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Asymptotic analysis of a thin interface: The case involving similar rigidity

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This study deals with a linear elastic body consisting of two solids connected by a thin adhesive interphase with a small thickness ε . The three parts have similar elastic moduli. It is proposed to model the limit behavior of the interphase when $\varepsilon \rightarrow 0$. It has been established [1], using matched asymptotic expansions, that at order zero, the interphase reduces to a perfect interface, while at order one, the interphase behaves like an imperfect interface, with a transmission condition involving the displacement and the traction vectors at order zero. The perfect interface model is exactly recovered using a Γ -convergence argument. At a higher order, a new model of imperfect interface is obtained by studying the properties of a suitable (weakly converging) sequence of equilibrium solutions. Some analytical examples are given to illustrate the results obtained.

1. Introduction

Interphases play a crucial role in the analysis of structure assemblies, especially in glue-bonding processes [27]. However, due to their small thickness (typically in the 1 μm to 1 mm range), it is difficult to account directly for these interfaces in a complete finite element analysis of a structure. One possible strategy consists in undertaking an asymptotic analysis (where the small parameter is the thickness ε of the interphase) to eliminate the interphase geometrically and to obtain an equivalent interface model which will be simpler to implement in numerical simulations. This idea has been used in many studies to obtain interface laws linking the stress vector to the jump in the displacement vector at the interface [2,3,5,7,13–15,19–21,23–25,28–30,32,35,36], similar to the phenomenological laws described in the literature [16–18]. In many of these studies, a soft interphase, has often been assumed to exist, with a much lower stiffness than that of the adherents (the stiffness is another small parameter). Fewer studies have focused on joints consisting of adherents and an interphase with a comparable level of rigidity [1,8,26,31]. It has been established [1], using matched asymptotic expansions [12,33], that at the first order ($\varepsilon \rightarrow 0$) one obtains a perfect interface model, which prescribes the vanishing of the jumps in the stress and the displacement vectors. At an higher order (the second term in the expansion), an imperfect interface model is obtained, with a transmission condition involving the first order displacement and traction vectors and their derivatives. The presence of these derivatives reflects non-locality.

The aim of this study was to conduct a rigorous asymptotic analysis via variational convergence (Γ -convergence [6,9,11]) of the equilibrium problem in the case of a body comprising interphase and adherents with comparable rigidities. The results obtained are compared with those presented in [1] and the imperfect interface law is illustrated by presenting some simple examples.

This paper is organized as follows: In Section 2, the mechanical problem is presented. In Section 3, the problem is analyzed at order zero, using Γ -convergence theory. In Section 4, an example of the results obtained in Section 3 is given. In

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Section 5, the asymptotic study is performed at a higher order and the results are presented in Section 6. The proof of these results is given in Section 7.

2. The mechanical problem

We consider a body occupying an open bounded set Ω of \mathbb{R}^3 , with a smooth boundary $\partial\Omega$. Let (O, e_1, e_2, e_3) be an orthonormal frame in the three-dimensional space. The set Ω is assumed to have a non-empty intersection S with the plane $\{x_3 = 0\}$. Introducing a small parameter $\varepsilon > 0$, we define the following domains:

$$\begin{aligned}
B^\varepsilon &= \left\{ (x_1, x_2, x_3) \in \Omega : |x_3| < \frac{\varepsilon}{2} \right\}, \\
\Omega^\varepsilon &= \left\{ (x_1, x_2, x_3) \in \Omega : |x_3| > \frac{\varepsilon}{2} \right\}, \\
\Omega_\pm^\varepsilon &= \left\{ (x_1, x_2, x_3) \in \Omega : \pm x_3 > \frac{\varepsilon}{2} \right\}, \\
S_\pm^\varepsilon &= \left\{ (x_1, x_2, x_3) \in \Omega : \pm x_3 = \frac{\varepsilon}{2} \right\}, \\
\Omega_\pm &= \{ (x_1, x_2, x_3) \in \Omega : \pm x_3 > 0 \}, \\
S &= \{ (x_1, x_2, x_3) \in \Omega : x_3 = 0 \}, \\
\Omega_0 &= \Omega_+ \cup \Omega_-.
\end{aligned} \tag{1}$$

The sets B^ε and Ω^ε are the domains occupied by the adhesive and the adherents, respectively, the set Ω^0 is the geometrical limit of Ω^ε , S_\pm^ε are the interfaces between the adhesive and the adherents and S is the surface to which the adhesive tends geometrically (Fig. 1).

The structure is subjected to a body force density φ and a surface force density g on part Γ_1 of the boundary, whereas it is clamped on the remaining part Γ_0 of the boundary. We take σ^ε to denote the stress tensor and u^ε to denote the displacement field. In the infinitesimal case, the strain tensor is the symmetric part of the displacement gradient

$$e_{ij}(u^\varepsilon) = \frac{1}{2} \left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i} \right). \tag{2}$$

The adherents and the adhesive are assumed to be linearly elastic and a_{ijkl}^\pm is taken to denote the elasticity coefficients of the adherents and a_{ijkl}^m to denote the elasticity coefficients of the adhesive.

For a given function $f : \Omega \rightarrow \mathbb{R}^3$, we define the restrictions of f to the adherents by f_ε^\pm and to the adhesive by f_ε^m . Taking $\hat{x} = (x_1, x_2)$ to denote the in-plane coordinates of the adhesive, we define the following jumps of f

$$[f]_\varepsilon^+(\hat{x}) := f_\varepsilon^+ \left(x_1, x_2, \left(\frac{\varepsilon}{2} \right)^+ \right) - f_\varepsilon^m \left(x_1, x_2, \left(\frac{\varepsilon}{2} \right)^- \right), \tag{3}$$

$$[f]_\varepsilon^-(\hat{x}) := f_\varepsilon^- \left(x_1, x_2, \left(-\frac{\varepsilon}{2} \right)^- \right) - f_\varepsilon^m \left(x_1, x_2, \left(-\frac{\varepsilon}{2} \right)^+ \right), \tag{4}$$

$$[f]_\varepsilon(\hat{x}) := f_\varepsilon^m \left(x_1, x_2, \left(\frac{\varepsilon}{2} \right)^- \right) - f_\varepsilon^m \left(x_1, x_2, \left(-\frac{\varepsilon}{2} \right)^+ \right). \tag{5}$$

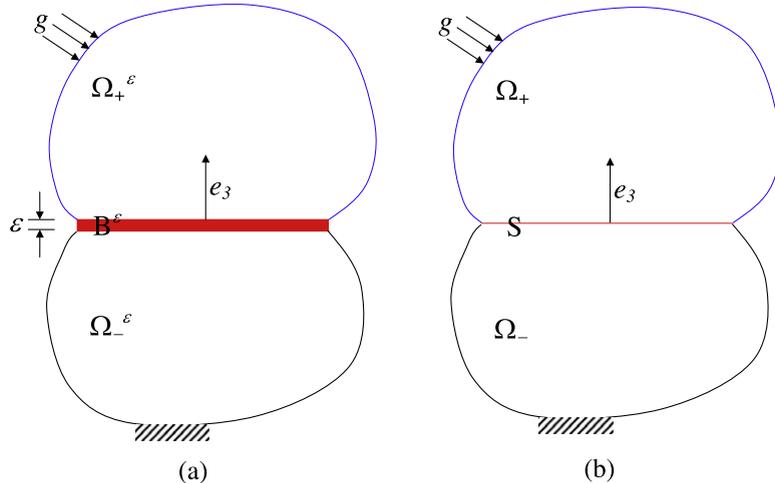


Fig. 1. (a) Reference and (b) limit configurations of the joint.

