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Claire Amiot, Osamu Iyama, Idun Reiten, Gordana Todorov

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# PREPROJECTIVE ALGEBRAS AND C-SORTABLE WORDS

CLAIRE AMIOT, OSAMU IYAMA, IDUN REITEN, AND GORDANA TODOROV

ABSTRACT. Let  $Q$  be an acyclic quiver and  $\Lambda$  be the completion of the preprojective algebra of  $Q$  over an algebraically closed field  $k$ . To any element  $w$  in the Coxeter group of  $Q$ , Buan, Iyama, Reiten and Scott have introduced and studied in [BIRS09a] a finite dimensional algebra  $\Lambda_w = \Lambda/I_w$ . In this paper we look at filtrations of  $\Lambda_w$  associated to any reduced expression  $\mathbf{w}$  of  $w$ . We are specially interested in the case where the word  $\mathbf{w}$  is  $c$ -sortable where  $c$  is a Coxeter element. In this situation, the consecutive quotients of this filtration can be related to tilting  $kQ$ -modules with finite torsionfree class. This nice description allows us to construct a triangle equivalence between the 2-Calabi-Yau triangulated category  $\mathbf{Sub}\Lambda_w$  and the generalized cluster category associated with an Auslander algebra.

## CONTENTS

Introduction	2
Notation	3
Acknowledgements	3
1. Background	4
1.1. 2-Calabi-Yau categories associated with reduced words	4
1.2. Mutation of tilting modules	5
1.3. Reflections and reflection functors	6
2. Generalities on the layers	6
2.1. Layers as images of simples	7
2.2. The dimension vectors of the layers	10
2.3. Reflection functors and ideals $I_i$	12
3. Tilting modules and $c$ -sortable words	13
3.1. Three series of $kQ$ -modules	15
3.2. Tilting modules with finite torsionfree class	18
3.3. Example	20
4. Categories as cluster categories associated with Auslander algebras	21
4.1. Canonical cluster-tilting object of $\mathbf{Sub}\Lambda_w$	22
4.2. Generalized cluster categories	23
4.3. Computing endomorphism algebras	24
4.4. Triangle equivalence	27
4.5. Example	30
5. Problems and examples	32
References	34

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## INTRODUCTION

Attempts to categorify the cluster algebras of Fomin and Zelevinsky [FZ02] have led to the investigation of categories with the 2-Calabi-Yau property (2-CY for short) and their cluster-tilting objects. Main early classes of examples were the cluster categories associated with finite dimensional path algebras [BMR<sup>+</sup>06] and the preprojective algebras of Dynkin type [GLS06]. This paper is centered around the more general class of stably 2-CY and triangulated 2-CY categories associated with elements in Coxeter groups [BIRS09a] (the adaptable case was done independently in [GLS08]), and their relationship to the generalized cluster categories from [Ami09a] (see Section 4 for definition).

Let  $Q$  be a finite connected quiver with vertices  $1, \dots, n$ , and  $\Lambda$  the completion of the preprojective algebra of the quiver  $Q$  over a field  $k$ . Denote by  $s_1, \dots, s_n$  the distinguished generators in the corresponding Coxeter group  $W_Q$ . To an element  $w$  in  $W_Q$ , there is associated a stably 2-CY category  $\mathbf{Sub} \Lambda_w$  and a triangulated 2-CY category  $\underline{\mathbf{Sub}} \Lambda_w$ . The definitions are based on first associating an ideal  $I_i$  in  $\Lambda$  to each  $s_i$ , hence to any reduced word by taking products. This way we also get a finite dimensional algebra  $\Lambda_w := \Lambda/I_w$ . Objects of the category  $\mathbf{Sub} \Lambda_w$  are submodules of finite dimensional free  $\Lambda_w$ -modules. The cluster category is then equivalent to  $\underline{\mathbf{Sub}} \Lambda_w$  with  $w = c^2$ , where  $c$  is a Coxeter element such that  $c^2$  is a reduced expression [BIRS09a, GLS08]. When  $\Lambda$  is a preprojective algebra of Dynkin type, then the category  $\mathbf{mod} \Lambda$  as investigated in [GLS06] is also obtained as  $\mathbf{Sub} \Lambda_w$  where  $w$  is the longest word [BIRS09a, III 3.5].

Using the construction of ideals we get for each reduced expression  $\mathbf{w} = s_{u_1} s_{u_2} \dots s_{u_l}$  a chain of ideals

$$\Lambda \supset I_{u_1} \supset I_{u_1 u_2} \supset \dots \supset I_w,$$

which gives rise to an interesting set of  $\Lambda$ -modules:

$$L_{\mathbf{w}}^1 := \frac{\Lambda}{I_{u_1}}, L_{\mathbf{w}}^2 := \frac{I_{u_1}}{I_{u_1 u_2}}, \dots, L_{\mathbf{w}}^l := \frac{I_{u_1 \dots u_{l-1}}}{I_w}$$

which all turn out to be indecomposable and to lie in  $\mathbf{Sub} \Lambda_w$ .

The investigation of this set of modules, which we call *layers*, from different points of view, including connections with tilting theory, is one of the main themes of this paper, especially for a class of words called  $c$ -sortable.

The modules  $L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^l$  provide a natural filtration for the cluster-tilting object  $M_{\mathbf{w}}$  associated with the reduced expression  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  (see Section 1). These modules can be used to show that the endomorphism algebras  $\mathbf{End}_{\Lambda}(M_{\mathbf{w}})$  are quasi-hereditary [IR10]. Here we show that these modules are rigid (Theorem 2.2), that is  $\mathbf{Ext}_{\Lambda}^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^j) = 0$  and that their dimension vectors are real roots (Theorem 2.6), so that there are unique associated indecomposable  $kQ$ -modules  $(L_{\mathbf{w}}^j)_Q$  (which are not necessarily rigid).

The situation is especially nice when all layers are indecomposable  $kQ$ -modules, so that  $L_{\mathbf{w}}^j = (L_{\mathbf{w}}^j)_Q$ . This is the case for  $c$ -sortable words. An element  $w$  of  $W_Q$  is  $c$ -sortable when there exists a reduced expression of  $w$  of the form  $\mathbf{w} = c^{(0)} c^{(1)} \dots c^{(m)}$  with  $c^{(m)} \subseteq \dots \subseteq c^{(1)} \subseteq c^{(0)} \subseteq c$  where  $c$  is a Coxeter element, that is, a word containing each generator  $s_i$  exactly once, and in an order admissible with respect to the orientation of  $Q$ .

Starting with the tilting  $kQ$ -module  $kQ$  (when  $c^{(0)} = c$ ), there is a natural way of performing exchanges of complements of almost complete tilting modules, determined by the given reduced expression. We denote the final tilting module by  $T_{\mathbf{w}}$ , and the indecomposable  $kQ$ -modules used in the sequence of constructions by  $T_{\mathbf{w}}^j$  for  $j = 1, \dots, l$ . We show that  $L_{\mathbf{w}}^j \simeq T_{\mathbf{w}}^j$  for all  $j$  (Theorem 3.8) and that the indecomposable modules in the torsionfree class  $\text{Sub}(T_{\mathbf{w}})$  are exactly the  $T_{\mathbf{w}}^j$  (Theorem 3.10). In particular this gives a one-one correspondence between  $c$ -sortable words and torsionfree classes, as first shown in [Tho] using different methods.

There is another sequence  $U_{\mathbf{w}}^1, \dots, U_{\mathbf{w}}^l$  of indecomposable  $kQ$ -modules, defined using restricted reflection functors, which coincide with the above sequences. This is both interesting in itself, and provides a method for proving  $L_{\mathbf{w}}^j \simeq T_{\mathbf{w}}^j$  for  $j = 1, \dots, l$ .

In another paper [AIRT], we give a description of the layers from a functorial point of view. When the  $c$ -sortable word is  $c^m$ , and  $c = s_1 \dots s_n$ , then the successive layers are given by

$$P_1, \dots, P_n, \tau^- P_1, \dots, \tau^- P_n, \tau^{-2} P_1, \dots, \tau^{-m} P_n$$

for the indecomposable projective  $kQ$ -modules  $P_i$ , where  $\tau$  denotes the AR-translation. In the general case we will give a description of the layers using specific factor modules of the above modules.

The generalized cluster categories  $\mathcal{C}_A$  for algebras  $A$  of global dimension at most two were introduced in [Ami09a]. It was shown that for a special class of words  $w$ , properly contained in the dual of the  $c$ -sortable words, the 2-CY category  $\underline{\text{Sub}}\Lambda_w$  is triangle equivalent to some  $\mathcal{C}_A$ . We show that the procedure for choosing  $A$  works more generally for any (dual of a)  $c$ -sortable word (Theorem 4.10), with a simpler proof due to developments in the meantime.

The paper is organized as follows. We start with some background material on 2-CY categories associated with reduced words, on complements of almost complete tilting modules and on reflection functors. In Section 2 we show that for any reduced word  $w$ , the associated layers are indecomposable rigid modules, which also are real roots. Hence there are unique associated indecomposable  $kQ$ -modules. In Section 3 we show that our three series of indecomposable modules  $\{L_{\mathbf{w}}^j\}$ ,  $\{T_{\mathbf{w}}^j\}$  and  $\{U_{\mathbf{w}}^j\}$  coincide in the  $c$ -sortable case. The description of the layers as specific factor modules of the  $\tau^{-i}P$  for  $P$  indecomposable projective is given in Section 4. In Section 5 we show the relationship with generalized cluster categories in the  $c$ -sortable case. Section 6 is devoted to examples and questions beyond the  $c$ -sortable case.

Some of this work was presented at a conference in Trondheim in August 2009.

**Notation.** Throughout  $k$  is an algebraically closed field. The tensor product  $- \otimes -$ , when not specified, will be over the field  $k$ . For a  $k$ -algebra  $A$ , we denote by  $\text{mod}A$  the category of finitely presented right  $A$ -modules, and by  $\text{f.l.}A$  the category of finite length right  $A$ -modules. For a quiver  $Q$  we denote by  $Q_0$  the set of vertices and by  $Q_1$  the set of arrows, and for  $a \in Q_1$  we denote by  $s(a)$  its source and by  $t(a)$  its target.

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## 1. BACKGROUND

**1.1. 2-Calabi-Yau categories associated with reduced words.** Let  $Q$  be a finite quiver without oriented cycles and with vertices  $Q_0 = \{1, \dots, n\}$ . For  $i, j \in Q_0$  we denote by  $m_{ij}$  the positive integer

$$m_{ij} := \#\{a \in Q_1 \mid s(a) = i, t(a) = j\} + \#\{a \in Q_1 \mid s(a) = j, t(a) = i\}.$$

The *Coxeter group* associated to  $Q$  is defined by the generators  $s_1, \dots, s_n$  and relations

- $s_i^2 = 1$ ,
- $s_i s_j = s_j s_i$  if  $m_{ij} = 0$ ,
- $s_i s_j s_i = s_j s_i s_j$  if  $m_{ij} = 1$ .

In this paper  $\mathbf{w}$  will denote a word (*i.e.* an expression in the free abelian group generated by  $s_i, i \in Q_0$ ), and  $w$  will be its equivalence class in the Coxeter group  $W_Q$ .

An expression  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  is *reduced* if  $l$  is smallest possible. An element  $c = s_{u_1} \dots s_{u_l}$  is called *Coxeter element* if  $l = n$  and  $\{u_1, \dots, u_l\} = \{1, \dots, n\}$ . We say that a Coxeter element  $c = s_{u_1} \dots s_{u_n}$  is *admissible* with respect to the orientation of  $Q$  if  $i < j$  when there is an arrow  $u_i \rightarrow u_j$ .

The preprojective algebra associated to  $Q$  is the algebra

$$k\overline{Q} / \langle \sum_{a \in Q_1} aa^* - a^*a \rangle$$

where  $\overline{Q}$  is the double quiver of  $Q$ , which is obtained from  $Q$  by adding for each arrow  $a : i \rightarrow j$  in  $Q_1$  an arrow  $a^* : i \leftarrow j$  pointing in the opposite direction. We denote by  $\Lambda$  the completion of the preprojective algebra associated to  $Q$  and by  $\text{f.l.}\Lambda$  the category of right  $\Lambda$ -modules of finite length.

The algebra  $\Lambda$  is selfinjective finite-dimensional if  $Q$  is a Dynkin quiver. Then the stable category  $\underline{\text{mod}}\Lambda$  satisfies the *2-Calabi-Yau property* (2-CY for short), that is, there is a functorial isomorphism

$$D\underline{\text{Hom}}_\Lambda(X, Y) \simeq \underline{\text{Hom}}_\Lambda(Y, X[2]),$$

where  $D := \text{Hom}_k(-, k)$  and  $[1] := \Omega^{-1}$  is the suspension functor.

When  $Q$  is not Dynkin, then  $\Lambda$  is infinite dimensional and of global dimension 2. In this case the triangulated category  $\mathcal{D}^b(\text{f.l.}\Lambda)$  is 2-CY.

We now recall some work from [IR08, BIRS09a]. For each  $i = 1, \dots, n$  we have an ideal  $I_i := \Lambda(1 - e_i)\Lambda$ , where  $e_i$  is the idempotent of  $\Lambda$  associated with the vertex  $i$ . We write  $I_{\mathbf{w}} := I_{u_1} \dots I_{u_2} I_{u_1}$  when  $\mathbf{w} = s_{u_1} s_{u_2} \dots s_{u_l}$  is a reduced expression of  $w \in W_Q$ .

We collect the following information which is useful for Section 2:

**Proposition 1.1.** [BIRS09a] *Let  $\Lambda$  be a preprojective algebra.*

- (a) *If  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  and  $\mathbf{w}' = s_{v_1} \dots s_{v_l}$  are two reduced expression of the same element in the Coxeter group, then  $I_{\mathbf{w}} = I_{\mathbf{w}'}$ .*
- (b) *If  $\mathbf{w} = \mathbf{w}' s_i$  with  $\mathbf{w}'$  reduced, then  $I_{\mathbf{w}} \subseteq I_{\mathbf{w}'}$ . Moreover  $\mathbf{w}$  is reduced if and only if  $I_{\mathbf{w}} \subsetneq I_{\mathbf{w}'}$ . And for  $j \neq i$  we have  $e_j I_{\mathbf{w}} = e_j I_{\mathbf{w}'}$ .*

*If  $\Lambda$  is not of Dynkin type we have moreover:*

- (c) *Any finite product  $I$  of the ideals  $I_j$  is a tilting module of projective dimension at most one, and  $\text{End}_\Lambda(I) \simeq \Lambda$ .*

- (d) If  $S$  is a simple  $\Lambda$ -module and  $I$  is a tilting module of projective dimension at most one, then  $S \otimes_{\Lambda} I = 0$  or  $\mathrm{Tor}_1^{\Lambda}(S, I) = 0$ .
- (e) If  $S_i := \Lambda/I_i$  and  $\mathrm{Tor}_1^{\Lambda}(S_i, I) = 0$ , then  $I_i \otimes_{\Lambda}^{\mathbf{L}} I = I_i \otimes_{\Lambda} I = I_i I$  for a tilting ideal  $I$  of projective dimension at most one.

By (a) the ideal  $I_{\mathbf{w}}$  does not depend on the choice of the reduced expression  $\mathbf{w}$  of  $w$ . Therefore we write  $I_w$  for the ideal  $I_{\mathbf{w}}$  and denote  $\Lambda_w := \Lambda/I_w$ . This is a finite dimensional algebra. We denote by  $\mathbf{Sub} \Lambda_w$  the category of submodules of free  $\Lambda_w$ -modules. This is a *Frobenius category*, that is an exact category with enough projectives and injectives, and the projectives and injectives coincide. Its stable category  $\underline{\mathbf{Sub}} \Lambda_w$  is a triangulated category which satisfies the 2-Calabi-Yau property [BIRS09a]. The category  $\mathbf{Sub} \Lambda_w$  is then said to be *stably 2-Calabi-Yau*.

Recall that a *cluster-tilting object* in a Frobenius stably 2-CY category  $\mathcal{C}$  with finite dimensional morphisms spaces is an object  $T \in \mathcal{C}$  such that

- $\mathrm{Ext}_{\mathcal{C}}^1(T, T) = 0$
- $\mathrm{Ext}_{\mathcal{C}}^1(T, X) = 0$  implies that  $X \in \mathrm{add} T$ .

For any reduced word  $\mathbf{w} = s_{u_1} \dots s_{u_l}$ , we write  $M_{\mathbf{w}}^j := e_{u_j} \frac{\Lambda}{I_{u_j \dots u_1}}$ .

**Theorem 1.2.** [BIRS09a, Thm III.2.8] *For any reduced expression  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  of  $w \in W_Q$ , the object  $M_{\mathbf{w}} := \bigoplus_{j=1}^l M_{\mathbf{w}}^j$  is a cluster-tilting object in the stably 2-CY category  $\mathbf{Sub} \Lambda_w$ .*

For any reduced word  $\mathbf{w} = s_{u_1} \dots s_{u_l}$ , we have the chain of ideals

$$\Lambda \supset I_{u_1} \supset I_{u_1 u_2} \supset \dots \supset I_w,$$

which is strict by Proposition 1.1 (b). For  $j = 1, \dots, l$  we define the *layer*

$$L_{\mathbf{w}}^j := \frac{I_{u_{j-1}} \dots I_{u_1}}{I_{u_j} \dots I_{u_1}}.$$

Using Proposition 1.1 (b) it is immediate to see the following

**Proposition 1.3.** *We have isomorphisms in f.l.  $\Lambda$ :*

$$L_{\mathbf{w}}^j \simeq e_{u_j} L_{\mathbf{w}}^j \simeq e_{u_j} \frac{I_{u_i} \dots I_{u_1}}{I_{u_j} \dots I_{u_1}} \simeq \mathrm{Ker}( M_{\mathbf{w}}^j \twoheadrightarrow M_{\mathbf{w}}^i ),$$

where  $i$  is the greatest integer  $< j$  satisfying  $u_i = u_j$ .

Therefore the layers  $L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^l$  give a filtration of the cluster-tilting object  $M_{\mathbf{w}}$ .

**1.2. Mutation of tilting modules.** Let  $Q$  be finite quiver with vertices  $\{1, \dots, n\}$  and without oriented cycles.

**Definition 1.4.** A *tilting  $kQ$ -module*  $T$  is a basic module with  $n$  indecomposable summands such that  $\mathrm{Ext}_{kQ}^1(T, T) = 0$ .

For each indecomposable summand  $T_i$  of  $T$ , it is known that there is at most one indecomposable  $T_i^* \not\cong T_i$  such that  $T/T_i \oplus T_i^*$  is a tilting module [RS90, Ung90], and that there is exactly one if and only if  $T/T_i$  is a sincere  $kQ$ -module [HU89]. We then say that

$T_i$  (and possibly  $T_i^*$ ) is a *complement* for the almost complete tilting module  $T/T_i$ . The (possibly) other complement of  $T/T_i$  can be obtained using the following result:

- Proposition 1.5.** (a) *If the minimal left  $\text{add}(T/T_i)$ -approximation  $T_i \xrightarrow{f} B$  is a monomorphism, then  $\text{Coker} f$  is a complement for  $T/T_i$ .*  
 (b) *If the minimal right  $\text{add}(T/T_i)$ -approximation  $B' \xrightarrow{g} T_i$  is an epimorphism, then  $\text{Ker} g$  is a complement for  $T/T_i$ .*

There is a one-one correspondence between tilting modules  $T$  and contravariantly finite torsionfree classes  $\mathcal{F} = \text{Sub} T$  containing the projective modules.

