

# Mathematical algorithms in Sanskrit prosody and music treatises

François Patte

Université Paris Descartes

Kyoto University

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1, 2, 3, 6, 10, 19, 33, 60, 106, 191, 340, 610, ...

# Śārṅgadeva: *Samgītaratnākara* (13th century)

## Chapter five: *Tālādhyaya*

# Śārṅgadeva: *Samgītaratnākara* (13th century)





Chapter five: *Tālādhyaya*

*Com.: Kalānidhi* by Kallinātha (15th century)

(ed. Ānandāśrama – 1896 and Adyar – 1951)

*Sudhākara* by Siṃhabhūpāla (14th century)

(ed. Adyar – 1951)

<i>druta</i>	●	unit of measure		semi-short
<i>laghu</i>	■	= 2 <i>druta</i>		short
<i>guru</i>	■	= 2 <i>laghu</i> = 4 <i>druta</i>		long
<i>pluta</i>	■	= 3 <i>laghu</i> = 6 <i>druta</i>		protracted

# Prastāra

How to build exhaustively all variations (*bheda*) for one musical measure of a given length.

# Prastāra

How to build exhaustively all variations (*bheda*) for one musical measure of a given length.

Example:

**o**S (♪ ♪)    **llo** (♪ ♪ ♪)    **oloo** (♪ ♪ ♪ ♪)    **ooooo** (♪ ♪ ♪ ♪ ♪)

Four variations for a five *druta* (**o**, ♪) measure

nyasyālpam ādyān mahato  
 'dhaścāc cheṣaṃ yathopari |  
 prāg ūne vāmasaṃsthāṃs tu  
 saṃbhava mahato likhet |  
 alpān asaṃbhava tāla-  
 pūrtyai bhūyo 'py ayaṃ vidhiḥ ||  
 sarvadrutāvadhiḥ kāryaḥ  
 prastāro 'yaṃ laghau gurau |  
 plute vyaste samaste ca  
 na tu vyaste drute 'sti saḥ ||

A smaller one being set down under the first greater one, what is left is as above.

If there is a deficiency in the opposite way, in order to complete the measure, one will write greater ones on the left, if possible; if impossible, smaller ones. This rule, the end of which is [a musical measure] wholly [made of] *drutas*, must be carried out many times.



o I = 2 o S = 4 o Š = 6 o

o S 5

$$\bullet \quad \mathbf{I} = 2 \bullet \quad \mathbf{S} = 4 \bullet \quad \mathbf{\dot{S}} = 6 \bullet$$

A smaller one being set down under the first greater one,  
what is left is as above.

$$\bullet \mathbf{S} \quad 5$$

$$\bullet \quad \mathbf{I} = 2 \bullet \quad \mathbf{S} = 4 \bullet \quad \mathbf{\dot{S}} = 6 \bullet$$

A smaller one being set down under the first greater one,  
what is left is as above.

$$\begin{array}{cc} \bullet \mathbf{S} & 5 \\ \mathbf{I} & 2 \end{array}$$

$$\bullet \quad \mathbf{I} = 2 \bullet \quad \mathbf{S} = 4 \bullet \quad \mathbf{\dot{S}} = 6 \bullet$$

If there is a deficiency in the opposite way, in order to complete the measure, one will write greater ones on the left, if possible; if impossible, smaller ones.

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If there is a deficiency in the opposite way, in order to complete the measure, one will write greater ones on the left, if possible; if impossible, smaller ones.

$$\begin{array}{ccc} & \bullet \mathbf{S} & 5 \\ \bullet & \mathbf{I} \mathbf{I} & 5 \end{array}$$

$$\bullet \quad \mathbf{I} = 2\bullet \quad \mathbf{S} = 4\bullet \quad \mathbf{\dot{S}} = 6\bullet$$

This rule, the end of which is [a musical measure] wholly [made of] *drutas*, must be carried out many times.

$$\begin{array}{cc} \bullet & \mathbf{S} \\ \bullet & \mathbf{I} \end{array} \quad \begin{array}{c} 5 \\ 5 \end{array}$$

$$\bullet \quad \mathbf{I} = 2\bullet \quad \mathbf{S} = 4\bullet \quad \mathbf{\dot{S}} = 6\bullet$$

A smaller one being set down under the first greater one, what is left is as above.

$$\begin{array}{r} \bullet \mathbf{S} \\ \bullet \mathbf{I} \mathbf{I} \\ \bullet \end{array} \quad \begin{array}{r} 5 \\ 5 \\ 1 \end{array}$$

$$\bullet \quad \mathbf{I} = 2 \bullet \quad \mathbf{S} = 4 \bullet \quad \mathbf{\dot{S}} = 6 \bullet$$