For a given function $f : \Omega_0 \mapsto \mathbb{R}^3$, we define the restrictions of f to Ω_{\pm} by f^{\pm} and we also define the following jump of f on S

$$[f](\hat{x}) := f^+(x_1, x_2, 0^+) - f^-(x_1, x_2, 0^-). \quad (6)$$

Writing the local expression for the equilibrium problem corresponds to finding the displacement u^{ε} , strain $e(u^{\varepsilon})$ and stress σ^{ε} fields which solve the following problem:

$$(P_{\varepsilon}) \quad \begin{cases} \text{Find } (u^{\varepsilon}, \sigma^{\varepsilon}) \text{ such that :} \\ \sigma_{ijj}^{\varepsilon} = -\varphi_i \quad \text{in } \Omega, \\ \sigma_{ij}^{\varepsilon} = a_{ijkh}^{\pm} e_{kh}(u^{\varepsilon}) \quad \text{in } \Omega_{\pm}^{\varepsilon}, \\ \sigma_{ij}^{\varepsilon} = a_{ijkh}^m e_{kh}(u^{\varepsilon}) \quad \text{in } B^{\varepsilon}, \\ u^{\varepsilon} = 0 \quad \text{on } \Gamma_0, \\ \sigma_{\varepsilon} n = g \quad \text{on } \Gamma_1, \\ [u^{\varepsilon}]_{\varepsilon}^{\pm} = 0, \quad [\sigma^{\varepsilon} e_3]_{\varepsilon}^{\pm} = 0. \end{cases}$$

We now introduce the space of kinematically admissible displacements

$$V^{\varepsilon} = \{u \in (W^{1,2}(\Omega))^3 : u = 0 \text{ on } \Gamma_0\}, \quad (7)$$

and the work of the internal and external loads, respectively,

$$\begin{aligned} A^{\varepsilon}(u, v) &= \int_{\Omega_{\pm}^{\varepsilon}} a^+ e(u) : e(v) dx + \int_{\Omega_{\pm}^{\varepsilon}} a^- e(u) : e(v) dx + \int_{B^{\varepsilon}} a^m e(u) : e(v) dx, \\ I^{\varepsilon}(v) &= \int_{\Omega} \phi \cdot v dx + \int_{\Gamma_1} g \cdot v ds. \end{aligned}$$

Under the following regularity assumptions

$$\begin{aligned} (H1) \quad & \begin{cases} a_{ijkl} \in L^{\infty}(\Omega), \\ a_{ijkl} = a_{klij} = a_{jilk}, \\ \exists \eta > 0 : a_{ijkl} e_{ij} e_{kl} \geq \eta e_{ij} e_{ij} \quad \forall e_{ij} = e_{ji}, \end{cases} \\ (H2) \quad & \exists \varepsilon_0 : B_{\varepsilon} \cap (\Gamma_1 \cup \text{supp}(\phi)) = \emptyset, \quad \forall \varepsilon < \varepsilon_0, \\ (H3) \quad & \phi \in (L^2(\Omega))^3, \quad g \in (L^2(\Gamma_1))^3, \end{aligned}$$

we can reformulate (P_{ε}) as follows:

$$(\bar{P}_{\varepsilon}) \quad \begin{cases} \text{Find } u^{\varepsilon} \in V^{\varepsilon} : \\ A(u^{\varepsilon}, v) = I^{\varepsilon}(v) \quad \forall v \in V^{\varepsilon}. \end{cases} \quad (8)$$

In view of the Lax–Milgram Lemma, this problems has a unique solution, which is also the unique solution of the following minimum problem:

$$(\tilde{P}_{\varepsilon}) \quad \begin{cases} \text{Find } u^{\varepsilon} \in V^{\varepsilon} : \\ J^{\varepsilon}(u^{\varepsilon}) \leq J^{\varepsilon}(v) \quad \forall v \in V^{\varepsilon}, \end{cases} \quad (9)$$

where

$$J^{\varepsilon}(v) = \frac{1}{2} A^{\varepsilon}(v, v) - I^{\varepsilon}(v) \quad (10)$$

is the potential energy associated with the displacement field v . We define the functional spaces

$$V^{\varepsilon} = \{u \in (W^{1,2}(\Omega))^3 : u = 0 \text{ on } \Gamma_0\}, \quad (11)$$

$$X = (L^2(\Omega))^3. \quad (12)$$

Finally, we introduce the strain energy functional

$$F^{\varepsilon}(v) = \begin{cases} \frac{1}{2} A^{\varepsilon}(v, v) & \text{if } v \in V^{\varepsilon}, \\ +\infty & \text{if } v \in X \setminus V^{\varepsilon}. \end{cases} \quad (13)$$

3. Zero order results

We introduce the limit functional

$$F^0(v) = \begin{cases} \int_{\Omega_0} a^\pm e(v) : e(v) dx & \text{if } v \in V^0, \\ +\infty & \text{if } v \in X \setminus V^0, \end{cases} \quad (14)$$

where

$$V^0 = \{u \in (W^{1,2}(\Omega_0))^3 : u = 0 \text{ on } \Gamma_0, [v] = 0 \text{ on } S\}. \quad (15)$$

Theorem 1. *The sequence of functionals F^ε Γ -converges for the strong topology of X to F^0 .*

To prove Theorem 1, we need the two following Lemmas, in which we take C to denote any constant independent of ε . The first Lemma is that obtained by Licht and Michaille [30]. Lemma 2 is proved in Appendix 1.

Lemma 1. *For all $v \in V^\varepsilon$, there exists a constant $C > 0$ independent of ε such that*

$$\begin{aligned} \int_{B^\varepsilon} |v(x)|^2 dx &\leq C \left(\varepsilon^2 \int_{B^\varepsilon} |e(v(x))|^2 dx + \varepsilon \int_{\Omega^\varepsilon} |e(v(x))|^2 dx \right), \\ \int_{\Omega^\varepsilon} |v(x)|^2 dx &\leq C \int_{\Omega^\varepsilon} |e(v(x))|^2 dx, \\ \int_{\Gamma_1} |v(x)|^2 dx &\leq C \int_{\Omega^\varepsilon} |e(v(x))|^2 dx. \end{aligned}$$

Lemma 2. *There exist constants $C > 0$ such that*

$$\int_{\Omega} |u^\varepsilon(x)|^2 dx \leq C, \quad (16)$$

$$\int_{\Omega} |e(u^\varepsilon(x))|^2 dx \leq C, \quad (17)$$

i.e., there exists a non relabeled subsequence, such that $u^\varepsilon \rightharpoonup u^0$ in V^ε , and $u^\varepsilon \rightarrow u^0$ in X .

We now proceed to prove Theorem 1.

Proof of Theorem 1. We need to show that $\forall u_0 \in X$

- *step 1:* $\exists u^\varepsilon \rightarrow u^0$ in X such that $\limsup_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon) \leq F^0(u^0)$;
- *step 2:* $\forall u^\varepsilon \rightarrow u^0$ in X $\liminf_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon) \geq F^0(u^0)$.

To prove step 1, take the constant sequence $u^\varepsilon = u^0$ and go to the limit in $F^\varepsilon(u^0)$. To prove step 2, let k^m be a positive constant and introduce the following functional

$$F^{\varepsilon, k^m}(v) = \begin{cases} \int_{\Omega_\pm^\varepsilon} a^\pm e(v) : e(v) dx + k^m \varepsilon \int_{B^\varepsilon} e(v) : e(v) dx & \text{if } v \in V_\varepsilon, \\ +\infty & \text{if } v \in X \setminus V_\varepsilon. \end{cases} \quad (18)$$

We also take $u^\pm \in C^\infty(\Omega^\pm)$ and introduce the regularizing sequence [34]

$$(R^\varepsilon u)(\hat{x}, x_3) = \begin{cases} \frac{1}{2} S^\varepsilon u(\hat{x}) + \frac{x_3}{\varepsilon} [u]_\varepsilon(\hat{x}) & \text{if } |x_3| \leq \frac{\varepsilon}{2}, \\ u^\pm(\hat{x}, x_3) & \text{if } |x_3| > \frac{\varepsilon}{2} \end{cases} \quad (19)$$

with $S^\varepsilon u(\hat{x}) = u^+(\hat{x}, \frac{\varepsilon}{2}) + u^-(\hat{x}, -\frac{\varepsilon}{2})$. This sequence shows the following property

$$\lim_{\varepsilon \rightarrow 0} \left| e(R^\varepsilon u) - \frac{1}{\varepsilon} [u]_\varepsilon \otimes_s e_3 \right|_{L^2(B^\varepsilon)}^2 = 0, \quad (20)$$

where \otimes_s denotes the symmetrical tensorial product. We introduce a smoothing sequence $u_n^0 \in (C^\infty(\Omega))^3$ converging strongly to u^0 in $W^{1,2}(\Omega^+ \cup \Omega^-)$.

