The lattice of embedded subsets

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Abstract

In cooperative game theory, games in partition function form are real-valued function on the set of so-called embedded coalitions, that is, pairs (S, π) where S is a subset (coalition) of the set N of players, and π is a partition of N containing S. Despite the fact that many studies have been devoted to such games, surprisingly nobody clearly defined a structure (i.e., an order) on embedded coalitions, resulting in scattered and divergent works, lacking unification and proper analysis. The aim of the paper is to fill this gap, thus to study the structure of embedded coalitions (called here embedded subsets), and the properties of games in partition function form.

Keywords: partition, embedded subset, game, valuation, k-monotonicity

1 Introduction

The Boolean lattice of subsets and the lattice of partitions are two well-known posets, with numerous applications in decision, game theory, classification, etc.

We consider in this paper a more complex structure, which is in some sense a combination of the above two. Its origin comes from cooperative game theory. Let us consider a set N of n players, and define a real-valued function v on 2^N, such that $v(\emptyset) = 0$. Such a function is called a *game* on N, and for any coalition (subset) $S \subseteq N$, the quantity $v(S)$ represents the "worth" or "power" of coalition S. Hence a game is a function on the Boolean lattice 2^N vanishing at the bottom element. Having a closer look at the meaning of $v(S)$, we could say more precisely: suppose that players in S form a coalition, the other players in $N \setminus S$ forming the "opponent" group. Then $v(S)$ is the amount of money earned by S (or the power of S) in such a dichotomic situation.

A more realistic view would be to consider that the opponent group may also be divided into groups, say $S_2, \dots, S_k$, so that $\{S_2, \dots, S_k\}$ form a partition of $N \setminus S$. In this case it is likely that the value earned by S may depend on the partition of $N \setminus S$.

Hence we are lead to define the quantity $v(S, \pi)$, where π is a partition of N containing S as a block, i.e., $\pi = \{S, S_2, \ldots, S_k\}$.

Such games are called *games in partition function form*, while (S, π) is called an *embedded coalition*, and have been introduced by Thrall and Lucas [21]. Despite the fact that many works have been undertaken on this topic (let us cite, among others, Myerson [17], Bolger [6], Do and Norde [9], Fujinaka [10], Clippel and Serrano [8], Albizuri et al. [2], Macho-Stadler et al. [16], who all propose various definitions and axiomatizations of the Shapley value, and Funaki and Yamato [11] who deal with the core, etc.), surprisingly nobody has clearly defined a structure for embedded coalitions. As a consequence, most of these works have divergent point of views, lack unification, and do not provide a good mathematical analysis of the concept of game in partition function form. This paper tends to fill this gap. We propose a natural structure for embedded coalitions (which we call *embedded subsets*), which is a lattice, study it, and provide a variety of results described below. Our analysis will consider in particular the following points, all motivated by game theory but also commonly considered in the field of posets:

(i) The number of maximal chains between two given embedded subsets. Many concepts in game theory are defined through maximal chains, like the Shapley value [19] and the core of convex games [20]. This is addressed in Section 3, where a thorough analysis of the poset of embedded subsets is done. Based on these results, Grabisch and Funaki propose in [14] a definition of the Shapley value, different from the ones cited above, and having good properties.

(ii) The Möbius function on the lattice of embedded subsets. Usually the function obtained by the Möbius inversion on a game is called the *Möbius transform* of this game, or *the dividends*. It is a fundamental notion, permitting to express a game in the basis of unanimity games. This topic is addressed in Section 5, and is a new achievement in the theory of games in partition function form.

(iii) Particular classes of games, like additive games, super- and submodular games (Section 4), ∞-monotone games, also called, up to some differences, positive games or belief functions, and minitive games (Section 6). Additive games, corresponding to valuations on lattices, are basic in game theory since they permit to define the core and all procedures of sharing. Super- and submodular functions are also very common in game theory and combinatorial optimization. ∞-monotone games are especially important because in the classical Boolean case they have nonnegative dividends (Möbius transform), and they are well-known in artificial intelligence and decision theory under the name of belief functions. Lastly, minitive games, also called necessity functions in artificial intelligence, are particular belief functions. In poset language, they are inf-preserving mappings. Our analysis brings many new results, establishing the existence of these particular classes of games.

2 Background on partitions

In this section, we introduce our notation and recall useful results on the geometric lattice of partitions (see essentially Aigner [1], and [12]), and prove some results needed in the following.

We consider the set $[n] =: N$, and denote its subsets by $S, T, S', T', \ldots$, and by $s, t, s', t', \ldots$ their respective cardinalities. The set of partitions of $[n]$ is denoted by $\Pi(n)$. Partitions are denoted by $\pi, \pi', \ldots$, and $\pi = \{S_1, \ldots, S_k\}$, $S_1, \ldots, S_k \in 2^N$. Subsets $S_1, \ldots, S_k$ are called *blocks* of π. A partition into k blocks is a k-partition.

