Phi-entropy inequalities and Fokker-Planck equations

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Abstract

We present new Φ-entropy inequalities for diffusion semigroups under the curvature-dimension criterion. They include the isoperimetric function of the Gaussian measure. Applications to the long time behaviour of solutions to Fokker-Planck equations are given.

Keywords: Functional inequalities, logarithmic Sobolev inequality, Poincaré inequality, Φ-entropies, Bakry-Emery criterion, diffusion semigroups, Fokker-Planck equation.
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1 Introduction

We consider a Markov semigroup $(\mathbf{P}_t)_{t\geq 0}$ on $\mathbb{R}^n$, acting on functions on $\mathbb{R}^n$ by $\mathbf{P}_t f(x) = \int_{\mathbb{R}^n} f(y)\, p_t(x,dy)$ for x in $\mathbb{R}^n$. The kernels $p_t(x,dy)$ are probability measures on $\mathbb{R}^n$ for all x and $t \geq 0$, called transition kernels. We assume that the Markov infinitesimal generator $L = \left.\dfrac{\partial \mathbf{P}_t}{\partial t}\right|_{t=0^+}$ is given by

$$Lf(x) = \sum_{i,j=1}^{n} D_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{i=1}^{n} a_i(x) \frac{\partial f}{\partial x_i}(x)$$

where $D(x) = (D_{ij}(x))_{1\leq i,j\leq n}$ is a symmetric $n \times n$ matrix, nonnegative in the sense of quadratic forms on $\mathbb{R}^n$ and with smooth coefficients, and where the $a_i, 1 \leq i \leq n$, are smooth. Such a semigroup or generator is called a *diffusion*, and we refer to Refs. [5], [6], [13] for backgrounds on them.

If μ is a Borel probability measure on $\mathbb{R}^n$ and f a μ-integrable map on $\mathbb{R}^n$ we let $\mu(f) = \int_{\mathbb{R}^n} f(x)\, \mu(dx)$. If, moreover, Φ is a convex map on an interval I of $\mathbb{R}$ and f an I-valued map with f and $\Phi(f)$ μ-integrable, we let

$$\mathbf{Ent}_\mu^\Phi(f) = \mu(\Phi(f)) - \Phi(\mu(f))$$

be the Φ-entropy of f under μ (see Ref. [10] for instance). Two fundamental examples are $\Phi(x) = x^2$ on $\mathbb{R}$, for which $\mathbf{Ent}_\mu^\Phi(f)$ is the variance of f, and $\Phi(x) = x \ln x$ on $]0, +\infty[$, for which $\mathbf{Ent}_\mu^\Phi(f)$ is the Boltzmann entropy of f. By Jensen's inequality, $\mathbf{Ent}_\mu^\Phi(f)$ is always nonnegative and, if Φ is strictly convex, it is positive unless f is a constant, equal to $\mu(f)$. The semigroup $(\mathbf{P_t})_{t\geq 0}$ is said *μ-ergodic* if $\mathbf{P_t} f$ tends to $\mu(f)$ as t tends to infinity in $L^2(\mu)$, for all f.

In Section 2 we shall derive bounds on $\mathbf{Ent}_\mu^\Phi(f)$ and $\mathbf{Ent}_{\mathbf{P}_t}^\Phi(f)(x)$ which will measure the convergence of $\mathbf{P_t} f$ to $\mu(f)$ in the ergodic setting. This is motivated by the study of the long time behaviour of solutions to Fokker-Planck equations, which will be discussed in Section 3.

Some results of this note with their proofs are detailled in Ref. [9].

2 Phi-entropy inequalities

Bounds on $\mathbf{Ent}^{\Phi}_{\mathbf{P}_t}(f)$ and assumptions on L will be given in terms of the *carré du champ* and Γ_2 operators associated to L, defined by

$$\Gamma(f,g) = \frac{1}{2}\Big(L(fg) - f\,Lg - g\,Lf\Big),\ \Gamma_2(f) = \frac{1}{2}\Big(L\Gamma(f) - 2\Gamma(f,Lf)\Big).$$

If ρ is a real number, we say that the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the $CD(\rho,\infty)$ *curvature-dimension* (or Bakry-Émery) *criterion* (see Ref. [7]) if

$$\Gamma_2(f) \geq \rho\,\Gamma(f)$$

for all functions f, where $\Gamma(f) = \Gamma(f,f)$.