**1.3. Reflections and reflection functors.** Let  $Q$  be finite quiver with vertices  $\{1, \dots, n\}$  and without oriented cycles. Let  $i \in Q_0$  be a source. Then the quiver  $Q' := \mu_i(Q)$  is obtained by replacing all arrows starting at the vertex  $i$  by arrows in the opposite direction.

Write  $kQ = P_1 \oplus \dots \oplus P_n$  where  $P_j$  is the indecomposable projective  $kQ$ -module associated with the vertex  $j$ . Then using results of [BGP73] and [APR79] we have functors:

$$\text{mod } kQ \begin{array}{c} \xrightarrow{R_i} \\ \xleftarrow{R_i^-} \end{array} \text{mod } kQ'$$

where  $R_i := \text{Hom}_{kQ}(M, -)$ ,  $R_i^- := - \otimes_{kQ} M$  and  $M := \tau^- P_i \oplus kQ/P_i$  which induce inverse equivalences

$$(\text{mod } kQ)/[e_i kQ] \begin{array}{c} \xrightarrow{R_i} \\ \xleftarrow{R_i^-} \end{array} (\text{mod } kQ')/[e_i DkQ'],$$

where  $\text{mod } kQ/[e_i kQ]$  (resp.  $\text{mod } kQ'/[e_i DkQ']$ ) is obtained from the module category  $\text{mod } kQ$  (resp.  $\text{mod } kQ'$ ) by annihilating morphisms factoring through  $P_i = e_i kQ$  (resp.  $e_i DkQ'$ ). Since  $i$  is a source (resp. a sink) of  $Q$  (resp.  $Q'$ ) the category  $\text{mod } kQ/[e_i kQ]$  (resp.  $\text{mod } kQ'/[e_i DkQ']$ ) is also a full subcategory of  $\text{mod } kQ$  (resp.  $\text{mod } kQ'$ ).

When the vertex  $i$  is not a sink or source, there is still defined a reflection on the level of the Grothendieck group  $K_0(\text{mod } kQ)$ . It is constructed using the semigroup with generators  $[X]$  for  $X \in \text{mod } kQ$  and relations  $[X] + [Z] = [Y]$  if there is a short exact sequence  $X \rightarrow Y \rightarrow Z$ . This is a free abelian group with basis  $\{[S_1], \dots, [S_n]\}$ , where  $S_1, \dots, S_n$  are the simple  $kQ$ -modules. With respect to this basis we define

$$R_i([S_j]) = [S_j] + (m_{ij} - 2\delta_{ij})[S_i],$$

where  $m_{ij}$  is the number of edges of the underlying graph of  $Q$  as before.

This definition is coherent with the previous one. Indeed if  $i$  is a source and  $M$  is an indecomposable in  $\text{mod } kQ$  which is not isomorphic to  $P_i$ , then we have

$$R_i([M]) = [R_i(M)].$$

## 2. GENERALITIES ON THE LAYERS

Let  $w$  be an element in the Coxeter group of an acyclic quiver  $Q$ , and fix  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  a reduced expression of  $w$ . For  $j = 1, \dots, l$  we have defined in Section 1 the layer  $L_{\mathbf{w}}^j$  as

the quotient

$$L_{\mathbf{w}}^j := \frac{I_{u_{j-1}} \cdots I_{u_1}}{I_{u_j} \cdots I_{u_1}}.$$

In this section, we investigate some main properties of these layers. We show that each layer can be seen as the image of a simple  $\Lambda$ -module under an auto-equivalence of  $\mathcal{D}^b(\text{f.l.}\Lambda)$ . Hence they are rigid indecomposable  $\Lambda$ -modules of finite length, and we compute explicitly their dimension vectors and show that they are real roots. Hence to each layer we can associate a unique indecomposable  $kQ$ -module with the same dimension vector, but which is not necessarily rigid.

Note that some of the results of this section have been proven independently in [GLS10] but with different proofs.

### 2.1. Layers as images of simples.

**Proposition 2.1.** *Let  $Q$  a non Dynkin quiver and  $\Lambda$  the completion of the preprojective algebra. For  $j = 1, \dots, l$  we have isomorphisms in  $\mathcal{D}(\text{Mod } \Lambda)$ :*

$$L_{\mathbf{w}}^j \simeq S_{u_j} \overset{\mathbf{L}}{\otimes}_{\Lambda} (I_{u_{j-1}} \cdots I_{u_1}) \simeq S_{u_j} \overset{\mathbf{L}}{\otimes}_{\Lambda} I_{u_{j-1}} \overset{\mathbf{L}}{\otimes}_{\Lambda} \cdots \overset{\mathbf{L}}{\otimes}_{\Lambda} I_{u_1}$$

where  $S_{u_j}$  is the simple  $\Lambda$ -module associated to the vertex  $u_j$ .

*Proof.* We set  $\mathbf{w}' := s_{u_1} \cdots s_{u_j}$  and  $\mathbf{w}'' := s_{u_1} \cdots s_{u_{j-1}}$ . Since  $\mathbf{w}''$  is reduced, by Proposition 1.1(e) we have

$$I_{\mathbf{w}''} \simeq I_{u_{j-1}} \otimes_{\Lambda} \cdots \otimes_{\Lambda} I_{u_1} \simeq I_{u_{j-1}} \overset{\mathbf{L}}{\otimes}_{\Lambda} \cdots \overset{\mathbf{L}}{\otimes}_{\Lambda} I_{u_1},$$

and hence we get the second isomorphism.

Since  $\mathbf{w}' = \mathbf{w}'' s_{u_j}$  is reduced, we have  $I_{\mathbf{w}'} = I_{u_j} I_{\mathbf{w}''} \not\subseteq I_{\mathbf{w}''}$ , and therefore  $\text{Tor}_1^{\Lambda}(S_{u_j}, I_{\mathbf{w}''}) = 0$  by Proposition 1.1 (d). Thus we have

$$S_{u_j} \overset{\mathbf{L}}{\otimes}_{\Lambda} I_{\mathbf{w}''} \simeq S_{u_j} \otimes_{\Lambda} I_{\mathbf{w}''} \simeq \frac{\Lambda}{I_{u_j}} \otimes_{\Lambda} I_{\mathbf{w}''} \simeq \frac{I_{\mathbf{w}''}}{I_{u_j} I_{\mathbf{w}''}} = L_{\mathbf{w}}^j.$$

□

Immediately we have the following result, which implies that  $L_{\mathbf{w}}^j$  is an indecomposable rigid  $\Lambda$ -module of finite length.

**Theorem 2.2.** *For  $j = 1, \dots, l$  we have*

- if  $\Lambda$  is of non Dynkin type:

$$\dim \text{Ext}_{\Lambda}^i(L_{\mathbf{w}}^j, L_{\mathbf{w}}^j) = \begin{cases} 1 & i = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

- if  $\Lambda$  is of Dynkin type:

$$\dim \text{Ext}_{\Lambda}^i(L_{\mathbf{w}}^j, L_{\mathbf{w}}^j) = \begin{cases} 1 & i = 0, 2, \\ 0 & i = 1. \end{cases}$$

Note that there can be higher extensions in the Dynkin case. In the non Dynkin case,  $L_{\mathbf{w}}^j$  is then said to be *2-spherical* in the sense of Seidel-Thomas [ST01].



*Proof.* We separate the proof when  $\Lambda$  is of non Dynkin type and when  $\Lambda$  is of Dynkin type.

*Non Dynkin case:*

By Proposition 1.1 (c),  $I_{w''}$  is a tilting  $\Lambda$ -module with  $\text{End}_\Lambda(I_{w''}) \simeq \Lambda$ . Hence the functor  $-\overset{\mathbf{L}}{\otimes}_\Lambda I_{w''}$  is an autoequivalence of  $\mathcal{D}(\text{Mod } \Lambda)$ . We have  $\text{End}_\Lambda(S_j) \simeq k$  and hence  $\text{Ext}_\Lambda^2(S_j, S_j) \simeq k$  since  $\mathcal{D}^b(\text{f.l. } \Lambda)$  is 2-CY. Moreover since  $Q$  has no loops,  $\text{Ext}_\Lambda^1(S_j, S_j)$  vanishes and since  $\Lambda$  is known to have global dimension 2,  $\text{Ext}_\Lambda^n(S_j, S_j)$  vanishes for  $n \geq 3$ . Hence  $S_j$  is 2-spherical. Since by Proposition 2.1 the layer  $L_{\mathbf{w}}^j$  is the image of the simple  $S_j$  by an autoequivalence of  $\mathcal{D}^b(\text{f.l. } \Lambda)$ , it follows that  $L_{\mathbf{w}}^j$  is also 2-spherical.

*Dynkin case:*

Let  $Q$  be a Dynkin quiver and  $\tilde{Q}$  be an acyclic extended Dynkin quiver containing  $Q$  as a subquiver. Let  $\Lambda := \Lambda_Q$  and  $\tilde{\Lambda} := \Lambda_{\tilde{Q}}$  be the corresponding (completion of) their preprojective algebras. Then we have  $\Lambda \simeq \tilde{\Lambda}/\tilde{\Lambda}e\tilde{\Lambda}$  where  $e$  is the idempotent associated to the additional vertex of  $\tilde{Q}$ . The restriction functor  $R : \text{mod } \Lambda \rightarrow \text{mod } \tilde{\Lambda}$  is fully faithful and  $\text{mod } \Lambda$  can be seen as an extension closed subcategory of  $\text{mod } \tilde{\Lambda}$ .

It is immediate to check that for a reduced expression  $\mathbf{w}$  of  $w \in W_Q$  we have  $L_{\mathbf{w}, \Lambda}^j \simeq L_{\mathbf{w}, \tilde{\Lambda}}^j$ . Using the first part of the proof, we get

$$\text{End}_\Lambda(L_{\mathbf{w}}^j) \simeq \text{End}_{\tilde{\Lambda}}(L_{\mathbf{w}}^j) \simeq k \text{ and } \text{Ext}_\Lambda^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^j) \simeq \text{Ext}_{\tilde{\Lambda}}^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^j) = 0.$$

Finally using the fact that  $\text{mod } \Lambda$  is stably 2-CY we get  $\text{Ext}_\Lambda^2(L_{\mathbf{w}}^j, L_{\mathbf{w}}^j) \simeq k$ . □

Here we state a property about two consecutive layers of the same type, which gives rise to special non split short exact sequences in  $\text{f.l. } \Lambda$ .

**Proposition 2.3.** *Let  $1 \leq i < j < k \leq l$  be integers such that  $u_i = u_j = u_k$  and such that  $j$  is the only integer satisfying  $i < j < k$  and  $u_i = u_j = u_k$ . Then we have*

$$\dim_k \text{Ext}_\Lambda^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^k) = 1.$$

In order to prove this proposition, we first need a lemma. For  $1 \leq h \leq l$ , we denote as before by  $M_{\mathbf{w}}^h$  the  $\Lambda$ -module  $M_{\mathbf{w}}^h := e_{u_h} \frac{\Lambda}{I_{u_h} \dots I_{u_1}}$ .

**Lemma 2.4.** *Let  $i < j < k$  be as in Proposition 2.3.*

- (a) *The map  $\text{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^j) \rightarrow \text{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^i)$  induced by the irreducible map  $M_{\mathbf{w}}^j \rightarrow M_{\mathbf{w}}^i$  is an epimorphism.*
- (b) *The image of the map  $\text{Hom}_\Lambda(M_{\mathbf{w}}^i, M_{\mathbf{w}}^j) \rightarrow \text{Hom}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j)$  induced by the irreducible map  $M_{\mathbf{w}}^j \rightarrow M_{\mathbf{w}}^i$  is in  $\text{Rad}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j)$ .*

*Proof.* (a) Since  $i < j < k$ , then by Lemma III.1.14 of [BIRS09a], we have isomorphisms

$$\text{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^j) \simeq e \frac{\Lambda}{I_{u_j} \dots I_{u_1}} e \quad \text{and} \quad \text{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^i) \simeq e \frac{\Lambda}{I_{u_i} \dots I_{u_1}} e,$$

where  $e$  is the idempotent  $e := e_{u_i} = e_{u_j} = e_{u_k}$ . Then the map  $\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^j) \rightarrow \mathbf{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^i)$  is the epimorphism  $e \frac{\Lambda}{I_{u_j \dots I_{u_1}}} e_{u_k} \rightarrow e \frac{\Lambda}{I_{u_i \dots I_{u_1}}} e_{u_k}$  induced by the inclusion  $I_{u_j} \dots I_{u_1} \subset I_{u_i} \dots I_{u_1}$ .

- (b) It is clear that the image is contained in the radical. By Lemma III.1.14 of [BIRS09a], we have isomorphisms

$$\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^i, M_{\mathbf{w}}^j) \simeq e \frac{I_{u_j} \dots I_{u_{i+1}}}{I_{u_j} \dots I_{u_1}} e \quad \text{and} \quad \mathbf{Rad}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j) \simeq e \frac{I_{u_j}}{I_{u_j} \dots I_{u_1}} e.$$

The map  $\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^i, M_{\mathbf{w}}^j) \rightarrow \mathbf{Rad}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j)$  is induced by the inclusion of ideals  $I_{u_j} \dots I_{u_{i+1}} \subset I_{u_j}$ . But since  $j$  is the only integer satisfying  $i < j < k$  and  $u_i = u_j = u_k$ , we have  $e I_{u_j} \dots I_{u_{i+1}} e \simeq e I_{u_j} e$  and hence the map  $\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^i, M_{\mathbf{w}}^j) \rightarrow \mathbf{Rad}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j)$  is an isomorphism.  $\square$

*Proof of Proposition 2.3.* By definition of the layers, we have the following short exact sequences

$$(j) \quad L_{\mathbf{w}}^j \twoheadrightarrow M_{\mathbf{w}}^j \twoheadrightarrow M_{\mathbf{w}}^i \quad \text{and} \quad (k) \quad L_{\mathbf{w}}^k \twoheadrightarrow M_{\mathbf{w}}^k \twoheadrightarrow M_{\mathbf{w}}^j$$

Let  $K$  be the kernel of the composition map  $M_{\mathbf{w}}^k \rightarrow M_{\mathbf{w}}^j \rightarrow M_{\mathbf{w}}^i$ . Then we have a short exact sequence

$$(l) \quad K \twoheadrightarrow M_{\mathbf{w}}^k \twoheadrightarrow M_{\mathbf{w}}^i$$

which gives rise to the following long exact sequence in  $\mathbf{mod} \mathbf{End}_\Lambda(M_{\mathbf{w}})$ , where  $M_{\mathbf{w}} = \bigoplus_{h=1}^l M_{\mathbf{w}}^h$

$$D\mathbf{Ext}_\Lambda^1(M_{\mathbf{w}}^i, M_{\mathbf{w}}) \longrightarrow D\mathbf{Hom}_\Lambda(K, M_{\mathbf{w}}) \longrightarrow D\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}) \longrightarrow D\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^i, M_{\mathbf{w}}) \longrightarrow \dots$$

The space  $D\mathbf{Ext}_\Lambda^1(M_{\mathbf{w}}^i, M_{\mathbf{w}})$  is zero by Lemma III.2.1 of [BIRS09a], and the  $\mathbf{End}_\Lambda(M_{\mathbf{w}})$ -module  $D\mathbf{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}})$  is indecomposable injective. Therefore the module  $D\mathbf{Hom}_\Lambda(K, M_{\mathbf{w}})$  has simple socle, and hence  $K$  is indecomposable.

Moreover from the sequences (j), (k) and (l), we deduce that we have a short exact sequence  $L_{\mathbf{w}}^k \twoheadrightarrow K \twoheadrightarrow L_{\mathbf{w}}^j$  which is non split since  $K$  is indecomposable. Hence we get

$$\dim_k \mathbf{Ext}_\Lambda^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^k) \geq 1$$

From (j) we deduce the following long exact sequence

$$\dots \longrightarrow \mathbf{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^j) \longrightarrow \mathbf{Hom}_\Lambda(M_{\mathbf{w}}^k, M_{\mathbf{w}}^i) \longrightarrow \mathbf{Ext}_\Lambda^1(M_{\mathbf{w}}^k, L_{\mathbf{w}}^j) \longrightarrow \mathbf{Ext}_\Lambda^1(M_{\mathbf{w}}^k, M_{\mathbf{w}}^j) = 0.$$

Hence by Lemma 2.4 (a) we get  $\mathbf{Ext}_\Lambda^1(M_{\mathbf{w}}^k, L_{\mathbf{w}}^j) = 0$ .

From (j) we also deduce the following long exact sequence

$$0 \longrightarrow \mathbf{Hom}_\Lambda(M_{\mathbf{w}}^i, M_{\mathbf{w}}^j) \longrightarrow \mathbf{Hom}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j) \longrightarrow \mathbf{Hom}_\Lambda(L_{\mathbf{w}}^j, M_{\mathbf{w}}^j) \longrightarrow \mathbf{Ext}_\Lambda^1(M_{\mathbf{w}}^i, M_{\mathbf{w}}^j) = 0.$$

Hence by Lemma 2.4 (b) we get  $\mathbf{Hom}_\Lambda(L_{\mathbf{w}}^j, M_{\mathbf{w}}^j) \simeq \mathbf{Hom}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j) / \mathbf{Rad}_\Lambda(M_{\mathbf{w}}^j, M_{\mathbf{w}}^j)$  which is one dimensional since  $M_{\mathbf{w}}^j$  is indecomposable.

Finally using (k) we get the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_\Lambda^1(M_{\mathbf{w}}^k, L_{\mathbf{w}}^j) \longrightarrow \mathrm{Ext}_\Lambda^1(L_{\mathbf{w}}^k, L_{\mathbf{w}}^j) \longrightarrow \mathrm{Ext}_\Lambda^2(M_{\mathbf{w}}^j, L_{\mathbf{w}}^j) \longrightarrow \cdots$$

By the 2-CY property and the previous remarks we have

$$\mathrm{Ext}_\Lambda^1(M_{\mathbf{w}}^k, L_{\mathbf{w}}^j) = 0 \quad \text{and} \quad \mathrm{Ext}_\Lambda^2(M_{\mathbf{w}}^j, L_{\mathbf{w}}^j) \simeq D\mathrm{Hom}_\Lambda(L_{\mathbf{w}}^j, M_{\mathbf{w}}^j) \simeq k$$

and therefore

$$\dim_k \mathrm{Ext}_\Lambda^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^k) \leq 1.$$

□

**2.2. The dimension vectors of the layers.** In this section we investigate the action of the functor  $-\overset{\mathbf{L}}{\otimes}_\Lambda I_w$  at the level of the Grothendieck group of  $\mathcal{D}^b(\mathrm{f.l.}\Lambda)$  when  $\Lambda$  is not of Dynkin type. We show that this action has interesting connections with known actions. We denote by  $[-\overset{\mathbf{L}}{\otimes}_\Lambda I_w]$  the induced automorphism of  $K_0(\mathcal{D}^b(\mathrm{f.l.}\Lambda))$ .