Taking π, π' partitions in $\Pi(n)$, we say that π is a *refinement* of π' (or π' is a *coarsening* of π), denoted by $\pi \leq \pi'$, if any block of π is contained in a block of π' (or every block of π' fully decomposes into blocks of π). When endowed with the refinement relation, $(\Pi(n), \leq)$ is a lattice, called the *partition lattice*.

We use the following shorthands: $\pi^\top := \{N\}$, $\pi^\perp := \{\{1\}, \ldots, \{n\}\}$. For any $\emptyset \neq S \subset N$, $\pi_S^\top := \{S, N \setminus S\}$, $\pi_S^\perp := \{S, \{i_1\}, \ldots, \{i_{n-s}\}\}$, with $N \setminus S =: \{i_1, \ldots, i_{n-s}\}$. Also, for any two partitions π, π' such that $\pi \leq \pi'$, the notation $[\pi, \pi']$ means as usual the set of all partitions π'' such that $\pi \leq \pi'' \leq \pi'$.

The following facts on $\Pi(n)$ will be useful in the following:

(i) The number of partitions of k blocks is $S_{n,k}$ (Stirling number of the second kind), with

$$S_{n,k} := \frac{1}{k!} \sum_{i=0}^{n} (-1)^{k-i} \binom{k}{i} i^n, \quad n \geq 0, k \leq n.$$

(ii) Each partition π covers $\sum_{S \in \pi} 2^{|S|-1} - |\pi|$ partitions. Each k-partition is covered by $\binom{k}{2}$ partitions.

(iii) Let $\pi := \{S_1, \ldots, S_k\}$ be a k-partition. Then we have the following isomorphisms:

$$\begin{aligned}
[\pi, \pi^\top] &\cong \Pi(k) \\
[\pi^\perp, \pi] &\cong \prod_{i=1}^{k} \Pi(s_i) \\
[\pi, \pi'] &\cong \prod_{i=1}^{|\pi'|} \Pi(m_i) \text{ for some } m_i\text{'s with } \sum_{i=1}^{|\pi'|} m_i = k.
\end{aligned}$$

We give the following results (up to our knowledge, some of them have not yet been investigated), which will be used in the following. We use the notation $\mathcal{C}(P)$ to denote the set of maximal chains from bottom to top in the poset P, whenever this makes sense.

Proposition 1. Let $\pi, \pi' \in \Pi(n)$ such that $\pi' < \pi$, with $\pi := \{S_1, \ldots, S_k\}$ and $\pi' := \{S_{11}, \ldots, S_{1l_1}, S_{21}, \ldots, S_{2l_2}, \ldots, S_{kl_k}\}$, with $\{S_{i1}, \ldots, S_{il_i}\}$ a partition of S_i, $i = 1, \ldots, k$, and $k' := \sum_{i=1}^{k} l_i$.

(i) The number of maximal chains of $\Pi(n)$ from bottom to top is

$$|\mathcal{C}(\Pi(n))| = \frac{n!(n-1)!}{2^{n-1}}.$$

(ii) The number of maximal chains from $\pi^\perp$ to π is

$$|\mathcal{C}([\pi^\perp, \pi])| = \frac{(n-k)!}{2^{n-k}} s_1! s_2! \cdots s_k!.$$

(iii) The number of maximal chains from π to $\pi^\top$ is

$$|\mathcal{C}([\pi, \pi^\top])| = |\mathcal{C}(\Pi(k))|.$$

(iv) The number of maximal chains from π' to π is

$$|\mathcal{C}([\pi', \pi])| = \frac{(k'-k)!}{2^{k'-k}} l_1! l_2! \cdots l_k!$$

Proof. (i) See Barbut and Monjardet [3, p. 103].

(ii) We use the fact that for any k-partition $\pi = \{S_1, \dots, S_k\}$, $[\pi^\perp, \pi] = \prod_{i=1}^k \Pi(s_i)$. We have the general following fact: if $L = L_1 \times \cdots \times L_k$, then to obtain all maximal chains in L, we select a k-uple of maximal chains $C_1, \dots, C_k$ in $L_1, \dots, L_k$ respectively, say of length $c_1, \dots, c_k$. Then the number of chains induced by $C_1, \dots, C_k$ in L is equal to the number of chains in the lattice $C_1 \times \cdots \times C_k$, isomorphic to the lattice $c_1 \times \cdots \times c_k$ (c_i denotes the linear lattice of c_i elements). This is known to be

$$\frac{(\sum_{i=1}^k c_i)!}{\prod_{i=1}^k (c_i!)}.$$

Applied to our case, this gives

$$|\mathcal{C}([\pi^\perp, \pi])| = \prod_{i=1}^k |\mathcal{C}(\Pi(s_i)| \frac{\left(\sum_{i=1}^k (s_i - 1)\right)!}{\prod_{i=1}^k (s_i - 1)!}$$

which after simplification gives the desired result, using the fact that $\sum_{i=1}^k (s_i - 1) = n - k$.