The carré du champ is explicitely given by

$$\Gamma(f,g)(x) = < \nabla f(x), D(x)\,\nabla g(x) > .$$

Expressing Γ_2 is more complex in the general case but, for instance, if D is constant, then L satisfies the $CD(\rho,\infty)$ criterion if and only if

$$\frac{1}{2}\big(Ja(x)D + (Ja(x)D)^*\big) \geq \rho\,D \tag{1}$$

for all x, as quadratic forms on $\mathbb{R}^n$, where Ja is the Jacobian matrix of a and M^* denotes the transposed matrix of a matrix M (see Ref. [2, 4]) .

Poincaré and logarithmic Sobolev inequalities for the semigroup $(\mathbf{P}_t)_{t\geq 0}$ are known to be implied by the $CD(\rho,\infty)$ criterion. More generally, and following Ref. [6, 7, 10], let $\rho > 0$ and Φ be a C^4 strictly convex function on an interval I of $\mathbb{R}$ such that $-1/\Phi''$ is convex. If $(\mathbf{P_t})_{t\geq 0}$ is μ-ergodic and satisfies the $CD(\rho,\infty)$ criterion, then μ satisfies the Φ-entropy inequality

$$\mathbf{Ent}^{\Phi}_{\mu}(f) \leq \frac{1}{2\rho}\,\mu(\Phi''(f)\,\Gamma(f)) \tag{2}$$

for all I-valued functions f.

The main instances of such Φ's are the maps $x \mapsto x^2$ on $\mathbb{R}$ and $x \mapsto x\ln x$ on $]0,+\infty[$ or more generally, for $1 \leq p \leq 2$

$$\Phi_p(x) = \begin{cases} \frac{x^p - x}{p(p-1)}, & x > 0 \quad \text{if } p \in]1,2] \\ x\ln x, & x > 0 \quad \text{if } p = 1. \end{cases} \tag{3}$$

For this Φ_p with p in $]1,2]$ the Φ-entropy inequality (2) becomes

$$\frac{\mu(g^2) - \mu(g^{2/p})^p}{p-1} \leq \frac{2}{p\rho}\,\mu(\Gamma(g)) \tag{4}$$

for all positive functions g. For given g the map $p \mapsto \frac{\mu(g^2)-\mu(g^{2/p})^p}{p-1}$ is nonincreasing with respect to $p > 0, p \neq 1$. Moreover its limit for $p \to 1$ is $\mathbf{Ent}_{\mu}\big(g^2\big)$, so that the so-called *Beckner inequalities* (4) for p in $]1,2]$ give a natural monotone interpolation between the weaker Poincaré inequality (for $p = 2$), and the stronger logarithmic Sobolev inequality (for $p \to 1$).

Long time behaviour of the semigroup

The Φ-entropy inequalities provide estimates on the long time behaviour of the associated diffusion semigroups. Indeed, let $(\mathbf{P_t})_{t\geq 0}$ be such a semigroup, ergodic for the measure μ. If Φ is a $\mathcal{C}^2$ function on an interval I, then

$$\frac{d}{dt}\mathbf{Ent}_\mu^\Phi(\mathbf{P_t}f) = -\mu\big(\Phi''(\mathbf{P_t}f)\,\Gamma(\mathbf{P_t}f)\big) \tag{5}$$

for all $t \geq 0$ and all I-valued functions f. As a consequence, if C is a positive number, then the semigroup converges in Φ-entropy with exponential rate:

$$\mathbf{Ent}_\mu^\Phi(\mathbf{P_t}f) \leq e^{-\frac{t}{C}}\mathbf{Ent}_\mu^\Phi(f) \tag{6}$$

for all $t \geq 0$ and all I-valued functions f, if and only if the measure μ satisfies the Φ-entropy inequality for all I-valued functions f,

$$\mathbf{Ent}_\mu^\Phi(f) \leq C\mu\big(\Phi''(f)\,\Gamma(f)\big). \tag{7}$$

2.1 Refined Φ-entropy inequalities

We now give and study improvements of (2) for the Φ_p maps given by (3):