A smaller one being set down under the first greater one, what is left is as above.

$$\begin{array}{rcl} & \bullet \mathbf{S} & 5 \\ \bullet & \mathbf{I} \mathbf{I} & 5 \\ & \bullet \mathbf{I} & 3 \end{array}$$



$$\bullet \quad \mathbf{I} = 2 \bullet \quad \mathbf{S} = 4 \bullet \quad \mathbf{\dot{S}} = 6 \bullet$$

If there is a deficiency in the opposite way, in order to complete the measure, one will write greater ones on the left, if possible; if impossible, smaller ones.

$$\begin{array}{r} \bullet \mathbf{S} \quad 5 \\ \bullet \mathbf{I} \mathbf{I} \quad 5 \\ \bullet \mathbf{I} \quad 3 \end{array}$$

$$\bullet \quad \mathbf{I} = 2\bullet \quad \mathbf{S} = 4\bullet \quad \mathbf{\dot{S}} = 6\bullet$$

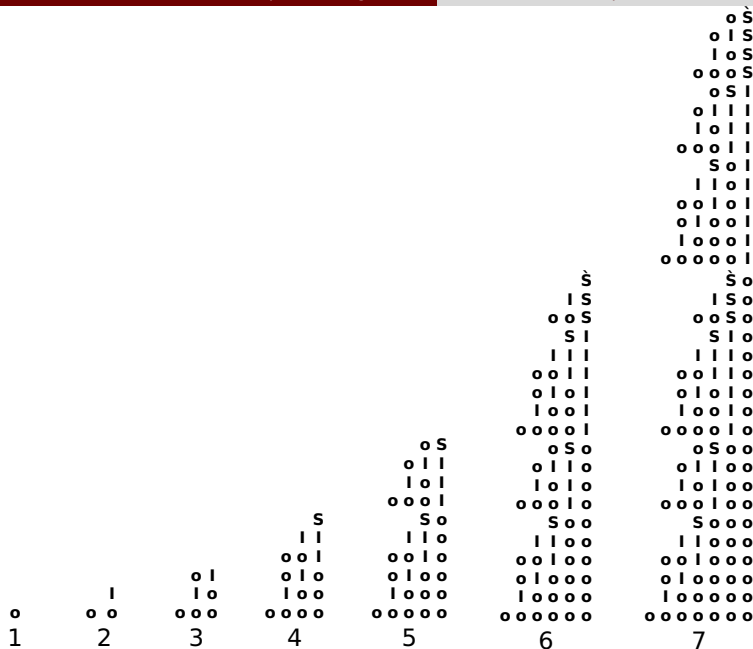
If there is a deficiency in the opposite way, in order to complete the measure, one will write greater ones on the left, if possible; if impossible, smaller ones.

$$\begin{array}{rcccl} & \bullet & \mathbf{S} & & 5 \\ \bullet & \mathbf{I} & \mathbf{I} & & 5 \\ \mathbf{I} & \bullet & \mathbf{I} & & 5 \end{array}$$

$$\circ \quad \text{I} = 2\circ \quad \text{S} = 4\circ \quad \text{Ṣ} = 6\circ$$

This rule, the end of which is [a musical measure] wholly [made of] *drutas*, must be carried out many times.

o S	5
o I I	5
I o I	5
o o o I	5
S o	5
I I o	5
o o I o	5
o I o o	5
I o o o	5
o o o o o	5



# Samkhyā

## Counting

Or how to count the number of variations (*bheda*) built with the *prastāra*.

ekadvyaṅkau kramān nyasya  
 yuñjītāntyaṃ purātanaḥ |  
 dvitīyaturyaṣaṣṭhāṅkair  
 abhave turyaṣaṣṭhayoḥ ||  
 tṛtīyapañcamāṅkābhyāṃ  
 kramāt taṃ yogam agrataḥ |  
 likhed dakṣiṇasaṃsthaivam  
 aṅkaśreṇīvidhiyate ||

Having laid down successively the numbers one and two, the last one is added, as far as possible, to the second, the fourth and the sixth preceeding numbers; in the absence of the fourth and sixth numbers, to the third and fifth ones. One will write this sum gradually in the beginning. A sequence of numbers standing together on the right is thus established.

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1, 2, 3, 6, 10, 19,

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1, 2, 3, 6, **10**, **19**,

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1, 2, 3, 6, 10, 19, 33,

Having laid down successively the numbers one and two, the last one is added, as far as possible, to the second, the fourth and the sixth preceeding numbers; in the absence of the fourth and sixth numbers, to the third and fifth ones. One will write this sum gradually in the beginning. A sequence of numbers standing together on the right is thus established.