In view of the convexity of the functional F^{ε, k^m} , we can write:

$$F^{\varepsilon, k^m}(u^\varepsilon) \geq F^{\varepsilon, k^m}(R^\varepsilon u_n^0) + \int_{\Omega_\pm^\varepsilon} a^\pm e(R^\varepsilon u_n^0) : e(u^\varepsilon - R^\varepsilon u_n^0) dx + k^m \varepsilon \int_{B^\varepsilon} e(R^\varepsilon u_n^0) : e(u^\varepsilon - R^\varepsilon u_n^0) dx. \quad (21)$$

The second term on the right-hand side tends to zero because of the definition of the regularizing sequence (19) and because of the convergence properties of sequences u^ε and u_n^0 . Using (20), and [30]

$$\left| u^\varepsilon \left(\hat{x}, \pm \frac{\varepsilon}{2} \right) - u^0(\hat{x}, 0^\pm) \right|_{L^2(S)} \rightarrow 0, \quad (22)$$

we also have

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \varepsilon \int_{B^\varepsilon} |e(R^\varepsilon u_n^0)|^2 dx = \lim_{\varepsilon \rightarrow 0} \int_S |[u^0]_\varepsilon \otimes_s e_3|^2 ds = \int_S |[u^0] \otimes_s e_3|^2 ds. \quad (23)$$

Again using (20) and the regularity of u^ε , we obtain

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \varepsilon \int_{B^\varepsilon} e(R^\varepsilon u_n^0) \cdot e(u^\varepsilon) dx = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{B^\varepsilon} ([u_n^0]_\varepsilon \otimes_s e_3) \cdot (u_3^\varepsilon \otimes_s e_3) dx = \int_S |[u^0] \otimes_s e_3|^2 ds. \quad (24)$$

Therefore, the third term on the right-hand side of (21) tends to zero. Consider

$$F^{\varepsilon, k^m}(R^\varepsilon u_n^0) = \int_{\Omega_\pm^\varepsilon} a^\pm e(u_n^0) : e(u_n^0) dx + k^m \varepsilon \int_{B^\varepsilon} e(R^\varepsilon u_n^0) : e(R^\varepsilon u_n^0) dx. \quad (25)$$

The first term on the right-hand side converges to $F^0(u^0)$. For the second term on the right-hand side, we use (23) and substituting into (21) gives

$$\liminf_{\varepsilon \rightarrow 0} F^{\varepsilon, k^m}(u^\varepsilon) \geq F^0(u^0) + k^m \int_S |[u^0] \otimes_s e_3|^2 ds. \quad (26)$$

The coercivity of the elasticity tensor ensures that $F^\varepsilon(u) \geq F^{\varepsilon, n^m \varepsilon^{-1}}(u) = F^{\varepsilon, k^m}(u)$ for all $u \in V^\varepsilon$, and $k^m > 0$. Therefore,

$$\liminf_{\varepsilon \rightarrow 0} F^\varepsilon(u^\varepsilon) \geq F^0(u^0), \quad (27)$$

and this completes step 2. \square

Comment 1. The Γ -convergence of $J^\varepsilon(v)$ to $F^0(v) - l(v)$ follows from Theorem 1 and from the convergence of $F^\varepsilon(v)$ to $l(v)$ [30].

Comment 2. Using standard arguments, we obtain the following limit equilibrium problem

$$(P_0) \begin{cases} \text{Find } (u^0, \sigma^0) \text{ such that :} \\ \sigma_{ijj}^0 = -\varphi_i \quad \text{in } \Omega_0, \\ \sigma_{ij}^0 = a_{ijkh}^\pm e_{kh}(u^0) \quad \text{in } \Omega^\pm, \\ u^0 = 0 \quad \text{on } \Gamma_0, \\ \sigma^0 n = g \quad \text{on } \Gamma_1, \\ [u^0] = 0, \quad [\sigma^0 e_3] = 0. \end{cases}$$

Note that this limit problem involves only the adherents. The adhesive has vanished geometrically (as was to be expected) as well as mechanically, contrary to what happens in other models [15,19,24,30]. We also observe that, since the displacement and the traction vectors on S , have to be continuous, a very thin interphase will behave like a perfect interface.

The case of a soft adhesive is known to give a spring-type interface model, which is governed by a linear relation between the stress vector and the displacement vector jump [4]. One might intuitively expect the perfect interface model to be the limit case of the spring-type interface model when the spring parameters become sufficiently large, and our results confirm this prediction. With adhesives which are stiffer than the adherents, the limit model may not be so straightforward. A complete study of this case is presented in [8], where it is assumed that the elastic coefficients of the adhesive, a_{ijkl}^m , scales like ε^{-a} , with $a > 0$. It was established in the latter study that there exist various regimes, depending on a . In particular, the perfect interface is the limit of the thin adhesive model as long as a is equal to less than one. Theorem 1 extends this result to the case $a = 0$.

4. Illustration in one dimension

We take the example of a bar AB divided into three parts, AC , CD and DB (see Fig. 2). The bar is fixed at point A , and a given displacement δ is imposed at point B . We have

$$AC = L_1 - \varepsilon, \quad CD = \varepsilon, \quad DB = L_2, \quad L = L_1 + L_2, \\ u(0) = 0, \quad u(L) = \delta.$$

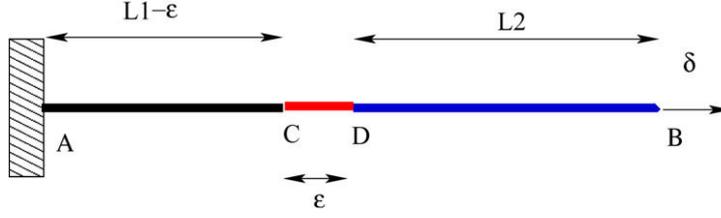


Fig. 2. Illustration: a simple bar under traction loading.

The elastic moduli of AC, CD and DB are denoted E_1 , E_2 and E_3 , respectively. At equilibrium, the displacement field is given by:

$$u(x) = \frac{\sigma}{E_1}x \quad \text{if } 0 \leq x \leq L_1 - \varepsilon, \quad (28)$$

$$u(x) = \frac{\sigma}{E_2}(x - L_1 + \varepsilon) + \frac{\sigma}{E_1}(L_1 - \varepsilon) \quad \text{if } L_1 - \varepsilon \leq x \leq L_1, \quad (29)$$

$$u(x) = \frac{\sigma}{E_3}(x - L_1) + \frac{\sigma}{E_2}\varepsilon + \frac{\sigma}{E_1}(L_1 - \varepsilon) \quad \text{if } L_1 \leq x \leq L_2. \quad (30)$$

The stress, which is constant, is given by $\sigma = \frac{\delta}{\frac{L_2 + \varepsilon}{E_3} + \frac{L_1 - \varepsilon}{E_2} + \frac{L_1 - \varepsilon}{E_1}}$.

At order zero, we obtain:

$$u^0(x) = \frac{\sigma^0}{E_1}x \quad \text{if } 0 \leq x \leq L_1, \quad (31)$$

$$u^0(x) = \frac{\sigma^0}{E_3}(x - L_1) + \frac{\sigma^0}{E_1}L_1 \quad \text{if } L_1 \leq x \leq L_2, \quad (32)$$

where $\sigma^0 = \frac{\delta}{\frac{L_2 + \varepsilon}{E_3} + \frac{L_1 - \varepsilon}{E_1}}$. In conclusion, we confirm that $[u^0] = [\sigma^0] = 0$.

