Entangling gates in even Euclidean lattices such as Leech lattice

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Abstract

We point out a organic relationship between real entangling n-qubit gates of quantum computation and the group of automorphisms of even Euclidean lattices Λ of the corresponding dimension $2n$. The type of entanglement that is found in the gates/generators of $\mathrm{Aut}(\Lambda)$ depends on the lattice. In particular, we investigate $\mathbb{Z}^n$ lattices, Barnes-Wall lattices D_4, E_8, Λ_{16} (associated to $n = 2$, 3 and 4 qubits), and the Leech lattices $h\Lambda_{24}$ and Λ_{24} (associated to a 3-qubit/qutrit system). Balanced tripartite entanglement is found to be a basic feature of $\mathrm{Aut}(\Lambda)$, a finding that bears out our recent work related to the Weyl group of E_8 [1, 2].

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1. Introduction

Bipartite (or multipartite) entanglement is a form of correlation that prevents two parties (or many parties) of controlling quantum states at their local sites. Entanglement leads to strange properties such as action at a distance, no-cloning, teleportation, and paradoxes between the algebraic description of (quantum) observables and (classical) measurements. Entangled states play a key role in quantum-error correction and quantum computational speedup.

Some properties of entangled states may be mimicked from differential geometry (two- and three-qubit Hilbert space geometry corresponds to Hopf

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fibrations, that are entanglement sensitive [3]), from topological spaces concepts (for instance the 3-qubit GHZ state corresponds to a borromean ring [4]) and from finite geometries (by using the hyperplanes of the relevant generalized quadrangles [5] or in the black-hole analogy [6]). Besides, group theoretical concepts have been repetitively put forward for describing classical [7] and quantum error correction [8]. In this paper, we exploit the unoticed but organic link between real entangling gates of quantum computation and the group of automorphisms of some dense even Euclidean group lattices. In particular, we display incomplete tripartite entanglement from $\mathbb{Z}^4$ (in Sec. 2) and various forms of balanced tripartite entanglement from Barnes-Wall and Leech lattices (in Secs 3 and 4).

Let us define a n-dimensional Euclidean lattice Λ as a discrete additive subgroup of the real vector space $\mathbb{R}^n$, endowed with the standard Euclidean product, and spanned by a generator matrix M with rows in $\mathbb{R}^n$. The automorphism group $\mathrm{Aut}(\Lambda)$ is the set of orthogonal matrices B such that under the conjugation action $U = MBM^{-1}$ by the generating matrix M, one has (i) $\det U = \pm 1$ and (ii) U is an integer matrix (see [9], p. 90). It will be shown that for a suitable choice of even Euclidean lattices Λ such as $\mathbb{Z}^n$ lattices, the root lattices D_4 and E_8, the Barnes-Wall lattice Λ_{16} and the Leech lattice Λ_{24}, the orthogonal group $\mathrm{Aut}(\Lambda)$ is a group of *quantum gates* acting on a two-, three-, four- and a three-qubit/qutrit system, respectively. In retrospect, such lattices bear out our recent work relating D_4 and E_8 Weyl groups to entangled states arising from the joint basis of specific mutually commuting set of generalized Pauli observables [1, 2]. In Magma [10], the line of code "aut:=AutomorphismGroup(L:NaturalAction); " has been used to explicit the generators of the automorphism group of the lattice L.

To illustrate our topic, let us start with the $\mathbb{Z}^n$ lattice whose lattice points are the integers and whose basis M is the unitary matrix. The elementary cell is the n-dimensional hypercube and the automorphism group is a wreath product [11]

$$\mathrm{Aut}(\mathbb{Z}^n) \cong \mathbb{Z}_2 \wr S_n = \mathbb{Z}_2^n \rtimes S_n, \tag{1}$$

of order $2^n n!$, where the symbol $\rtimes$ denotes the semidirect product and the wreath product $\wr$ corresponds to a permutation action of the symmetric group S_n on the n copies of the two-letter group $\mathbb{Z}_2$.