1, 2, 3, 6, 10, 19, 33, 60, 106, 191, 340, 610, ...

# Some mathematics

Problem: how to count all the possible ways to split a given musical measure lasting 7 *drutas* into a combination of the four note values: *druta*, *laghu* (=2*d.*), *guru* (=4*d.*) and *pluta* (=6*d.*).

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Mathematical point of view: how to count all possible ways to write a given number (7) as a sum of the integers: 1, 2, 4 and 6 (partitions of the integer 7).

# Some mathematics

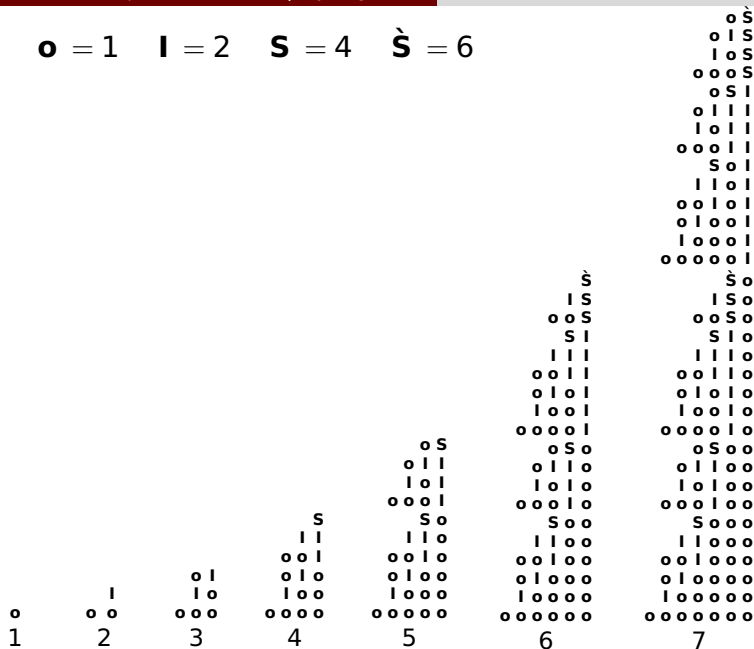
Problem: how to count all the possible ways to split a given musical measure lasting 7 *drutas* into a combination of the four note values: *druta*, *laghu* (=2d.), *guru* (=4d.) and *pluta* (=6d.).

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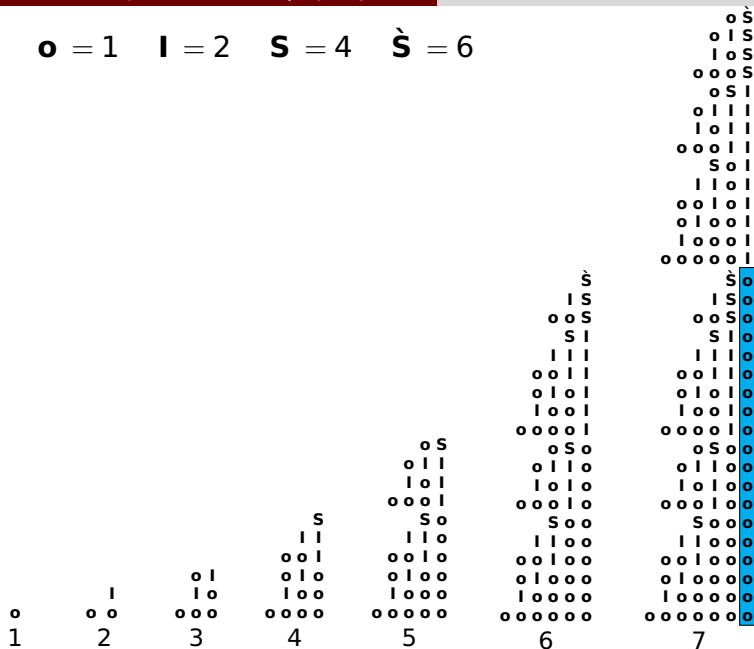
$$7 = 1 + 2 + 4 = 6 + 1 = 1 + 1 + 1 + 2 + 1 + 1 = \dots$$