To check whether the solution at order zero provides a good approximation of the exact solution, the error $(u - u^0)/u$ in $(0, L)$ is given in Fig. 3 in an a-dimensionalized case.

The relative tension error can be easily computed:

$$\frac{\sigma - \sigma^0}{\sigma^0} = \frac{\varepsilon}{L} \left(\frac{E_1/E_2 - 1}{E_1/E_3 + 1} \right). \quad (33)$$

Note that this error depends on the ratios between the elastic moduli of the materials. For example, consider the three cases of a lead joint with thickness $\varepsilon = 0.01L$ and rigidity $E_2 = 14$ GPa between a steel bar with $E_1 = 210$ GPa and a bar made of

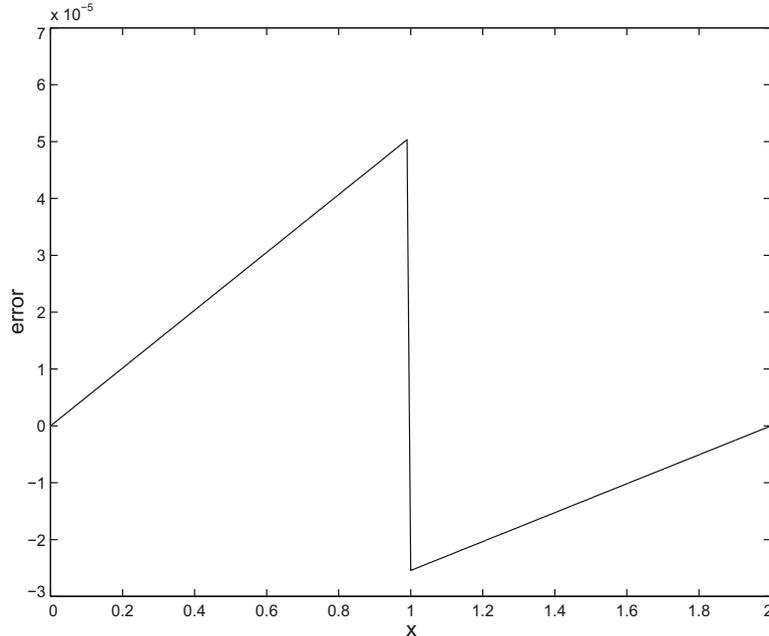


Fig. 3. Plot of the error $(u - u^0)/u$ in $(0, L)$ in an a-dimensionalized case $L_1 = L_2 = 1$, $\varepsilon = 0.01$, $E_1 = 1$, $E_2 = 1.5$, $E_3 = 2$, $\delta = 1$.

either steel ($E_3 = 210$ GPa), tungsten carbide ($E_3 = 640$ GPa) or aluminium ($E_3 = 70$ GPa). The error (33) in these three cases is equal to 7%, 10.5% and 3.5%, respectively. This simple example shows that in some cases, it can be necessary to improve the model, and it turns out that the most natural solution is to go to a higher order in the asymptotic analysis.

The situation is even more difficult in the three-dimensional case, because of the singularities which appear on the surfaces of the adhesive. If the adherents are made of the same material, then the singularities disappear at order zero and this gives a further reason for studying the equilibrium problem at a higher order. This was done in the following section.

5. First order results

In this section, the results obtained in [1,24] are improved and generalized to the three-dimensional case, and we focus on the case of an isotropic adhesive with Lamé's coefficients λ and μ . We introduce the displacement field u^1 and the stress field σ^1 as the limits, as defined by the following Lemma, of suitable sub-sequences of $\frac{u^\varepsilon - u^0}{\varepsilon}$ and $\frac{\sigma^\varepsilon - \sigma^0}{\varepsilon}$, respectively. We take $D(A)$ to denote the space of the C^∞ functions with compact support on the open set A and $D'(A)$ its dual space.

Lemma 3. *Let u^ε , u^0 , σ^ε and σ^0 be the fields defined in Sections 2 and 3, then there exist sub-sequences, non relabeled, such that*

$$\begin{aligned} \frac{u^\varepsilon - u^0}{\varepsilon} &\rightharpoonup u^1 \quad \text{in } L^2(\Omega) \text{ (weak),} \\ \frac{\sigma^\varepsilon - \sigma^0}{\varepsilon} &\rightharpoonup \sigma^1 \quad \text{in } D'(\Omega) \text{ (weak).} \end{aligned} \quad (34)$$

This Lemma is proved in the Appendix. Contrary to the fields at order zero u^0 and $\sigma^0 e_3$, which are continuous on the surface S , the fields u^1 and $\sigma^1 e_3$ show discontinuities on S . The following Theorem, involving restrictions of the traction and displacement vectors $\sigma^0 e_3$ and u^0 on S , relates their jumps.

Theorem 2.

$$\begin{aligned} [u_\alpha^1] &= \frac{1}{\mu} \sigma_{\alpha 3}^0(\hat{x}, 0) - u_{3,\alpha}^0(\hat{x}, 0) - \frac{1}{2}(u_{\alpha,3}^0(\hat{x}, 0^+) + u_{\alpha,3}^0(\hat{x}, 0^-)), \quad \alpha = 1, 2, \\ [u_3^1] &= \frac{1}{\lambda + 2\mu} \sigma_{33}^0(\hat{x}, 0) - \frac{\lambda}{\lambda + 2\mu} (u_{1,1}^0(\hat{x}, 0) + u_{2,2}^0(\hat{x}, 0)) - \frac{1}{2}(u_{3,3}^0(\hat{x}, 0^+) + u_{3,3}^0(\hat{x}, 0^-)), \\ [\sigma_{13}^1] &= -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{1,11}^0(\hat{x}, 0) - \mu u_{1,22}^0(\hat{x}, 0) - \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{2,21}^0 - \frac{\lambda}{\lambda + 2\mu} \sigma_{33,1}^0(\hat{x}, 0) - \frac{1}{2}(\sigma_{13,3}^0(\hat{x}, 0^+) + \sigma_{13,3}^0(\hat{x}, 0^-)), \\ [\sigma_{23}^1] &= -\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{2,22}^0(\hat{x}, 0) - \mu u_{2,11}^0(\hat{x}, 0) - \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{1,12}^0(\hat{x}, 0) - \frac{\lambda}{\lambda + 2\mu} \sigma_{33,2}^0(\hat{x}, 0) - \frac{1}{2}(\sigma_{23,3}^0(\hat{x}, 0^+) + \sigma_{23,3}^0(\hat{x}, 0^-)), \\ [\sigma_{33}^1] &= -\sigma_{13,1}^0(\hat{x}, 0) - \sigma_{23,2}^0(\hat{x}, 0) - \frac{1}{2}(\sigma_{33,3}^0(\hat{x}, 0^+) + \sigma_{33,3}^0(\hat{x}, 0^-)), \end{aligned} \quad (35)$$

in $D'(S)$.

The proof of this Theorem will be given to Section 7.

Comment 3. Using standard arguments, we obtain the equilibrium equations

$$(P_1) \begin{cases} \text{Find } (u^1, \sigma^1) \text{ such that :} \\ \sigma_{ij,j}^1 = 0 \quad \text{in } \Omega_0, \\ \sigma_{ij}^1 = a_{ijkh}^\pm e_{kh}(u^1) \quad \text{in } \Omega^\pm, \\ u^1 = 0 \quad \text{on } \Gamma_0, \\ \sigma^1 n = 0 \quad \text{on } \Gamma_1, \\ [u^1] \text{ given by } (35)_{1,2}, \quad [\sigma^1 e_3] \text{ given by } (35)_{3-5}. \end{cases}$$

As with (P_0) , the limit problem (P_1) involves only the adherents, since the adhesive has vanished geometrically. However, the adhesive has not disappeared from the mechanical point of view: it has been replaced by a mechanical constraint linking the jump in the displacement and traction vectors to the displacement and traction vectors at order zero and their derivatives. In particular, we note the presence of the tangential derivatives of u^0 , showing non-local character of (35). Relations (35) suggest that at higher orders, a thin interphase behaves like an imperfect interface which has memorized the elastic behavior of the interphase.

