Exponential stability and stabilization of sampled-data systems with time-varying period

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Abstract: This article proposes a novel approach to assess the exponential stability of linear systems with sampled-data inputs. The paper considers both uncertainties in the model parameters and in the sampling period. Inspired by the input-delay approach and the stability of impulsive systems, the proposed method provides easy tractable stability conditions. Sufficient stability and stabilization conditions are provided to deal with both cases of constant and time-varying sampling periods. The period-dependent conditions are expressed using computable linear matrix inequalities. Several examples show the efficiency and the limitation of such stability criteria.

Keywords: Time-delay systems, time-varying delay, Lyapunov-Krasovskii functionals, linear systems.

1. INTRODUCTION

In the last decades, a large attention has been taken to Networked Control Systems (NCS) (see Hespanha et al. (2007), or Zampieri (2008)). Such systems are controlled systems containing several distributed plants which are connected through a communication network. In such applications, a heavy temporary load of computation in a processor can corrupt the sampling period of a certain controller. The variations of the sampling period will affect the stability properties. In this article, our objective is to guarantee the exponential stability in the case of time varying sampling period.

Sampled-data systems have already been studied in the literature Chen and Francis (1995); Zhang and Branicky (2001); Zhang et al. (2001) and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However the case of asynchronous samplings still leads to several open problems such as the guarantee of stability whatever the sampling period lying in an interval. Recently, several articles drive the problem of time-varying periods based on a discrete-time approach, Yue et al. (2008); Fujioka (2009). Note that discrete-time approaches do not fit with the case of uncertain systems or systems with time-varying parameters. Recent papers considered continuous-time modelling of systems with sampled-data control. The idea is to represent the discrete-time control law as a continuous but delayed input. This was proposed in Fridman et al. (2004) where the stability conditions are derived on a Lyapunov-Krasovskii approach. Improvements are provided in Mirkin (2007), using the small gain theorem and in Naghshtabrizi et al. (2008) based on an impulsive systems analysis. These approaches are very relevant because it cope with the problem of time-varying sampling periods and also with uncertain systems (see Fridman et al. (2004) and Naghshtabrizi et al. (2008)). Nevertheless, these sufficient conditions are still conservative for time-varying period. This means that the sufficient conditions obtained by continuous time approaches are not able to guarantee asymptotic stability whereas the systems is stable. Note that recent improvements are obtained in Liu and Fridman (2009) and in Seuret (2009).

The article proposes a novel approach to obtain sufficient conditions to exponential stability of linear time-varying systems. Those conditions are based on the continuous-time approach and the stability of impulsive systems. The proposed theorems provide larger upper-bounds of the allowable sampling period than the existing ones (based on the continuous time approach).

This article is organized as follows. The next section formulates the problem. Section 3 deals with the analysis of exponential stability. Section 4 concerns the stabilization of linear systems under sampled inputs. Several examples and simulations are provided in Section 5 and show the efficiency of the method.

Notations. *Throughout the article, for a n-dimensional state vector x and a non-negative delay τ, W represent the set of functions x_t such that $x_t(\theta) = x(t-\theta)$ for all $\theta \in [-\tau,\ 0]$. For any x_t, the notation $|x_t(0)|$ represents the classical Euclidean norm of the vector $x_t(0) = x(t)$ and we define the norm $\|x_t\| = \sup_{\theta\in[-\tau,\ 0]} |x_t(\theta)|$. The superscript '$T$' stands for the matrix transposition. The notation $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is a symmetric positive definite matrix. The symbols I and 0 represent the identity and the zero matrices of appropriate dimension.*

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2. PROBLEM FORMULATION

Consider the linear system with a sampled-data input:

$$\dot{x}(t) = Ax(t) + Bu(t_k) \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state variable and the input vector. The matrices A and B are constant and of appropriate dimension. We are looking for a piecewise-constant control law of the form $u(t) = u_d(t_k),\ t_k \leq t < t_{k+1}$, where u_d is a discrete-time control signal and $0 = t_0 < t_1 < ... < t_k < ...$ are the sampling instants. Our objective is to ensure the stability of the system together with a state-feedback controller of the form

$$u(t) = Kx(t_k),\ t_k \leq t < t_{k+1}, \tag{2}$$

where the gain K in $\mathbb{R}^{n\times m}$ is given. Assume that there exists a positive scalar T such that the sampling interval $t_{k+1} - t_k = T_k$ satisfies

$$\forall k \geq 0, \quad 0 < T_k \leq T. \tag{3}$$

Several authors investigated in guaranteing the stability of such a system. Substituting (2) into (1), we obtain the following closed-loop system:

$$\begin{array}{l} \dot{x}(t) = Ax(t) + A_d x(t - \tau(t)), \\ \tau(t) = t - t_k,\ t_k \leq t < t_{k+1}. \end{array} \tag{4}$$

where $A_d = BK$. From (3), it follows that $\tau(t) \leq T$ since $\tau(t) \leq t_{k+1} - t_k$. For the sake of simplicity, the notation τ stands for the time-varying sampling delay $\tau(t)$. We will further consider (4) as a linear system with uncertain and bounded delay.