(iii) Immediate from $[\pi, \pi^\top] \cong \Pi(k)$.

(iv) Simply consider $S_{11}, \dots, S_{1l_1}, S_{21}, \dots, S_{2l_2}, \dots, S_{kl_k}$, and use (ii) for $\Pi(k')$.

□

3 The structure of embedded subsets

An *embedded subset* is a pair (S, π) where $S \in 2^N \setminus \{\emptyset\}$ and $\pi \ni S$, where $\pi \in \Pi(n)$. We denote by $\mathfrak{C}(N)$ (or by $\mathfrak{C}(n)$) the set of embedded coalitions on N. For the sake of concision, we often denote by $S\pi$ the embedded coalition (S, π), and omit braces and commas for subsets (example with $n = 3$: $12\{12, 3\}$ instead of $(\{1, 2\}, \{\{1, 2\}, \{3\}\})$). Remark that $\mathfrak{C}(N)$ is a proper subset of $2^N \times \Pi(N)$.

As mentionned in the introduction, works on games in partition function form do not explicitly define a structure (that is, some order) on embedded coalitions. A natural choice is to take the product order on $2^N \times \Pi(N)$:

$$(S, \pi) \sqsubseteq (S', \pi') \Leftrightarrow S \subseteq S' \text{ and } \pi \leq \pi'.$$

Evidently, the top element of this ordered set is $(N, \pi^\top)$ (denoted more simply by $N\{N\}$ according to our conventions). However, due to the fact that the empty set is not allowed in (S,π), there is no bottom element in the poset $(\mathfrak{C}(N),\sqsubseteq)$, since all elements of the form $(\{i\},\pi^\perp)$ are minimal elements. For mathematical convenience, we introduce an artificial bottom element $\perp$ to $\mathfrak{C}(N)$ (it could be considered as $(\emptyset,\pi^\perp)$), and denote $\mathfrak{C}(N)_\perp := \mathfrak{C}(N)\cup\{\perp\}$. We give as illustration the partially ordered set $(\mathfrak{C}(N)_\perp,\sqsubseteq)$ with $n=3$ (Fig. 1).

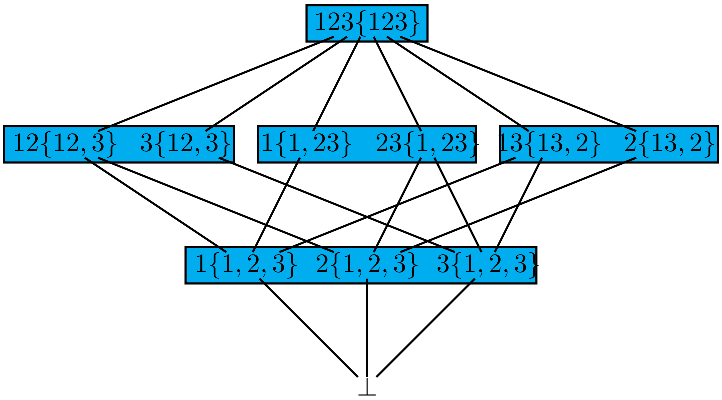

Figure 1: Hasse diagram of $(\mathfrak{C}(N)_\perp,\sqsubseteq)$ with $n=3$. Elements with the same partition are framed in grey.

As the next proposition will show, $(\mathfrak{C}(n)_\perp,\sqsubseteq)$ is a lattice, whose main properties are given below. We investigate in particular the number of maximal chains between two elements of this lattice. The reason is that, in game theory, many notions are defined through maximal chains (e.g., the Shapley value, the core, etc.). The cases $n=1,2$ are discarded since trivial ($\mathfrak{C}(2)_\perp = 2^2$). The standard terminology used hereafter can be found in any textbook on lattices and posets (e.g., [1, 7, 15]).

Proposition 2. For any $n>2$, $(\mathfrak{C}(n)_\perp,\sqsubseteq)$ is a lattice, with the following properties:

(i) Supremum and infimum are given by

$$(S,\pi)\vee(S',\pi') = (T\cup T',\rho)$$
$$(S,\pi)\wedge(S',\pi') = (S\cap S',\pi\wedge\pi') \text{ if } S\cap S'\neq\emptyset, \text{ and } \perp \text{ otherwise},$$

where T,T' are blocks of $\pi\vee\pi'$ containing respectively S and S', and ρ is the partition obtained by merging T and T' in $\pi\vee\pi'$.