**Lemma 2.5.** *Let  $Q$  be a non Dynkin quiver. For all  $i, j$  in  $Q_0$  we have*

$$[S_j \overset{\mathbf{L}}{\otimes}_\Lambda I_i] = [S_j] + (m_{ij} - 2\delta_{ij})[S_i]$$

in  $K_0(\mathcal{D}^b(\mathrm{f.l.}\Lambda))$ , where  $m_{ij}$  is the number of arrows between  $i$  and  $j$  in  $Q$ .

*Proof.* Since  $S_j$  is a simple  $\Lambda$ -bimodule, we have  $DS_i \simeq S_i$  as  $\Lambda$ -bimodules. Hence we have the following isomorphisms in  $\mathrm{Mod}(\Lambda^{op} \otimes \Lambda)$ :

$$\begin{aligned} S_j \overset{\mathbf{L}}{\otimes}_\Lambda S_i &\simeq D\mathrm{Hom}_k(S_j \overset{\mathbf{L}}{\otimes}_\Lambda S_i, k) \\ &\simeq DR\mathrm{Hom}_\Lambda(S_j, \mathrm{Hom}_k(S_i, k)) \\ &\simeq DR\mathrm{Hom}_\Lambda(S_j, DS_i) \\ &\simeq DR\mathrm{Hom}_\Lambda(S_j, S_i). \end{aligned}$$

Therefore we have

$$[S_j \overset{\mathbf{L}}{\otimes}_\Lambda S_i] = \left( \sum_t (-1)^t \dim \mathrm{Ext}_\Lambda^t(S_j, S_i) \right) [S_i] = (2\delta_{ij} - m_{ij})[S_i].$$

From the triangle  $S_i[-1] \longrightarrow I_i \longrightarrow \Lambda \longrightarrow S_i$  we get a triangle

$$S_j \overset{\mathbf{L}}{\otimes}_\Lambda S_i[-1] \longrightarrow S_j \overset{\mathbf{L}}{\otimes}_\Lambda I_i \longrightarrow S_j \longrightarrow S_j \overset{\mathbf{L}}{\otimes}_\Lambda S_i.$$

Hence we have  $[S_j \overset{\mathbf{L}}{\otimes}_\Lambda I_i] = [S_j] - [S_j \overset{\mathbf{L}}{\otimes}_\Lambda S_i] = [S_j] - (2\delta_{ij} - m_{ij})[S_i]$ . □

From Lemma 2.5, we deduce the following results.

**Theorem 2.6.** *Let  $\Lambda$  be the completion of a preprojective algebra of any type.*

- (1) *For  $j = 1, \dots, l$  we have  $[L_{\mathbf{w}}^j] = R_{u_1} \dots R_{u_{j-1}}([S_{u_j}])$ , where the  $R_t$  are the reflection functors defined in Section 1.*
- (2) *For  $j = 1, \dots, l$ , there exists a unique indecomposable  $kQ$ -module  $(L_{\mathbf{w}}^j)_Q$  such that  $[L_{\mathbf{w}}^j] = [(L_{\mathbf{w}}^j)_Q]$ .*

*Proof.* (1) As in the previous subsection we treat separately the Dynkin and the non Dynkin case. The non Dynkin case is a direct consequence of Lemma 2.5 and Proposition 2.1.

For the Dynkin case, we can follow the strategy of the proof of Proposition 2.3. We introduce an extended Dynkin quiver containing  $Q$  as subquiver. Then applying reflection functors associated to the vertices of  $Q$  to modules whose support do not contain the additional vertex is the same as applying the reflection functors of  $Q$ . Thus the equality coming from the non Dynkin quiver gives us the equality for  $Q$ .

(2) From (1) it follows that the dimension vector of the layer  $L_w^j$  is a positive real root, and we get the result applying Kac's Theorem.  $\square$

The layer  $L_w^j$  is always rigid as  $\Lambda$ -module, but the associated indecomposable  $kQ$ -module  $(L_w^j)_Q$  is not always rigid as shown in the following.

*Example 2.1.* Let  $Q$  be the quiver  $1 \begin{matrix} \xrightarrow{2} \\ \xrightarrow{2} \end{matrix} 3$ , and  $\mathbf{w} := s_1 s_2 s_3 s_2 s_1 s_3$ . Then we have

$$L_w^1 = 1, \quad L_w^2 = \begin{matrix} 2 \\ 1 \end{matrix}, \quad L_w^3 = \begin{matrix} 3 \\ 1 \\ 2 \\ 1 \end{matrix}, \quad L_w^4 = \begin{matrix} 3 \\ 1 \end{matrix}, \quad L_w^5 = \begin{matrix} 3 \\ 1 \\ 3 \\ 2 \\ 1 \end{matrix}, \quad \text{and} \quad L_w^6 = \begin{matrix} 3 \\ 1 \\ 3 \\ 2 \\ 1 \\ 3 \end{matrix}.$$

Thus the associated indecomposable  $kQ$ -modules are the following:

$$(L_w^j)_Q = L_w^j \text{ for } j = 1, \dots, 4, \quad (L_w^5)_Q = \begin{matrix} 3 \\ 1 \\ 3 \\ 2 \\ 1 \end{matrix}, \quad \text{and} \quad (L_w^6)_Q = \begin{matrix} 3 \\ 1 \\ 3 \\ 2 \\ 1 \\ 3 \end{matrix}.$$

The module  $(L_w^6)_Q$  lies in the tube of rank 2, with indecomposable objects  $\begin{matrix} 3 \\ 1 \end{matrix}$  and  $\begin{matrix} 2 \\ 1 \end{matrix}$  on the border of the tube. Since  $(L_w^6)_Q$  is not on the border of the tube, it is not rigid.

**Definition 2.7.** [BB05] Let  $Q$  be an acyclic quiver with  $n$  vertices, and  $W_Q$  be the Coxeter group of  $Q$ . Let  $V$  be the vector space with basis  $v_1, \dots, v_n$ . The *geometric representation*  $W \rightarrow \text{GL}(V)$  of  $W$  is defined by

$$s_i v_j := v_j + (m_{ij} - 2\delta_{ij})v_i.$$

The *contragredient of the geometric representation*  $W \rightarrow \text{GL}(V)$  is then

$$s_i v_j^* = \begin{cases} v_j^* & i \neq j \\ -v_j^* + \sum_{t \neq j} m_{tj} v_t^* & i = j \end{cases}$$

The Grothendieck group  $K_0(\mathcal{D}^b(\text{f.l. } \Lambda))$  has a basis consisting of the simple  $\Lambda$ -modules, and  $K_0(\mathcal{K}^b(\text{proj } \Lambda))$  has a basis consisting of the indecomposable projective  $\Lambda$ -modules.

**Proposition 2.8.** (a) *The Coxeter group  $W$  acts on  $K_0(\mathcal{D}^b(\text{f.l. } \Lambda))$  by  $w \mapsto [-\otimes_{\Lambda}^{\mathbf{L}} I_w]$  as the geometric representation.*

(b) *The Coxeter group  $W$  acts on  $K_0(\mathcal{K}^b(\text{proj } \Lambda))$  by  $w \mapsto [-\otimes_{\Lambda}^{\mathbf{L}} I_w]$  as the contragredient of the geometric representation.*

*Proof.* (a) This follows directly from Lemma 2.5.

(b) This is shown in [IR08, Theorem 6.6]. It is assumed in [IR08] that  $Q$  is extended Dynkin, but this assumption is not used in the proof for this statement.  $\square$

**2.3. Reflection functors and ideals  $I_i$ .** In this subsection, we state some basic properties of the first layers. In particular we show that the equivalence  $-\overset{\mathbf{L}}{\otimes}_{\Lambda} I_i$ , when  $Q$  is not Dynkin, can be interpreted as a reflection functor of the category  $\mathcal{D}^b(\text{f.l.}\Lambda)$ .

**Lemma 2.9.** *Let  $Q$  be an acyclic quiver, and  $\Lambda = \Lambda_Q$ . Let  $c \in W_Q$  be a Coxeter element admissible with respect to the orientation of  $Q$ . Let  $i \in Q_0$  be a source of  $Q$ . Then we have the following isomorphisms in  $\text{mod } \Lambda$ :*

- (1)  $\Lambda/I_c \simeq kQ$ ,
- (2)  $I_i/I_{cs_i} \simeq \tau^{-1}P_i \oplus kQ/P_i = R_i^-(kQ)$  where  $P_i = e_i kQ$  is the indecomposable projective  $kQ$ -module associated to  $i$  and  $\tau$  is the AR-translation of  $\text{mod } kQ$ .
- (3)  $I_{c^n}/I_{c^{n+1}} \simeq \tau^{-n}(kQ)$ .

*Proof.* (1) This is Propositions II.3.2 and II.3.3 of [BIRS09a].

- (2) We separate the case whether  $Q$  is of Dynkin type and of non Dynkin type. Note that by Proposition 1.1 (b) we have  $e_j I_i = e_j \Lambda$  and  $e_j I_{cs_i} = e_j I_i I_c = e_j I_c$  if  $j \neq i$ . Therefore by (1) it is enough to prove that  $e_i I_i/I_{cs_i} \simeq \tau^{-1}(e_i kQ)$ .

Assume first that  $Q$  is of non Dynkin type. The projective resolution of  $e_i I_i$  in  $\text{mod } \Lambda$  has the form:

$$(*) \quad 0 \longrightarrow e_i \Lambda \longrightarrow \bigoplus_{a \in \bar{Q}_1, s(a)=i} e_{t(a)} \Lambda \longrightarrow e_i I_i \longrightarrow 0$$

Applying the functor  $-\otimes_{\Lambda} I_c$  to the exact sequence  $(*)$ , we get an exact sequence

$$(**) \quad 0 \longrightarrow e_i I_c \longrightarrow \bigoplus_{a \in \bar{Q}_1, s(a)=i} e_{t(a)} I_c \longrightarrow e_i I_i \otimes_{\Lambda} I_c \longrightarrow 0.$$

By Proposition 1.1 (e), we have  $I_i \otimes_{\Lambda} I_c = I_i I_c = I_{cs_i}$ . Hence we deduce from  $(*)$  and  $(**)$  the short exact sequence

$$0 \longrightarrow e_i \frac{\Lambda}{I_c} \longrightarrow \bigoplus_{a \in \bar{Q}_1, s(a)=i} e_{t(a)} \frac{\Lambda}{I_c} \longrightarrow e_i \frac{I_i}{I_{cs_i}} \longrightarrow 0.$$

Since  $i$  is a source in  $Q$ , we have the set equality

$$\{a \in \bar{Q}_1, \text{ with } s(a) = i\} = \{a \in Q_1, \text{ with } s(a) = i\}.$$

Therefore by (1) this short exact sequence is

$$0 \longrightarrow e_i kQ \longrightarrow \bigoplus_{a \in Q_1, s(a)=i} e_{t(a)} kQ \longrightarrow e_i \frac{I_i}{I_{cs_i}} \longrightarrow 0.$$

Hence we have  $e_i \frac{I_i}{I_{cs_i}} \simeq \tau^{-1}(e_i kQ)$ .

Let  $Q$  be of Dynkin type. Denote by  $\tilde{Q}$  an acyclic extended Dynkin quiver containing  $Q$  as a subquiver and such that the additional vertex is a sink. Let  $\Lambda := \Lambda_Q$  and  $\tilde{\Lambda} := \Lambda_{\tilde{Q}}$  be the corresponding (completion of) their preprojective algebras. Denote by  $c_Q$  the Coxeter element of  $W_Q$  admissible with respect to the orientation of  $Q$ . Using the above argument for the quiver  $\tilde{Q}$  and for  $\tilde{I}_{c_Q}$  we get a short exact sequence

$$0 \longrightarrow e_i k\tilde{Q} \longrightarrow \bigoplus_{a \in \bar{Q}_1, s(a)=i} e_{t(a)} k\tilde{Q} \longrightarrow e_i \frac{\tilde{I}_i}{\tilde{I}_{c_Q s_i}} \longrightarrow 0.$$

Since the additional vertex  $i_0$  is a sink, we get that  $e_j k\tilde{Q} \simeq e_j kQ$  for  $j \neq i_0$  and  $\tilde{\Lambda}/\tilde{I}_{c_Q} \simeq \Lambda/I_{c_Q} \simeq kQ$ . Hence we have

$$e_i \frac{I_i}{I_i I_{c_Q}} \simeq e_i I_i \otimes_{\Lambda} \frac{\Lambda}{I_{c_Q}} \simeq e_i \tilde{I}_i \otimes_{\tilde{\Lambda}} \frac{\tilde{\Lambda}}{\tilde{I}_{c_Q}} \simeq e_i \frac{\tilde{I}_i}{\tilde{I}_{c_Q s_i}} \simeq \tau_{\tilde{Q}}^{-1}(e_i kQ) \simeq \tau_Q^{-1}(e_i kQ).$$

(3) This is a direct consequence of (1) and (2). □

From Lemma 2.9 we deduce the following result which gives another interpretation of the tilting ideals  $I_i$ .

**Corollary 2.10.** *Let  $Q$  be an acyclic quiver which is not Dynkin, and  $\Lambda = \Lambda_Q$ . Let  $i \in Q_0$  be a sink of  $Q$ . Denote  $Q' := \mu_i(Q)$ . Then the following diagram commute*

$$\begin{array}{ccc} \text{mod } kQ/[e_i DkQ] & \xrightarrow{R_i^-} & \text{mod } kQ'/[e_i kQ'] \\ \downarrow & & \downarrow \\ \mathcal{D}^b(\text{f.l. } \Lambda) & \xrightarrow{-\mathbb{L}_{\otimes_{\Lambda} I_i}} & \mathcal{D}^b(\text{f.l. } \Lambda) \end{array}$$

where the vertical functors are the natural inclusions.

*Proof.* Denote by  $c$  the Coxeter element admissible with respect to the orientation of  $Q$ , and by  $c' = s_i c s_i$  the Coxeter element admissible with respect to the orientation of  $Q'$ . We have the following isomorphisms in  $\mathcal{D}^b(\text{f.l. } \Lambda)$ .

$$\begin{aligned} kQ \otimes_{\Lambda}^{\mathbb{L}} I_i &\simeq \Lambda/I_c \otimes_{\Lambda}^{\mathbb{L}} I_i && \text{by Lemma 2.9 (1)} \\ &\simeq \Lambda/I_c \otimes_{\Lambda} I_i && \text{by Proposition 1.1 (e)} \\ &\simeq I_i/I_c I_i \simeq I_i/I_i I_{c'} && \\ &\simeq kQ' && \text{by Lemma 2.9 (2)} \end{aligned}$$

□

### 3. TILTING MODULES AND $c$ -SORTABLE WORDS

In this section  $Q$  is a finite acyclic quiver,  $\Lambda$  is the completion of the preprojective algebra associated with  $Q$  and  $c$  a Coxeter element admissible with respect to the orientation of  $Q$ . The purpose of this section is to investigate the layers for words  $\mathbf{w}$  satisfying a certain property called  $c$ -sortable.

**Definition 3.1.** [Rea07] Let  $c$  be a Coxeter element of the Coxeter group  $W_Q$ . An element  $w$  of  $W_Q$  is called  $c$ -sortable if there exists a reduced expression  $\mathbf{w}$  of  $w$  of the form  $\mathbf{w} = c^{(0)} c^{(1)} \dots c^{(m)}$  where all  $c^{(t)}$  are subwords of  $c$  whose supports satisfy

$$\text{supp}(c^{(m)}) \subseteq \text{supp}(c^{(m-1)}) \subseteq \dots \subseteq \text{supp}(c^{(1)}) \subseteq \text{supp}(c^{(0)}) \subseteq Q_0.$$

For  $i \in Q_0$ , if  $s_i$  is in the support of  $c^{(t)}$ , by abuse of notation, we will write  $i \in c^{(t)}$ .

Here is an immediate result [Rea07].

**Lemma 3.2.** *Let  $w$  be a  $c$ -sortable element of  $W_Q$ . Then the expression  $\mathbf{w} = c^{(0)}c^{(1)} \dots c^{(m)}$  is unique.*

Let  $w$  be an element of  $W_Q$ , and  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  a reduced expression. Recall from Section 1 that for  $j = 1, \dots, l$  the layer  $L_{\mathbf{w}}^j$  is defined to be the  $\Lambda$ -module:

$$L_{\mathbf{w}}^j = e_{u_j} \frac{I_{u_k} \dots I_{u_1}}{I_{u_j} \dots I_{u_1}} = \frac{I_{u_{j-1}} \dots I_{u_1}}{I_{u_j} \dots I_{u_1}}$$

where  $k < j$  satisfies  $u_k = u_j$  and is maximal with this property.

Here is a Theorem giving a nice characterization of  $c$ -sortable words.

**Theorem 3.3.** *Let  $w$  be an element of  $W_Q$  and  $\mathbf{w} = s_{u_1} s_{u_2} \dots s_{u_l}$  be a reduced expression of  $w$ . Then we have the following:*

- (1) *if there exists a Coxeter  $c$  such that  $w$  is  $c$ -sortable and  $\mathbf{w}$  is the  $c$ -sortable expression of  $w$ , then for all  $j = 1, \dots, l$   $L_{\mathbf{w}}^j$  is in  $\text{mod } kQ$ , where  $Q$  is admissible for the Coxeter element  $c$ ;*
- (2) *if for all  $j = 1, \dots, l$  the layer  $L_{\mathbf{w}}^j$  is in  $\text{mod } kQ$  for a certain orientation of  $Q$ , then  $w$  is  $c$ -sortable, where  $c$  is the Coxeter element admissible for the orientation of  $Q$ .*

*Proof.* (1) Assume that  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  is a  $c$ -sortable word. Let  $j \geq 1$ , and  $k$  be the (possibly) last index  $< j$  such that  $u_j = u_k$ . Since  $\mathbf{w}$  is  $c$ -sortable, the word  $s_{u_1} \dots s_{u_j}$  is a subsequence of  $cs_{u_1} \dots s_{u_k}$ . Therefore we have an inclusion

$$e_{u_j} I_{u_1 \dots u_k} I_c = e_{u_j} I_{cu_1 \dots u_k} \subseteq e_{u_j} I_{u_1 \dots u_j}$$

Hence there is a surjection

$$e_{u_j} \frac{I_{u_1 \dots u_k}}{I_{u_1 \dots u_k} I_c} \twoheadrightarrow e_{u_j} \frac{I_{u_1 \dots u_k}}{I_{u_1 \dots u_j}} = L_{\mathbf{w}}^j.$$

The left term is a  $kQ$ -module, indeed it is isomorphic to

$$e_{u_j} I_{u_1 \dots u_k} \otimes_{\Lambda} \frac{\Lambda}{I_c} = e_{u_j} I_{u_1 \dots u_k} \otimes_{\Lambda} kQ$$

by Lemma 2.9 (1). Thus the right term  $L_{\mathbf{w}}^j$  is also a  $kQ$ -module.

