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ON THE UNIMODALITY OF INVERSE POSITIVE STABLE LAWS

THOMAS SIMON

ABSTRACT. We observe that the function $F_\alpha(x) = (1 + \alpha x^\alpha)e^{-x^\alpha}$ is completely monotone iff $\alpha \leq \alpha_0$ for some $\alpha_0 \in]2/3, 3/4[$. This property is equivalent to the unimodality of the inverse positive α -stable law. The random variable associated with F_α appears then in two different factorizations of the positive α -stable distribution. Furthermore, it is infinitely divisible iff $\alpha \leq \alpha_1$ for some $\alpha_1 \in]2/3, \alpha_0[$ and self-decomposable iff $\alpha \leq \alpha_2$ for some $\alpha_2 \in]2/3, \alpha_1[$.

The purpose of this note is to point out a curious phenomenon for positive stable distributions for which we did not find any trace in the literature, though it could be of some interest. For every $\alpha \in]0, 1[$, let f_α be the positive α -stable density and Z_α the corresponding random variable, normalized such that

$$(1) \quad \int_0^\infty e^{-\lambda t} f_\alpha(t) dt = \mathbb{E}[e^{-\lambda Z_\alpha}] = e^{-\lambda^\alpha}, \quad \lambda \geq 0.$$

The function f_α is known to be smooth, which can also be easily retrieved in inverting (1). Consider now the function $g_\alpha : t \mapsto 2f_\alpha(t) + tf'_\alpha(t)$.

Theorem. *There exists a critical value $\alpha_0 \in]2/3, 3/4[$ such that the following are equivalent:*

- (a) *One has $\alpha \leq \alpha_0$.*
- (b) *The function $F_\alpha : x \mapsto (1 + \alpha x^\alpha)e^{-x^\alpha}$ is completely monotone.*
- (c) *The random variable Z_α^{-1} is unimodal.*
- (d) *The function g_α is a probability density over \mathbb{R}^+ .*
- (e) *There exists a factorization*

$$Z_\alpha \stackrel{d}{=} U^{-1} \times \tilde{Z}_\alpha$$

where U is a uniform variable in $[0, 1]$ and \tilde{Z}_α an independent variable with density g_α .

Proof: We first consider the simple equivalence (b) \Leftrightarrow (c) \Leftrightarrow (d), which was the starting point of this work. Recall the Humbert-Pollard representation for f_α which may be found e.g. in [7], formula (14.31):

$$(2) \quad f_\alpha(t) = \sum_{n \geq 1} \frac{(-1)^{n-1} \Gamma(1 + \alpha n) \sin \pi \alpha n}{\pi n!} t^{-\alpha n - 1} = \sum_{n \geq 1} \frac{(-1)^n}{n! \Gamma(-\alpha n)} t^{-\alpha n - 1} \quad t > 0.$$

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Differentiating (2) term by term shows that g_α is bounded and integrable. Since on the other hand $f_\alpha(0) = 0$, an integration by parts gives

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-\lambda t} g_\alpha(t) dt &= e^{-\lambda^\alpha} + \lambda \int_{\mathbb{R}^+} t e^{-\lambda t} f_\alpha(t) dt \\ &= e^{-\lambda^\alpha} + \lambda \frac{d}{d\lambda} (e^{-\lambda^\alpha}) = (1 + \alpha \lambda^\alpha) e^{-\lambda^\alpha} = F_\alpha(\lambda) \end{aligned}$$

for every $\lambda \geq 0$. Specifying to $\lambda = 0$ yields

$$\int_{\mathbb{R}^+} g_\alpha(t) dt = 1$$

so that g_α is a probability density iff it is everywhere non-negative. By the above computation and Bernstein's theorem, this is equivalent to the complete monotonicity of F_α . On the other hand, the density of the random variable Z_α^{-1} is $h_\alpha(t) = t^{-2} f_\alpha(t^{-1}) \rightarrow +\infty$ as $t \rightarrow 0$, the latter asymptotic coming from (2). In particular, we see that Z_α^{-1} is unimodal iff h'_α is everywhere non-increasing. Computing

$$h'_\alpha(t) = -t^{-3} g_\alpha(t^{-1}), \quad t > 0$$

and putting everything together entails the equivalence (b) \Leftrightarrow (c) \Leftrightarrow (d).

