\frac12})}\leq C_0. \tag{3.14}$$

Assuming (3.14), an $\dot H^{\frac12}_h$ energy estimate (joint with the fact that $w^3_{\varepsilon|t=0}(\cdot,0)=0$) gives directly that

$$\|w^3_\varepsilon(\cdot,0)\|_{L^\infty(\mathbb{R}^+;\dot H^{\frac12}_h)}+\|\nabla^h w^3_\varepsilon(\cdot,0)\|_{L^2(\mathbb{R}^+;\dot H^{\frac12}_h)}\leq C\varepsilon.$$

But by Proposition 3.1 we know that w^3_ε is uniformly bounded, say in $L^\infty(\mathbb{R}^+; L^\infty_v \dot{H}^{-\frac{1}{2}}_h)$ and $\nabla^h w^3_\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^+; L^\infty_v \dot{H}^{-\frac{1}{2}}_h)$, so we get by interpolation that

$$\|w^3_\varepsilon(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2_h)} + \|\nabla^h w^3_\varepsilon(\cdot,0)\|_{L^2(\mathbb{R}^+;L^2_h)} \leq C\varepsilon^{\frac{1}{2}}.$$

It remains to prove the claim (3.14). On the one hand, Proposition 3.1 and Lemma 3.2 of [3] imply that

$$\|\partial_3^2 w^3_\varepsilon\|_{L^2(\mathbb{R}^+;L^\infty_v \dot{H}^{-\frac{1}{2}}_h)} \leq C_0. \tag{3.15}$$

The estimate on the pressure seems slightly more delicate, but we notice as in [3] that

$$-(\varepsilon^2\partial_3^2 + \Delta_h)p_{\varepsilon,1} = \mathrm{div}_h\big(v^h \cdot \nabla_h w^h_\varepsilon + \partial_3(w^3_\varepsilon v^h)\big).$$

Since $\varepsilon\partial_3 \,\mathrm{div}_h(\varepsilon^2\partial_3^2 + \Delta_h)^{-1}$ is a uniformly bounded Fourier multiplier, this implies easily using the estimates of Proposition 3.1 and Sobolev embeddings that that

$$\|\varepsilon\partial_3 p_{\varepsilon,1}\|_{L^2(\mathbb{R}^+;L^\infty_v \dot{H}^{-\frac{1}{2}}_h)} \leq C\|\varepsilon\partial_3 p_{\varepsilon,1}\|_{L^2(\mathbb{R}^+;H^1_v \dot{H}^{-\frac{1}{2}}_h)} \leq C_0. \tag{3.16}$$

The combination of (3.15) and (3.16) proves the claim, hence Proposition 3.2.

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