Theorem 1 ([[9])] *Let $\rho \in \mathbb{R}$ and $p \in]1,2[$. Then the following assertions are equivalent, with $\big(1-e^{-2\rho t}\big)/\rho$ and $\big(e^{2\rho t}-1\big)/\rho$ replaced by $2t$ if $\rho = 0$:*

(i) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the $CD(\rho,\infty)$ criterion;

(ii) $(\mathbf{P}_t)_{t\geq 0}$ satisfies the refined local Φ_p-entropy inequality

$$\frac{1}{(p-1)^2}\left[\mathbf{P_t}(f^p) - \mathbf{P_t}(f)^p\left(\frac{\mathbf{P_t}(f^p)}{\mathbf{P_t}(f)^p}\right)^{\frac{2}{p}-1}\right] \leq \frac{1-e^{-2\rho t}}{\rho}\mathbf{P_t}\big(f^{p-2}\,\Gamma(f)\big)$$

for all positive t and all positive functions f;

(iii) $(\mathbf{P}_t)_{t\geq 0}$ satisfies the reverse refined local Φ_p-entropy inequality

$$\frac{1}{(p-1)^2}\left[\mathbf{P_t}(f^p) - \mathbf{P_t}(f)^p\left(\frac{\mathbf{P_t}(f^p)}{\mathbf{P_t}(f)^p}\right)^{\frac{2}{p}-1}\right] \geq \frac{e^{2\rho t}-1}{\rho}\left(\frac{(\mathbf{P_t}f)^p}{\mathbf{P_t}(f^p)}\right)^{\frac{2}{p}-1}(\mathbf{P_t}f)^{p-2}\,\Gamma(\mathbf{P_t}f)$$

for all positive t and all positive functions f.

If, moreover, $\rho > 0$ and the measure μ is ergodic for the semigroup $(\mathbf{P_t})_{t\geq 0}$, then μ satisfies the refined Φ_p-entropy inequality

$$\frac{p^2}{(p-1)^2}\left[\mu(g^2) - \mu(g^{2/p})^p\left(\frac{\mu(g^2)}{\mu(g^{2/p})^p}\right)^{\frac{2}{p}-1}\right] \leq \frac{4}{\rho}\mu(\Gamma(g)) \tag{8}$$

for all positive maps g.

The bound (8) has been obtained in Ref. [3] for the generator L defined by $Lf = \mathrm{div}(D\nabla f) - < D\nabla V, \nabla f >$ with $D(x)$ a scalar matrix and for the ergodic measure $\mu = e^{-V}$, and under the corresponding $CD(\rho,\infty)$ criterion.

It improves on the Beckner inequality (4) since

$$\frac{\mu(g^2) - \mu(g^{2/p})^p}{p-1} \leq \frac{p}{2(p-1)^2}\Big[\mu(g^2) - \mu(g^{2/p})^p\left(\frac{\mu(g^2)}{\mu(g^{2/p})^p}\right)^{\frac{2}{p}-1}\Big]. \tag{9}$$

We have noticed that for all g the map $p \mapsto \frac{\mu(g^2)-\mu(g^{2/p})^p}{p-1}$ is continuous and nonincreasing on $]0,+\infty[$, with values $\mathbf{Ent}_\mu\big(g^2\big)$ at $p=1$ and $\mathbf{Var}_\mu(g)$ at $p=2$. Similarly, for the larger functional introduced in (9), the map

$$p \mapsto \frac{p}{2(p-1)^2}\Big[\mu(g^2)-\mu(g^{2/p})^p\,\Big(\frac{\mu(g^2)}{\mu(g^{2/p})^p}\Big)^{\frac{2}{p}-1}\Big]$$

is nonincreasing on $]1,+\infty[$ (see [9, Prop. 11]). Moreover its value is $\mathbf{Var}_\mu(g)$ at $p=2$ and it tends to $\mathbf{Ent}_\mu\big(g^2\big)$ as $p \to 1$, hence providing a new monotone interpolation between Poincaré and logarithmic Sobolev inequalities.