3. EXPONENTIAL STABILITY OF SYSTEMS WITH SAMPLED INPUTS

In this section, a study of the convergence rate of the solutions of sampled-data systems is provided. The objective is to ensure that the LKF is decreasing exponentially fast with a known decay rate $2\alpha > 0$ during a sampling period. Instead of ensuring $\dot{V} < 0$, we use the following lemma:

Lemma 1. Liu and Fridman (2009) Assume there exist positive numbers α, β and δ and a function $V : \mathbb{R}\times W \to \mathbb{R}$ such that the function $V(t, x_t)$ is continuous with respect to the time argument t and for $t \neq t_k$ and x_t from (4) and V satisfies

$$\beta|x_t(0)|^2 \leq V(t, x_t) \leq \delta\|x_t\|^2, \tag{5}$$

$$\dot{V} + 2\alpha V < 0, \tag{6}$$

$$\lim_{t\to t_k^-} V(t, x_t) \geq V(t_k, x_{t_k}), \tag{7}$$

Then (4) is exponentially stable with the decay rate α.

If the conditions from Lemma 1 are satisfied, the LKF is decreasing faster than $e^{-2\alpha t}$. This expression still makes sense if α is negative. In this situation, it means that the LKF is not increasing faster than $e^{2|\alpha|t}$. In the following, several theorems provide sufficient conditions to ensure (6) for the cases with constant and time-varying periods and systems with bounded time-varying uncertainties. Our objective here is to design a new type of Lyapunov-Krasovskii functionals dedicated to exponential stability with a guaranteed exponential decay rate.

3.1 Constant sampling periods

Theorem 1. For a given $\alpha \in \mathbb{R}$, assume that there exist symmetric positive definite matrices P, P_0, R and $S_1 \in \mathbb{R}^{n\times n}$ and two matrices $S_2 \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{2n\times n}$ that satisfy

$$\Pi_1 + f_\alpha(T, 0)\Pi_2 < 0, \tag{8}$$

$$\begin{bmatrix} \Pi_1 & g_\alpha(T,T)N \\ * & -g_\alpha(T,T)R \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{array}{rl} \Pi_1 = & M_1^T P M_0 + M_0^T P M_1 - M_3^T S_1 M_3 \\ & -M_2^T S_2 M_3 - M_3^T S_2^T M_2 \\ & -N M_3 - M_3^T N^T + 2\alpha M_1^T P M_1, \\ \Pi_2 = & M_3^T S_1 M_0 + M_0^T S_1 M_3 + M_0^T R M_0 \\ & +M_2^T S_2 M_0 + M_0^T S_2^T M_2, \end{array}$$

and $M_0 = [\, A\ A_d \,]$, $M_1 = [\, I\ 0 \,]$, $M_2 = [\, 0\ I \,]$, $M_3 = [\, I\ {-I} \,]$. The functions f_α and g_α are

$$\begin{array}{ll} \text{if } \alpha \neq 0, & f_\alpha(T,\tau) = (e^{2\alpha(T-\tau)} - 1)/2\alpha, \\ \text{if } \alpha = 0, & f_0(T,\tau) = T - \tau, \\ \text{if } \alpha > 0, & g_\alpha(T,\tau) = e^{2\alpha T}(1 - e^{-2\alpha\tau})/2\alpha, \\ \text{if } \alpha < 0, & g_\alpha(T,\tau) = (1 - e^{-2\alpha\tau})/2\alpha, \\ \text{if } \alpha = 0, & g_0(T,\tau) = \tau. \end{array} \tag{10}$$

System (4) is thus exponentially stable with an exponential decay rate α for a constant period T.

Proof. Consider $\alpha \in \mathbb{R}$ and the novel type of functional

$$\begin{array}{rl} V_\alpha(t, x_t) = & x^T(t) P x(t) \\ & +f_\alpha(T,\tau)\zeta_0^T(t)[S_1\zeta_0(t) + 2S_2 x(t_k)] \\ & +f_\alpha(T,\tau) \displaystyle\int_{t_k}^{t} \xi^T(s) M_0^T R M_0 \xi(s) ds, \end{array} \tag{11}$$

where $\zeta_0(t) = x(t) - x(t_k)$, $\xi(s) = [x^T(s)\ x^T(t_k)]^T$. The objective is here to ensure that $\Delta V_\alpha = V_\alpha(t_k, x_{t_k}) - V_\alpha(t_{k-1}, x_{t_{k-1}})$ is negative definite as in the Lyapunov approach for discrete-time systems. The functional V_α is composed by a quadratic term and two others depending on the delay function τ. At each sampling instant t_k, V_α is equal to the quadratic term $x^T(t_k)Px(t_k)$. The other terms in V_α are introduced to take into account the behavior of the system between two successive sampling instants. Consider a positive scalar $0 < \epsilon < T$ and the functional V_α at time $t_k - \epsilon$ and $t_k + \epsilon$. Since $\zeta_0(t_k + \epsilon)$ and $f_\alpha(T, T - \epsilon)$ tend to 0 as $\epsilon \to 0$ for all $\alpha \in \mathbb{R}$, the following equalities are satisfied

$$\begin{array}{l} \lim\limits_{\epsilon\to 0} V_\alpha(t_k - \epsilon, x_{t_k-\epsilon}) = x^T(t_k) P x(t_k), \\ \lim\limits_{\epsilon\to 0} V_\alpha(t_k + \epsilon, x_{t_k+\epsilon}) = x^T(t_k) P x(t_k). \end{array} \tag{12}$$

The LKF V_α is thus continuous with respect to t at all sampling instants. However as no additional conditions are introduced on S_2, V_α is not necessary positive definite within two sampling instants. This is the reason why the functional can only be considered to deal with the particular the case of constant sampling period.