Let us now prove (c) \Leftrightarrow (e). From the above discussion, we know that Z_α^{-1} is unimodal iff it is unimodal with mode at zero. By Khintchine's theorem, the latter is equivalent to the factorization

$$(3) \quad Z_\alpha^{-1} \stackrel{d}{=} U \times Y_\alpha$$

where U is uniform on $[0, 1]$ and Y_α some independent positive random variable. This yields (e) \Rightarrow (c), and the reverse inclusion will be also true once it is proved that the density of the variable $\tilde{Z}_\alpha = Y_\alpha^{-1}$ is g_α . This can be done in evaluating the moments: from (3) and the classical computation made for Z_α - see e.g. formula (25.5) in [7], one gets first

$$(4) \quad \mathbb{E}[\tilde{Z}_\alpha^s] = \frac{(1-s)\Gamma(1-s/\alpha)}{\Gamma(1-s)}, \quad s < \alpha.$$

Setting $\mathcal{R}(z) = \mathbb{E}[\tilde{Z}_\alpha^z]$ for every $z \in \mathbb{C}$ such that $\operatorname{Re}(z) < \alpha$ and inverting this Mellin transform, one sees that \tilde{Z}_α has a continuous bounded density \tilde{f}_α given by

$$\tilde{f}_\alpha(t) = \frac{1}{2\pi t} \int_{\mathbb{R}} \mathcal{R}(iz) t^{-iz} dz, \quad t \geq 0.$$

Computing the integral on the right-hand side with the help of a semi-circular contour in the lower half-plane and an evaluation of the residues at $-i\alpha n$, $n \geq 1$, entails

$$(5) \quad \tilde{f}_\alpha(t) = \sum_{n \geq 1} \frac{(-1)^n (1 - \alpha n)}{n! \Gamma(-\alpha n)} t^{-\alpha n - 1}, \quad t > 0,$$

and we see from (2) that the right-hand side is $g_\alpha(t)$. We will leave the details to the interested reader since an alternative route which is much simpler consists in deriving from

(4) the identity

$$\mathbb{E}[\tilde{Z}_\alpha^s] = 2\mathbb{E}[Z_\alpha^s] - (1+s)\mathbb{E}[Z_\alpha^s] = 2 \int_{\mathbb{R}^+} t^s f_\alpha(t) dt + \int_{\mathbb{R}^+} t^{s+1} f'_\alpha(t) dt = \int_{\mathbb{R}^+} t^s g_\alpha(t) dt,$$

where the second equality comes after an integration by parts, noticing that (2) and the condition $s < \alpha$ let the boundary term at infinity vanish. It is then sufficient to invoke the injectivity of the Mellin transform to show that \tilde{Z}_α has density g_α .

It remains to prove the equivalence (a) \Leftrightarrow (b) and to estimate the location of the threshold α_0 , which will be slightly more involved. We first remark that from the decomposition

$$F_\beta(\lambda) = (1 - \beta/\alpha)e^{-\lambda^\beta} + (\beta/\alpha)F_\alpha(\lambda^{\beta/\alpha})$$

and Criterion 2 p. 417 in [3], the function F_β is completely monotone (CM) as soon as F_α is CM and $0 \leq \beta \leq \alpha$. On the other hand, it is easily seen that F_α is not CM whenever $\alpha > 3/4$. Indeed, if it were, then it would be also log-convex as the Laplace transform of a positive function (this latter assertion follows immediately from the Hölder inequality) so that the function $G_\alpha(\lambda) = \log(1 + \lambda^\alpha) - \lambda^\alpha/\alpha$ would be convex. We compute

$$G''_\alpha(\lambda) = \frac{\lambda^{\alpha-2}((\alpha-1)^2 + (2-3\alpha)\lambda^\alpha + (1-\alpha)\lambda^{2\alpha})}{(1+\lambda^\alpha)^2}$$

and we see after some discussion on the sign of the trinomial that the right-hand side is everywhere non-negative over \mathbb{R}^+ iff $\alpha \leq 3/4$.