(2) For this statement we again have to treat separately the Dynkin and the non Dynkin case. Assume first that  $Q$  is not Dynkin. We prove this assertion by induction on the length of the word  $w$ . For  $l(w) = 1$  the result is immediate.

Assume that (2) is true for any word  $\mathbf{w}$  of length  $\leq l - 1$  and let  $\mathbf{w} := s_{u_1} \dots s_{u_l}$  be a reduced expression such that  $L_{\mathbf{w}}^j$  is a  $kQ$ -module for all  $j = 1, \dots, l$ . Without loss of generality we can assume that the support of  $\mathbf{w}$  contains all the vertices of  $Q$ . We first show that  $u_1$  is a source of  $Q$ . Assume it is not, then there exists  $k \geq 2$  such that there is an arrow  $u_k \rightarrow u_1$  in  $Q$ . Take the smallest such number. It is then not hard to check that the top of  $L_{\mathbf{w}}^k$  is the simple  $S_{u_k}$  and that the kernel of the map  $L_{\mathbf{w}}^k \rightarrow S_{u_k}$  contains  $S_{u_1}$  in its top. Thus  $L_{\mathbf{w}}^k$  is not a  $kQ$ -module, which is a contradiction.

Therefore  $u_1$  is a source of the quiver  $Q$  and we have

$$L_{\mathbf{w}}^j = e_{u_j} \frac{I_{u_k} \dots I_{u_1}}{I_{u_j} \dots I_{u_1}} \simeq (e_{u_j} \frac{I_{u_k} \dots I_{u_2}}{I_{u_j} \dots I_{u_2}}) \otimes_{\Lambda} I_{u_1} \quad \text{by Proposition 1.1 (e).}$$

Hence we have  $L_{\mathbf{w}}^j = L_{\mathbf{w}'}^{j-1} \otimes_{\Lambda}^{\mathbf{L}} I_{u_1}$  for  $j = 2, \dots, l$ , where  $\mathbf{w}' := s_{u_2} \dots s_{u_l}$ . By Theorem 2.6 (1) we have  $[L_{\mathbf{w}}^j] = R_{u_1} \circ \dots \circ R_{u_{j-1}}([S_{u_j}])$  in the Grothendieck group  $K_0(\mathcal{D}(\text{f.l.}\Lambda))$ . Since  $w$  is reduced we then have  $[L_{\mathbf{w}}^j] \neq [S_{u_1}]$  for  $j \geq 2$ . Thus  $L_{\mathbf{w}}^j$  is not isomorphic to the simple projective  $e_{u_1} kQ = S_{u_1}$  if  $j \geq 2$ . Then by Corollary 2.10, we get

$$L_{\mathbf{w}'}^{j-1} \simeq R_{u_1}(L_{\mathbf{w}}^j) \in \text{mod } kQ' / [e_{u_1} DkQ']$$

where  $Q' = \mu_{u_1}(Q)$ . By induction hypothesis we get that  $\mathbf{w}'$  is  $c'$ -sortable where  $c'$  is the Coxeter element admissible for the orientation of  $Q'$ , *i.e.*  $c' = s_{u_1} c s_{u_1}$ . We get the conclusion using the following criterion which detects  $c$ -sortability:

**Lemma 3.4.** [Rea07, Lemma 2.1] *Let  $c := s_{u_1} \dots s_{u_n}$  be a Coxeter element. If  $l(s_{u_1} w) < l(w)$ , then  $w$  is  $c$ -sortable if and only if  $s_{u_1} w$  is  $s_{u_1} c s_{u_1}$ -sortable.*

If  $Q$  is Dynkin, we introduce an extended Dynkin quiver  $\tilde{Q}$  such that the additional vertex is a source. And then we conclude by the above argument for non Dynkin quivers.  $\square$

**3.1. Three series of  $kQ$ -modules.** To the  $c$ -sortable word  $\mathbf{w} = s_{u_1} \dots s_{u_l}$ , we associate three different series of  $kQ$ -modules, and show that they coincide.

For  $j = 1, \dots, l$ , we define  $kQ$ -modules  $T_{\mathbf{w}}^j$ . For  $1 \leq j \leq l(c^{(0)})$ ,  $T_{\mathbf{w}}^j$  is the projective  $kQ$ -module  $e_{u_j} kQ$ . For  $j > l(c^{(0)})$ , let  $k$  be the maximal integer such that  $k < j$  and  $u_k = u_j$ . We define  $T_{\mathbf{w}}^j$  as the cokernel of the map

$$f : T_{\mathbf{w}}^k \rightarrow E$$

where  $f$  is a minimal left  $(T_{\mathbf{w}}^{k+1} \oplus \dots \oplus T_{\mathbf{w}}^{j-1})$ -approximation.

**Definition 3.5.** An *admissible triple* is a triple  $(Q, c, \mathbf{w})$  consisting of an acyclic quiver  $Q$ , a Coxeter element  $c$  admissible with respect to the orientation of  $Q$ , and a  $c$ -sortable word  $\mathbf{w} = c^{(0)} c^{(1)} \dots c^{(m)}$  such that  $c = c^{(0)} \mathbf{v}$  for some  $\mathbf{v}$  as words.

We denote by  $Q^{(j)}$  the quiver  $Q$  restricted to the support of  $c^{(j)}$ .

**Definition 3.6.** Let  $(Q, c, \mathbf{w})$  be an admissible triple, with  $\mathbf{w} = s_{u_1} \mathbf{w}'$ . The *reduction* of  $(Q, c, \mathbf{w})$  at  $s_{u_1}$  is the triple  $(Q', c', \mathbf{w}')$  with  $Q' = \mu_{u_1}(Q^{(0)})$ , where  $\mu_{u_1}$  is the reflection at  $u_1$  and  $c' = s_{u_1} c^{(0)} s_{u_1}$ .

It is not hard to check the following property:

**Lemma 3.7.** *The triple  $(Q', c', \mathbf{w}')$  is admissible.*

Note that since  $u_1$  is a source on the restriction of  $Q$  to  $\text{supp}(c^{(0)})$ , it is always possible to apply the reflection functor  $\mu_{u_1}$ .

Let  $(Q, c, \mathbf{w})$  be an admissible triple with  $\mathbf{w} = s_{u_1} s_{u_2} \dots s_{u_l}$ . For  $j = 1, \dots, l$ , we define  $kQ$ -modules  $U_{\mathbf{w}}^j$  by induction on  $l$ .

If  $l = 1$  then we define  $U_{\mathbf{w}}^1 = e_{u_1} kQ$ , the projective indecomposable  $kQ$ -module associated to the vertex  $u_1$ .



Assume  $l \geq 2$ . Then we write  $\mathbf{w} = s_{u_1} \mathbf{w}'$ , and by Lemma 3.7 the triple  $(Q' = \mu_{u_1}(Q^{(0)}), s_{u_1} c s_{u_1}, \mathbf{w}')$  is an admissible triple with  $l(w') = l - 1$ . Therefore by the induction hypothesis we have  $kQ$ -modules  $U_{\mathbf{w}'}^1, \dots, U_{\mathbf{w}'}^{l-1}$ . For  $j = 2, \dots, l$  we define

$$U_{\mathbf{w}}^j = \tilde{R}_{u_1}^-(U_{\mathbf{w}'}^{j-1})$$

where  $\tilde{R}_{u_1}^-$  is the composition

$$\text{mod } kQ' = \text{mod } k(\mu_{u_1} Q^{(0)}) \xrightarrow{R_{u_1}^-} \text{mod } kQ^{(0)} \hookrightarrow \text{mod } kQ.$$

**Theorem 3.8.** *Let  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  be a  $c$ -sortable word where  $c$  is admissible for the orientation of  $Q$ . Then for  $j = 1, \dots, l$ , we have  $L_{\mathbf{w}}^j \simeq U_{\mathbf{w}}^j \simeq T_{\mathbf{w}}^j$ , where the  $L_{\mathbf{w}}^j$  are the layers and the  $kQ$ -modules  $T_{\mathbf{w}}^j$  and  $U_{\mathbf{w}}^j$  are defined as above.*

*Proof.* We first prove that  $L_{\mathbf{w}}^j \simeq U_{\mathbf{w}}^j$ . By definition  $L_{\mathbf{w}}^1 = e_{u_1} \Lambda / I_{u_1} = S_{u_1}$ . Since by assumption  $c = c^{(0)} \mathbf{v}$ , we have  $e_{u_1} kQ^{(0)} = e_{u_1} kQ = S_{u_1}$ . Hence we get  $U_{\mathbf{w}}^1 = L_{\mathbf{w}}^1$ .

Let  $\mathbf{w}'$  be the word  $s_{u_2} \dots s_{u_l}$ . We will prove that  $L_{\mathbf{w}}^j = \tilde{R}_{u_1}^-(L_{\mathbf{w}'}^{j-1})$  for  $j \geq 2$ .

By Lemma 2.9 (2) we have  $R_{u_1}^-(-) = - \otimes_{kQ'} \frac{I_{u_1}}{I_{c^{(0)} s_{u_1}}}$ . Hence we can write

$$L_{\mathbf{w}'}^{j-1} = \frac{e_{u_j} I_{u_k} \dots I_{u_2}}{e_{u_j} I_{u_j} \dots I_{u_2}} =: \frac{Y}{X}.$$

We have the following exact commutative diagram:

$$\begin{array}{ccccccc} X \otimes_{\Lambda} I_{u_1} I_{c^{(0)}} & \longrightarrow & X \otimes_{\Lambda} I_{u_1} & \longrightarrow & X \otimes_{\Lambda} \frac{I_{u_1}}{I_{u_1} I_{c^{(0)}}} & \longrightarrow & 0 \\ \downarrow & \nearrow & \downarrow a & & \downarrow & & \\ Y \otimes_{\Lambda} I_{u_1} I_{c^{(0)}} & \xrightarrow{f} & Y \otimes_{\Lambda} I_{u_1} & \longrightarrow & Y \otimes_{\Lambda} \frac{I_{u_1}}{I_{u_1} I_{c^{(0)}}} & \longrightarrow & 0 \\ \downarrow g & & \downarrow b & & \downarrow & & \\ \frac{Y}{X} \otimes_{\Lambda} I_{u_1} I_{c^{(0)}} & \xrightarrow{d} & \frac{Y}{X} \otimes_{\Lambda} I_{u_1} & \xrightarrow{e} & \frac{Y}{X} \otimes_{\Lambda} \frac{I_{u_1}}{I_{u_1} I_{c^{(0)}}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

By Proposition 1.1 (b) we have the inclusion  $Y I_{c'} \subset X$  since  $u_2 \dots u_j$  is a subword of  $c' u_2 \dots u_k$ . By definition  $c^{(0)} s_{u_1}$  is  $s_{u_1} c'$ , so the map  $f$  factors through  $a$ . Therefore the composition  $dg$  vanishes and since  $g$  is epi,  $d$  vanishes and hence  $e$  is an isomorphism.

Moreover since  $w = s_{u_1} s_{u_2} \dots s_{u_k}$  is a subword of  $s_{u_1} \dots s_{u_j}$ , then  $X I_{u_1}$  is contained in  $Y I_{u_1}$  by Proposition 1.1 (b). Hence  $a$  is mono. Finally we get an isomorphism

$$\frac{Y}{X} \otimes_{\Lambda} \frac{I_{u_1}}{I_{u_1} I_{c^{(0)}}} \simeq \frac{Y \otimes_{\Lambda} I_{u_1}}{X \otimes_{\Lambda} I_{u_1}} \simeq L_{\mathbf{w}}^j.$$

We will now prove that  $U_{\mathbf{w}}^j \simeq T_{\mathbf{w}}^j$ . For  $j \leq l(c^{(0)})$  this is clear because of a basic property of reflection functors.

Assume  $j > l(c^{(0)})$ . Let  $k$  be the maximal integer  $< j$  such that  $u_k = u_j$ . It exists because  $j > l(c^{(0)})$  and because  $\mathbf{w}$  is  $c$ -sortable. We define the subwords  $\mathbf{w}'' = s_{u_1} \dots s_{u_{k-1}}$  and  $\mathbf{w}' = s_{u_k} \dots s_{u_j}$  of  $\mathbf{w}$ . Let  $c'$  be  $s_{u_k} \dots s_{u_{j-1}}$ , and  $Q'$  be the quiver  $\mu_{\mathbf{w}''}(Q) = \mu_{u_{k-1}} \circ \dots \circ \mu_{u_1}(Q)$ . Then  $(Q', c', \mathbf{w}')$  is an admissible triple. We have  $U_{\mathbf{w}'}^1 = S_{u_k}$  and  $U_{\mathbf{w}'}^{j-k+1} = R_{c'}^-(S_{u_k}) = \tau_{kQ'}^{-1}(S_{u_k})$ , thus we have an almost split sequence:

$$0 \rightarrow U_{\mathbf{w}'}^1 \rightarrow E \rightarrow U_{\mathbf{w}'}^{j-k+1} \rightarrow 0$$

Applying the reflection functor  $R_{\mathbf{w}''}^-$  to this short exact sequence we still get a short exact sequence:

$$0 \rightarrow R_{\mathbf{w}''}^-(U_{\mathbf{w}'}^1) \rightarrow R_{\mathbf{w}''}^-(E) \rightarrow R_{\mathbf{w}''}^-(U_{\mathbf{w}'}^{j-k+1}) \rightarrow 0$$

which is

$$0 \rightarrow U_{\mathbf{w}}^k \rightarrow R_{\mathbf{w}''}^-(E) \rightarrow U_{\mathbf{w}}^j \rightarrow 0$$

and the left map is a left  $\mathbf{add}(R_{\mathbf{w}''}^-(U_{\mathbf{w}'}^2) \oplus \dots \oplus R_{\mathbf{w}''}^-(U_{\mathbf{w}'}^{j-k}))$ -approximation, thus a left  $\mathbf{add}(U_{\mathbf{w}}^{k+1} \oplus \dots \oplus U_{\mathbf{w}}^{j-1})$ -approximation. Note moreover that this approximation is always mono.  $\square$

**Corollary 3.9.** *Let  $\mathbf{w}$  be a  $c$ -sortable word, where  $c$  is admissible with respect to the orientation of  $Q$ . Then the  $kQ$ -modules  $L_{\mathbf{w}}^j$  satisfy the following properties:*

- (1) *They are non zero.*
- (2) *They are pairwise non-isomorphic.*
- (3) *The space  $\mathbf{Hom}_{kQ}(L_{\mathbf{w}}^j, L_{\mathbf{w}}^k)$  vanishes if  $j > k$ .*
- (4) *The minimal left  $\mathbf{add}\{L_{\mathbf{w}}^{k+1}, \dots, L_{\mathbf{w}}^{j-1}\}$ -approximation map  $f : L_{\mathbf{w}}^k \rightarrow E$  is a monomorphism, where  $k$  and  $j$  are consecutive of same type.*

*Proof.* (1) This is Proposition 1.1 (c).

- (2) This is clearly true for the  $U_{\mathbf{w}}^j$  because reflection functors preserve isoclasses.
- (3) Using reflection functors, we can assume that  $U_{\mathbf{w}}^k$  is simple projective, and then this is clear.
- (4) The fact that the approximation map is mono comes from the fact that reflection functors preserves short exact sequences.  $\square$

**Theorem 3.10.** *Let  $\mathbf{w} = s_{u_1} \dots s_{u_l}$  be a  $c$ -sortable word, where  $c$  is admissible for the orientation of  $Q$ . For  $i \in Q_0^{(0)}$ , denote by  $t_{\mathbf{w}}(i)$  the maximal integer such that  $u_{t_{\mathbf{w}}(i)} = i$ . Then the  $kQ$ -module*

$$T_{\mathbf{w}} := \bigoplus_{i \in Q_0^{(0)}} L_{\mathbf{w}}^{t_{\mathbf{w}}(i)}$$

*is a  $kQ^{(0)}$ -tilting module and we have  $\mathbf{Sub}(T_{\mathbf{w}}) = \{L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^l\}$ .*

*Proof.* The fact that  $T_{\mathbf{w}}$  is a  $kQ^{(0)}$ -tilting module can easily be seen using the fact that  $L_{\mathbf{w}}^j = T_{\mathbf{w}}^j$ , and Corollary 3.9 (3)-(4).

We prove that  $\mathbf{Sub}(T_{\mathbf{w}}) = \{L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^l\}$  by induction on  $l = l(\mathbf{w})$ .

If  $l(\mathbf{w}) = 1$ , then the assertion is clear.

Assume that  $l \geq 2$  and write  $\mathbf{w} = s_{u_1} \mathbf{w}'$ .

*Case 1:  $u_1$  is in the support of  $\mathbf{w}'$ :* this means that  $t_{\mathbf{w}}(u_1) \geq 2$ . Thus we have

$$\begin{aligned} T_{\mathbf{w}} &= \bigoplus_{i \in Q_0^{(0)}} U_{\mathbf{w}}^{t_{\mathbf{w}}(i)} \\ &= \bigoplus_{i \in Q_0^{(0)}} R_{u_1}^-(U_{\mathbf{w}'}^{t_{\mathbf{w}}(i)-1}) \\ &= \bigoplus_{i \in Q_0^{(0)}} R_{u_1}^-(U_{\mathbf{w}'}^{t_{\mathbf{w}'}(i)}) \\ &= R_{u_1}^-(T_{\mathbf{w}}') \end{aligned}$$

Then using the induction hypothesis we get

$$\{R_{u_1}^-(U_{\mathbf{w}'}^1), \dots, R_{u_1}^-(U_{\mathbf{w}'}^{l(\mathbf{w}')})\} \subset \text{Sub} T_{\mathbf{w}} \subset \{S_{u_1}, R_{u_1}^-(U_{\mathbf{w}'}^1), \dots, R_{u_1}^-(U_{\mathbf{w}'}^{l(\mathbf{w}')})\}$$

By definition of the  $T_{\mathbf{w}}^j$  there exists a short exact sequence:

$$S_{u_1} = T_{\mathbf{w}}^1 \longrightarrow E \longrightarrow T_{\mathbf{w}}^j \longrightarrow 0$$

where  $E$  is in  $\text{add}(T_{\mathbf{w}}^2 \oplus \dots \oplus T_{\mathbf{w}}^{j-1})$  and where  $j$  is the minimal integer  $\leq 2$  such that  $u_j = u_1$ . It exists since  $u_1$  is in the support of  $\mathbf{w}'$ .

The approximation map is a monomorphism, thus  $S_{u_1}$  is in  $\text{Sub}(E) \subset \text{Sub}(T_{\mathbf{w}}^2 \oplus \dots \oplus T_{\mathbf{w}}^{j-1}) \subset \text{Sub} T_{\mathbf{w}}$ .