It is straightforward to check that the automorphism group $\mathrm{Aut}(\mathbb{Z}^4)$ of

the four-dimensional lattice contains the gate CNOT=$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, that is well known to feature two-qubit entanglement. More precisely, taking the input qubit states as $|0\rangle$ and $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$, the action CNOT$(|0\rangle\,|\psi\rangle)$ ends up into the non-separable Bell state $a\,|00\rangle + b\,|11\rangle$. The CNOT gate is a primitive of universal quantum computing as shown in the reference book [12].

The wreath product $\mathbb{Z}_2 \wr S_5$, corresponding to the automorphisms of the $\mathbb{Z}^5$ lattice, plays a role in the context of topologically protected two-qubit states [13] and its derived subgroup is the so-called Mathieu group M_{20} of order 960. The largest finite group of automorphisms of a K_3 surface turns out to be isomorphic to the group extension $\mathbb{Z}_4.M_{20}$ and the proof involves the Leech roots mentioned at the end of the present paper [14].

2. Entanglement arising from $\mathbb{Z}^n$ lattices

A exhaustive finite geometrical picture of two-qubit Pauli spin observables [5] tells us that two-qubit entanglement is encapsulated as a 3×3 grid (or Mermin square) having a single real row $r = \{\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z\}$ and a single real column $s = \{\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y\}$, where σ_x, σ_y and σ_z are the ordinary Pauli spin matrices. One may use the common eigenstate basis $M_r = \{(1,0,0,1), (1,0,0,-1), (0,1,1,0), (0,1,-1,0)\}$ of the set r for generating a lattice isomorphic to $\mathbb{Z}^4$. The natural action of the automorphism group on the basis M_r reads

$$\mathrm{Aut}(\Lambda) = \left\langle S = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, S' \right\rangle, \tag{2}$$

where the two generators S and S' only depart from the (opposite) sign of the last row. The four rows in S and S' build the common eigenstate basis of the set s. A matrix similar to S (with rows in a different order) is used in our previous work aimed at a three-qubit representation of the Weyl group of E_8 and its subgroups [1, 2].

Then, following a similar reasoning as before, we construct the $\mathbb{Z}^8$ lattice by means a generating basis built from the eigenstates of the following triple

of three-qubit observables

$$s_3 = \sigma_z \otimes (\{\sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_y \otimes \sigma_y\}), \tag{3}$$

that follows from s by adjoining the left tensor product σ_z. The basis we select consists of the rows of the matrix S_3 first introduced in [1] (Eq. (9)]. As for the two-qubit case above, the natural action of the automorphism group on the basis is obtained as

$$\mathrm{Aut}(\Lambda) = \langle g_1, g_2 \rangle,$$
$$\text{with } g_1 = \frac{1}{2}\begin{pmatrix} . & . & 1 & 1 & . & 1 & . & 1 \\ . & . & 1 & 1 & . & -1 & . & -1 \\ -1 & 1 & . & . & 1 & . & -1 & . \\ 1 & -1 & . & . & 1 & . & -1 & . \\ 1 & 1 & . & . & . & 1 & . & -1 \\ 1 & 1 & . & . & . & -1 & . & 1 \\ . & . & -1 & 1 & 1 & . & 1 & . \\ . & . & 1 & -1 & 1 & . & 1 & . \end{pmatrix}. \tag{4}$$

Let us now investigate the type of entanglement contained in the 3-qubit generators/gates g_1 and g_2 (g_2 is not made explicit but it has a similar form and properties than g_1). We single out the state encoded by the fourth row of g_1

$$|\psi\rangle = \frac{1}{2}(|000\rangle - |001\rangle + |100\rangle - |110\rangle). \tag{5}$$