The pointwise $CD(\rho,\infty)$ criterion can be replaced by the integral criterion

$$\mu\Big(g^{\frac{2-p}{p-1}}\Gamma_2(g)\Big) \geq \rho\,\mu\Big(g^{\frac{2-p}{p-1}}\Gamma(g)\Big)$$

for all positive functions g, and one can still get the refined Φ_p-entropy inequality (2), even in the case of non-reversible semigroups (see [9, Prop. 14]).

Remark 2 *For $\rho=0$, and following Ref. [3], the convergence of $\mathbf{P_t}f$ towards $\mu(f)$ can be measured on $H(t)=\mathbf{Ent}_\mu^\Phi(\mathbf{P_t}f)$ as*

$$|H'(t)| \leq \frac{|H'(0)|}{1+\alpha t}, \quad t \geq 0$$

where $\alpha = \frac{2-p}{p}|H'(0)|/H(0)$. This illustrates the improvement offered by (8) instead of (4), which does not give here any convergence rate.

2.2 The case of the Gaussian isoperimetry function

Let F be the distribution function of the one-dimensional standard Gaussian measure. The map $\mathcal{U}=F'\circ F^{-1}$, which is the isoperimetry function of the Gaussian distribution, satisfies $\mathcal{U}''=-1/\mathcal{U}$ on the set $[0,1]$, so that the map $\Phi=-\mathcal{U}$ is convex with $-1/\Phi''$ also convex on $[0,1]$.

Theorem 3 *Let ρ be a real number. Then the following three assertions are equivalent, with $\big(1-e^{-2\rho t}\big)/\rho$ and $\big(e^{2\rho t}-1\big)/\rho$ replaced by $2t$ if $\rho=0$:*

(i) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the $CD(\rho,\infty)$ criterion;

(ii) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the local Φ-entropy inequality

$$\mathbf{Ent}_{\mathbf{P_t}}^\Phi(f) \leq \frac{1}{\Phi''(\mathbf{P_t}f)}\log\left(1+\frac{1-e^{-2\rho t}}{2\,\rho}\Phi''(\mathbf{P_t}f)\,\mathbf{P_t}(\Phi''(f)\Gamma(f))\right)$$

for all positive t and all $[0,1]$-valued functions f;

(iii) the semigroup $(\mathbf{P}_t)_{t\geq 0}$ satisfies the reverse local Φ-entropy inequality

$$\mathbf{Ent}_{\mathbf{P_t}}^\Phi(f) \geq \frac{1}{\Phi''(\mathbf{P_t}f)}\log\left(1+\frac{e^{2\rho t}-1}{2\,\rho}\Phi''(\mathbf{P_t}f)^2\Gamma(\mathbf{P_t}f))\right)$$

for all positive t and all $[0,1]$-valued functions f.

If, moreover, $\rho>0$ and the measure μ is ergodic for the semigroup $(\mathbf{P_t})_{t\geq 0}$, then μ satisfies the Φ-entropy inequality for all $[0,1]$-valued functions f:

$$\mathbf{Ent}_\mu^\Phi(f) \leq \frac{1}{\Phi''(\mu(f))}\log\left(1+\frac{\Phi''(\mu(f))}{2\rho}\mu(\Phi''(f)\Gamma(f))\right).$$

The proof is based on [9, Lemma 4]. For $\Phi = -\mathcal{U}$ it improves on the general Φ-entropy inequality (2) since $\log(1+x) \leq x$. Links with the isoperimetric bounds of Ref. [8] for instance will be addressed elsewhere.