The rest of the proof consists in ensuring that V_α is decreasing exponentially within each period. To do so, we consider $W_\alpha(t, x_t) = \dot{V}_\alpha(t, x_t) + 2\alpha V_\alpha(t, x_t)$ as suggested in Lemma 1. From (10), we have, for all $\alpha \in \mathbb{R}$ and for all $\tau \in [0, T]$, $\dot{f}_\alpha(T,\tau) + 2\alpha f_\alpha(T,\tau) = -1$. This leads to

$$
\begin{aligned}
W_\alpha(t,x_t) = & 2x^T(t)P\dot{x}(t) + 2\alpha x^T(t)Px(t) \\
& +2f_\alpha(T,\tau)\dot{x}(t)^T[S_1\zeta_0(t) + S_2x(t_k)] \\
& +f_\alpha(T,\tau)\xi^T(t)M_0^TRM_0\xi(t) \\
& -\zeta_0^T(t)[S_1\zeta_0(t) + 2S_2x(t_k)] \\
& -\int_{t_k}^t \xi^T(s)M_0^TRM_0\xi(s)ds.
\end{aligned} \tag{13}
$$

Consider a matrix $N \in \mathbb{R}^{2n\times n}$ and the following equality

$$
2N\left[x(t) - x(t_k)\right] = \int_{t_k}^t \left[2N\dot{x}(s)\right]ds = \int_{t_k}^t \left[2NM_0\xi(s)\right]ds. \tag{14}
$$

Since R is positive definite, a classical bounding ensures that for all $t \in [t_k,\ t_{k+1}[$ and for all $s \in [t_k,\ t]$:

$$
2\xi^T(t)NM_0\xi(s) \le \xi^T(t)NR^{-1}N^T\xi(t) + \xi^T(s)M_0^TRM_0\xi(s).
$$

Integrating the previous inequality over $[t_k,\ t]$, the following inequality is obtained

$$
\begin{aligned}
-\int_{t_k}^t \xi^T(s)M_0^TRM_0\xi(s)ds \le & -2\xi^T(t)NM_3\xi(t) \\
& +\tau\xi^T(t)NR^{-1}N^T\xi(t),
\end{aligned} \tag{15}
$$

Noting that

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + A_dx(t_k) = M_0\xi(t), \quad x(t) = M_1\xi(t), \\
&x(t_k) = M_2\xi(t), \quad \zeta_0(t) = x(t) - x(t_k) = M_3\xi(t),
\end{aligned}
$$

and adding (15) to (13), the following inequality is obtained for all $t \in [t_k,\ t_{k+1}[$:

$$
W_\alpha(t,x_t) \le \xi^T(t)[\Pi_1 + f_\alpha(T,\tau)\Pi_2 + \tau NR^{-1}N^T]\xi(t)
$$

The right hand term does not depend linearly on τ but on both τ and a non linear function of τ, $f_\alpha(T,\tau) = \frac{e^{2\alpha(T-\tau)}-1}{2\alpha}$. A first possibility to obtain sufficient conditions for exponential stability would be to consider that these two terms are independent. Then sufficient conditions can be derived by taking the right hand side term at each vertices $\tau = 0, T$ and $f_\alpha(T,\tau) = f_\alpha(T,0), f_\alpha(T,T)$. However this adds some conservatism to the conditions since the cases $(0, f_\alpha(T,T))$ and $(T, f_\alpha(T,0))$ never happen. The solution proposed here is to use the properties of the exponential function to avoid this situation. The convexity of the exponential function ensures that $e^{2\alpha\tau} \ge 1 + 2\alpha\tau$ and $e^{-2\alpha\tau} \ge 1 - 2\alpha\tau$. Consequently, the following upper-bounds are obtained

$$
\begin{aligned}
&\text{if } \alpha > 0, \ \ \tau \le (e^{2\alpha\tau} - 1)/2\alpha \le g_\alpha(T,\tau), \\
&\text{if } \alpha < 0, \ \ \tau \le g_\alpha(T,\tau), \\
&\text{if } \alpha = 0, \ \ \tau = g_0(T,\tau).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
W_\alpha(t,x_t) \le & \xi^T(t)[\Pi_1 + f_\alpha(T,\tau)\Pi_2 \\
& +g_\alpha(T,\tau)NR^{-1}N^T]\xi(t).
\end{aligned}
$$

Then this new upper-bound depends linearly only on the time-varying gain $e^{-2\alpha\tau}$. It is now sufficient to ensure that the right-hand side of the previous equation is negative definite for $e^{-2\alpha\tau} = 1$ and for $e^{-2\alpha\tau} = e^{-2\alpha T}$. This leads to conditions (8) and (9). The last step of the proof consists in integrating the differential inequality over each sampling interval. This leads to

$$
\forall k > 0, \quad V(t_k, x_{t_k}) < V(t_{k-1}, x_{t_{k-1}})e^{-2\alpha(t_k - t_{k-1})}. \tag{16}
$$

Consequently, we have $V(t_k, x_{t_k}) < V(0, x_0)e^{-2\alpha t_k}$ which implies the exponential stability of the solutions of (4).■

Remark 1. Note that, when α tends to 0, f_α and g_α tend to f_0 and g_0. This means that the conditions of Theorem 1 are continuous with respect to α. Thus choosing $\alpha = 0$ in Theorem 1 leads to the asymptotic stability conditions given in Seuret (2009).