A quantitative measure of tripartite entanglement is the 3-tangle [15, 16]

$$\tau^{(3)} = 4\left|(T^{001} - T^{000})^2 - 4P^{00}_{B_1}P^{00}_{B_0}\right|, \tag{6}$$

which contains the four determinants
$T^{000} = \det\begin{pmatrix} a_{000} & a_{011} \\ a_{100} & a_{111} \end{pmatrix}$, $T^{001} = \det\begin{pmatrix} a_{001} & a_{010} \\ a_{101} & a_{110} \end{pmatrix}$,
$P^{00}_{B_0} = \det\begin{pmatrix} a_{000} & a_{001} \\ a_{100} & a_{101} \end{pmatrix}$ and $P^{00}_{B_1} = \det\begin{pmatrix} a_{010} & a_{011} \\ a_{110} & a_{111} \end{pmatrix}$.

To quantify bipartite entanglement the strategy is as follows [16]. Take the density matrix $|\psi\rangle\langle\psi|$ of the 3-qubit system and trace out over the bipartite subsystems to obtain a reduced density matrix ρ. A measure of two-qubit entanglement is the tangle $\tau = C^2$, with the concurrence $C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$, where the λ_i are non-negative eigenvalues of the product $\rho\tilde{\rho}$ arranged in decreasing order, $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$ is the spin-flipped density matrix and $*$ denotes the complex conjugate.

Two important families are the GHZ family with $\tau^{(3)}=1$ and all 2-tangles τ vanishing, and the W family with $\tau^{(3)}=0$ and all 2-tangles equal to 1. Both families are unequivalent under SLOCC transformations [16].

The 3-tangle of the state (5) is $\tau^{(3)}=\frac{1}{4}$ and the reduced density matrices for subsystems $A-B$, $A-C$ and $B-C$ are $\rho_{AB}=\frac{1}{4}\begin{pmatrix} 2 & . & 1 & -1 \\ . & . & . & . \\ 1 & . & 1 & -1 \\ -1 & . & -1 & 1 \end{pmatrix}$, $\rho_{AC}=\frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & . \\ -1 & 1 & -1 & . \\ 1 & -1 & 2 & . \\ . & . & . & . \end{pmatrix}$ and $\rho_{BC}=\frac{1}{4}\begin{pmatrix} 2 & -1 & -1 & . \\ -1 & 1 & . & . \\ -1 & . & 1 & . \\ . & . & . & . \end{pmatrix}$. The set of eigenvalues of $\rho\tilde{\rho}$ associated to subsystems $A-B$ and $A-C$ are $\{\frac{1}{16}(3+2\sqrt{2}),\frac{1}{16}(3-2\sqrt{2}),0,0\}$ leading to the tangles $\tau_{AB}=\tau_{AC}=\frac{1}{4}$ and the set of eigenvalues of $\rho\tilde{\rho}$ associated to the subsystem $B-C$ is $\{\frac{1}{16},\frac{1}{16},0,0\}$ leading to a vanishing tangle τ_{BC}.

All states encoded by rows of g_1 and g_2 behaves as state (5). We denote a tripartite entanglement characterized by the tangles $\tau^{(3)}=\tau_{AB}=\tau_{AC}=\frac{1}{4}$ and $\tau_{BC}=0$ a *incomplete balanced entanglement*. Fully balanced tripartite entanglement is found in [1, 2] and in the next section below devoted to the E_8 lattice and other dense lattices.

3. Entanglement from Barnes-Wall lattices

There exists a remarkable relationship between protected real n-qubits (from $n=1$ to 4) and the densest known $2n$-dimensional lattices [17, 7, 8], that are $\mathbb{Z}^2$, D_4, E_8 and Λ_{16} (the smallest lattices of the Barnes-Wall family L_n). The automorphism group of L_n (in dimension 2^n, $n\neq 3$) corresponds to the real Clifford group $\mathcal{C}_n^+$ (defined as the normalizer in the general orthogonal group $O(2^n)$ of the generalized Pauli group on n real qubits: the notations comes from [2]). For $n=3$ qubits, L_3 is E_8 and $\mathcal{C}_3^+$ is the second largest maximal subgroup of $W'(E_8)$ (where $W'((E_8)$ is the derived subgroup of the Weyl group $W(E_8)$). In the sequel, we concentrate on the entanglement arising from the lattices embodying two, three and four qubits, respectively.