*Case 2:  $u_1$  is not in the support of  $\mathbf{w}'$ .*

Then it is easy to see that

$$T_{\mathbf{w}} = S_{u_1} \oplus R_{u_1}^-(T_{\mathbf{w}}').$$

And we get

$$\text{Sub} T_{\mathbf{w}} = \{S_{u_1}, R_{u_1}^-(U_{\mathbf{w}'}^1), \dots, R_{u_1}^-(U_{\mathbf{w}'}^{l(\mathbf{w}')})\} = \{U_{\mathbf{w}}^1, U_{\mathbf{w}}^2, \dots, U_{\mathbf{w}}^{l(\mathbf{w}')}\}.$$

□

*Remark 3.1.* (a) The short exact sequence  $L_{\mathbf{w}}^k \xrightarrow{f} E \twoheadrightarrow L_{\mathbf{w}}^j$  in  $\text{mod } kQ$  is an almost split sequence of the category  $\text{Sub}(T_{\mathbf{w}})$ .

(b) This almost split sequence is an element of  $\text{Ext}_{\Lambda}^1(L_{\mathbf{w}}^j, L_{\mathbf{w}}^k)$ , which is the ‘2-Calabi-Yau complement’ of the short exact sequence  $L_{\mathbf{w}}^j \twoheadrightarrow K \twoheadrightarrow L_{\mathbf{w}}^k$  of Proposition 2.3.

**3.2. Tilting modules with finite torsionfree class.** In this section we establish a converse of Theorem 3.10. Hence we get a natural bijection between tilting  $kQ$ -module with finite torsionfree class and  $c$ -sortable elements in  $W_Q$ .

**Proposition 3.11.** *Let  $\mathbf{w} = s_{u_1} \dots s_{u_l} = c^{(0)} \dots c^{(m)}$  be a  $c$ -sortable word where  $c$  is a Coxeter word admissible for the orientation of  $Q$ . We define  $T_{\mathbf{w}} := \bigoplus_{j \in c^{(0)}} T_{\mathbf{w}}^{t_{\mathbf{w}}(j)}$  as in Theorem 3.10. Let  $i \in Q_0$  such that  $c^{(m)} s_i$  is a subword of  $c^{(m-1)}$  or  $i \in c^{(m)}$ . We*

*define  $L' := e_i \frac{I_{\mathbf{w}}}{I_i I_{\mathbf{w}}}$  and  $T'$  as the cokernel of  $T_{\mathbf{w}}^{t_{\mathbf{w}}(i)} \xrightarrow{f} E$  where  $f$  is a minimal left- $\text{add}(T_{\mathbf{w}}^{t_{\mathbf{w}}(i)+1} \oplus \dots \oplus T_{\mathbf{w}}^l)$ -approximation.*

*Then we have the following*

- (1)  $T' \simeq L'$ ;
- (2) *the  $kQ^{(0)}$ -module  $T' \oplus \bigoplus_{j \neq i} T_{\mathbf{w}}^{t_{\mathbf{w}}(j)}$  is a tilting module if and only if the expression  $\mathbf{w} s_i$  is  $c$ -sortable.*

*Proof.* If  $c^{(m)}s_i$  is a subword of  $c^{(m-1)}$  then we can write  $\mathbf{w}s_i = c^{(0)} \dots c^{(m-1)}c'$  with  $c' := c^{(m)}s_i$  and we have

$$\text{supp}(c') \subseteq \text{supp}(c^{(m-1)}) \subseteq \dots \subseteq \text{supp}(c^{(1)}) \subseteq \text{supp}(c^{(0)}).$$

If  $i \in c^{(m)}$  then we write  $\mathbf{w}s_i = c^{(0)} \dots c^{(m-1)}c^{(m+1)}$  with  $c^{(m+1)} := s_i$  and then

$$\text{supp}(c^{(m+1)}) \subseteq \text{supp}(c^{(m)}) \subseteq \dots \subseteq \text{supp}(c^{(1)}) \subseteq \text{supp}(c^{(0)}).$$

To prove (1) it is then enough to observe that the proof of Theorem 3.8 does not use the fact that the expression  $\mathbf{w}$  is reduced.

By Theorem 3.10 it is enough to check that if  $T' \oplus \bigoplus_{j \neq i} T_{\mathbf{w}}^{t_{\mathbf{w}}(j)}$  is tilting then  $\mathbf{w}s_i$  is reduced. If  $\mathbf{w}s_i$  is not reduced we have  $L' = 0$  by Proposition 1.1 (b), and therefore  $T' = 0$  by (1). Since  $\mathbf{w}$  is  $c$ -sortable all  $T_{\mathbf{w}}^j = L_{\mathbf{w}}^j$  are non zero indecomposable modules by Theorem 2.2. Therefore the module  $T' \oplus \bigoplus_{j \neq i} T_{\mathbf{w}}^{t_{\mathbf{w}}(j)}$  has  $l(c^{(0)}) - 1$  indecomposable summands, so it can not be a tilting module over  $kQ^{(0)}$ .  $\square$

From Proposition 3.11 we deduce a nice consequence.

**Theorem 3.12.** *Let  $Q$  be an acyclic quiver. Let  $c$  be a Coxeter element admissible with respect to the orientation of  $Q$ . Let  $T$  be a tilting module over  $kQ$ . Assume that  $\mathbf{Sub}T$  has finitely many indecomposable modules. Then there exists a unique  $c$ -sortable word  $\mathbf{w}$  such that  $T_{\mathbf{w}} \simeq T$ .*

*Proof.* Assume that the orientation of  $Q$  is admissible for the Coxeter element  $s_1s_2 \dots s_n$ . The category  $\mathbf{Sub}T$  has almost split sequences. Denote by  $\tau$  the AR-translation of this category. Since  $\mathbf{Sub}T$  is finite, then for any  $i \in Q_0$  there exists  $m_i \geq 1$  such that  $\tau^{-m_i-1}(e_i kQ) = 0$ . And for each indecomposable  $X$  in  $\mathbf{Sub}T$ , there exist unique  $t \geq 0$  and  $i \in Q_0$  such that  $X \simeq \tau^{-t}(e_i kQ)$ . Indeed since  $\mathbf{Sub}T$  is finite, the AR quiver of  $\mathbf{Sub}T$  is connected and since the algebra  $kQ$  is hereditary it is not hard to see that there are no periodic modules. Then for  $t \geq 0$  we look at the set

$$\{i \in Q_0 \mid \tau^{-t}(e_i kQ) \neq 0\} = \{i_1^{(t)} < i_2^{(t)} < \dots < i_{p_t}^{(t)}\}$$

and set  $c^{(t)} := s_{i_1^{(t)}}s_{i_2^{(t)}} \dots s_{i_{p_t}^{(t)}}$ . It then clear that the word  $\mathbf{w} := c^{(0)}c^{(1)} \dots c^{(m)}$  where  $m := \max\{m_i \mid i \in Q_0\}$  satisfies

$$\text{supp}(c^{(m)}) \subseteq \dots \subseteq \text{supp}(c^{(1)}) \subseteq \text{supp}(c^{(0)}).$$

We have to check that  $\mathbf{w}$  is reduced. Assume it is not and write  $\mathbf{w} = \mathbf{w}'s_i\mathbf{v}$  where  $\mathbf{w}'$  is reduced and  $\mathbf{w}'s_i$  is not reduced. The word  $\mathbf{w}'$  is again  $c$ -sortable so can be written as  $\mathbf{w}' := c^{(0)} \dots c^{(m')}$ . For  $j \in Q_0$  denote by  $m'_j$  the integer such that  $j \in c^{(m'_j)}$  and  $j \notin c^{(m'_j+1)}$ . Then by hypothesis  $m'_i < m_i$ . Using the almost split sequences of  $\mathbf{Sub}T$ , it is immediate that

$$T_{\mathbf{w}'} \simeq \bigoplus_{i \in Q_0} \tau^{-m'_i}(e_i kQ).$$

Then by Proposition 3.11 the cokernel  $T'$  of the minimal left  $\text{add}\{T_{\mathbf{w}'}^{t_{\mathbf{w}'}(i)+1} \oplus T_{\mathbf{w}'}^{t_{\mathbf{w}'}(i)+2} \dots \oplus T_{\mathbf{w}'}^{l(\mathbf{w}')} \}$ -approximation map  $T_{\mathbf{w}'}^{t_{\mathbf{w}'}(i)} \rightarrow E$  is  $L_{\mathbf{w}'}^{l(\mathbf{w}')+1}$  which is zero by Proposition 1.1

(b). Therefore we have  $\tau^-(T_{\mathbf{w}'}^{t_{\mathbf{w}'}(i)}) = 0$  which is a contradiction since  $\tau^-(T_{\mathbf{w}'}^{t_{\mathbf{w}'}(i)}) = \tau^{-m'_i-1}(e_i kQ)$  and  $m'_i + 1 \leq m_i$ .  $\square$

As a consequence we get the following:

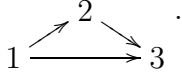
**Corollary 3.13.** *If  $T$  is a tilting  $kQ$ -module such that  $\text{Sub}T$  is of finite type, then all indecomposables  $\text{Sub}T$  are rigid as  $kQ$ -modules.*

Combining Theorem 3.12 with Theorem 3.10 we get the following result which was first proved using other methods in [Tho].

**Corollary 3.14.** *There is a 1-1 correspondence*

$$\{\text{finite torsionfree class of mod } kQ\} \xleftrightarrow{1:1} \{c\text{-sortable words with } c^{(0)} = c\}$$

3.3. **Example.** Let  $Q$  be the following graph  $\begin{array}{ccc} & 2 & \\ & \swarrow \quad \searrow & \\ 1 & \xrightarrow{\quad} & 3 \end{array}$ , and let  $\mathbf{w}$  be the word  $s_1 s_2 s_3 s_1 s_2 s_1$  in the Coxeter group  $W_Q$ . An admissible orientation for  $Q$  is the following



The canonical cluster-tilting object  $M_{\mathbf{w}}$  in  $\text{Sub} \Lambda_{\mathbf{w}}$  has the following direct summands

$$M_{\mathbf{w}}^1 = 1, \quad M_{\mathbf{w}}^2 = \begin{array}{c} 2 \\ 1 \end{array}, \quad M_{\mathbf{w}}^3 = \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \end{array}, \quad M_{\mathbf{w}}^4 = \begin{array}{c} 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array}, \quad M_{\mathbf{w}}^5 = \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array}, \quad M_{\mathbf{w}}^6 = \begin{array}{c} 1 \\ 3 \quad 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array}.$$

Then we can easily compute the layers  $L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^6$ . They are the indecomposable summands of the  $M_{\mathbf{w}}^i$  as  $kQ$ -modules:

$$L_{\mathbf{w}}^1 = 1, \quad L_{\mathbf{w}}^2 = \begin{array}{c} 2 \\ 1 \end{array}, \quad L_{\mathbf{w}}^3 = \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \end{array}, \quad L_{\mathbf{w}}^4 = \begin{array}{c} 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array}, \quad L_{\mathbf{w}}^5 = \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array}, \quad \text{and} \quad L_{\mathbf{w}}^6 = \begin{array}{c} 3 \\ 1 \end{array}.$$

Let us compute the  $T_{\mathbf{w}}^j$ . For  $j \leq 3$  the  $T_{\mathbf{w}}^j$  are the projective  $kQ$ -modules, thus we have

$$T_{\mathbf{w}}^1 = 1, \quad T_{\mathbf{w}}^2 = \begin{array}{c} 2 \\ 1 \end{array}, \quad \text{and} \quad T_{\mathbf{w}}^3 = \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \end{array}.$$

Then we have to compute approximations. We have a short exact sequence

$$0 \longrightarrow 1 \longrightarrow \begin{array}{c} 2 \\ 1 \end{array} \oplus \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \end{array} \longrightarrow \begin{array}{c} 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array} \longrightarrow 0,$$

where the map  $1 \longrightarrow \begin{array}{c} 2 \\ 1 \end{array} \oplus \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \end{array}$  is the minimal left  $\text{add}(T_{\mathbf{w}}^2 \oplus T_{\mathbf{w}}^3)$ -approximation of  $T_{\mathbf{w}}^1$ . Hence we have  $T_{\mathbf{w}}^4 = \begin{array}{c} 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array}$ . We have an exact sequence

$$0 \longrightarrow \begin{array}{c} 2 \\ 1 \end{array} \longrightarrow \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array} \longrightarrow \begin{array}{c} 3 \\ 1 \quad 2 \\ 1 \quad 3 \\ 2 \\ 1 \end{array} \longrightarrow 0,$$

where  $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 & 3 \\ 1 & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{smallmatrix}$  is the minimal left  $\mathbf{add}(T_{\mathbf{w}}^3 \oplus T_{\mathbf{w}}^4)$ -approximation of  $T_{\mathbf{w}}^2$ .

Hence we have  $T_{\mathbf{w}}^5 = \begin{smallmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 3 \\ 1 & 2 \\ 1 \end{smallmatrix}$ . There is an exact sequence

$$0 \longrightarrow \begin{smallmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \longrightarrow 0,$$

hence  $T_{\mathbf{w}}^6 = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ . So we have  $T_{\mathbf{w}}^j = L_{\mathbf{w}}^j$  as in Theorem 3.8.

The module  $T_{\mathbf{w}}$  is by definition  $T_{\mathbf{w}}^3 \oplus T_{\mathbf{w}}^5 \oplus T_{\mathbf{w}}^6$ . It is easy to check Theorem 3.10. The module  $T_{\mathbf{w}}$  is a tilting module over  $kQ$ , and we have

$$\mathbf{Sub}T_{\mathbf{w}} = \left\{ 1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 3 \\ 1 & 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{smallmatrix}, \begin{smallmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right\}.$$

Let us now compute the  $U_{\mathbf{w}}^j$ 's. By definition  $U_{\mathbf{w}}^1 = 1$ . Then we have

$$\begin{aligned} U_{\mathbf{w}}^2 &= R_1^-(2) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, & U_{\mathbf{w}}^3 &= R_1^- R_2^-(3) = R_1^-\left(\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right) = \begin{smallmatrix} 2 & 1 \\ 1 & 3 \\ 2 \end{smallmatrix} \\ \text{and } U_{\mathbf{w}}^4 &= R_1^- R_2^- R_3^-(1) = R_1^- R_2^-\left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right) = R_1^-\left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \\ 2 \end{smallmatrix}\right) = \begin{smallmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \end{smallmatrix}. \end{aligned}$$

And finally we have  $U_{\mathbf{w}}^6 = R_1^- R_2^- R_3^- \tilde{R}_1^- \tilde{R}_2^-(1)$  where  $\tilde{R}_i^-$  is the reflection functor associated to the quiver  $1 \longrightarrow 2$ . Therefore we have

$$U_{\mathbf{w}}^6 = R_1^- R_2^- R_3^- \tilde{R}_1^-\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) = R_1^- R_2^- R_3^-(2) = R_1^- R_2^-\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) = R_1^-(3) = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}.$$

#### 4. CATEGORIES AS CLUSTER CATEGORIES ASSOCIATED WITH AUSLANDER ALGEBRAS

In this section  $Q$  is an acyclic quiver,  $c$  is the Coxeter element admissible with respect to the orientation of  $Q$  and  $\mathbf{w} = c^{(0)}c^{(1)}\dots c^{(m)}$  is a  $c$ -sortable word with  $c^{(0)} = c$ . We denote by  $M_{\mathbf{w}}$  the canonical cluster-tilting object of  $\mathbf{Sub}\Lambda_w$  associated with the  $c$ -sortable expression  $\mathbf{w}$  of  $w$ .

This section is devoted to proving that the triangulated category  $\mathbf{Sub}\Lambda_w$  is triangle equivalent to a generalized cluster category associated to an algebra of global dimension at most two. Note that the result also holds in the case of general words [ART09], but with a very different construction. A link between the construction given in this paper and the construction of [ART09] is given in [Ami09b].

The first subsection is devoted to recalling results on Jacobian algebras defined in [DWZ08], and on the endomorphism algebra of the cluster-tilting object  $M_{\mathbf{w}}$  from [BIRS09a] and [BIRS09b]. In the second subsection we recall some definitions and basic properties for generalized cluster categories. In the third subsection we construct an algebra  $A$  of global dimension at most two such that the endomorphism algebra of the canonical cluster-tilting object in the generalized cluster category  $\mathcal{C}_A$  is isomorphic to the endomorphism algebra of  $M_{\mathbf{w}}$  in the category  $\mathbf{Sub}\Lambda_w$  (Proposition 4.9). In the fourth subsection we construct a triangle functor from  $\mathcal{C}_A$  to the category  $\mathbf{Sub}\Lambda_w$  using a consequence of the universal property of the generalized cluster category (see Proposition 4.4). Using a criterion of [KR08] (Proposition 4.14), we show that this functor is an equivalence. In the last subsection we describe an example.

**4.1. Canonical cluster-tilting object of  $\text{Sub } \Lambda_w$ .** Quivers with potentials and their associated Jacobian algebras have been investigated in [DWZ08]. Let  $Q$  be a finite quiver. For each arrow  $a$  in  $Q$ , the *cyclic derivative*  $\partial_a$  with respect to  $a$  is the unique linear map

$$\partial_a : kQ/[kQ, kQ] \rightarrow kQ$$

which takes the class of a path  $p$  to the sum  $\sum_{p=uv} vu$  taken over all decompositions of the path  $p$  (where  $u$  and  $v$  are possibly idempotent elements  $e_i$  associated to the vertex  $i$ ). An element  $W$  in  $kQ/[kQ, kQ]$  is a *potential* on  $Q$ , and is given by a linear combination of cycles in  $Q$ . The associated *Jacobian algebra*  $\text{Jac}(Q, W)$  is by definition the algebra

$$kQ/\langle \partial_a W; a \in Q_1 \rangle.$$

There is a generalization of quivers with potentials  $(Q, W)$  to *frozen quivers with potentials*  $(Q, W, F)$  in [BIRS09b] (see also [ART09]), where  $F = (F_0, F_1)$  is a pair of a subset  $F_0$  of vertices of  $Q$  (called *frozen vertices*) and a subset  $F_1$  of arrows contained in the set  $\{a \in Q_1, s(a) \in F_0 \text{ and } t(a) \in F_0\}$  (called *frozen arrows*). The associated *frozen Jacobian algebra* is by definition the algebra

$$\text{Jac}(Q, W, F) = kQ/\langle \partial_a W, a \notin F_1 \rangle.$$

Let  $\mathbf{w} = c^{(0)}c^{(1)} \dots c^{(m)}$  be a  $c$ -sortable word. Assume that the orientation of  $Q$  is admissible with respect to  $c$  and that  $c^{(0)} = c$ . For  $t \geq 0$ , we define  $Q^{(t)}$  to be the full subquiver of  $Q$  with vertices in the support of  $c^{(t)}$ . For each  $i$  in  $Q_0$  we denote by  $m_i$  the integer such that  $i \in c^{(m_i)}$  and  $i \notin c^{(m_i+1)}$ . Let  $Q_{\mathbf{w}}$  be the following quiver:

- the vertices are  $\{(i, r), r = 0, \dots, m, \quad i \in c^{(r)}\}$ .
- for each  $r \geq 1$ , for each  $i$  in  $Q_0^{(r+1)}$ , one arrow  $p_r^i : (i, r+1) \rightarrow (i, r)$
- for each  $a : i \rightarrow j \in Q_1$ , if  $r < m_i$  and  $r \leq m_j$ , one arrow  $a_r : (i, r) \rightarrow (j, r)$ ,
- for each  $a : i \rightarrow j \in Q_1$ , if  $m_i \leq m_j$ , one arrow  $a_{m_i} : (i, m_i) \rightarrow (j, m_j)$ ,
- for each  $a : i \rightarrow j \in Q_1$ , if  $r < m_i$  and  $r < m_j$  then one arrow  $a_r^* : (j, r) \rightarrow (i, r+1)$ ,
- for each  $a : i \rightarrow j \in Q_1$ , if  $m_j < m_i$ , one arrow  $a_{m_j}^* : (j, m_j) \rightarrow (i, m_i)$ .