Remark 2. Theorem 1 provides sufficient conditions for exponential stability. It allows estimating the maximal exponential decay rate α_m by solving the following optimization problem:

$$
\alpha_m = \max \alpha \text{ such that (8) and (9) are satisfied}
$$

Liu and Fridman (2009) proposes a method to relaxe the conditions on S_1. This is exposed in the following theorem.

Theorem 2. For a given $\alpha \in \mathbb{R}$, assume that there exist symmetric positive definite matrices P, R and three matrices $S_1 = S_1^T \in \mathbb{R}^{n\times n}$, $S_2 \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{2n\times n}$ such that (8) and (9) are satisfied and

$$
\Pi_3 = \begin{bmatrix} P + f_\alpha(T,0)S_1 & f_\alpha(T,0)(S_2 - S_1) \\ * & f_\alpha(T,0)(S_1 - S_2 - S_2^T) \end{bmatrix} > 0, \tag{17}
$$

System (4) is thus exponentially stable with an exponential decay rate α for all time-varying period less than T.

Proof. The proof follows the line of Liu and Fridman (2009). Consider V_α in (11), which can be rewritten as

$$
\begin{aligned}
V_\alpha(t,x_t) = & f_\alpha(T,\tau)\int_{t_k}^t \xi^T(s)M_0^TRM_0\xi(s)ds \\
& +\xi^T(t)\begin{bmatrix} P + f_\alpha(T,\tau)S_1 & f_\alpha(T,\tau)(S_2 - S_1) \\ * & f_\alpha(T,\tau)(S_1 - S_2 - S_2^T) \end{bmatrix}\xi(t).
\end{aligned}
$$

The continuity of V_α implies (7). We are now looking for conditions to ensure that (5) is satisfied. As the term of the previous equation is linear with respect to $f_\alpha(T,\tau)$, it is sufficient to consider the positivity of this term for $f_\alpha(T,\tau) = f_\alpha(T,T)$ and $f_\alpha(T,\tau) = 0$, which lead to $P > 0$ and $\Pi_3 > 0$. ■

The previous theorem can also be relaxed as follows

Theorem 3. For a given $\alpha \in \mathbb{R}$, assume that there exist symmetric positive definite matrices P, R and three matrices $S_1 = S_1^T \in \mathbb{R}^{n\times n}$, $S_2 \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{2n\times n}$ such that (8) and (9) are satisfied and

$$
\Pi_3' = \begin{bmatrix} P + f_\alpha(T,0)U & f_\alpha(T,0)(S_2 - U) \\ * & f_\alpha(T,0)(U - S_2 - S_2^T) \end{bmatrix} > 0, \tag{18}
$$

where $U = S_1 + R/T$. System (4) is thus exponentially stable with an exponential decay rate α for all time-varying period less than T.

Proof. Following the proof of Theorem 2, we apply the Jensen's inequality to the integral term and we use the inequality $1/\tau \ge 1/T$. This leads to

$$
\int_{t_k}^t \xi^T(s)M_0^TRM_0\xi(s)ds \ge \xi^T(t)M_3^TR/TM_3\xi(t)
$$

which implies

$$
V_\alpha(t,x_t) \ge \xi^T\begin{bmatrix} P + f_\alpha(T,\tau)U & f_\alpha(T,\tau)(S_2 - U) \\ * & f_\alpha(T,\tau)(U - S_2 - S_2^T) \end{bmatrix}\xi
$$

ensuring (5) only requires Π_3' to be satisfied.■

3.2 Time-varying sampling periods

The case of time-varying periods leads to additional constraints on the LKF. In the literature, researchers are looking for a unique LKF of the form (11) where T_0 is the upper-bound of the sampling periods. However it leads to additional difficulties since the LKF becomes discontinuous at the sampling instant and (12) is not satisfied any more. In the following, we prove that the exponential stability conditions for constant sampling period also holds for time-varying periods.

Theorem 4. Theorems 1, 2 and 3 considered with $T = T_0$ ensure the exponential stability for sampled systems with time-varying periods less than T_0.