We can also check that $F_{3/4}$ itself is not CM: although $(-1)^n F_{3/4}^{(n)}(\lambda) \geq 0$ for every $n \leq 4$, some further computation together with the change of variable $\mu = \lambda^{-3/4}$ gives

$$(-1)^5 F_{3/4}^{(5)}(\lambda) = \frac{9e^{-\lambda^{3/4}}}{4^6 \lambda^{1/2}} (195\mu^5 + 35\mu^4 - 150\mu^3 - 135\mu^2 - 27\mu + 81),$$

and the right-hand side is negative for $\mu = 4/5$. However, we will now exhibit some cases where F_α is CM. Let us write

$$F_\alpha(\lambda) = e^{-H_\alpha(\lambda)}$$

with $H_\alpha(\lambda) = \lambda^\alpha - \log(1 + \alpha\lambda^\alpha)$ a non-negative function. Again, from Criterion 2 p. 417 in [3] the function F_α will be CM as soon as H'_α is CM. Moreover, since H_α is non-negative and $H_\alpha(0) = 0$, the latter condition is known to be equivalent to the fact that H_α is a so-called Bernstein function, in other words the Laplace exponent of some subordinator. Changing the variable $\mu = \lambda\alpha^{1/\alpha}$ yields the formulae

$$\lambda^\alpha = \frac{1}{\Gamma(1-\alpha)} \int_{\mathbb{R}^+} (1 - e^{-\mu x}) \frac{dx}{x^{1+\alpha}} \quad \text{and} \quad \log(1 + \alpha\lambda^\alpha) = \alpha \int_{\mathbb{R}^+} (1 - e^{-\mu x}) E_\alpha(-x^\alpha) \frac{dx}{x}$$

where

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1 + \alpha n)}$$

is the Mittag-Leffler function of index α . We refer to Remark 2.2 in [6] (correcting $x^k \rightarrow x^{\alpha k}$ therein) for the second formula, which actually follows [3] p. 429. We also recall the

asymptotic

$$(6) \quad E_\alpha(-x^\alpha) \sim \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$$

as $x \rightarrow +\infty$ (see (18.1.7) in [2]) which ensures that the integral in the second formula does converge. We deduce from these two formulae the representation

$$(7) \quad H_\alpha(\lambda) = \frac{1}{\Gamma(1-\alpha)} \int_{\mathbb{R}_+} (1 - e^{-\mu x})(1 - U_\alpha(x)) \frac{dx}{x^{1+\alpha}}$$

with the notation $U_\alpha(x) = \alpha\Gamma(1-\alpha)x^\alpha E_\alpha(-x^\alpha)$, and the Lévy-Khintchine formula for subordinators entails that H_α is a Bernstein function iff

$$(8) \quad U_\alpha(x) \leq 1$$

for every $x \geq 0$. We now discuss this latter inequality with the help of some features of the Mittag-Leffler functions. From (1) and Exercise 29.18 in [7], one gets first the representation

$$(9) \quad E_\alpha(-x^\alpha) = \mathbb{E}[e^{-x^\alpha/Z_\alpha^\alpha}] = \mathbb{E}[e^{-x(\bar{Z}_\alpha/Z_\alpha)}]$$

where \bar{Z}_α is an independent copy of Z_α . Evaluating the moments with the help of Formula (25.5) in [7] entails