We define the potential  $W_{\mathbf{w}}$  to be the sum

$$\begin{aligned} W_{\mathbf{w}} = & \sum_{a:i \rightarrow j} \left( \sum_{r < m_i, r < m_j} p_r^i a_r^* a_r - \sum_{r \leq m_i, r < m_j} p_r^j a_{r+1} a_r^* \right) \\ & - \sum_{a:i \rightarrow j, m_i \leq m_j} p_{m_i-1}^j \dots p_{m_j-1}^j a_{m_i} a_{m_i-1}^* + \sum_{a:i \rightarrow j, m_i > m_j} p_{m_j} \dots p_{m_i-1} a_{m_j}^* a_{m_j} \end{aligned}$$

Let us denote by  $\bar{Q}_{\mathbf{w}}$  the full subquiver of  $Q_{\mathbf{w}}$  with vertices  $(i, r)$  where  $r \neq m_i$ . And let  $\bar{W}_{\mathbf{w}}$  be the potential

$$\bar{W}_{\mathbf{w}} = \sum_{a:i \rightarrow j} \left( \sum_{r < m_i, r < m_j} p_r^i a_r^* a_r - \sum_{r \leq m_i, r < m_j} p_r^j a_{r+1} a_r^* \right)$$

Then we have the following result:

**Theorem 4.1.** [BIRS09b, Theorem 6.8] *Let  $\mathbf{w} = c^{(0)} \dots c^{(m)}$  be a  $c$ -sortable word. Then the endomorphism algebra  $\text{End}_{\text{Sub}\Lambda_{\mathbf{w}}}(M_{\mathbf{w}})$  of the standard cluster-tilting object  $M_{\mathbf{w}}$  is the frozen Jacobian algebra  $\text{Jac}(Q_{\mathbf{w}}, W_{\mathbf{w}}, F)$  with frozen vertices being  $F_0 = \{(i, m_i), i \in Q_0\}$  and frozen arrows being  $F_1 = \{a \in Q_1, s(a) \in F_0 \text{ and } t(a) \in F_1\}$ .*

*And the endomorphism algebra  $\text{End}_{\text{Sub}\Lambda_{\mathbf{w}}}(M_{\mathbf{w}})$  is the Jacobian algebra  $\text{Jac}(\bar{Q}_{\mathbf{w}}, \bar{W}_{\mathbf{w}})$ .*

**4.2. Generalized cluster categories.** In this subsection we recall some basic facts on the generalized cluster categories associated to algebras of global dimension at most two introduced in [Ami09a].

Let  $A$  be a finite dimensional  $k$ -algebra of global dimension at most two. We denote by  $\mathcal{D}^b(A)$  the bounded derived category of finitely generated  $A$ -modules. It has a Serre functor that we denote by  $\mathbb{S}$ , which coincides with  $\tau[1]$ . We denote by  $\mathbb{S}_2$  the composition  $\mathbb{S}[-2] = \tau[-1]$ .

The generalized cluster category  $\mathcal{C}_A$  of  $A$  has been defined in [Ami09a] as the triangulated hull in the sense of [Kel05] of the orbit category  $\mathcal{D}^b(A)/\mathbb{S}_2$ . There is a triangle functor

$$\pi_A : \mathcal{D}^b(A) \longrightarrow \mathcal{D}^b(A)/\mathbb{S}_2 \xrightarrow{\hookrightarrow} \mathcal{C}_A$$

**Theorem 4.2.** [Ami09a, Theorem 4.10] *Let  $A$  be a finite dimensional algebra of global dimension  $\leq 2$ , and assume that the endomorphism algebra  $\text{End}_{\mathcal{C}_A}(\pi(A))$  is finite dimensional. Then  $\mathcal{C}_A$  is a Hom-finite, 2-CY category and  $\pi(A) \in \mathcal{C}_A$  is a cluster-tilting object.*

The following result obtained from Theorem 6.11 a) of [Kel09] shows that the 2-CY-tilted algebra given by the canonical cluster-tilting object in a generalized cluster category is Jacobian. Recall that a 2-CY-tilted algebra is by definition the endomorphism algebra of a cluster-tilting object in a Hom-finite 2-CY triangulated category.

**Theorem 4.3** (Keller). *Let  $A = kQ/I$  be an algebra of global dimension  $\leq 2$ , such that  $I$  is generated by a finite set of minimal relations  $(r_i)$ . The relation  $r_i$  starts at the vertex  $s(r_i)$  and ends at the vertex  $t(r_i)$ . Then we have an isomorphism of algebras:*

$$\text{End}_{\mathcal{C}_A}(\pi(A)) \simeq \text{Jac}(\tilde{Q}, W)$$

*where the quiver  $\tilde{Q}$  is the quiver  $Q$  with additional arrows  $a_i : t(r_i) \rightarrow s(r_i)$ , and the potential  $W$  is  $\sum_i a_i r_i$ .*

There is the following criterion for constructing triangle functors from the generalized cluster category to some stable category. It can be deduced from the universal property of the generalized cluster category (see [Kel05], subsection 1.3.1 of [Ami08] or appendix [IO09] for more details).

**Proposition 4.4.** *Let  $A$  be an algebra of global dimension  $\leq 2$  such that the algebra  $\text{End}_{\mathcal{C}_A}(\pi(A))$  is finite dimensional. Let  $\mathcal{E}$  be a Frobenius category, stably 2-CY with a cluster tilting object  $M$ . Assume that  $M$  has a structure of left  $A$ -module. Then if there is a morphism in  $\mathcal{D}^b(A^{\text{op}} \otimes \mathcal{E})$*

$$M \longrightarrow \text{RHom}_A(DA, A) \overset{\mathbf{L}}{\otimes}_A M[2]$$



whose cone lies in  $\mathcal{D}^b(A^{op} \otimes \mathcal{P})$ , where  $\mathcal{P}$  is the subcategory of  $\mathcal{E}$  of projective-injectives, then there exists a triangle functor  $F : \mathcal{C}_A \rightarrow \underline{\mathcal{E}}$  such that we get following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}^b(A) & \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A M} & \mathcal{D}^b(\mathcal{E}) \\ \downarrow \pi & & \downarrow \\ \mathcal{C}_A & \xrightarrow{F} & \underline{\mathcal{E}}. \end{array}$$

Here the category  $\mathcal{D}^b(A^{op} \otimes \mathcal{E})$  denotes the bounded derived category of  $A^{op} \otimes \mathcal{E}$  as defined in [Kel94]. Objects are bounded complexes of objects in  $\mathcal{E}$  with a structure of left  $A$ -modules. Note that the endofunctor  $-\overset{\mathbf{L}}{\otimes}_A R\mathrm{Hom}_A(DA, A)[2] \simeq R\mathrm{Hom}_A(DA, -)[2]$  of  $\mathcal{D}^b(A)$  is isomorphic to the functor  $\mathbb{S}_2^{-1}$ . Hence this universal property requires that the image of  $A$  and of  $\mathbb{S}_2^{-1}A$  under the composition

$$\mathcal{D}^b(A) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_A M} \mathcal{D}^b(\mathcal{E}) \longrightarrow \mathcal{D}^b(\mathcal{E})/\mathcal{D}^b(\mathcal{P}) \simeq \underline{\mathcal{E}}$$

are isomorphic. Here the category  $\mathcal{D}^b(\mathcal{P})$  is the thick subcategory of  $\mathcal{D}^b(\mathcal{E})$  generated by  $\mathcal{P}$ . The localization of  $\mathcal{D}^b(\mathcal{E})$  by  $\mathcal{D}^b(\mathcal{P})$  is equivalent to the stable category  $\underline{\mathcal{E}}$  by [KV87].

**4.3. Computing endomorphism algebras.** We define a quiver  $\Gamma_{\mathbf{w}}$  by

- $\Gamma_{\mathbf{w},0} = \{(i, t), i \in Q_0^{(t)}, 0 \leq t \leq m\}$
- for any arrow  $a : i \rightarrow j$  in  $Q$ , we put an arrow  $a^{(t)} : (i, t) \rightarrow (j, t)$  if  $i$  and  $j$  are in  $Q_0^{(t)}$ ;
- for any arrow  $a : i \rightarrow j$  in  $Q$ , we put an arrow  $\bar{a}^{(t)} : (j, t) \rightarrow (i, t+1)$  if  $i$  is in  $Q_0^{(t+1)}$  and  $j$  is in  $Q_0^{(t)}$ .

The quiver  $\Gamma_{\mathbf{w}}$  is a translation quiver in a natural way.

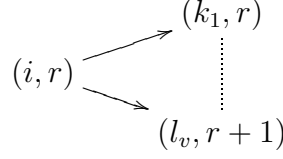
**Proposition 4.5.** *The translation quiver  $\Gamma_{\mathbf{w}}$  is isomorphic to the Auslander-Reiten quiver of  $\mathrm{Sub}T_{\mathbf{w}}$ , where  $T_{\mathbf{w}}$  is the tilting  $kQ$ -module defined in Section 3.*

*Proof.* We prove this by induction on  $l(w)$ , starting with  $c^{(0)}$ . For  $c^{(0)}$  both quivers are clearly the quiver  $Q^{(0)} = Q$ , which is a translation quiver with trivial translation.

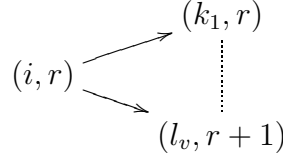
Assume that the claim holds for the subword  $\mathbf{w}'$ , containing  $c^{(r)}$ , but not  $c^{(r+1)}$ , for some  $r < m$ . So  $\Gamma_{\mathbf{w}'}$  is isomorphic to the AR-quiver of  $\mathrm{Sub}T_{\mathbf{w}'} = \mathrm{add}\{T_{\mathbf{w}'}^1, \dots, T_{\mathbf{w}'}^{l(\mathbf{w}')}\} = \mathrm{add}\{T_{\mathbf{w}'}^1, \dots, T_{\mathbf{w}'}^{l(\mathbf{w}')}\}$ . In this proof, for  $i \in Q_0$  and  $r \geq 0$  we write  $T_{(i,r)} = T_{\mathbf{w}}^k$  if  $u_k = i$  and  $\#\{t < k | u_t = i\} = r$ . We omit the index  $\mathbf{w}$  since  $T_{\mathbf{w}}^k = T_{\mathbf{w}'}^k$ . Consider the word  $\mathbf{w}'' = \mathbf{w}'s_i$  where  $i \in c^{(r+1)}$ . Then we have exactly one new mesh in  $\Gamma_{\mathbf{w}''}$ , compared to  $\Gamma_{\mathbf{w}'}$ , namely

$$\begin{array}{ccccc} & & (k_1, r) & & \\ & & \vdots & & \\ & & (k_u, r) & & \\ & \nearrow & & \searrow & \\ (i, r) & \longrightarrow & (l_1, r+1) & \longrightarrow & (i, r+1) \\ & \searrow & \vdots & \nearrow & \\ & & (l_v, r+1) & & \end{array}$$

By the induction assumption



corresponds to a minimal left almost split map in  $\text{Sub } T_{\mathbf{w}'} = \text{add}\{T_{\mathbf{w}'}^1, \dots, T_{\mathbf{w}'}^{l(w')}\}$ . Since all  $k_1, \dots, k_u, l_1, \dots, l_v$  are the last vertices of their type in  $\Gamma_{\mathbf{w}'}$ , the corresponding indecomposable modules are all in  $\text{add}(T_{\mathbf{w}'})$ . Hence



also corresponds to a minimal left  $\text{add}(T_{\mathbf{w}'}/T_{(i,r)})$ -approximation. Hence we have an exact sequence

$$(*) \quad 0 \longrightarrow N_{(i,r)} \xrightarrow{g} \left(\bigoplus_{j=1}^u N_{(k_j,r)}\right) \oplus \left(\bigoplus_{j=1}^v N_{(l_j,r+1)}\right) \longrightarrow N_{(i,r+1)} \longrightarrow 0,$$

where  $N_{(s,t)}$  denotes the indecomposable module associated with the vertex  $s$ . By the induction hypothesis we have  $N_{(s,t)} = T_{(s,t)}$  for  $(s,t) \neq (i,r+1)$ . And by the short exact sequence  $(*)$  we have  $N_{(i,r+1)} = T_{(i,r+1)} \in \text{Sub } T_{\mathbf{w}''}$ , which is a summand of the tilting module  $T_{\mathbf{w}''}$ .

We have the exact sequence

$$\text{Hom}_{kQ}(T_{(i,r)}, \left(\bigoplus_{j=1}^u T_{(k_j,r)}\right) \oplus \left(\bigoplus_{j=1}^v T_{(l_j,r+1)}\right)) \rightarrow \text{Hom}_{kQ}(T_{(i,r)}, T_{(i,r)}) \rightarrow \text{Ext}_{kQ}^1(T_{(i,r)}, T_{(i,r)}),$$

where  $\text{Ext}_{kQ}^1(T_{(i,r)}, T_{(i,r)}) = 0$  since  $T_{(i,r)}$  is a summand of the tilting module  $T_{\mathbf{w}'}$ . Hence  $g$  is a minimal left almost split map also in  $\text{Sub } T_{\mathbf{w}''}$ , and therefore  $(*)$  is an almost split sequence in  $\text{Sub } T_{\mathbf{w}''}$ .

Since there is no nonzero map from  $T_{(i,r+1)}$  to an indecomposable module in  $\text{Sub } T_{\mathbf{w}'}$ , the irreducible maps in  $\text{Sub } T_{\mathbf{w}'}$  stay irreducible in  $\text{Sub } T_{\mathbf{w}''}$ . The irreducible maps to  $T_{(i,r+1)}$  in  $\text{Sub } T_{\mathbf{w}''}$  are given by the above almost split sequence, and correspond to the new arrows in  $\Gamma_{\mathbf{w}''}$  compared to  $\Gamma_{\mathbf{w}'}$ . Hence  $\Gamma_{\mathbf{w}''}$  and the AR quiver of  $\text{Sub } T_{\mathbf{w}''}$  are isomorphic as translation quivers. □

For  $\mathbf{w}$  a  $c$ -sortable word, we define

$$A_{\mathbf{w}} := \text{End}_{kQ}\left(\bigoplus_{j=1}^{l(w)} L_{\mathbf{w}}^j\right).$$

**Corollary 4.6.** *We have an isomorphism of algebras  $A_{\mathbf{w}} \simeq k\Gamma_{\mathbf{w}}/I_{\mathbf{w}}$ , where  $I_{\mathbf{w}}$  is the ideal generated by the mesh relations*

$$\sum_{s(a)=i} \bar{a}^{(t)} a^{(t)} - \sum_{\epsilon(b)=i} b^{(t+1)} \bar{b}^{(t)} = 0 \quad \text{for any } i \in c^{(t+1)}.$$

**Proposition 4.7.** *There is an algebra morphism*

$$A_{\mathbf{w}} \longrightarrow \text{End}_{\text{Sub } \Lambda_w}(M_{\mathbf{w}}).$$

*Proof.* We define an algebra map  $G : k\Gamma_{\mathbf{w}} \rightarrow kQ_{\mathbf{w}}$  by

- $G(i, r) = (i, r)$  for  $i \in Q^{(r)}$ ,
- for  $a : i \rightarrow j$  in  $Q_1$ , if  $r < m_i$  and  $r \leq m_j$ , we define  $G(a^{(r)}) = a_r$ ,
- for  $a : i \rightarrow j$  in  $Q_1$ , if  $r < m_i$  and  $r < m_j$ , we define  $G(\bar{a}^{(r)}) = a_r^*$ ,
- for  $a : i \rightarrow j$  in  $Q_1$ , if  $m_i \leq m_j$  then  $G(a^{(m_i)})$  is defined to be the composition  $G(a^{(m_i)}) = p_{m_i}^j \cdots p_{m_j-1}^j a_{m_i}$
- for  $a : i \rightarrow j$  in  $Q_1$ , if  $m_j < m_i$  then  $G(\bar{a}^{(m_j)})$  is defined to be the composition  $G(\bar{a}^{(m_j)}) = p_{m_j+1}^i \cdots p_{m_i-1}^i a_{m_j}^*$ .

Then one can check that for any  $i \in c^{(t+1)}$ ,

$$G\left(\sum_{s(a)=i} \bar{a}^{(t)} a^{(t)} - \sum_{e(b)=i} b^{(t+1)} \bar{b}^{(t)}\right) = \partial_{p_i} W_{\mathbf{w}}.$$

Since all arrows of  $Q_{\mathbf{w}}$  of type  $p_i^j$  are not in  $F_1$ , the morphism  $G$  yields a morphism of algebras  $A_{\mathbf{w}} \rightarrow \text{Jac}(Q_{\mathbf{w}}, W_{\mathbf{w}}, F)$ . Hence we get the result applying Theorem 4.1.  $\square$

Let  $\mathbf{w}'$  be the subword  $\mathbf{w}' := c^{(1)} \dots c^{(m)}$  of  $w$ . The word  $\mathbf{w}'$  is also  $c$ -sortable.

**Corollary 4.8.** *We have isomorphisms of algebras*

$$A_{\mathbf{w}'} \simeq \text{End}_{kQ}\left(\bigoplus_{j=l(c^{(0)})+1}^{l(w)} L_{\mathbf{w}'}^j\right) \simeq \text{End}_{kQ}\left(\bigoplus_{j=1}^{l(w)} L_{\mathbf{w}'}^j\right) / [\text{add } kQ]$$

and  $A_{\mathbf{w}'}$  is an algebra of global dimension  $\leq 2$ .

*Proof.* Morphisms  $L_{\mathbf{w}'}^j \rightarrow L_{\mathbf{w}'}^k$  where  $j, k \geq l(c^{(0)}) + 1$  do not factor through  $kQ$ . Thus we get immediately that

$$\text{End}_{kQ}\left(\bigoplus_{j=1}^{l(w)} L_{\mathbf{w}'}^j\right) / [\text{add } kQ] \simeq \text{End}_{kQ}\left(\bigoplus_{j=l(c^{(0)})+1}^{l(w)} L_{\mathbf{w}'}^j\right).$$

For the same reason we have an isomorphism  $\text{End}_{kQ}\left(\bigoplus_{j=l(c^{(0)})+1}^{l(w)} L_{\mathbf{w}'}^j\right) \simeq k\Gamma_{\mathbf{w}'} / I_{\mathbf{w}'}$ . We get the first isomorphism applying Corollary 4.6.