Proof. Consider now that the sampling period is not constant, the difference between two sampling instants $T_k = t_{k+1} - t_k$ is time varying but satisfies $T_k \leq T_0$. Consider now the following functional for all $t \in [t_k,\ t_{k+1}[$

$$\begin{aligned} V_\alpha^*(t, x_t) &= x^T(t)Px(t) \\ &+f_\alpha(T_k, \tau)\zeta_0^T(t)[S_1\zeta_0(t) + 2S_2x(t_k)], \\ &+f_\alpha(T_k, \tau)\int_{t_k}^{t} \xi^T(s)M_0^T RM_0\xi(s)ds \end{aligned} \tag{19}$$

This is the same functional as in Theorem 1, 2 and 3 but the time-varying period appears in the functional. Since T_k is constant over $t \in [t_k,\ t_{k+1}[$, the same stability analysis as in Theorem 1 leads to

$$\begin{aligned} W_\alpha^*(t, x_t) &= \dot{V}_\alpha^*(t, x_t) + 2\alpha V_\alpha^*(t, x_t), \\ W_\alpha^*(t, x_t) &\leq \xi^T(t)[\Pi_1 + f_\alpha(T_k, \tau)\Pi_2 \\ &\quad +g_\alpha(T_k, \tau)NR^{-1}N^T]\xi(t) \end{aligned}$$

As the right hand side of the previous inequality depends linearly on $e^{-2\alpha\tau}$, the conditions derived when $e^{-2\alpha\tau} = 1$ and $e^{-2\alpha\tau} = e^{-2\alpha T_k}$, lead to

$$\Pi_1 + f_\alpha(T_k, 0)\Pi_2 < 0, \tag{20}$$

$$\Pi_1 + g_\alpha(T_k, T_k)NR^{-1}N^T < 0, \tag{21}$$

ensure that $W_\alpha^* < 0$ for all constant sampling periods T_k. We can also apply the same argument to $e^{-2\alpha T_k}$ to (20). From its definition, $f_\alpha(T_k, 0)$ is linear with respect to $e^{2\alpha T_k}$. Ensuring that (20) is thus satisfied for $e^{2\alpha T_k} = 1$ and $e^{2\alpha T_k} = e^{2\alpha T_0}$. This leads to (8) and $\Pi_1 > 0$. Consider now the second inequality (21). If $\alpha \leq 0$, the same argument holds for $e^{-2\alpha T_k} = 1$ and leads directly to (9). If $\alpha > 0$, we note that R is positive definite and that

$$g_\alpha(T_k, T_k) \leq e^{2\alpha T_0}(1 - e^{-2\alpha T_k})/2\alpha.$$

Then we have

$$\begin{aligned} &\Pi_1 + g_\alpha(T_k, T_k)NR^{-1}N^T \leq \\ &\qquad \Pi_1 + e^{2\alpha T}(1 - e^{-2\alpha T_k})/2\alpha NR^{-1}N^T, \end{aligned}$$

which is linear with respect to $e^{-2\alpha T_k}$. We can apply the same argument as for (20) which leads to $\Pi_1 < 0$ which is already included in (9) and to (9) using the Schur complement. To conclude the proof, the same argument is employed to $\Pi_3 > 0$ with $T = T_0$ and to $\Pi_3' > 0$ using the inequality $1/T_k \geq 1/T_0$. ■

Remark 3. Based on Theorem 4, we can prove that Theorems 2 and 3 also deal with time-varying sampling periods.

A corollary is provided to cope with asymptotic stability.

Corollary 1. Assume that there exist symmetric positive definite matrices P, R and $S_1 \in \mathbb{R}^{n\times n}$ and two matrices $S_2 \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{2n\times n}$ that satisfy

$$\Pi_1 + T_0\Pi_2 < 0, \quad \begin{bmatrix} \Pi_1 & T_0N \\ * & -T_0R \end{bmatrix} < 0,$$

$$\begin{bmatrix} P + U + T_0S_1 & -U + T_0(S_2 - S_1) \\ * & U + T_0(S_1 - S_2 - S_2^T) \end{bmatrix} > 0.$$

System (4) is thus asymptotically stable for all time-varying period less than T_0.

3.3 Parameter uncertainties

An extension to the case of systems with parameter uncertainties can be dealt by considering system (4) and with A and A_d from the time-varying uncertain polytope given by

$$\forall t \in \mathbb{R}^+, \quad \Omega(t) = \sum_{k=1}^{M} \lambda_i(t)\Omega_i$$

where for all $t \in \mathbb{R}^+$ and $i = 1, .., M$, $\sum_{i=1}^{M} \lambda_i(t) = 1$, and $\lambda_i(t) \geq 0$. The Ω vertices of the polytope are described by $\Omega_i = [A(i) \quad A_d(i)]$. The conditions of Theorems 1 and 2 are not linear with respect matrices A and A_d because of the term $M_3^T RM_3$. Thus a direct extension to the case of polytopic systems is not straightforward and requires some attention provided in the sequel:

Theorem 5. Assume that there exist symmetric positive definite matrices P, R and $S \in \mathbb{R}^{n\times n}$ and a matrix $N \in \mathbb{R}^{2n\times n}$ such that $\Pi_3' > 0$ and for all $I = 1, \ldots, M$

$$\begin{bmatrix} \Pi_1^i + f_\alpha(T_0, 0)\Pi_2^i & f_\alpha(T_0, 0){M_3^i}^T R \\ * & -f_\alpha(T_0, 0)R \end{bmatrix} < 0, \tag{22}$$

$$\begin{bmatrix} \Pi_1^i & g_\alpha(T_0, T_0)N^i \\ * & -g_\alpha(T_0, T_0)R \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned} \Pi_1^i &= M_1^T PM_0^i + {M_0^i}^T PM_1 - M_3^T S_1M_3 \\ &\quad -2M_2^T S_2M_3 - M_3^T S_2^T M_2 \\ &\quad -N^iM_3 - M_3^T {N^i}^T + 2\alpha M_1^T PM_1, \\ \Pi_4^i &= M_3^T S_1M_0^i + {M_0^i}^T S_1M_3 \\ &\quad +M_2^T S_2M_0^i + {M_0^i}^T S_2^T M_2, \end{aligned}$$

and M_k, for $k = 0, 1, 2, 3$ are given in Theorem 1 and $M_0^i = [A^i \quad A_d^i]$. System (4) with polytopic uncertainties is thus exponentially stable with an exponential decay rate α for all time-varying sampling periods less than $\bar{T}$.