$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

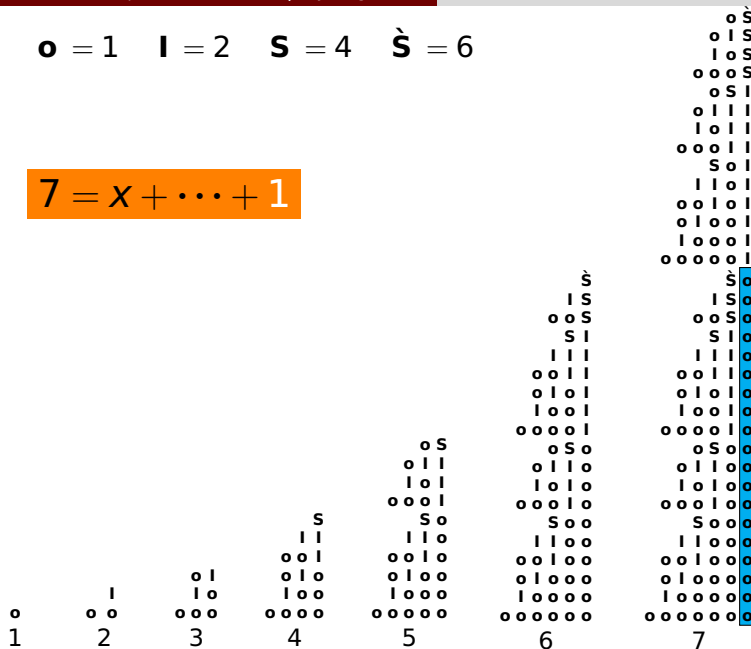


**o** = 1    **l** = 2    **s** = 4    **ṣ** = 6



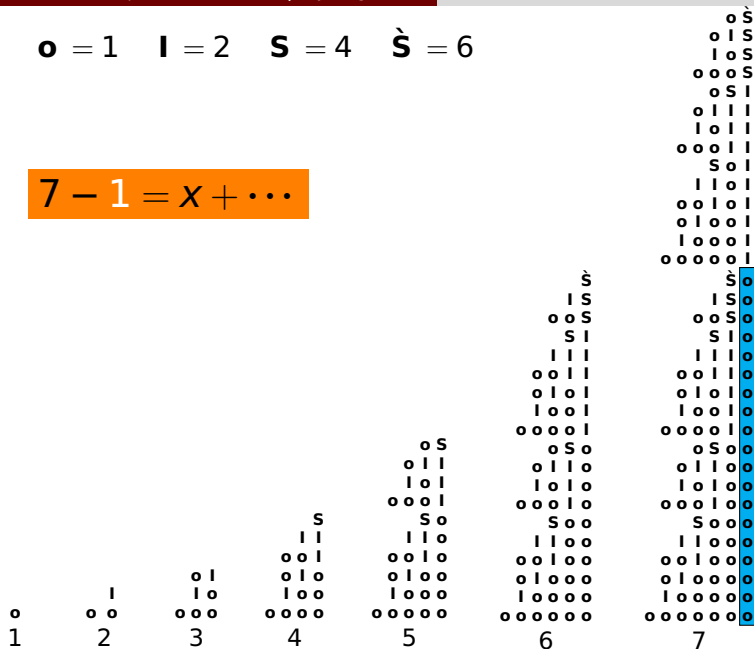
$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$7 = x + \dots + 1$$



**o** = 1    **l** = 2    **s** = 4    **š** = 6

$$7 - 1 = x + \dots$$



**o** = 1    **l** = 2    **s** = 4    **š** = 6

$$u_7 = u_6 +$$

$$7 - 1 = x + \dots$$

```

      oS
     oIS
    IoS
   ooS
  oSI
 oIII
IoII
oOoII
  Sol
   IloI
  oloI
 oloo
 llooI
oooooI

```

```

      o S
      o l l
      l o l
      o o o l
      S
      l l
      l l o
      o o l
      o l o
      l o o
      o o o o
      o
      o o
      o o o
      o o o o
      o o o o o
1      2      3      4      5

```

```

      $
    IS
  ooS
  SI
  III
    oII
  oIoI
  looI
oo o o I
  oSo
    oII
  loIo
oo oIo
  So
  IIoo
ooIo
oIo
  Iooo
ooooo