Lastly, we note that (35) contain some extra terms in comparison with the analogous relations obtained in [1]. These terms are related to the normal derivatives of u^0 and $\sigma^0 e_3$ in $x_3 = 0^\pm$.

6. Numerical examples

6.1. Composite bar

Let us take a bar composed of two identical adherents connected by an adhesive, which are subjected to symmetrical forces. Because of the symmetry, we will study only half of the bar, AB , which is fixed at one extremity, A , as shown in Fig. 4.

Part AB is composed of the adhesive AC and the adherent CB , consisting of two different materials with elastic moduli E_1 and E_2 , respectively. A force density $\phi(x) = \frac{b}{L-\varepsilon}(x-\varepsilon) + c$ is applied to CB . We have

$$AC = \varepsilon, \quad CD = L - \varepsilon,$$

$$u(0) = 0, \quad E_2 \frac{du}{dx}(L) = 0.$$

At equilibrium, the displacement field is given by:

$$u(x) = (b/2 + c)(L - \varepsilon) \frac{x}{E_1} \quad \text{if } 0 \leq x \leq \varepsilon, \quad (36)$$

$$u(x) = -\frac{b(x-\varepsilon)^3}{6(L-\varepsilon)E_2} - \frac{c(x-\varepsilon)^2}{2E_2} + \frac{(b/2 + c)(L-\varepsilon)}{E_2}(x-\varepsilon) + (b/2 + c)(L-\varepsilon) \frac{x}{E_1} \quad \text{if } L-\varepsilon \leq x \leq L. \quad (37)$$

The stress, which is not constant, is given by:

$$\sigma(x) = (b/2 + c)(L - \varepsilon) \quad \text{if } 0 \leq x \leq \varepsilon, \quad (38)$$

$$\sigma(x) = -\frac{b(x-\varepsilon)^2}{2(L-\varepsilon)} - c(x-\varepsilon) + (b/2 + c)(L - \varepsilon) \quad \text{if } L - \varepsilon \leq x \leq L. \quad (39)$$

Note that the reaction at the fixed end is $R = (b/2 + c)L - (b/2 + c)\varepsilon$.

At order zero, we obtain:

$$u^0(x) = -\frac{bx^3}{6LE_2} - \frac{cx^2}{2E_2} + \frac{(b/2 + c)Lx}{E_1}, \quad (40)$$

$$\sigma^0(x) = -\frac{b}{2L}x^2 - cx + (b/2 + c)L. \quad (41)$$

It can easily be confirmed that $[u^0] = [\sigma^0] = 0$. At order one, we obtain:

$$u^1(x) = -\frac{b(x^3/L - 3x^2)}{6LE_2} + \frac{cx}{E_2} - \frac{(b/2 + c)(L+x)}{E_2} + \frac{(b/2 + c)L}{E_1}, \quad (42)$$

$$\sigma^1(x) = \frac{b(2x - x^2/L)}{2L} + b/2. \quad (43)$$

It can be confirmed that $[u^1] = (b/2 + c)L\left(\frac{1}{E_1} - \frac{1}{E_2}\right) = \frac{1}{E_1}\sigma^0(0) - u_x^0(0)$ and $[\sigma^1] = c = -\frac{1}{2}(\sigma_x^0(0^+) + \sigma_x^0(0^-))$. The exact stress σ and its approximation at order one $\sigma^0 + \varepsilon\sigma^1$ can be seen in Fig. 5. The relative error here is less than 0.7%.

6.2. Simple shear loads in a composite block

In this example, we take a composite solid consisting of three blocks with the same constant rectangular cross-section $S = (-l_1, l_1) \times (-l_2, l_2)$ and we set (Fig. 6)

$$\begin{aligned} \Omega_-^e &= \{(x_1, x_2, x_3) \in \Omega : x_3 \in (-h^-, -\varepsilon/2)\}, \\ B^e &= \{(x_1, x_2, x_3) \in \Omega : x_3 \in (-\varepsilon/2, -\varepsilon/2)\}, \\ \Omega_+^e &= \{(x_1, x_2, x_3) \in \Omega : x_3 \in (\varepsilon/2, h^+)\}. \end{aligned} \quad (44)$$

The base of the block is kept fixed

$$u(\hat{x}, -h^-) = 0, \quad \hat{x} \in S. \quad (45)$$

The base at $x_3 = h^+$ and the lateral surfaces at $x_2 = \pm l_2$ are subjected to a given constant tangential load τ

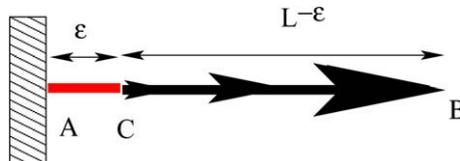


Fig. 4. A composite bar subjected to body forces.

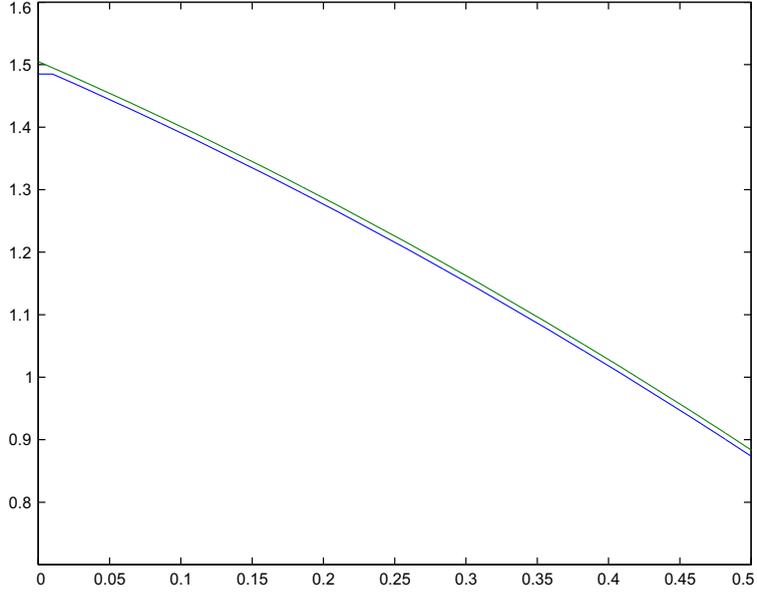


Fig. 5. Exact and order one stresses in an a-dimensionalized case $L = 1$, $\varepsilon = 0.01$, $b = c = 1$, $E_1 = 89$, $E_2 = 210$.

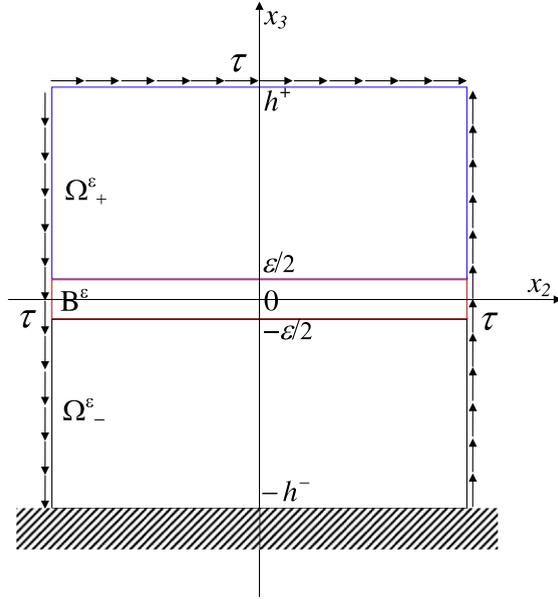


Fig. 6. Composite block subjected to tangential loading.

$$\sigma^\varepsilon(\hat{x}, h^+)e_3 = \tau e_2, \quad \hat{x} \in S, \quad (46)$$

$$\sigma^\varepsilon(x_1, \pm l_2, x_3)e_2 = \pm \tau e_3, \quad -l_1 \leq x_1 \leq l_1, \quad h^- \leq x_3 \leq h^+. \quad (47)$$

The adherents Ω_\pm^ε are assumed to consist of isotropic, possibly different elastic materials and we take μ_\pm, λ_\pm to denote their elastic constants.