The word  $\mathbf{w}' = c^{(1)} c^{(2)} \dots c^{(m)}$  is  $c^{(1)}$ -sortable. Therefore  $A_{\mathbf{w}'}$  is the Auslander algebra of a category which is stable under kernels, hence it is of global dimension at most two.  $\square$

**Proposition 4.9.** *There is an isomorphism of algebras*

$$\text{End}_{\mathcal{C}_{\mathbf{w}'}}(\pi(A_{\mathbf{w}'})) \simeq \text{End}_{\text{Sub } \Lambda_w}(M_{\mathbf{w}}),$$

where  $\mathcal{C}_{\mathbf{w}'}$  is the generalized cluster category associated with  $A_{\mathbf{w}'}$  and  $\pi : \mathcal{D}^b(A_{\mathbf{w}'}) \rightarrow \mathcal{C}_{\mathbf{w}'}$  is the canonical map.

*Proof.* If  $i$  is in  $c^{(t+1)}$  and  $t \geq 1$ , the set  $\sum_{s(a)=i} a^{(t)}\bar{a}^{(t)} - \sum_{e(b)=i} \bar{b}^{(t)}b^{(t+1)}$  forms a set of minimal relations of  $A_{\mathbf{w}'}$  between the vertices  $(i, t)$  and  $(i, t+1)$ . These relations form a basis of minimal relations in  $A_{\mathbf{w}'}$ . Now using Proposition 4.3, we know that the algebra  $\text{End}_{\mathcal{C}_{\mathbf{w}'}}(\pi(A_{\mathbf{w}'}))$  is a Jacobian algebra  $\text{Jac}(\tilde{\Gamma}, W)$ . The quiver  $\tilde{\Gamma}$  is the same quiver as  $\Gamma_{\mathbf{w}'}$  with extra arrows  $q_i^t : (i, t+1) \rightarrow (i, t)$  for  $t \leq 1$ , and the potential  $W$  is

$$W = \sum_{t \geq 1} \sum_{i \in c^{(t+1)}} q_i^t \left( \sum_{s(a)=i} \bar{a}^{(t)}a^{(t)} - \sum_{e(b)=i} b^{(t+1)}\bar{b}^{(t)} \right)$$

Now we define an algebra morphism  $G : k\tilde{\Gamma} \rightarrow k\overline{Q}_{\mathbf{w}}$  by:

- $G(i, r) = (i, r-1)$  for  $i \in c^{(r)}$ ;
- $G(a^{(r)}) = a_{r-1}$  and  $G(\bar{a}^{(r)}) = a_{r-1}^*$ .

It is not hard to check that  $G(W) = \overline{W}_{\mathbf{w}}$ . Thus the Jacobian algebras  $\text{Jac}(\tilde{\Gamma}, W)$  and  $\text{Jac}(\overline{Q}_{\mathbf{w}}, \overline{W}_{\mathbf{w}})$  are isomorphic.  $\square$

**4.4. Triangle equivalence.** The aim of this subsection is to prove the following theorem.

**Theorem 4.10.** *Let  $Q$  be an acyclic quiver. Let  $\mathbf{w} = c^{(0)} \dots c^{(m)}$  be a  $c$ -sortable word with  $c^{(0)} = c$  admissible for the orientation of  $Q$ . Let  $A_{\mathbf{w}'} := \text{End}_{kQ}(\bigoplus_{j=1}^{l(\mathbf{w})} L_{\mathbf{w}'}^j) / [\text{add } kQ]$ , where the  $L_{\mathbf{w}'}^j$  are defined in Section 3. Then there is a triangle equivalence*

$$\mathcal{C}_{\mathbf{w}'} \simeq \underline{\text{Sub}}\Lambda_w,$$

where  $\mathcal{C}_{\mathbf{w}'}$  is the generalized cluster category associated to the algebra  $A_{\mathbf{w}'}$ .

In order to prove this result, we will use the universal property (Proposition 4.4) of the generalized cluster category associated to an algebra of global dimension  $\leq 2$ .

Let  $A_{\mathbf{w}} \rightarrow A_{\mathbf{w}'}$  be the canonical projection sending the vertices  $(i, 0)$  to zero. It yields a restriction functor

$$\mathcal{D}^b(A_{\mathbf{w}'}) \xrightarrow{\text{Res}} \mathcal{D}^b(A_{\mathbf{w}})$$

Let us denote by  $\mathcal{S}$  the subcategory  $\text{Sub}(T_{\mathbf{w}})$  of  $\text{mod } kQ$ , where  $T_{\mathbf{w}}$  is the tilting module defined in Theorem 3.10. The projective (resp. injective) indecomposable  $A_{\mathbf{w}}$ -modules are of the form  $\mathcal{S}(-, X)$  (resp.  $D\mathcal{S}(X, -)$ ), where  $X$  is indecomposable in  $\mathcal{S}$ . The restriction in  $\text{mod } A_{\mathbf{w}}$  of the projective (resp. injective)  $A_{\mathbf{w}'}$ -modules are of the form  $\mathcal{S}(-, X) / [\text{add } (kQ)]$  (resp.  $D\mathcal{S}(X, -) / [\text{add } kQ]$ ) where  $X$  is an indecomposable non projective. The category  $\mathcal{S} = \text{Sub } T_{\mathbf{w}}$  is a category with almost split sequences. In this section, we will denote by  $\tau$  the Auslander-Reiten translation in  $\mathcal{S}$ , (which is not the same as the AR translation in  $\text{mod } kQ$ ). The category  $\mathcal{S}$  is finite and contains all projective modules  $e_i kQ$ . Hence as we already noticed in the proof of Theorem 3.12, for  $X \in \mathcal{S}$ , there exist unique  $p \geq 0$  and  $i \in Q_0$  such that  $X = \tau^{-p}(e_i kQ)$ .

As before we denote by  $M_{\mathbf{w}}$  the standard cluster-tilting object of  $\text{Sub } \Lambda_w$ . Since there is a canonical bijection between the indecomposable objects of  $\mathcal{S} = \text{Sub } T_{\mathbf{w}}$  and the direct summands of  $M_{\mathbf{w}}$ , if  $X = \tau^{-p}(e_i kQ)$  is an indecomposable object of  $\mathcal{S}$ , then we will denote by  $M_X$  the summand  $M_{(i,p)}$  of  $M_{\mathbf{w}}$ .

By Proposition 4.7 we have a morphism of algebras  $A_{\mathbf{w}} \rightarrow \mathbf{End}_{\mathbf{Sub} \Lambda_{\mathbf{w}}}(M_{\mathbf{w}})$ , thus  $M_{\mathbf{w}}$  has a structure of left  $A_{\mathbf{w}}$ -module. Let  $F$  be the following composition

$$F : \mathcal{D}^b(A_{\mathbf{w}'}) \xrightarrow{Res} \mathcal{D}^b(A_{\mathbf{w}}) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_{A_{\mathbf{w}}} M_{\mathbf{w}}} \mathcal{D}^b(\mathbf{Sub} \Lambda_{\mathbf{w}}) \hookrightarrow \mathcal{D}^b(\Lambda)$$

**Lemma 4.11.** *Let  $X$  be an indecomposable object in  $\mathcal{S}$  which is not projective. There exists an exact sequence in  $\mathbf{mod} \Lambda$*

$$0 \longrightarrow M_{H_0} \longrightarrow M_{H_1} \longrightarrow M_X \longrightarrow M_{\tau X} \longrightarrow 0$$

where  $0 \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X \longrightarrow 0$  is the projective resolution of  $X$  as  $kQ$ -module.

*Proof.* The object  $X$  is of the form  $\tau^{-p}(e_i kQ)$  where  $p \geq 1$ . By the previous part, it is  $L_{\mathbf{w}}^j$ , where  $j$  is the  $p^{\text{th}}$  index in the word  $\mathbf{w}$  of type  $i$ . By definition  $L_{\mathbf{w}}^j$  is the kernel of the canonical map  $M_{(i,p)} \rightarrow M_{(i,p-1)}$ . Hence we have a short exact sequence in  $\mathbf{mod} \Lambda$

$$0 \longrightarrow X \longrightarrow M_X \longrightarrow M_{\tau X} \longrightarrow 0.$$

Let  $0 \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X \longrightarrow 0$  be the projective resolution of  $X$  as  $kQ$ -module. Since the  $H_i$ 's are projective  $kQ$ -modules,  $M_{H_i}$  is equal to  $H_i$  for  $i = 0, 1$ . Thus we have a short exact sequence in  $\mathbf{mod} \Lambda$

$$0 \longrightarrow M_{H_0} \longrightarrow M_{H_1} \longrightarrow X \longrightarrow 0.$$

□

**Lemma 4.12.** *Let  $X$  be an indecomposable non projective object in  $\mathcal{S}$ . The objects  $F(\mathcal{S}(-, X)/[\mathbf{add}(kQ)])$  and  $F(D\mathcal{S}(X, -)/[\mathbf{add} kQ])$  of  $\mathcal{D}^b(\mathbf{Sub} \Lambda_{\mathbf{w}})$  are quasi-isomorphic to complexes concentrated in degree 0. Moreover there exists a short exact sequence in  $\mathbf{mod} \Lambda$  functorial in  $X$*

$$(*) \quad 0 \longrightarrow F(\mathcal{S}(-, X)/[\mathbf{add}(kQ)]) \longrightarrow R_0 \longrightarrow R_1 \longrightarrow F(D\mathcal{S}(X, -)/[\mathbf{add}(kQ)]) \longrightarrow 0$$

where  $R_0$  and  $R_1$  are projective-injective objects in  $\mathbf{Sub} \Lambda_{\mathbf{w}}$ .

*Proof.* Let  $0 \longrightarrow H_0 \longrightarrow H_1 \longrightarrow X \longrightarrow 0$  be the projective resolution of  $X$  as  $kQ$ -module. It induces a short exact sequence in  $\mathbf{mod} A_{\mathbf{w}}$

$$0 \longrightarrow \mathcal{S}(-, H_0) \longrightarrow \mathcal{S}(-, H_1) \longrightarrow \mathcal{S}(-, X) \longrightarrow \mathcal{S}(-, X)/[\mathbf{add}(kQ)] \longrightarrow 0$$

Thus the complex  $F(\mathcal{S}(-, X)/[\mathbf{add}(kQ)])$  is by definition

$$\cdots \longrightarrow 0 \longrightarrow M_{H_0} \longrightarrow M_{H_1} \longrightarrow M_X \longrightarrow 0 \longrightarrow \cdots$$

By Lemma 4.11 it is quasi-isomorphic to the stalk complex  $M_{\tau X}$ .

Since  $\tau X$  is not zero and in  $\mathcal{S} = \mathbf{Sub} T_{\mathbf{w}}$ , there exists a short exact sequence

$$0 \longrightarrow \tau X \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

where  $T_i$  is in  $\mathbf{add}(T_{\mathbf{w}})$  for  $i = 0, 1$ . It yields a long exact sequence in  $\mathbf{mod} A_{\mathbf{w}}$ :

$$0 \longrightarrow \mathcal{S}(-, \tau X) \longrightarrow \mathcal{S}(-, T_0) \longrightarrow \mathcal{S}(-, T_1) \longrightarrow \mathrm{Ext}_{kQ}^1(-, \tau X)|_{\mathcal{S}} \longrightarrow \mathrm{Ext}_{kQ}^1(-, T_0)|_{\mathcal{S}} \longrightarrow \cdots$$

The functor  $\text{Ext}_{kQ}^1(-, T_0)$  vanishes on the category  $\mathcal{S}$  by definition. And by the Auslander-Reiten formula we have an isomorphism of functors

$$\text{Ext}_{kQ}^1(-, \tau X) \simeq D\text{Hom}_{kQ}(X, -)/[\text{add}(kQ)]$$

thus we have a short exact sequence in  $\text{mod } A_{\mathbf{w}}$

$$0 \longrightarrow \mathcal{S}(-, \tau X) \longrightarrow \mathcal{S}(-, T_0) \longrightarrow \mathcal{S}(-, T_1) \longrightarrow D\mathcal{S}(X, -)/[\text{add}(kQ)] \longrightarrow 0.$$

Hence by definition the complex  $F(D\mathcal{S}(X, -)/[\text{add}(kQ)])$  is the complex

$$\cdots \longrightarrow 0 \longrightarrow M_{\tau X} \longrightarrow M_{T_0} \longrightarrow M_{T_1} \longrightarrow 0 \longrightarrow \cdots.$$

This is a stalk complex whose homology is in degree zero and isomorphic to  $\Omega^{-2}M_{\tau X}$ , where  $\Omega$  is the syzygy functor.

Since the sequence

$$0 \longrightarrow \tau X \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

is functorial in  $X$  we get a short exact sequence in  $\text{mod } \Lambda$  functorial in  $X$ :

$$(*) \quad 0 \longrightarrow F(\mathcal{S}(-, X)/[\text{add}(kQ)]) \longrightarrow M_{T_0} \longrightarrow M_{T_1} \longrightarrow F(D\mathcal{S}(X, -)/[\text{add}(kQ)]) \longrightarrow 0$$

The objects  $M_{T_i}$  ( $i = 0, 1$ ) are projective-injective since the  $T_i$ 's are in  $\text{add}(T_{\mathbf{w}})$ .  $\square$

**Corollary 4.13.** *There exists a morphism  $F(DA_{\mathbf{w}'}) \rightarrow F(A_{\mathbf{w}'})[2]$  in the category  $\mathcal{D}^b(A_{\mathbf{w}'}^{op} \otimes \Lambda)$ , whose cone is in  $\mathcal{D}^b(A_{\mathbf{w}'}^{op} \otimes \mathcal{P})$ , where  $\mathcal{P}$  is the subcategory of  $\text{Sub } \Lambda_w$  of the projective-injectives.*

*Proof.* In the above lemma, if we take the sum of all  $X$  non projective in  $\mathcal{S}$ , then we get an exact sequence in  $\text{mod } \Lambda$

$$0 \longrightarrow F(A_{\mathbf{w}'}) \longrightarrow M_{T_0} \longrightarrow M_{T_1} \longrightarrow F(DA_{\mathbf{w}'}) \longrightarrow 0.$$

Hence we get a morphism  $f : F(DA_{\mathbf{w}'}) \rightarrow F(A_{\mathbf{w}'})[2]$  in  $\mathcal{D}^b(\Lambda)$  whose cone is quasi-isomorphic to a bounded complex of projective-injective objects of  $\text{Sub } \Lambda_w$ , namely is in  $\mathcal{D}^b(\mathcal{P})$ , where  $\mathcal{P}$  is the subcategory of  $\text{Sub } \Lambda_w$  of the projective-injectives. Since the sequence  $(*)$  is functorial in  $X$  and since  $F(A_{\mathbf{w}'})$  and  $F(DA_{\mathbf{w}'})$  are stalk complexes, the morphism  $f$  can be lifted to a morphism in  $\mathcal{D}^b(A_{\mathbf{w}'}^{op} \otimes \Lambda)$ . Its cone is in  $\mathcal{D}^b(A_{\mathbf{w}'}^{op} \otimes \mathcal{P})$ .  $\square$

We are now able to prove Theorem 4.10. First note that we have

$$F(A_{\mathbf{w}'}) = \bigoplus_{X \in \text{ind}(\mathcal{S}), \text{non projective}} M_{\tau X} = \bigoplus_{Y \in \text{ind}(\mathcal{S}), \text{not in } \text{add}(T_{\mathbf{w}})} M_Y = M_{\mathbf{w}}/P,$$

where  $P$  is the sum of the indecomposable projective-injective objects of  $\text{Sub } \Lambda_w$ . By Proposition 4.4 and Lemma 4.12, the functor  $F : \mathcal{D}^b(A_{\mathbf{w}'}) \rightarrow \mathcal{D}^b(\text{Sub } \Lambda_w)$  induces a triangle functor

$$F : \mathcal{C}_{A_{\mathbf{w}'}} \rightarrow \underline{\text{Sub}} \Lambda_w.$$

It sends the cluster-tilting object  $A_{\mathbf{w}'}$  to the cluster-tilting object  $M_{\mathbf{w}}/P$  in  $\text{Sub } \Lambda_w$ . By Proposition 4.9 we have an isomorphism of algebras

$$\text{End}_{\mathcal{C}_{A_{\mathbf{w}'}}}(\pi(A_{\mathbf{w}'})) \simeq \text{End}_{\underline{\text{Sub}} \Lambda_w}(M_{\mathbf{w}}).$$

Hence by the following proposition, we get the result.

**Proposition 4.14.** [KR08, Lemma 4.5] *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be 2-Calabi-Yau triangulated categories. Let  $T$  (resp.  $T'$ ) be a cluster-tilting object in  $\mathcal{T}$  (resp.  $\mathcal{T}'$ ). If we have a triangle functor  $F : \mathcal{T} \rightarrow \mathcal{T}'$  which sends  $T$  to  $T'$  and which induces an equivalence between  $\text{add}(T)$  and  $\text{add}(T')$ , then  $F$  is an equivalence.*

We also have the dual result which is a general version of a result in [Ami09a]. In [Ami09a] the author proves it for a certain type of co- $c$ -sortable words ( $\mathbf{w}$  is co- $c$ -sortable if  $\mathbf{w}^{-1}$  is  $c^{-1}$ -sortable) which are associated to tilting modules in the preinjective component.

**Theorem 4.15.** *Let  $\mathbf{w} = c^{(r)} \dots c^{(0)}$  be a co- $c$ -sortable word. Let  $\mathbf{w}'$  be the subword  $w' = c^{(r)} \dots c^{(1)}$ . Then the algebra  $A_{\mathbf{w}'} = k\Gamma_{\mathbf{w}'} / I_{\mathbf{w}'}$  is of global dimension at most 2 and we have a triangle equivalence:*

$$\mathcal{C}_{A_{\mathbf{w}'}} \simeq \underline{\text{Sub}}\Lambda_{\mathbf{w}}$$

sending the cluster-tilting object  $\pi(A_{\mathbf{w}'})$  to the cluster-tilting object  $M_{\mathbf{w}} \in \underline{\text{Sub}}\Lambda_{\mathbf{w}}$ .