```

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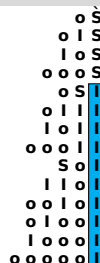
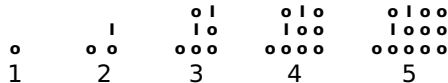
      S o
    I S o
   ooS o
  S l o
 i l l o
o l l o
o l o l o
l o o l o
ooo l o
  oS o o
  o l l o
  l o l o
ooo l o o
   S o o o
  l l o o
oo l o o o
o l o o o
l o o o o
ooooooo o

```

$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$u_7 = u_6 +$$

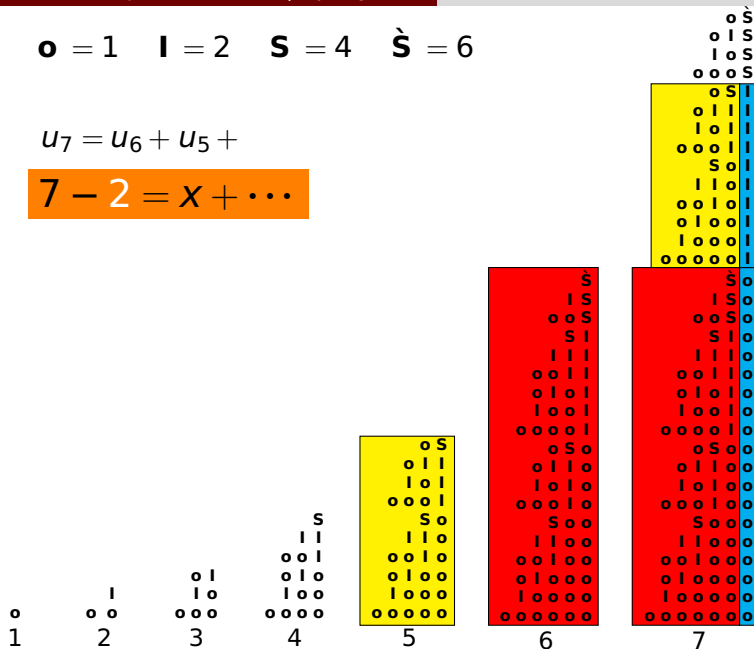
$$7 = x + \dots + 2$$



$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$u_7 = u_6 + u_5 +$$

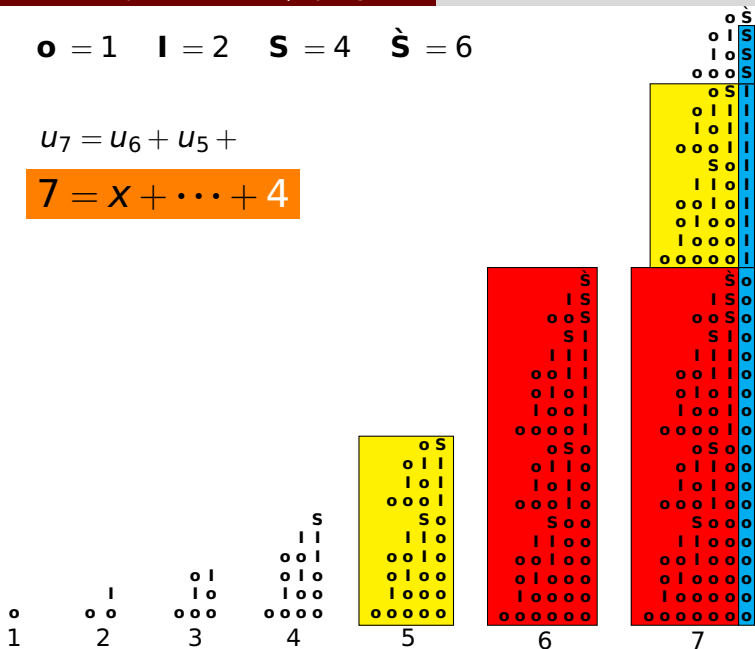
$$7 - 2 = x + \dots$$



$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$u_7 = u_6 + u_5 +$$

$$7 = x + \dots + 4$$

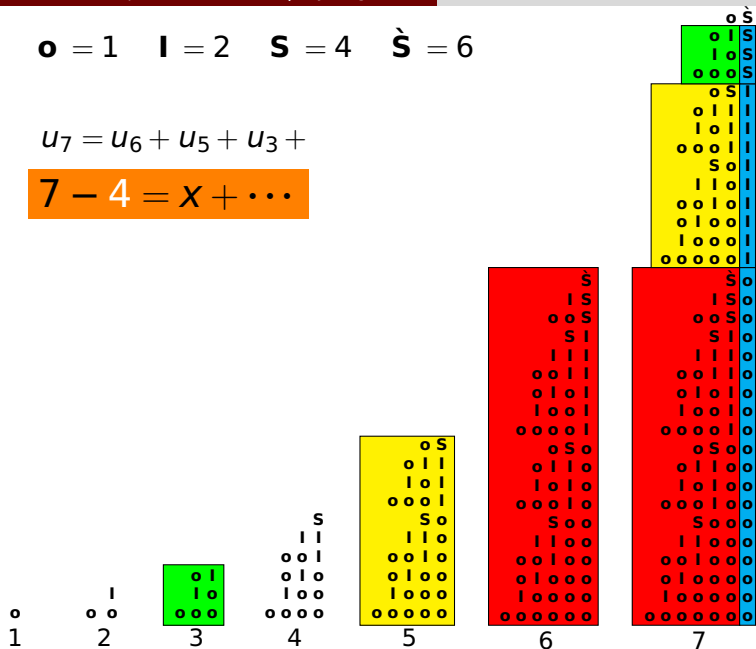




**o** = 1    **l** = 2    **s** = 4    **š** = 6

$$u_7 = u_6 + u_5 + u_3 +$$

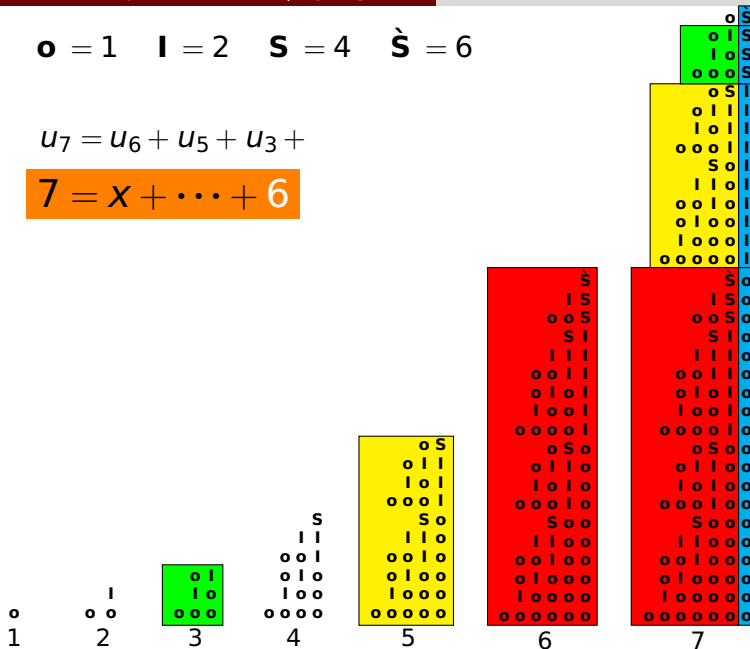
$$7 - 4 = x + \dots$$



$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$u_7 = u_6 + u_5 + u_3 +$$

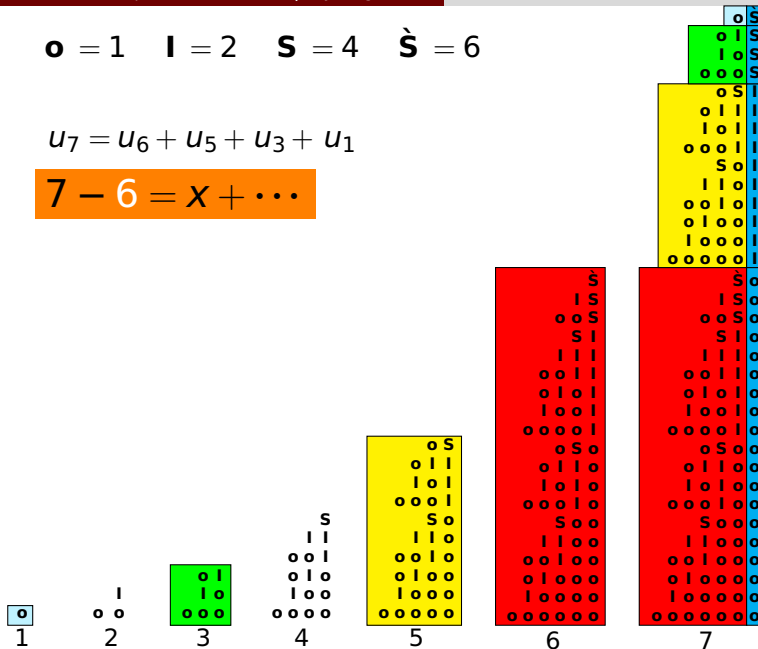
$$7 = x + \dots + 6$$



$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$u_7 = u_6 + u_5 + u_3 + u_1$$

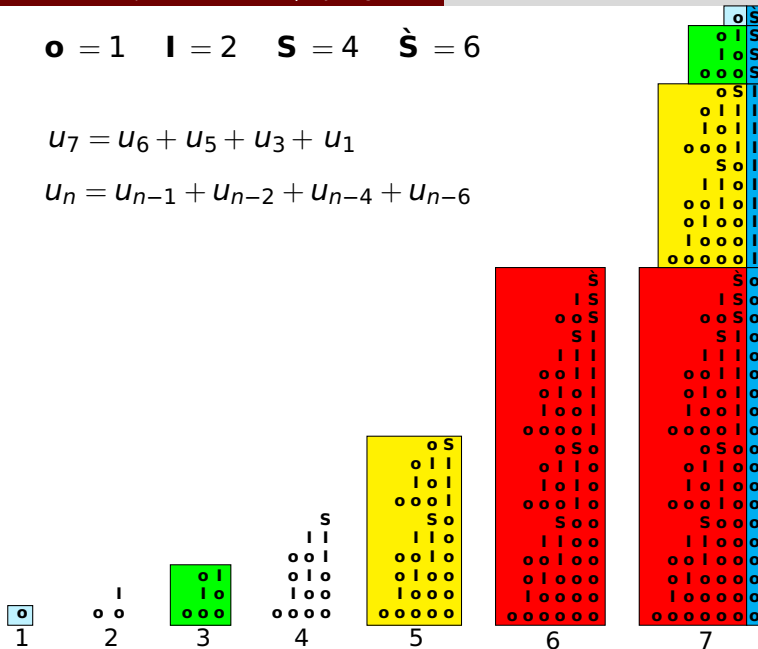