The displacement solution to the equilibrium problem is the field

$$u^\varepsilon = \begin{cases} \left(\frac{\tau}{\mu_-} (x_3 + h^- + \frac{\varepsilon}{2}) \right) e_2 & \text{in } \Omega_-^\varepsilon, \\ \left(\frac{\tau}{\mu} (x_3 + \frac{\varepsilon}{2}) + \frac{\tau}{\mu_-} h^- \right) e_2 & \text{in } B^\varepsilon, \\ \left(\frac{\tau}{\mu_+} (x_3 - \frac{\varepsilon}{2}) + \frac{\tau}{\mu} \varepsilon + \frac{\tau}{\mu_-} h^- \right) e_2 & \text{in } \Omega_+^\varepsilon, \end{cases} \quad (48)$$

corresponding to a piecewise homogeneous deformation obtained by applying a simple shear load to each block. We have

$$u^0 = \begin{cases} \left(\frac{\tau}{\mu_-}(x_3 + h^-)\right)e_2 & \text{in } \Omega_-, \\ \left(\frac{\tau}{\mu_+}x_3 + \frac{\tau}{\mu_-}h^-\right)e_2 & \text{in } \Omega_+, \end{cases} \quad (49)$$

$$u^1 = \begin{cases} \frac{\tau}{2\mu_-}e_2 & \text{in } \Omega_-, \\ \left(-\frac{\tau}{2\mu_+} + \frac{\tau}{\mu_-}\right)e_2 & \text{in } \Omega_+. \end{cases} \quad (50)$$

The limit u^1 is discontinuous at $x_3 = 0$ and

$$\begin{aligned} [u_1^1] &= 0, \\ [u_2^1] &= \frac{\tau}{\mu} - \frac{1}{2} \left(\frac{\tau}{\mu_+} + \frac{\tau}{\mu_-} \right), \\ [u_3^1] &= 0, \end{aligned} \quad (51)$$

these relations are in agreement with (35)_{1,2}. Lastly, note that $\sigma^\varepsilon = \tau(e_2 \otimes e_3 + e_3 \otimes e_2) = \sigma^0$. We therefore obtain $\sigma^1 = 0$, in line with relations (35)₃₋₅.

7. Proof of Theorem 2

In this section we give the proof of Theorem 2. For the sake of convenience, the proof is divided into two parts. The first part proves Eqs. (35)_{1,2} and the second one proves (35)₃₋₅.

7.1. Proof of the relation involving the displacement jump

The proof is subdivided here into three steps. First, we link the jump in u^1 to a new variable denoted \bar{u}^1 , which represents the ‘‘internal’’ jump mechanically in the thin layer.

Lemma 4

$$\frac{[u^\varepsilon]_\varepsilon}{\varepsilon} \rightharpoonup \bar{u}^1(\hat{x}) \quad \text{in } L^2(S) \text{ (weak)}. \quad (52)$$

$$\frac{[u^\varepsilon - u^0]_\varepsilon}{\varepsilon} \rightharpoonup \bar{u}^1(\hat{x}) - \frac{1}{2} S u_{,3}^0(\hat{x}, 0) \quad \text{in } L^2(S) \text{ (weak)}, \quad (53)$$

where $Sv(\cdot, 0) = v(\cdot, 0^+) + v(\cdot, 0^-)$

The proof of this Lemma is given in Appendix 3.

Comment 4. Since the displacement field u^1 is generally not regular enough for us to be able to define its traces on S_\pm , the jump in u^1 across S is defined via the limit given in Lemma 4. If the loads are sufficiently smooth, the trace of u^1 on S will exist and the left hand side in (53) converges to the jump in u^1 . Two examples, in which the equilibrium solution is sufficiently regular, have been presented in Section 6.

Comment 5. The result of Lemma 4 confirms the order one matching condition introduced into the matched asymptotic expansion (see for example [1, Eq. (5)]).

In the second step, the stresses resulting from the sequences u^ε and $R^\varepsilon u^\varepsilon$ have to be introduced:

$$\sigma_m^\varepsilon = a^m e(u^\varepsilon), \quad (54)$$

$$\sigma_\pm^\varepsilon = a^\pm e(u^\varepsilon) = a^\pm e(R^\varepsilon u^\varepsilon) = \bar{\sigma}_\pm^\varepsilon, \quad (55)$$

$$\bar{\sigma}_m^\varepsilon = a^m e(R^\varepsilon u^\varepsilon), \quad (56)$$

$$\bar{\sigma}^\varepsilon = \begin{cases} \bar{\sigma}_\pm^\varepsilon & \text{in } \Omega_\pm^\varepsilon, \\ \bar{\sigma}_m^\varepsilon & \text{in } B_\varepsilon, \end{cases} \quad (57)$$

and the limit of $\int_S \bar{\sigma}_m^\varepsilon(\hat{x}, x_3) \phi(\hat{x}) d\hat{x}$ can be shown to exist for any $\phi \in D(S)$. Denoting this limit $\int_S \bar{\sigma}^0(\hat{x}) \phi(\hat{x}) d\hat{x}$, we relate the weak limit $\bar{\sigma}^0$ to the weak limit \bar{u}^1 .

Proposition 1

$$\bar{\sigma}_{\alpha 3}^0 = \mu \left(\bar{u}_\alpha^1 + u_{3,\alpha}^0 \right), \quad \alpha = 1, 2, \quad (58)$$

$$\bar{\sigma}_{33}^0 = \lambda(u_{1,1}^0 + u_{2,2}^0) + (\lambda + 2\mu)\bar{u}_3^1, \quad (59)$$

$$\bar{\sigma}_{\alpha\alpha}^0 = \lambda(u_{1,1}^0 + u_{2,2}^0 + \bar{u}_3^1) + 2\mu u_{\alpha,\alpha}^0, \quad \alpha = 1, 2, \quad (60)$$

$$\bar{\sigma}_{12}^0 = \mu(u_{1,2}^0 + u_{2,1}^0) \quad (61)$$

in $D'(S)$.

The proof of this Proposition is given in Appendix 3. To conclude the first part of the proof of Theorem 2, we give the following Lemma relating the values of $\bar{\sigma}^0 e_3$ and $\sigma^0 e_3$.

Lemma 5

$$\bar{\sigma}^0 e_3 = \sigma^0 e_3 \quad \text{a.e. on } S. \quad (62)$$

This Lemma is proved in Appendix 3. The first two equations of Theorem 2 follow easily after collecting the results of Lemmas 4 and 5 and Proposition 1.

7.2. Proof of the relation involving the traction vector jump

Here the proof is subdivided into two steps. In the first step, the jump in $\sigma^1 e_3$ related to a new variable denoted $\bar{\sigma}^1 e_3$, which represents the ‘‘internal’’ jump in the traction vector in the thin layer mechanically.

Lemma 6

$$\left(\frac{[\sigma^e - \sigma^0]_e}{\varepsilon} \right) e_3 \rightharpoonup \bar{\sigma}^1(\hat{x}) e_3 - \frac{1}{2} S \sigma_{,3}^0(\hat{x}, 0) e_3 \quad \text{in } D'(S) \text{ (weak)}. \quad (63)$$

The proof of this Lemma is identical to the proof of Lemma 4 given in Appendix 3.

Comment 6. The vector field $\bar{\sigma}^1 e_3$ is generally not regular enough for us to be able to define its traces on S_\pm , and we define therefore the jump in $\bar{\sigma}^1 e_3$ across S as done above in the case of the jump in u^1 in the Comment on Lemma 4.