Proof. Consider the first condition of Theorem 1. This inequality is not linear with respect to the matrix A and A_d because of the term $M_3^T RM_3$. Noting that it can be rewritten as $(RM_0)^T R^{-1}(RM_0)$, the first condition Theorem 1 is obtained by application of the Schur complement. Consider the second LMI of Theorem 1 which is linear with respect to the system parameters A and A_d. Then the extension to polytopic systems is straightforward. As both conditions become linear with respect to the matrices A and A_d, one has to solve simultaneously the LMIs for all the Ω vertices. Finally instead of a single matrix N in (14), we considered

$$2\sum_{k=1}^{M} \lambda_i(t)N_i\,[x(t) - x(t_k)] = 2\sum_{k=1}^{M} \lambda_i(t)\int_{t_k}^{t} [NM_0\xi(s)]\,ds,$$

which leads to (22) and (23).■

4. STABILIZATION WITH A SAMPLED STATE FEEDBACK

Theorem 6. Assume that there exist symmetric positive definite matrices Q, W and $U \in \mathbb{R}^{n\times n}$ and two matrices $Z \in \mathbb{R}^{2n\times n}$ and $Y \in \mathbb{R}^{n\times m}$such that satisfy:

$$\begin{bmatrix} \Psi_1 + f_\alpha(T_0,0)\Psi_2 & f_\alpha(T_0,0)\Psi_3 \\ * & f_\alpha(T,0)\Psi_4 \end{bmatrix} < 0, \qquad (24)$$

$$\begin{bmatrix} \Psi_1 & g_\alpha(T_0,T_0)\bar{N} \\ * & -g_\alpha(T_0,T_0)\bar{R} \end{bmatrix} < 0, \qquad (25)$$

where M_k, for $k = 0, \dots, 3$ are given in Theorem 1 and

$$\begin{aligned} \Psi_1 &= M_1^T(A\bar{P} + \bar{P}A^T)M_1 - \epsilon_1 M_3^T \bar{P} M_3 \\ &\quad -\epsilon_2(M_2^T\bar{P}M_3 + M_3^T\bar{P}M_2) - \bar{N}M_3 - (\bar{N}M_3)^T \\ &\quad +M_1^T BYM_2 + M_2^T(BY)^T M_1 + 2\alpha M_1^T\bar{P}M_1, \\ \Psi_2 &= 2\epsilon_1(M_3^T A\bar{P}M_1 + M_3^T BYM_2) \\ &\quad +2\epsilon_2(M_2^T A\bar{P}M_1 + M_2^T BYM_2), \\ \Psi_3 &= M_1^T\bar{P}A^T + M_2^T(BY)^T, \ \Psi_4 = -2\bar{P} + \bar{R}. \end{aligned}$$

The sampled control law (2) with the gain $K = YQ^{-1}$ exponentially stabilizes system (1) with a decay rate α for any time-varying sampling period less than T.

Proof. Consider Theorem 5 with only one polytope. As P and R are positive definite, we can define $\bar{P} = P^{-1}$ and R^{-1}. Consider (22) and pre and post-multiply it by $\Xi_1 = \begin{bmatrix} \Xi & 0 \\ 0 & R^{-1} \end{bmatrix}$ where $\Xi = \begin{bmatrix} \bar{P} & 0 \\ 0 & \bar{P} \end{bmatrix}$. The first condition is rewritten as follows:

$$\begin{bmatrix} \Xi\Pi_1\Xi + f_\alpha(T_0,0)\Xi\Pi_4\Xi & f_\alpha(T_0,0)(M_3\Xi)^T \\ * & -f_\alpha(T_0,0)R^{-1} \end{bmatrix} < 0$$

Using the equalities

$$M_1\Xi = \bar{P}M_1, \quad M_2\Xi = \bar{P}M_2,$$
$$M_3\Xi = A\bar{P}M_1 + BK\bar{P}M_4,$$

and introducing the matrix variable $Y = K\bar{P}$, the development of $\Xi\Pi_4\Xi$ is given by

$$\begin{aligned} \Xi\Pi_4\Xi &= 2M_3^T\bar{P}S_1A\bar{P}M_1 + 2M_3^T\bar{P}S_1BYM_2 \\ &\quad +2M_2^T\bar{P}S_2A\bar{P}M_1 + 2M_2^T\bar{P}S_2BYM_2 \end{aligned}$$