$$\mathbb{E}\left[\frac{\bar{Z}_\alpha^s}{Z_\alpha^s}\right] = \frac{\Gamma(1-s/\alpha)\Gamma(1+s/\alpha)}{\Gamma(1-s)\Gamma(1+s)} = \frac{\sin(\pi s)}{\alpha \sin(\pi s/\alpha)} = \int_0^\infty \frac{(\sin \pi \alpha)t^{s+\alpha-1}}{\pi(t^{2\alpha} + 2t^\alpha \cos \pi \alpha + 1)} dt,$$

the last equality being a standard computation carried out e.g. in [1] Formula VIII.(10.2.4). This gives the expression of the density of the variable \bar{Z}_α/Z_α . Plugging it now into (9) yields

$$(10) \quad E_\alpha(-x^\alpha) = \frac{\sin \pi \alpha}{\pi} \int_{\mathbb{R}_+} \frac{t^{\alpha-1} e^{-xt}}{t^{2\alpha} + 2t^\alpha \cos \pi \alpha + 1} dt,$$

so that after a change of variable,

$$U_\alpha(x) = \frac{\alpha}{\Gamma(\alpha)} \int_{\mathbb{R}_+} \frac{t^{\alpha-1} e^{-t}}{(t/x)^{2\alpha} + 2(t/x)^\alpha \cos \pi \alpha + 1} dt$$

and the right-hand side is easily seen to be smaller than α if $\alpha \leq 1/2$, respectively smaller than $\alpha/\sin^2(\pi\alpha)$ if $\alpha > 1/2$. Putting everything together entails that H_α will be Bernstein and hence F_α will be CM whenever $\alpha \leq \max(1/2, \sin^2 \pi \alpha)$ i.e. $\alpha \in [0, \alpha^*]$ for some $\alpha^* \in]2/3, 3/4[$.

All in all, we showed that there exists a threshold $\alpha_0 \in]2/3, 3/4[$ such that F_α is not CM if $\alpha > \alpha_0$ and F_α is CM if $\alpha < \alpha_0$. A straightforward continuity argument entails that F_{α_0} itself is CM, and the proof is complete. \square

Remarks. (a) From the final part of our proof, one may wonder whether $U_\alpha \leq 1$ whenever F_α is CM, in other words whether \tilde{Z}_α is infinitely divisible (ID) as soon as g_α is a density function. This is however not the case, since the threshold variable \tilde{Z}_{α_0} is not ID. Indeed, we easily see from (5) that there exists some $t_0 > 0$ and a neighbourhood \mathcal{V}_0 of α_0 such that $g_\alpha(t) > 0$ for every $\alpha \in \mathcal{V}_0$ and every $t \geq t_0$. From the continuity of the application

$(\alpha, t) \mapsto g_\alpha(t)$ - which comes from (2) - and the fact that g_α is a density iff it is non-negative, one sees that g_{α_0} must vanish inside $]0, t_0[$. If \tilde{Z}_{α_0} were ID, then its Laplace exponent would be H_α and we see by (7) that \tilde{Z}_{α_0} would be driftless and that the support of its Lévy measure would contain zero. From Theorem 24.10 in [7] this would entail that the support of \tilde{Z}_{α_0} is \mathbb{R}^+ and since g_α is smooth, Theorem 3.3 in [8] entails that g_α would never vanish on $]0, +\infty[$, a contradiction.

On the other hand, our proof shows that Z_α is ID whenever $\alpha < \alpha_1$ for some $\alpha_1 \in]2/3, \alpha_0[$. Since (9) entails that $U_\alpha(x) \rightarrow \alpha$ as $x \rightarrow +\infty$ for every $\alpha \in]0, 1[$, similarly as above there exists $x_1 > 0$ and a neighbourhood \mathcal{V}_1 of α_1 such that $U_\alpha(x) < 1$ for every $x > x_1$ and every $\alpha \in \mathcal{V}_1$. By the continuity of $(\alpha, x) \mapsto U_\alpha(x)$, this entails that $U_{\alpha_1} \leq 1$ everywhere so that the critical variable \tilde{Z}_{α_1} is ID. In particular, from the above discussion we see that $\alpha_1 < \alpha_0$.