In the second step, the vector field $\bar{\sigma}^1 e_3$ is related to the stress tensor $\bar{\sigma}^0$. This is done in the following Proposition (which is proved in Appendix 4), based on the equilibrium equations related to the thin layer.

Proposition 2. Denoting $\bar{\sigma}_{\alpha 3}^1 := \mu \bar{u}_{\alpha,\alpha}^1$, $\alpha = 1, 2$, and $\bar{\sigma}_{33}^1 := \lambda(\bar{u}_{1,1}^1 + \bar{u}_{2,2}^1)$, we have

$$\bar{\sigma}_{i3}^1 = -\bar{\sigma}_{i1,1}^0 - \bar{\sigma}_{i2,2}^0, \quad i = 1, 2, 3 \quad (64)$$

in $D'(S)$.

To complete the second step, the results of Propositions 1 and 2 and Lemma 5 are combined, giving:

$$\begin{aligned} \bar{\sigma}_{13}^1 &= -\left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{1,11}^0 + \mu u_{1,22}^0 + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{2,21}^0 + \frac{\lambda}{\lambda + 2\mu} \sigma_{33,1}^0 \right), \\ \bar{\sigma}_{23}^1 &= -\left(\frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} u_{2,22}^0 + \mu u_{2,11}^0 + \frac{\mu(3\lambda + 2\mu)}{\lambda + 2\mu} u_{1,12}^0 + \frac{\lambda}{\lambda + 2\mu} \sigma_{33,2}^0 \right), \\ \bar{\sigma}_{33}^1 &= -\sigma_{31,1}^0 - \sigma_{32,2}^0, \end{aligned} \quad (65)$$

in $D'(S)$.

Comment 7. Conditions (65) can be interpreted mechanically as follows: Let us assume that the adhesive is in a state of generalized plane stress, where the average out-of-plane stress vector, which coincides with the stress vectors $\sigma^0 e_3$ [10], is given. Let us also assume that the body forces are nil.

Applying an averaging operation to the equilibrium equations across the adhesive, we obtain

$$\begin{aligned} \bar{\sigma}_{11,1} + \bar{\sigma}_{12,2} + [\sigma_{13}]_i &= 0, \\ \bar{\sigma}_{21,1} + \bar{\sigma}_{22,2} + [\sigma_{23}]_i &= 0, \\ \bar{\sigma}_{31,1} + \bar{\sigma}_{32,2} + [\sigma_{33}]_i &= 0, \end{aligned} \quad (66)$$

where the bar denotes the average. Taking the average constitutive equations of the adhesive and recalling that $\bar{\sigma}_{33} = \sigma_{33}^0$, we have

$$\bar{e}_{33} = \frac{1}{\lambda + 2\mu} \sigma_{33}^0 - \frac{\lambda}{\lambda + 2\mu} (\bar{e}_{11} + \bar{e}_{22}), \quad (67)$$

$$\bar{\sigma}_{11} = \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \bar{e}_{11} + \frac{2\mu\lambda}{\lambda + 2\mu} \bar{e}_{22} + \frac{\lambda}{\lambda + 2\mu} \sigma_{33}^0, \quad (68)$$

$$\bar{\sigma}_{22} = \frac{2\mu\lambda}{\lambda + 2\mu} \bar{e}_{11} + \frac{4\mu(\lambda + \mu)}{\lambda + 2\mu} \bar{e}_{22} + \frac{\lambda}{\lambda + 2\mu} \sigma_{33}^0, \quad (69)$$

$$\bar{\sigma}_{12} = 2\mu \bar{e}_{12}. \quad (70)$$

Substituting into (66), recalling that $\bar{\sigma}_{\alpha 3} = \sigma_{\alpha 3}^0$, $\alpha = 1, 2$, and identifying the jumps $[\sigma_{i3}], i = 1, 2, 3$, with the jumps in components of $\bar{\sigma}^1 e_3$, we obtain (65).

Note that Lemmas 4 and 3 introduce extra terms into the contact law (35). The jumps can be taken to result from superimposing the deformation of the adhesive, as described by (65) on the “matching” conditions (63 and 53) imposing continuous deformation between the adhesive and the adherents.

8. Conclusion

This study focuses on the asymptotic analysis of a joint consisting of two adherents and an adhesive, all of which having a similar rigidity. The aim of this paper was two-fold. First it was proposed to establish that the results obtained at order zero in [1] using matched asymptotic analysis can be exactly reproduced via Γ -convergence. Secondly, it was proposed to extend the results obtained at a higher order in [1] to the three-dimensional case, and this was accomplished by studying the properties of a (weakly converging) sequence of equilibrium solutions of the original three-dimensional problem.

The model obtained by studying the weak limit of this sequence, which is given by relations (65), gives a non-local model for the interface relating the jump in the stress vector to the jump in the displacement along the interface, along with their derivatives. It is worth noting that the relations (65) contain some extra terms in comparison with the corresponding relations obtained in [1]. These terms are seen to be necessary, as shown by the simple examples presented above.

This study should lead to many developments. It would be interesting to obtain an energy interpretation of (65) by performing the asymptotic Γ -expansion of the energy (13).

Another interesting extension would consist in taking the adhesive to be curved: an example of a composite cylindrical assemblage given in [22] shows that in this case, the elastic constants in the relations involving imperfect interfaces are different from those occurring in (65).

In glue-bonding processes, the presence of residual stresses may determine the resistance of the joint, and it would be of interest to extend relations (65) to account for the residual stress. Lastly, implementing this model in a finite element code might make it possible to apply the imperfect interface model (65) to a whole range of practical engineering problems.

Appendix 1. Proof of Lemma 2

For the solution to (\bar{P}_ε) , we have

$$\int_{\Omega^\varepsilon} a^\pm e(u^\varepsilon) : e(u^\varepsilon) dx + \int_{B^\varepsilon} a^m e(u^\varepsilon) : e(u^\varepsilon) dx = l(u^\varepsilon). \quad (71)$$

Using the definition of the linear form l , the Cauchy–Schwartz inequality, and $\varepsilon < 1$, we obtain

$$|l(u^\varepsilon)| \leq C \left\{ \left(\int_{B^\varepsilon} |e(u^\varepsilon(x))|^2 dx \right)^{1/2} + \left(\int_{\Omega^\varepsilon} |e(u^\varepsilon(x))|^2 dx \right)^{1/2} \right\}. \quad (72)$$

The coercivity of the elasticity tensors is then used to obtain:

$$\eta \left\{ \int_{B^\varepsilon} |e(u^\varepsilon(x))|^2 dx + \int_{\Omega^\varepsilon} |e(u^\varepsilon(x))|^2 dx \right\} \leq \int_{\Omega^\varepsilon} a^\pm e(u^\varepsilon) : e(u^\varepsilon) dx + \int_{B^\varepsilon} a^m e(u^\varepsilon) : e(u^\varepsilon) dx. \quad (73)$$

This gives (16). To obtain (17), use (16) combined with Lemma 3.1. The weak convergence in V^ε follows from the Trace Theorem, and the strong convergence in X follows from the Rellich–Kondrachev Theorem. \square

Appendix 2. Proof of Lemma 3

From Theorem 1, it can be said that

$$u^\varepsilon \rightharpoonup u^0 \quad \text{in } L^2(\Omega_0) \text{ (strong)}. \quad (74)$$

Using a result obtained by Licht and Michaille [30, Lemma 3.3] and the continuity of u_0 in problem (P_0) , we obtain

$$[u^\varepsilon]_\varepsilon \rightarrow 0 \quad \text{in } L^2(S) \text{ (strong)}. \quad (75)$$

A subsequence, non relabeled, can therefore be extracted such that

$$\frac{u^\varepsilon - u^0}{\varepsilon} \rightharpoonup u^1 \quad \text{in } L^2(\Omega_0) \text{ (weak)}. \quad (76)$$

$$\frac{[u^\varepsilon]_\varepsilon}{\varepsilon} \rightharpoonup \bar{u}^1 \quad \text{in } L^2(S) \text{ (weak)}. \quad (77)$$