In order to obtain an LMI, the following constraints $S_1 = \epsilon_1\bar{P}^{-1}$ and $S_2 = \epsilon_2\bar{P}^{-1}$ are introduced. Thus it is easy to see that $\Xi\Pi_4\Xi$ leads to Ψ_2. The next step of the proof uses the inequality $(\bar{P} - R^{-1})R(\bar{P} - R^{-1}) \geq 0$ since $\bar{P}$ and R are positive definite. This ensures that $-R^{-1} \leq -2\bar{P} + \bar{R}$. Define the variables $\bar{R} = \bar{P}R\bar{P}$ and $\bar{N} = \Xi_1 N\bar{P}$. We have $\Psi_1 = \Xi^T\Pi_1\Xi$ as defined in Theorem 6. Consider (23) with $M_3^i = M_3$, for all i. Pre and post-multiply it by $\Xi_2 = \begin{bmatrix} \Xi & 0 \\ 0 & \bar{P} \end{bmatrix}$. We have

$$\begin{bmatrix} \Xi^T\Pi_1\Xi & g_\alpha(T_0,T_0)\Xi N\bar{P} \\ * & -g_\alpha(T_0,T_0)\bar{P}R\bar{P} \end{bmatrix} = \begin{bmatrix} \Psi_1 & g_\alpha(T_0,T_0)\bar{N} \\ * & -g_\alpha(T_0,T_0)\bar{R} \end{bmatrix} < 0,$$

■

5. EXAMPLES

5.1 Example 1

Consider system (1) from Fridman et al. (2004), Naghshtabrizi et al. (2008) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0 & 0 \\ 0.375 & 1.15 \end{bmatrix}$$

The results for asymptotic stability ($\alpha = 0$) are summarized in Table 1. In this table, the acronym NDV means the number of decisions variables. One can see the the proposed Theorems requires less complexity. It can also be seen that the results from Theorems 1, 2 and 3 are less conservative than the ones from the literature. In Seuret (2009), the upper bound $T = 1.719$ was already obtained for constant sampling periods. Here we prove that the same upper-bound also holds for time-varying sampling periods. Figure 1 shows the relation between the maximal convergence rate α and the upper-bound of the time-varying sampling period given by Theorems 1,2,3. One can see that Theorem 3 leads to less conservative than the other two. Note that the conditions for all these theorems hold for negative α's. One can also see a discontinuity in the slope at $T_0 = 1.72$ which corresponds to the definitions of f_α and g_α.

Theorems	T_0	NDV
Fridman et al. (2004)	0.8696	$5n^2 + 2n$
Yue et al. (2004)	0.8696	$7n^2 + n$
Yue et al. (2005)	0.8871	$16n^2 + 3n$
Naghshtabrizi et al. (2008)	1.1137	$3.5n^2 + 1.5n$
Mirkin (2007)	1.3659	$0.5(n^2 + n) + 1$
Seuret (2009)	1.6894	$3.5n^2 + 1.5n$
Liu and Fridman (2009)	1.69	$11n^2 + 2n$
Th.1,2	1.719	$4.5n^2 + 1.5n$
Th.3	1.720	$4.5n^2 + 1.5n$

Table 1. Maximal allowable sampling period T_0 for Example 1

Theorems	T_0	NDV
Fridman et al. (2004)	0.8696	$5n^2 + 2n$
Yue et al. (2004)	0.8696	$7n^2 + n$
Yue et al. (2005)	0.8871	$16n^2 + 3n$
Naghshtabrizi et al. (2008)	1.9999	$3.5n^2 + 1.5n$
Liu and Fridman (2009)	2.53	$11n^2 + 2n$
Th.1	1.99	$4.5n^2 + 1.5n$
Th.2&3	2.51	$4.5n^2 + 1.5n$

Table 2. Maximal allowable sampling period T_0 for Example 2

5.2 Example 2

Consider system (1) from Fridman et al. (2004), Naghshtabrizi et al. (2008) with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

The results on asymptotic stability ($\alpha = 0$) are summarized in Table 2. It can be seen that the maximal allowable sampling periods provided by Theorem 1 and 2 are the same as the ones from Naghshtabrizi et al. (2008). In Liu and Fridman (2009), the authors derive less conservative result because of the introduction of additional terms in the LKF and slack matrices. Nevertheless, this example show the limits of those approaches in such case since the systems remain stable for some T_0 greater than 3.

5.3 Example 3

Consider the uncertain system from Fridman et al. (2004) defined by

$$A = \begin{bmatrix} 1 & 0.5 \\ g_1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 + g_2 \\ -1 \end{bmatrix}$$

where $|g_1| \leq 0.1$, and $|g_2| \leq 0.3$. In Fridman et al. (2004), Naghshtabrizi et al. (2008) and in Seuret (2009),

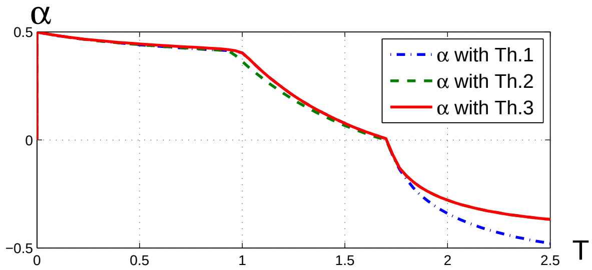

Fig. 1. Relation between exponential decay rate α and the upper-bound T_0 for Example 1