(b) The last part of our proof and the well-known characterization given e.g. by [7] Corollary 15.11, entail also that \tilde{Z}_α is self-decomposable (SD) iff the function

$$x \mapsto x^{-\alpha} - \alpha\Gamma(1 - \alpha)E_\alpha(-x^\alpha)$$

is non-increasing, which is equivalent to the fact that $V_\alpha(x) = \alpha x^{2\alpha}\Gamma(1 - \alpha)E'_\alpha(-x^\alpha) \leq 1$ for every $x \geq 0$. Differentiating (10) and changing the variable yield on the other hand

$$V_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^+} \frac{t^\alpha e^{-t}}{(t/x)^{2\alpha} + 2(t/x)^\alpha \cos \pi\alpha + 1} dt.$$

Discussions similar to the above show then that \tilde{Z}_α is SD iff $\alpha \leq \alpha_2$ for some $\alpha_2 > 2/3$. Notice also with the previous notations that since $1 - U_{\alpha_1}$ must vanish inside $]0, x_1[$ and not at infinity, the variable \tilde{Z}_{α_1} itself is not SD, so that $\alpha_2 < \alpha_1$.

(c) The two above remarks allow to give a characterization of the two thresholds α_1, α_2 in terms of extremal properties for the Mittag-Leffler function E_α and its derivative E'_α , through inequalities for the above functions U_α and V_α . But we could not find any simple characterization for the critical value α_0 , save of course by the CM property for F_α . One could also view α_0 as the largest index α for which the function

$$\frac{(1 + is)\Gamma(1 + is/\alpha)}{\Gamma(1 + is)}$$

is a characteristic function but this does not seem to give any valuable information on α_0 . Notice that formula (3.6) in [5] draws attention upon a characteristic function which is infinitely divisible (but not SD) and somewhat close to the above function.

(d) The function

$$x \mapsto \frac{F_\alpha(x)}{2\Gamma(1 + 1/\alpha)}$$

is a density over \mathbb{R}^+ for every $\alpha > 0$ and we know from our proof that it is log-convex iff $\alpha \leq 3/4$. In particular, Theorem 51.4 in [7] entails that the corresponding variable is ID whenever $\alpha \leq 3/4$. An easy computation entails that it is also ID for $\alpha = 1$, but we do not know whether it remains ID for $\alpha \in]3/4, 1[$, as for the completely monotone density $e^{-x^\alpha}/\Gamma(1 + 1/\alpha)$.

(e) When F_α is CM, the factorization $e^{-\lambda^\alpha} = (1 + \alpha\lambda^\alpha)^{-1}F_\alpha(\lambda)$ gives an other, additive identity in law

$$Z_\alpha \stackrel{d}{=} \alpha^{1/\alpha}M_\alpha + \tilde{Z}_\alpha$$

where M_α is a so-called Mittag-Leffler random variable [6] independent of \tilde{Z}_α . A straightforward computation - see also the final remark in [6] - gives also for every $\alpha \in]0, 1[$ the identity

$$M_\alpha \stackrel{d}{=} Z_\alpha \times L^{1/\alpha}$$

where L is an independent standard exponential variable. On the other hand, it is purposeless to search for a multiplicative factorization of Z_α through M_α since the latter is unimodal with mode at zero (this comes from the fact that its density is $\alpha x^{\alpha-1}E'_\alpha(-x^\alpha)$, a decreasing function over \mathbb{R}^+ - see (18.1.6) in [2]) whereas Z_α is not, and this would contradict Khintchine's theorem.

(f) When F_α is CM, the multiplicative factorization for Z_α can also be read $Z_\alpha \stackrel{d}{=} e^L \times \tilde{Z}_\alpha$ where L is a standard exponential variable. In view of Kanter's "polynomial" representation

$$Z_\alpha \stackrel{d}{=} L^{(\alpha-1)/\alpha} \times Y_\alpha$$

for some variable Y_α - see Corollary 4.1 in [4], our identity might seem a little bit surprising.

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