Thus, two protected real qubits may be described by means of the D_4 lattice (also called checkerboard lattice) whose a convenient basis is $M_{D_4}=\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(1/2,1/2,1/2,1/2)\}$. In this definition, it is clear that D_4 consists of the vectors (a,b,c,d) all in $\mathbb{Z}$ or all on $\mathbb{Z}+\frac{1}{2}$

(i.e. the lattice of Hurwitz integral quaternions). The natural action of the automorphism group on this basis

$$Aut(D_4) = \langle \mathrm{CNOT}, S, H \otimes H \rangle \tag{7}$$

involves the entangling gates CNOT and S of the previous sections and the Kronecker product of two Hadamard gates $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Group $Aut(D_4)$, of order 1152, is the same object than the two-qubit real Clifford group $\mathcal{C}_2^+$ and the Weyl group $W(F_4)$ of the 24-cell [2].

Gosset lattice E_8 is the unique even 8-dimensional unimodular lattice and may described in several ways [9]. For instance, it is defined as a discrete subgroup of $\mathbb{R}_8$ such that all the coordinates a_i ($i = 1..8$) are integers or all the coordinates are half-integers, and the sum of eight coordinates is a even integer

$$E_8 = \left\{ a_i \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 : \sum_i a_i \equiv 0 (\mathrm{mod}\ 2) \right\}. \tag{8}$$

The automorphism group attached to this description of E_8 reads

$$\mathrm{Aut}(E_8) = \langle g_1, g_2 \rangle,$$

$$\text{with } g_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 & . & . & . & . & 1 & 1 \\ 1 & 1 & . & . & . & . & -1 & -1 \\ -1 & 1 & . & . & . & . & 1 & -1 \\ . & . & -1 & 1 & 1 & -1 & . & . \\ . & . & -1 & -1 & 1 & 1 & . & . \\ . & . & -1 & 1 & -1 & 1 & . & . \\ . & . & -1 & -1 & -1 & -1 & . & . \\ -1 & 1 & . & . & . & . & -1 & 1 \end{pmatrix}$$

$$\text{and } g_2 = \frac{1}{2} \begin{pmatrix} -1 & 1 & . & . & -1 & 1 & . & . \\ . & . & 1 & 1 & . & . & -1 & 1 \\ 1 & 1 & . & . & 1 & -1 & . & . \\ . & . & -1 & 1 & . & . & 1 & 1 \\ -1 & 1 & . & . & 1 & 1 & . & . \\ . & . & -1 & -1 & . & . & -1 & 1 \\ . & . & 1 & -1 & . & . & 1 & 1 \\ 1 & 1 & . & . & -1 & 1 & . & . \end{pmatrix}. \tag{9}$$

Group $Aut(E_8)$ is of order $4!6!8! = 696\ 729\ 600$. Let consider the state encoded by the first row of g_1

$$|\psi\rangle = \frac{1}{2}(|000\rangle + |001\rangle + |110\rangle + |111\rangle). \tag{10}$$

The tangles attached to the state (10) are calculated, as we did for the state (5) of the previous section. They are such that $\tau^{(3)} = \tau_{AC} = \tau_{AB} = \tau_{BC} = \frac{1}{4}$ corresponding to fully *balanced tripartite entanglement*. The same result holds for all states encoded by the rows of g_1 and g_2. One can conclude that this type of entanglement, first featured in [1], is a specific property of the lattice E_8. As mentioned previously, the 3-qubit real Clifford group $\mathcal{C}_3^+$ of order 2 580 480 is a maximal subgroup of $\mathrm{Aut}'(E_8)$ (the second largest one); the largest maximal subgroup is the automorphism group $\mathrm{Aut}(E_7)$ of the lattice E_7.