$$7 - 6 = x + \dots$$



$$\mathbf{o} = 1 \quad \mathbf{l} = 2 \quad \mathbf{s} = 4 \quad \mathbf{\dot{s}} = 6$$

$$u_7 = u_6 + u_5 + u_3 + u_1$$

$$u_n = u_{n-1} + u_{n-2} + u_{n-4} + u_{n-6}$$



# Naṣṭa

Disappearance

Or how to retrieve a deleted line in the *prastāra*

# Some mathematics

Order allowing to compare words, using an ordered alphabet:

$$a < b < c < \dots < y < z$$

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Words are compared letter by letter in the same position.

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A word  $\mathcal{B}$  is greater than a word  $\mathcal{A}$  if:



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Words are compared letter by letter in the same position.

A word  $\mathcal{B}$  is greater than a word  $\mathcal{A}$  if:

- ▶ There is a letter in  $\mathcal{B}$  greater than a letter in  $\mathcal{A}$  in the **same position**,
- ▶ Letters before that position are the same in  $\mathcal{B}$  and  $\mathcal{A}$ ,
- ▶ Whatever the letters after this position are.

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$$a***** \prec b***** \prec \dots \prec g***** \prec \dots$$

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$$\text{alpha} \prec \text{alphab}et \prec \text{alphabetic} \prec \text{alphabetical} \prec \text{alphabetize}$$

# Some mathematics

In our case, the ordered alphabet is:

$$\mathbf{o} < \mathbf{l} < \mathbf{s} < \mathbf{\grave{s}}$$

# Some mathematics

In our case, the ordered alphabet is:

$$\mathbf{o} < \mathbf{l} < \mathbf{s} < \mathbf{\dot{s}}$$

Reading each "word" from right to left:

$$\mathbf{oooooooo} \prec \mathbf{looooo} \prec \mathbf{oloooo} \prec \dots \prec \mathbf{o\dot{s}}$$

# Some mathematics

In our case, the ordered alphabet is:

$$\mathbf{o} < \mathbf{l} < \mathbf{s} < \mathbf{\grave{s}}$$

Reading each "word" from right to left:

**oooooooo**  $\prec$  **looooo**  $\prec$  **oloooo**  $\prec \dots \prec$  **oṣ**

and from bottom to top:

oṣ  
o l s  
l o s  
o o o s  
o s l  
o l l l  
l o l l  
o o o l l  
s o l  
l l o l  
o o l o l  
o l o o l  
l o o o l  
o o o o l  
ṣ o  
l s o  
o s o  
s l o  
l l l o  
o o l l o  
o l o l o  
l o o l o  
o o o l o  
o s o o  
o l l o o  
l o l o o  
o o l o o  
s o o o  
l l o o o  
o o l o o o  
o l o o o o  
l o o o o o  
o o o o o o