3 Long time behaviour for Fokker-Planck equations

Let us consider the linear Fokker-Planck equation

$$\frac{\partial u_t}{\partial t} = \mathrm{div}[D(x)(\nabla u_t + u_t(\nabla V(x) + F(x)))], \quad t \geq 0,\, x \in \mathbb{R}^n \tag{10}$$

where $D(x)$ is a positive symmetric $n \times n$ matrix and F satisfies

$$\mathrm{div}\big(e^{-V} DF\big) = 0. \tag{11}$$

It is one of the purposes of Refs. [2] and [4] to rigorously study the asymptotic behaviour of solutions to (10)-(11). Let us formally rephrase the argument.

Assume that the Markov diffusion generator L defined by

$$Lf = \mathrm{div}(D\nabla f) - < D(\nabla V - F), \nabla f > \tag{12}$$

satisfies the $CD(\rho, \infty)$ criterion with $\rho > 0$, that is (1) if D is constant, etc.

Then the semigroup $(\mathbf{P_t})_{t\geq 0}$ associated to L is μ-ergodic with $d\mu = e^{-V}/Z dx$ where Z is a normalization constant. Moreover, a Φ-entropy inequality (7) holds with $C = 1/(2\rho)$ by (2), so that the semigroup converges to μ according to (6). However, under (11), the solution to (10) for the initial datum u_0 is given by $u_t = e^{-V}\,\mathbf{P_t}(e^V u_0)$. Then we can deduce the convergence of the solution u_t towards the stationary state e^{-V} (up to a constant) from the convergence estimate (6) for the semigroup, in the form

$$\mathbf{Ent}_\mu^\Phi\Big(\frac{u_t}{e^{-V}}\Big) \leq e^{-2\rho t}\mathbf{Ent}_\mu^\Phi\Big(\frac{u_0}{e^{-V}}\Big), \quad t \geq 0. \tag{13}$$

In fact such a result holds for the general Fokker-Planck equation

$$\frac{\partial u_t}{\partial t} = \mathrm{div}[D(x)(\nabla u_t + u_t a(x))], \quad t \geq 0,\, x \in \mathbb{R}^n \tag{14}$$

where again $D(x)$ is a positive symmetric $n \times n$ matrix and $a(x) \in \mathbb{R}^n$. Its generator is the dual (for the Lebesgue measure) of the generator

$$Lf = \mathrm{div}(D\nabla f) - < Da, \nabla f > . \tag{15}$$

Assume that the semigroup associated to L is ergodic and that its invariant probability measure μ satisfies a Φ-entropy inequality (7) with a constant $C \geq 0$: this holds for instance if L satisfies the $CD(1/(2C), \infty)$ criterion.

In this setting when $a(x)$ is not a gradient, the invariant measure μ is not explicit. Moreover the relation $u_t = e^{-V}\,\mathbf{P_t}(e^V u_0)$ between the solution of (14) and the semigroup associated to L does not hold, so that the asymptotic behaviour (13) for solutions to (14) can not be proved by using (6). However, this relation can be replaced by the following argument, for which the ergodic measure is only assumed to have a positive density u_∞ with respect to the Lebesgue measure.

Let u be a solution of (14) with initial datum u_0. Then, by [9, Lemma 7],

$$\frac{d}{dt}\mathbf{Ent}_\mu^\Phi\big(\frac{u_t}{u_\infty}\big) = \int \Phi'\big(\frac{u_t}{u_\infty}\big) L^* u_t dx = \int L\Big[\Phi'\big(\frac{u_t}{u_\infty}\big)\Big]\frac{u_t}{u_\infty} d\mu = -\int \Phi''\big(\frac{u_t}{u_\infty}\big)\Gamma\big(\frac{u_t}{u_\infty}\big) d\mu.$$

Then a Φ-Entropy inequality (7) for μ implies the exponential convergence:

Theorem 4 *With the above notation, assume that a Φ-entropy inequality* (7) *holds for μ and with a constant C. Then all solutions $u = (u_t)_{t\geq 0}$ to the Fokker-Planck equation* (14) *converge to u_∞ in Φ-entropy, with*

$$\mathbf{Ent}_\mu^\Phi\left(\frac{u_t}{u_\infty}\right) \leq e^{-t/C}\mathbf{Ent}_\mu^\Phi\left(\frac{u_0}{u_\infty}\right), \quad t \geq 0.$$

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