Four real qubits are protected from the real Clifford group $\mathcal{C}_4^+ \equiv \mathrm{Aut}(\Lambda_{16})$, of order 89 181 388 800, associated to the Barnes-Wall lattice Λ_{16} [7]. We use construction "B" associated to the $[16, 5, 8]$ Reed-Muller code (see [9], p. 141) (in Magma, the code C:=ReedMullerCode(1,4); L:=Lattice(C,"B"); aut:=AutomophismGroup(L:NaturalAction);).

One gets two 16×16 gates generating $\mathrm{Aut}(\Lambda_{16})$ with rows encoding entangled states such as the factorizable state

$$\begin{aligned} |\psi\rangle &= \tfrac{1}{2}(|0000\rangle - |0011\rangle - |0101\rangle + |0110\rangle) \\ &= \tfrac{1}{2}\,|0\rangle\,((|000\rangle - |011\rangle - |101\rangle + |110\rangle)). \end{aligned} \tag{11}$$

It is easy to quantify the residual tripartite entanglement of state (11). Using (6), one gets true tripartite entanglement $\tau^{(3)} = 1$ corresponding to the GHZ family. All states encoded by the rows in the generators of $\mathrm{Aut}(\Lambda_{16})$ display vanishing 4-tangle (the four-tangle is calculated from the expressions given in [15]). Presumably, all residual three-tangles equal unity as for GHZ-type states).

4. Entanglement from the Leech lattice

The 24-dimensional Leech lattice Λ_{24} is the densest known lattice in dimension 24 (see [9], p.131), it is unimodular, the kissing number is 196560 and the automorphism group is Conway group, denoted Co_0, of order $2^{22}3^9 5^4 7^2$.11.13.23. Group Co_0 contains the three sporadic Conway groups Co_1, Co_2 and Co_3.

The even part $h\Lambda_{24}$ of the lattice may be constructed by means of the construction B applied to the $[24, 12, 8]$ extended binary Golay code [in Magma one uses the following code C:=GolayCode(GF(2),true); L:=Lattice (C,"B"); and one restores a Euclidean lattice by taking a unity inner product matrix; in Magma for Linux one can also write D:=LatticeDatabase(); L:=Lattice(D,24,75);].

The automorphism group $\mathrm{Aut}(h\Lambda_{24}) \cong 2^{12}.M_{24}$ (where M_{24} is the 5-transitive Mathieu group), of order 1 002 795 171 840 may be generated with two 24×24 entangling gates. Rows in the generators encode several types of entangled states.

We select a first representative

$$\begin{aligned} |\psi\rangle = \tfrac{1}{3}[&- |0000\rangle + |0011\rangle + |0110\rangle \\ &- |1000\rangle - |1011\rangle + |1111\rangle \\ &- |2000\rangle - |2110\rangle + |2111\rangle]. \end{aligned} \tag{12}$$

In this form, state (12) belongs to a qutrit/3-qubit system and the rows corresponding to each qutrit $|0\rangle$, $|1\rangle$ and $|2\rangle$ encode the following 3-qubit states $|\psi_0\rangle = \frac{1}{\sqrt{3}}(-|000\rangle + |011\rangle + |110\rangle)$, $|\psi_1\rangle = \frac{1}{\sqrt{3}}(-|000\rangle - |011\rangle + |111\rangle)$ and $|\psi_2\rangle = \frac{1}{\sqrt{3}}(-|000\rangle + |110\rangle + |111\rangle)$, respectively. Three-tangles associated to the above 3-qubit sates are calculated as $\tau^{(3)}(\psi_0) = 0$ and $\tau^{(3)}(\psi_1) = \tau^{(3)}(\psi_2) = \frac{4}{9}$.