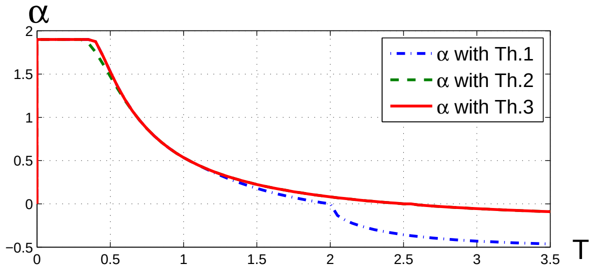

Fig. 2. Relation between exponential decay rate α and the upper-bound T_0 for Example 2

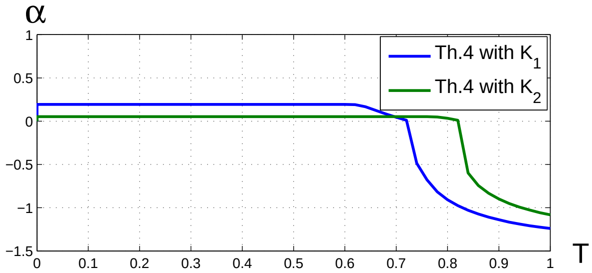

Fig. 3. Relation between exponential decay rate α and the upper-bound of the sampling period T_0 for Example 3 based on conditions from Theorem 5 and the controller gains K_1 and K_2

it was respectively proven that the state feedback gain $K_1 = -[2.6884\ 0.6649]$ stabilizes the system for any time-varying sampling periods smaller than $0.35s$, $0.4476s$, $0.602s$. Theorem 5 ensures that the closed-loop system is stable for all time-varying sampling period less than $0.720s$ which is greater than the others. Moreover based on an extension of Theorem 6 to polytopic systems with $\epsilon_1 = \epsilon_2 = 1$, the gain $K_2 = -[2.3719\ 0.5879]$ ensures stability for all time-varying sampling periods less than $0.624s$. This gain with Theorem 4 leads to stability for all time-varying sampling periods less than $0.821s$. Figure 2 shows the evolution of the convergence rate with respect to the maximum allowable sampling period T for this systems with K_1 and K_2. We can see that K_2 leads to a larger stability time-varying sampling $T_0 = 0.821$ than K_1 where stability is ensured for $\bar{T} \leq 0.72$. However the performances of the system with the control gain K_1 are better when $T_0 \leq 0.7$.

6. CONCLUSION

In this article, an analysis of linear invariant and time-varying systems with constant and time-varying sampling periods is provided. More especially, we prove that the conditions for asymptotic and exponential stability for constant and time-varying sampling periods are equivalent. Tractable conditions are derived to ensure exponential stability with an estimate of the convergence rate. The conditions are also valid for negative α which help to evaluate the exponential divergence rate of the solutions. The examples show the efficiency of the method and the reduction of the conservatism compared to others results from the literature. This has been treated by a continuous-time approach which helps to cope with uncertain or time-varying systems.

REFERENCES

Ariba, Y. and Gouaisbaut, F. (2007). Delay-dependent stability analysis of linear systems with time-varying delay. In *Proc. of the* 46^{th} *IEEE Conference on Decision and Control*. New Orleans, LA, USA.

Chen, T. and Francis, B. (1995). *Optimal sampled-data control systems*. Springer-Verlag, Berlin, Germany.

Fridman, E., Seuret, A., and Richard, J.P. (2004). Robust sampled-data stabilization of linear systems: An input delay approach. *Automatica*, 40(8), 1141–1446.

Fujioka, H. (2009). Stability analysis of systems with aperiodic sample-and-hold devices. *Automatica*, 45(3), 771–775.

Hespanha, J., Naghshtabrizi, P., and Xu, Y. (2007). A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1), 138–162.

Liu, K. and Fridman, E. (2009). Discontinuous lyapunov functionals for linear systems with sawtooth delays. In *Proceedings of the* 8^{th} *IFAC Workshop on Time-Delay Systems*.

Mirkin, L. (2007). Some remarks on the use of time-varying delay to model sample-and-hold circuits. *IEEE Trans. on Automatic Control*, 52(6), 1009–1112.

Naghshtabrizi, P., Hespanha, J., and Teel, A. (2008). Exponential stability of impulsive systems with application to uncertain sampled-data systems. *Systems and Control Letters*, 57(5), 378–385.

Seuret, A. (2009). Stability analysis for sampled-data systems with a time-varying period. In 48^{th} *IEEE Conference on Decision and Control*.

Yue, D., Han, Q.L., and Lam, J. (2005). Network-based robust H^∞ control of systems with uncertainty. *Automatica*, 41(6), 640–644.

Yue, D., Han, Q.L., and Lam, J. (2008). Stability and stabilization of nonuniform sampling systems. *Automatica*, 44(12), 3222–3226.

Yue, D., Han, Q.L., and Peng, C. (2004). State feedback controller design for networked control systems. *IEEE Trans. on Automatic Control*, 51(11), 640–644.

Zampieri, S. (2008). A survey of recent results in Networked Control Systems. In *Proc.* 17^{th} *IFAC World Congress*. Seoul, Korea.

Zhang, W. and Branicky, M. (2001). Stability of networked control systems with time-varying transmission period. In *Allerton Conf. Communication, Control, and Computing*.

Zhang, W., Branicky, M., and Phillips, S. (2001). Stability of networked control systems. *IEEE Control Systems Magazine*, (21).