Appendix 3. Proofs of Lemma 4, Proposition 1 and Lemma 5

Proof of Lemma 4. The first part of the Lemma is obtained using the same argument as for Lemma 3. The proof of the second part follows using (52), and noting that by density, using a Taylor's expansion at point zero and the continuity of u^0 across S ,

$$\frac{[u^0]_\varepsilon}{\varepsilon} \rightharpoonup \frac{1}{2} S u_3^0(\hat{x}, 0) \quad \text{in } L^2(S) \text{ (weak)}. \quad \square$$

Proof of Proposition 1. We write the constitutive relations as follows:

$$\begin{aligned} (\bar{\sigma}_m^\varepsilon)_{\alpha\alpha} &= \lambda \left(\frac{1}{2} S^\varepsilon u_{1,1}^\varepsilon + \frac{1}{2} S^\varepsilon u_{2,2}^\varepsilon + \frac{x_3}{\varepsilon} [u_{1,1}^\varepsilon + u_{2,2}^\varepsilon]_\varepsilon + \frac{1}{\varepsilon} [u_3^\varepsilon]_\varepsilon \right) + 2\mu \left(\frac{x_3}{\varepsilon} [u_{\alpha,\alpha}^\varepsilon]_\varepsilon + \frac{1}{2} S^\varepsilon u_{\alpha,\alpha}^\varepsilon \right), \\ (\bar{\sigma}_m^\varepsilon)_{12} &= \mu \left(\frac{x_3}{\varepsilon} [u_{1,2}^\varepsilon + u_{2,1}^\varepsilon]_\varepsilon + \frac{1}{2} S^\varepsilon u_{1,2}^\varepsilon + \frac{1}{2} S^\varepsilon u_{2,1}^\varepsilon \right), \\ (\bar{\sigma}_m^\varepsilon)_{\alpha 3} &= \mu \left(\frac{1}{\varepsilon} [u_{\alpha,\alpha}^\varepsilon]_\varepsilon + S^\varepsilon u_{\alpha,\alpha}^\varepsilon + \frac{x_3}{\varepsilon} [u_{3,\alpha}^\varepsilon]_\varepsilon \right), \\ (\bar{\sigma}_m^\varepsilon)_{33} &= \lambda \left(\frac{1}{2} S^\varepsilon u_{1,1}^\varepsilon + \frac{1}{2} S^\varepsilon u_{2,2}^\varepsilon + \frac{x_3}{\varepsilon} [u_{1,1}^\varepsilon + u_{2,2}^\varepsilon]_\varepsilon + \frac{1}{\varepsilon} [u_3^\varepsilon]_\varepsilon \right) + 2\mu \frac{1}{\varepsilon} [u_3^\varepsilon]_\varepsilon, \end{aligned}$$

where the derivatives are taken in the distributional sense. For a function $\phi \in D(S)$, we have

$$\begin{aligned} \left| \int_S \frac{x_3}{\varepsilon} [u_{i,\alpha}^\varepsilon]_\varepsilon \phi dx \right| &\leq \left| \int_S [u_{i,\alpha}^\varepsilon]_\varepsilon \phi dx \right| = \left| \int_S [u_{i,\alpha}^\varepsilon]_\varepsilon \phi_{,\alpha} dx \right| \rightarrow \left| \int_S [u_{i,\alpha}^0]_\varepsilon \phi_{,\alpha} dx \right| = 0, \quad \alpha = 1, 2, \quad i = 1, 2, 3, \\ \int_S \frac{1}{2} S^\varepsilon u_{i,\alpha}^\varepsilon \phi dx &= - \int_S \frac{1}{2} S^\varepsilon u_i^\varepsilon \phi_{,\alpha} dx \rightarrow - \int_S u_i^0 \phi_{,\alpha} d\hat{x} = \int_S u_{i,\alpha}^0 \phi d\hat{x}, \quad \alpha = 1, 2, \quad i = 1, 2, 3. \end{aligned}$$

The result is obtained using (52). \square

Proof of Lemma 5. From the zero order result, we have (see Theorem 4.1 in [34])

$$\sigma^\varepsilon \rightharpoonup \sigma^0 \quad \text{in } L^2(\Omega_0). \quad (78)$$

From the continuity of the trace operator, we obtain

$$\sigma_\pm^\varepsilon e_3 \rightharpoonup \sigma^0 e_3 \quad \text{in } L^2(S). \quad (79)$$

To prove the thesis in question, it is necessary to establish that the sequences $\sigma_\pm^\varepsilon e_3$ and $\bar{\sigma}_m^\varepsilon e_3$ have the same weak limit on S . Based on the continuity of the stress vector $\sigma^\varepsilon e_3$ on the surfaces S_\pm^ε , we obtain

$$0 = \int_{S_\pm^\varepsilon} (\sigma_\pm^\varepsilon e_3 - \sigma_m^\varepsilon e_3) \phi(\hat{x}) d\hat{x} = \int_{S_\pm^\varepsilon} (\sigma_\pm^\varepsilon e_3 - \bar{\sigma}_m^\varepsilon e_3) \phi(\hat{x}) d\hat{x} + \int_{S_\pm^\varepsilon} (\bar{\sigma}_m^\varepsilon e_3 - \sigma_m^\varepsilon e_3) \phi(\hat{x}) d\hat{x}. \quad (80)$$

Note that

$$(\bar{\sigma}_m^\varepsilon e_3 - \sigma_m^\varepsilon e_3)_\alpha = \mu \left(\frac{1}{\varepsilon} [u_{\alpha,\alpha}^\varepsilon]_\varepsilon - u_{\alpha,\alpha}^\varepsilon \right) \quad \alpha = 1, 2, \quad (81)$$

$$(\bar{\sigma}_m^\varepsilon e_3 - \sigma_m^\varepsilon e_3)_3 = (2\mu + \lambda) \left(\frac{1}{\varepsilon} [u_3^\varepsilon]_\varepsilon - u_{3,3}^\varepsilon \right), \quad (82)$$

on S_\pm^ε . Using a density argument, we conclude that the restriction of $\frac{1}{\varepsilon} [u^\varepsilon]_\varepsilon - u^\varepsilon$ on S_\pm^ε converges to zero in $D'(S)$, which, along with (81) and (82), means that the second term in (80) converges to zero. To complete the proof, we use (79) and recall that σ^0 is the weak limit of the sequence σ_m^ε on S .

Appendix 4. Proof of Proposition 2

From (78), we deduce $div \bar{\sigma}^\varepsilon \rightharpoonup 0$ in $D'(\Omega)$ (weak). We take $\phi(\hat{x}, x_3) = \phi_1(\hat{x}) \phi_2(x_3)$, $\phi_2(x_3) = 1$ on $]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$ to obtain $\int_S \phi_1(\hat{x}) \left(\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} div \bar{\sigma}^\varepsilon dx_3 \right) d\hat{x} \rightarrow 0$.

Therefore, up to a "rare" subsequence, $\int_S \phi_1(\hat{x}) \left(\frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} div \bar{\sigma}^\varepsilon dx_3 \right) d\hat{x} \rightarrow 0$.

Note that

$$\frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \bar{\sigma}_{ij}^{\varepsilon} dx_3 \rightharpoonup \bar{\sigma}_{ij}^0 \in D'(S), \text{ (weak)}, \quad i, j = 1, 2, 3,$$

and thus, after integrating by parts in the tangential derivatives,

$$\frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \bar{\sigma}_{i\alpha,\alpha}^{\varepsilon} dx_3 \rightharpoonup \bar{\sigma}_{i\alpha,\alpha}^0 \in D'(S), \text{ (weak)}, \quad i = 1, 2, 3, \quad \alpha = 1, 2.$$

To complete the proof, note also that

$$\frac{[\bar{\sigma}_{i3}^{\varepsilon}]_{\varepsilon}}{\varepsilon} \rightharpoonup \bar{\sigma}_{i3}^1 \in D'(S), \text{ (weak)}, \quad i = 1, 2, 3. \quad \square$$

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