The state $|\psi_0\rangle$ is approximately a W-state, with all 2-tangles equal to $\frac{4}{9}$. States $|\psi_1\rangle$ and $|\psi_2\rangle$ keep bipartite entanglement between parties $B-C$ and $A-B$, respectively, with a set of eigenvalues of $\rho\tilde{\rho}$ given by $\{\frac{1}{9}(3+2\sqrt{2}), \frac{1}{9}(3-2\sqrt{2}), 0, 0\}$. Thus $\tau_{BC}(\psi_1) = \tau_{AB}(\psi_2) = \frac{4}{9}$ and all other two-tangles of $|\psi_1\rangle$ and $|\psi_2\rangle$ vanish.

Another representative is

$$\begin{aligned} |\psi\rangle = \tfrac{1}{3}[&- |0101\rangle \\ &- |1000\rangle + |1001\rangle - |1100\rangle - |1101\rangle - |1110\rangle \\ &- |2000\rangle - |2010\rangle + |2011\rangle]. \end{aligned} \tag{13}$$

Let us look at the 3-qubit states factorizing the qutrits. Clearly the first state $\psi_0 = |101\rangle$ is fully separable. The third state $\psi_2 = -|000\rangle - |010\rangle + |011\rangle$ has $\tau^{(3)} = \tau_{AB} = \tau_{AC} = 0$ and $\tau_{BC} = \frac{4}{9}$, i.e. entanglement only lies at the parties B-C. The second state $\psi_1 = -|000\rangle + |001\rangle - |100\rangle - |101\rangle - |110\rangle$ is more interesting because entanglement is such that $\tau^{(3)} = \tau_{AB} = \tau_{BC} = \frac{4}{25}$ and $\tau_{AC} = \frac{16}{25}$. We name it *imperfectly balanced tripartite entanglement*.

There also exists other types of tripartite entanglement in $\mathrm{Aut}(h\Lambda_{24})$ that we do not describe further.

For the full Leech lattice Λ_{24} we use the so-called MOG (Miracle Octad Generator) coordinates so that $\det(\Lambda_{24}) = 1$ and the inner product matrix is one eight of the identity matrix (see [9], p 132 for details). The automorphism group preserving the lattice basis consists of two 24×24 entangling gates generating the Conway group Co_0. We restrict to a single representative state

$$\begin{gathered}|\psi\rangle = \tfrac{1}{4}[-|0000\rangle + |0001\rangle \\ -|1010\rangle + |1011\rangle + |1100\rangle - |1101\rangle - |1110\rangle + |1111\rangle \\ +|2001\rangle + 2|2011\rangle - 2|2100\rangle + |2101\rangle - 2|2110\rangle].\end{gathered} \tag{14}$$

Again we look at the 3-qubit states factorizing the qutrits. The state in factor of the first qutrit is fully separable. The state in factor of the second qutrit is such that $\tau^{(3)} = \tau_{AB} = \frac{4}{9}$ and $\tau_{AC} = \tau_{BC} = 0$. Finally, the state in factor of the third qutrit has $\tau^{(3)} = \tau_{AB} = \tau_{BC} = \frac{4}{25}$ and $\tau_{AC} = \frac{8}{25}$. Again *imperfectly balanced tripartite entanglement* is found.

Further types of entangled states occur in the automorphism group $\mathrm{Aut}(\Lambda_{24})$ of the Leech lattice, but we do not study them all in this paper.

5. Conclusion

The symmetry group acting on the generating basis of even Euclidean lattices has been described as a group of entangling quantum gates. Barnes-Wall lattices are known to be related to the Clifford group of quantum error correction [7, 8] but the occurrence of Leech lattice is novel in this context. It may be that real world systems, like quark systems, or codons, or even black holes [6], exploit dimension $24 = 2^8 \times 3$ as three-qubits and qutrits at the same time. Leech lattice may also be constructed from Leech roots in Lorentzian space $\mathbb{R}^{25,1}$ (see [9], chaps. 26-27) and dimension 26 corresponds to the original version of string theory through the no-ghost theorem. These new vistas (see also [18]), and their connection to tripartite entanglement, will be explored in future work.

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