```

      oŞ
      oIS
      loS
      oosS
      oSI
      oIII
      loII
      ooolI
      Sol
      lloI
      oolol
      olool
      loool
      oooooI
      Şo
      lSo
      oosS
      SIo
      lllol
      ? ? ? ? ? ?
      olool
      loolo
      ooolol
      oSoo
      olloo
      loloo
      oooloo
      Sooo
      lllooo
      oolooo
      oloooo
      looooo
      ooooooo

```

20

1, 2, 3, 6, 10, 19, 33

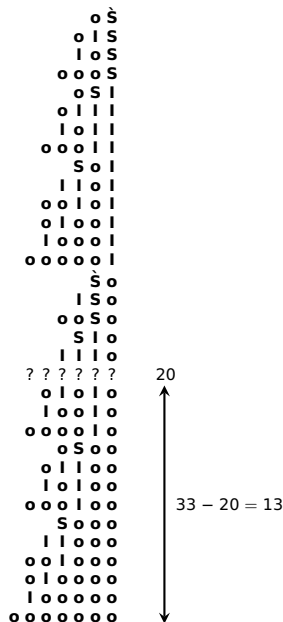
```

      o Š
      o l S
      l o S
      o o o S
      o S l
      o l l l
      l o l l
      o o o l l
      S o l
      l l o l
      o o l o l
      o l o o l
      l o o o l
      o o o o l
      Š o
      l S o
      o o S o
      S l o
      l l l o
      ? ? ? ? ?
      o l o l o
      l o o l o
      o o o l o
      o S o o
      o l l o o
      l o l o o
      o o l o o
      S o o o
      l l o o o
      o o l o o o
      o l o o o o
      l o o o o o
      o o o o o o

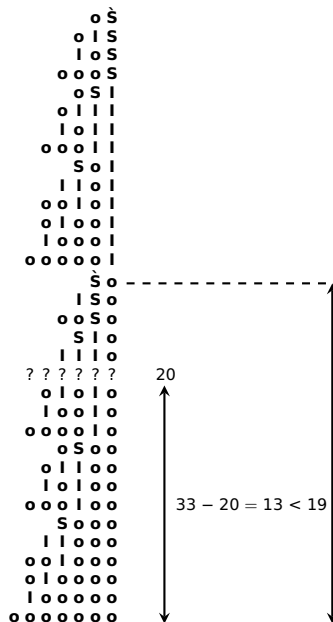
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20

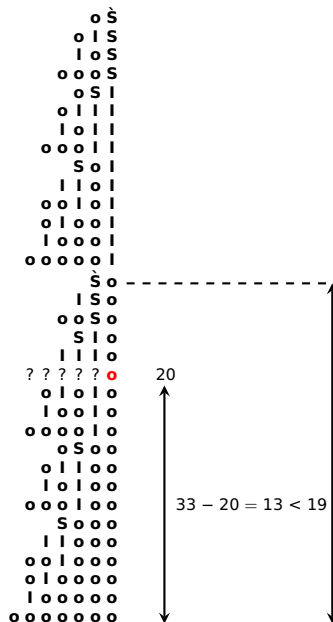
1, 2, 3, 6, 10, 19, 33



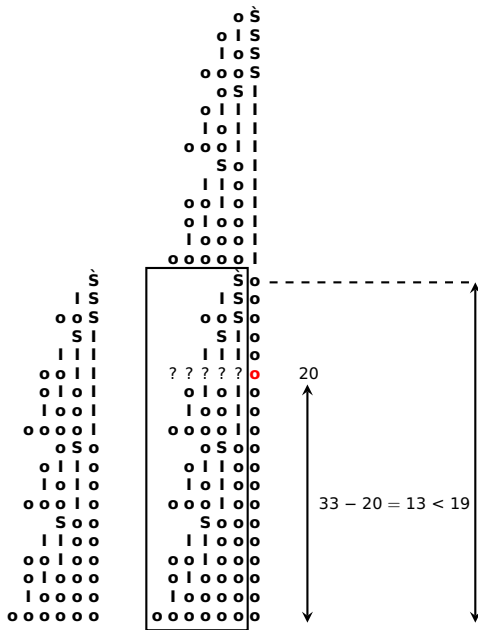
1, 2, 3, 6, 10, 19, 33



1, 2, 3, 6, 10, 19, 33



1, 2, 3, 6, 10, 19



1, 2, 3, 6

```

      S
     II
    ool
   olo
  loo
 oooo

```

```

      Š
     IS
    ooS
   SI
  III
 oolI
  oloI
 looI
 ooooI
  oSo
  oIIo
  lolo
 ooolo
   Soo
  lloo
  ooloo
  olooo
  loooo
 ooooo

```

```

      oŠ
     oIS
    loS
   ooS
   oSI
  oIII
 loII
 ooolI
  Sol
  lloI
 oolol
  olooI
 loooI
 oooooI
      Šo
     ISo
    ooS
   SIo
  IIIo
  ??? ? I o
  oloI
 looI
 ooooI
  oSoo
  oIIoo
  lolo
 ooolo
   Sooo
  llooo
  oolooo
  oloooo
  looooo
 oooooo

```

20

$$33 - 20 = 13 < 19$$

1, 2

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$$33 - 20 = 13 < 19$$

ありがとうございました。

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