**4.5. Example.** We take the same example as in Section 3.

Let  $Q$  be the following graph  $\begin{array}{ccc} & 2 & \\ & \swarrow \quad \searrow & \\ 1 & \text{---} & 3 \end{array}$ , and let  $\mathbf{w}$  be the word  $\mathbf{w} = s_1 s_2 s_3 s_1 s_2 s_1$  in

$$1 \text{---} 2 \text{---} 3$$

the Coxeter group  $W_Q$ . The admissible orientation for  $Q$  is the following

$$\begin{array}{ccc} & 2 & \\ & \swarrow \quad \searrow & \\ 1 & \longrightarrow & 3 \end{array}$$

The standard cluster-tilting object  $M_{\mathbf{w}}$  of  $\underline{\text{Sub}}\Lambda_{\mathbf{w}}$  has the following indecomposable direct summands

$$M^1 = 1, \quad M^2 = \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array}, \quad M^3 = \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \quad M^4 = \begin{array}{c} 2 \\ \downarrow \\ 1 \\ \downarrow \\ 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \quad M^5 = \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \\ \downarrow \\ 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \quad M^6 = \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \\ \downarrow \\ 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}.$$

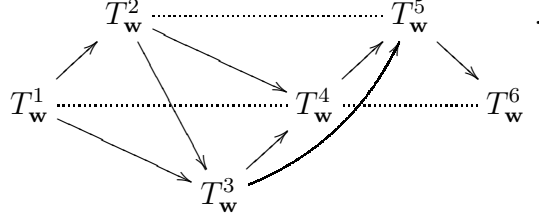
The indecomposable projective-injective objects are  $M^3$ ,  $M^5$  and  $M^6$ . The endomorphism algebra  $\text{End}_{\underline{\text{Sub}}\Lambda_{\mathbf{w}}}(M_{\mathbf{w}})$  has the following quiver

$$\begin{array}{ccccc} & M^2 & \longleftrightarrow & M^5 & \\ & \swarrow & & \searrow & \\ M^1 & \longleftarrow & M^4 & \longrightarrow & M^6 \\ & \searrow & & \swarrow & \\ & M^3 & & & \end{array}$$

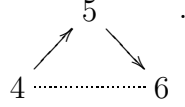
The layers  $L_{\mathbf{w}}^1, \dots, L_{\mathbf{w}}^6$  are the following, as we have seen before.

$$T_{\mathbf{w}}^1 = L_{\mathbf{w}}^1 = 1, \quad T_{\mathbf{w}}^2 = L_{\mathbf{w}}^2 = \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array}, \quad T_{\mathbf{w}}^3 = L_{\mathbf{w}}^3 = \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \\ T_{\mathbf{w}}^4 = L_{\mathbf{w}}^4 = \begin{array}{c} 2 \\ \downarrow \\ 1 \\ \downarrow \\ 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \quad T_{\mathbf{w}}^5 = L_{\mathbf{w}}^5 = \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \\ \downarrow \\ 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}, \quad T_{\mathbf{w}}^6 = L_{\mathbf{w}}^6 = \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array}.$$

As we already saw in section 3, the object  $T_{\mathbf{w}} = T_{\mathbf{w}}^3 \oplus T_{\mathbf{w}}^5 \oplus T_{\mathbf{w}}^6$  is a tilting  $kQ$ -module. The Auslander-Reiten quiver of the category  $\text{Sub}(T_{\mathbf{w}})$  is



which is the quiver of the algebra  $A_{\mathbf{w}}$ . The algebra  $A_{\mathbf{w}'}$  is the endomorphism algebra  $\text{End}_{kQ}(T_{\mathbf{w}}^4 \oplus T_{\mathbf{w}}^5 \oplus T_{\mathbf{w}}^6)$ . It has the following quiver



The projective  $A_{\mathbf{w}}$ -modules are

$$1, \quad \begin{matrix} 2 \\ 1 \end{matrix}, \quad \begin{matrix} 1 & 3 \\ & 2 \\ & 1 \end{matrix}, \quad \begin{matrix} 2 & 4 \\ 1 & 3 \\ & 2 \\ & 1 \end{matrix}, \quad \begin{matrix} 1 & 3 & 5 \\ & 2 & 4 \\ & 1 & 3 \\ & & 2 \\ & & 1 \end{matrix}, \quad \begin{matrix} 6 \\ 5 \\ 3 \\ 1 \end{matrix}.$$

Now, we will check that the images of  $e_5 A_{\mathbf{w}'}$  and  $e_5 D A_{\mathbf{w}'}[-2]$  through the functor

$$F : \mathcal{D}^b(A_{\mathbf{w}'}) \xrightarrow{\text{Res}} \mathcal{D}^b(A_{\mathbf{w}}) \xrightarrow{-\otimes_{A_{\mathbf{w}}}^{\mathbf{L}} M_{\mathbf{w}}} \mathcal{D}^b(\text{Sub } \Lambda_w) \longrightarrow \underline{\text{Sub}} \Lambda_w$$

are isomorphic.

Let  $X$  be the non projective module  $T_{\mathbf{w}}^5$ . The projective  $A_{\mathbf{w}'}$ -module  $e_5 A_{\mathbf{w}'} = \begin{matrix} 5 \\ 4 \end{matrix}$  viewed in  $\mathcal{D}^b(A_{\mathbf{w}})$  is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow 1 \longrightarrow \begin{matrix} 1 & 3 \\ & 2 \\ & 1 \end{matrix} \oplus \begin{matrix} 1 & 3 \\ & 2 \\ & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 3 & 5 \\ & 2 & 4 \\ & 1 & 3 \\ & & 2 \\ & & 1 \end{matrix} \longrightarrow 0 \longrightarrow \cdots$$

Hence its image through the functor

$$F : \mathcal{D}^b(A_{\mathbf{w}'}) \xrightarrow{\text{Res}} \mathcal{D}^b(A_{\mathbf{w}}) \xrightarrow{-\otimes_{A_{\mathbf{w}}}^{\mathbf{L}} M_{\mathbf{w}}} \mathcal{D}^b(\text{Sub } \Lambda_w) \hookrightarrow \mathcal{D}^b(\text{f.l. } \Lambda)$$

is the complex

$$\cdots \longrightarrow 0 \longrightarrow M^1 \longrightarrow M^3 \oplus M^3 \longrightarrow M^5 \longrightarrow 0 \longrightarrow \cdots$$

which is quasi-isomorphic to  $M^2 = M_{\tau X}$ . Note that the projective resolution of  $X$  in  $\text{mod } kQ$  is

$$0 \longrightarrow T_{\mathbf{w}}^1 \longrightarrow T_{\mathbf{w}}^3 \oplus T_{\mathbf{w}}^3 \longrightarrow X \longrightarrow 0$$

The injective  $A_{\mathbf{w}'}$ -module  $e_5 D A_{\mathbf{w}'} = \begin{matrix} 6 \\ 5 \\ 3 \\ 1 \end{matrix}$  viewed in  $\mathcal{D}^b(A_{\mathbf{w}})$  is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow \begin{matrix} 2 \\ 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 3 \\ & 2 \\ & 1 \end{matrix} \longrightarrow \begin{matrix} 6 \\ 5 \\ 3 \\ 1 \end{matrix} \longrightarrow 0 \longrightarrow \cdots$$



Hence its image through the functor

$$F : \mathcal{D}^b(A_{\mathbf{w}'}) \xrightarrow{Res} \mathcal{D}^b(A_{\mathbf{w}}) \xrightarrow{-\overset{\mathbf{L}}{\otimes}_{A_{\mathbf{w}}} M_{\mathbf{w}}} \mathcal{D}^b(\mathbf{Sub} \Lambda_w) \hookrightarrow \mathcal{D}^b(\text{f.l.} \Lambda)$$

is the complex

$$\cdots \longrightarrow 0 \longrightarrow M^2 \longrightarrow M^3 \longrightarrow M^6 \longrightarrow 0 \longrightarrow \cdots.$$

Since  $M^5$  and  $M^6$  are projective injective,  $F(e_5 D A_{\mathbf{w}'})$  is isomorphic to  $\Omega^2 M^2$  in  $\mathbf{Sub} \Lambda_w$ .

Note that we have an exact sequence in  $\mathbf{Sub}(T_{\mathbf{w}})$

$$0 \longrightarrow T_{\mathbf{w}}^2 \longrightarrow T_{\mathbf{w}}^3 \longrightarrow T_{\mathbf{w}}^6 \longrightarrow 0$$

Therefore we have an isomorphism in  $\mathbf{Sub} \Lambda_w$ :

$$F(e_5 D A_{\mathbf{w}'}) \simeq \Omega^2 F(e_5 A_{\mathbf{w}'}).$$

## 5. PROBLEMS AND EXAMPLES

In this section we discuss some possible generalizations of the description of the layers in terms of tilting modules, beyond the  $c$ -sortable case. We pose some problems and give some examples to illustrate limitations for what might be true.

Recall from Section 2 that to a reduced expression  $\mathbf{w}$  of an element  $w$  in  $W_Q$  we have associated a set  $\{L_{\mathbf{w}}^j\}$  of  $l(w)$  indecomposable rigid  $\Lambda$ -modules which we call layers, and which are indecomposable rigid  $kQ$ -modules when  $\mathbf{w}$  is  $c$ -sortable, where  $c$  is admissible with respect to the orientation of  $Q$ . Under the same assumption (*i.e.*  $\mathbf{w}$  is  $c$ -sortable), we constructed a set  $\{T_{\mathbf{w}}^j\}$  of  $l(w)$  indecomposable  $kQ$ -modules via minimal left approximations, starting with the tilting module  $kQ$ , and ending up with a tilting module  $T_{\mathbf{w}}$ . All minimal left approximations were monomorphisms. We showed that the two sets of indecomposable modules coincide. In particular, the module  $L_{\mathbf{w}} := L_{\mathbf{w}}^{t_{\mathbf{w}}(1)} \oplus \cdots \oplus L_{\mathbf{w}}^{t_{\mathbf{w}}(n)}$ , where for  $i \in Q_0 = \{1, \dots, n\}$  the integer  $t_{\mathbf{w}}(i)$  is the position of the last reflection  $s_i$  in the word  $\mathbf{w}$ , is a tilting module over  $kQ$ .

We now consider the case of words  $\mathbf{w}$  with the assumption that  $\mathbf{w} = c\mathbf{w}'$ , where  $c$  is a Coxeter element admissible with respect to the orientation of  $Q$ . When  $\mathbf{w} = cs_{u_{n+1}} \cdots s_{u_1}$  is a word, we define  $T_{\mathbf{w}}$  to be a tilting module associated with  $\mathbf{w}$  if it is possible to carry out the following. Start with  $kQ = P_1 \oplus \cdots \oplus P_n$ , where  $P_i$  is the indecomposable projective  $kQ$ -module associated with the vertex  $i$ . If possible, exchange  $P_{u_{n+1}}$  with a non isomorphic indecomposable  $kQ$ -module to get a tilting module  $T' = kQ/P_{u_{n+1}} \oplus P_{u_{n+1}}^*$ , then replace summand number  $i_2$  in  $T'$  by a non isomorphic indecomposable  $kQ$ -module to get a new tilting module  $T''$ , etc. If an exchange is possible at each step, we obtain a tilting module  $T_{\mathbf{w}}$ . We say that a word  $\mathbf{w} = c\mathbf{w}'$  is *tilting* if  $T_{\mathbf{w}}$  exists, and  $\mathbf{w}$  is *monotilting* if moreover  $T_{\mathbf{w}}$  is obtained by only using left approximations. Hence  $c$ -sortable words are examples of monotilting words.

It is natural to ask the following question about tilting and monotilting words.

### Problem 1:

- (a) Characterize the tilting words  $\mathbf{w}$ . In particular is every reduced word  $\mathbf{w} = c\mathbf{w}'$  tilting?
- (b) Characterize the monotilting words.

- (c) When do two tilting words  $\mathbf{w}_1$  and  $\mathbf{w}_2$  give rise to the same tilting module? Or formulated differently, for which tilting words  $\mathbf{w}$  do we have  $T_{\mathbf{w}} \simeq kQ$ ?

Note that all these questions can also be translated into combinatorial problems for acyclic cluster algebras.

Note that non reduced words may be monotilting as the following example shows.

*Example 5.1.* Let  $Q$  be the quiver  $\begin{array}{ccc} & 2 & \\ 1 & \nearrow & 4 \\ & 3 & \\ & \searrow & \end{array}$  and  $\mathbf{w} := s_1 s_2 s_3 s_4 s_3 s_1 s_4$ . Then  $\mathbf{w}$  is not reduced, but monotilting with  $T_{\mathbf{w}} = \begin{smallmatrix} 4 & & & \\ 2 & 3 & & \\ 1 & & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} \oplus 2$ .

Recall that in the  $c$ -sortable case, then  $\mathbf{w}$  is monotilting and  $\text{Sub} T_{\mathbf{w}}$  is of finite type. This not the case in general.

*Example 5.2.* Let  $Q$  be the quiver  $\begin{array}{ccc} & 2 & \\ 1 & \nearrow & 4 \\ & 3 & \\ & \searrow & \end{array}$  and  $\mathbf{w} := s_1 s_2 s_3 s_4 s_2 s_3 s_4 s_1$ . Then one can show that  $\mathbf{w}$  is monotilting and that  $T_{\mathbf{w}} = \begin{smallmatrix} 4 & & & \\ 3 & 2 & 4 & \\ 1 & & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} \oplus 4$ . Then one can check easily that all the modules of the form  $\begin{smallmatrix} 4 & & & \\ 2 & 3 & & \\ 1 & & 1 & \end{smallmatrix}, \begin{smallmatrix} 4 & & & \\ 2 & 3 & 2 & \\ 1 & & 1 & \end{smallmatrix}, \begin{smallmatrix} 4 & & & \\ 2 & 3 & 1 & \\ 1 & & 1 & \end{smallmatrix}, \begin{smallmatrix} 4 & & & \\ 2 & 3 & 1 & \\ 1 & & 1 & \end{smallmatrix}, \dots$  are in  $\text{Sub}(\begin{smallmatrix} 4 & & & \\ 3 & 2 & & \\ 1 & & 1 & \end{smallmatrix})$ .

However, it may happen that  $\text{Sub} T_{\mathbf{w}}$  is of finite type for a tilting word  $\mathbf{w}$  which is not  $c$ -sortable. It follows from Theorem 3.12 that there exists a unique  $c$ -sortable word  $\tilde{\mathbf{w}}$  such that  $T_{\mathbf{w}} = T_{\tilde{\mathbf{w}}}$ . We then pose the following.

**Problem 2:**

- (a) Characterize the tilting words  $\mathbf{w}$  with  $\text{Sub} T_{\mathbf{w}}$  finite.
- (b) For such words  $\mathbf{w}$ , how can we construct the unique  $\tilde{\mathbf{w}}$  such that  $T_{\mathbf{w}} = T_{\tilde{\mathbf{w}}}$ ?

When  $\mathbf{w}$  is monotilting, we have

$$\{T_{\mathbf{w}}^1, \dots, T_{\mathbf{w}}^{l(\mathbf{w})}\} \subseteq \text{Sub} T_{\mathbf{w}} = \text{Sub} T_{\tilde{\mathbf{w}}} = \text{add} \{T_{\tilde{\mathbf{w}}}^1, \dots, T_{\tilde{\mathbf{w}}}^{l(\tilde{\mathbf{w}})}\}.$$

Hence  $l(\mathbf{w}) \leq l(\tilde{\mathbf{w}})$  and we expect that  $\tilde{\mathbf{w}}$  is obtained by enlarging some rearrangement of  $\mathbf{w}$ .

*Example 5.3.* Let  $Q$  be the quiver  $\begin{array}{ccc} & 2 & \\ 1 & \nearrow & 4 \\ & 3 & \\ & \searrow & \end{array}$  and  $\mathbf{w} := s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_4$ . Then  $\mathbf{w}$  is monotilting and we have

$$T_{\mathbf{w}} = \begin{smallmatrix} 4 & & & \\ 2 & 3 & 2 & \\ 1 & & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 & & & \\ 2 & 3 & 2 & \\ 1 & & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 & & & \\ 2 & 3 & 2 & \\ 1 & & 1 & \end{smallmatrix}.$$

Then  $\mathbf{w}$  is not  $c$ -sortable,  $\text{Sub} T_{\mathbf{w}}$  is finite and one can check that  $\tilde{\mathbf{w}} = s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 s_2 s_3$ .

When  $\mathbf{w}$  is  $c$ -sortable,  $\mathbf{w}$  is a monotilting word and  $T_{\mathbf{w}}$  coincide with  $L_{\mathbf{w}}$  given by the layers. In general  $L_{\mathbf{w}}$  is not a  $kQ$ -module, but as we have seen there is an indecomposable  $kQ$ -module associated with each indecomposable summand of  $L_{\mathbf{w}}$ , and hence a  $kQ$ -module  $(L_{\mathbf{w}})_Q$  associated with  $L_{\mathbf{w}}$ . In this connection we have the following questions:

**Problem 3:**

- (1) For which  $\mathbf{w}$  does the following hold
- (a) each indecomposable summand of  $(L_{\mathbf{w}})_Q$  is rigid,
  - (b)  $(L_{\mathbf{w}})_Q$  is a tilting module,
  - (c)  $\mathbf{w}$  is tilting and  $T_{\mathbf{w}} = (L_{\mathbf{w}})_Q$ ,
  - (d)  $\mathbf{w}$  is monotilting and  $T_{\mathbf{w}} = (L_{\mathbf{w}})_Q$ .
- (2) If  $\mathbf{w}$  is monotilting and  $(L_{\mathbf{w}})_Q$  is rigid, do we have  $T_{\mathbf{w}} = (L_{\mathbf{w}})_Q$ ?

As we already saw in Example 2.1, it can happen that (a) fails. In this example, one can check that  $\mathbf{w}$  is monotilting.

*Example 5.4.* Let  $Q$  be the quiver  $\begin{array}{ccc} & 2 & \\ 1 & \rightleftarrows & 3 \end{array}$ , and  $\mathbf{w} := s_1 s_2 s_3 s_2 s_1 s_2$ . The word  $\mathbf{w}' = s_1 s_2 s_3 s_2 s_1$  is monotilting and we have  $T_{\mathbf{w}'} = \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix}$ . To exchange  $\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$  we have to use the minimal right approximation  $g : \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \longrightarrow \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$ . Hence  $\mathbf{w}$  is a tilting word which is not monotilting and we get  $T_{\mathbf{w}} = \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & \\ 1 & 3 & \\ & 1 & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix}$ . The cluster-tilting object  $M_{\mathbf{w}}$  of  $\mathbf{Sub} \Lambda_{\mathbf{w}}$  associated with  $\mathbf{w}$  has the indecomposable summands:

$$M_{\mathbf{w}} := 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 2 & \\ & 1 & 3 \end{smallmatrix}$$

We then see that  $T_{\mathbf{w}} = (L_{\mathbf{w}})_Q$ , even though  $\mathbf{w}$  is not a monotilting word.

*Example 5.5.* Let  $Q$  and  $\mathbf{w}$  be as in Example 5.3. Then we have

$$M_{\mathbf{w}} = 1 \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & \\ 1 & 4 & \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & \\ 2 & 4 & \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & \\ 3 & 4 & \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 4 & \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & \\ 1 & 4 & \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & \\ 1 & 4 & \\ & 3 & \end{smallmatrix}.$$

Therefore we obtain  $(L_{\mathbf{w}})_Q = \begin{smallmatrix} 3 & & \\ 1 & 4 & \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 3 & & \\ 1 & 4 & \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 1 \end{smallmatrix} \oplus 4$ . Each indecomposable summand is rigid, but  $(L_{\mathbf{w}})_Q$  is not a tilting module. Therefore we can have (a) without (b).

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT BONN  
*E-mail address:* amiot@math.uni-bonn.de

NAGOYA UNIVERSITY  
*E-mail address:* iyama@math.nagoya-u.ac.jp

INSITUTT FOR MATEMATISKE FAG, NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET, N-7491 TRONDHEIM, NORWAY  
*E-mail address:* idunr@math.ntnu.no

DEPARTEMENT OF MATHEMATICS, NORTHEASTERN UNIVERSITY, BOSTON, MA 02115, USA  
*E-mail address:* g.todorov@neu.edu