(ii) Top and bottom elements are $N\{N\}$ and $\perp$. Every element is complemented; for a given $S\pi$, any embedded subset of the form $\overline{S}\pi_{\overline{S}}$ with $\overline{S}$ the complement of S and $\pi_{\overline{S}}$ any partition containing $\overline{S}$, is a complement of $S\pi$.

(iii) Each element $S\pi$ where $\pi := \{S,S_2,\dots,S_k\}$ is a k-partition is covered by $\binom{k}{2}$ elements, and covers $\sum_{T\in\pi}2^{t-1}-|\pi|+2^{s-1}-1$ elements.

(iv) Its join-irreducible elements are $(i,\pi^\perp)$, $i\in N$ (atoms), and $(i,\pi^\perp_{jk})$, $i,j,k\in N$, $i\notin\{j,k\}$. Its meet-irreducible elements are (S,π) where π is any 2-partition (co-atoms).

(v) The lattice satisfies the Jordan-Dedekind chain condition (otherwise said, the lattice is ranked), and its height function is $h(S,\pi) = n - k + 1$, if π is a k-partition. The height of the lattice is n.

(vi) The lattice is not distributive (and even neither upper nor lower locally distributive), not atomistic (hence not geometric), not modular but upper semimodular.

(vii) The number of elements on level of height k is $kS_{n,k}$. The total number of elements is $\sum_{k=1}^{n} kS_{n,k} + 1$.

n	1	2	3	4	5	6	7	8
$\lvert\mathfrak{C}(n)_\perp\rvert$	2	4	11	38	152	675	3264	17008

(viii) Let $S\pi := S\{S, S_2, \ldots, S_k\}$ and $S'\pi' := S'\{S', S_{12}, \ldots, S_{1l_1}, S_{21}, \ldots, S_{2l_2}, \ldots, S_{k1}, \ldots, S_{kl_k}\}$, and $k' := \sum_{i=1}^{k} l_i$. We have the following isomorphisms:

$$\begin{aligned}
[(i,\pi^\perp), N\{N\}] &\cong \Pi(n) \\
[\perp, S\pi] &\cong \big(\mathfrak{C}(S) \times \Pi(S_2) \times \cdots \times \Pi(S_k)\big) \cup \{\perp\} \\
[S\pi, N\{N\}] &\cong [i\{i, i_2, \ldots, i_k\}, K\{K\}] \cong \Pi(k) \\
[S'\pi', S\pi] &\cong [i\pi^\perp, S\pi]_{\mathfrak{C}(k')_\perp},
\end{aligned}$$

where the subscript $\mathfrak{C}(k')_\perp$ means that elements in the brackets are understood to belong to $\mathfrak{C}(k')_\perp$.

(ix) The number of maximal chains from $\perp$ to $N\{N\}$ is $\frac{(n!)^2}{2^{n-1}}$, which is also the number of maximal longest chains in $\mathfrak{C}(n)$.

n	1	2	3	4	5	6	7	8
$\lvert\mathcal{C}(\mathfrak{C}(n)_\perp)\rvert$	1	2	9	72	900	16 200	396 900	12 700 800

(x) Let $S\pi$ be an embedded subset, with $\pi := \{S, S_2 \ldots, S_k\}$, and $|S| = s$. The number of maximal chains from $\perp$ to $S\pi$ is

$$|\mathcal{C}([\perp, S\pi])| = \frac{s(n-k)!}{2^{n-k}} s! s_2! \cdots s_k!.$$

The number of maximal chains from (S,π) to $N\{N\}$ is

$$|\mathcal{C}([(S,\pi), N\{N\}])| = \frac{1}{k} |\mathcal{C}(\mathfrak{C}(k)_\perp)| = \frac{k!(k-1)!}{2^{k-1}}.$$

(xi) Let $S'\pi' < S\pi$ with the above notation. The number of maximal chains from $S'\pi'$ to $S\pi$ is

$$|\mathcal{C}([S'\pi', S\pi])| = \frac{l_1(k'-k)}{2^{k'-k}} l_1! l_2! \ldots l_k!$$

Proof. Consider $(S, \pi), (S', \pi') \in \mathfrak{C}(n)_\perp$. Then (K, ρ) is an upper bound of both elements iff $K \supseteq S \cup S'$ and $\rho \geq \pi \vee \pi'$ (clearly exists). If $S \cup S' \in \pi \vee \pi'$ then $(S \cup S', \pi \vee \pi')$ is the least upper bound. If not, since $S \in \pi$ and $S' \in \pi'$ and by definition of $\pi \vee \pi'$, there exist blocks T, T' of $\pi \vee \pi'$ such that $T \supseteq S$ and $T' \supseteq S'$. Then $(T \cup T', \rho)$, where ρ is the partition obtained by merging T and T' in $\pi \vee \pi'$, is the least upper bound of $(S, \pi), (S', \pi')$.

Next, $(S \cap S', \pi \wedge \pi')$ would be the infimum if $S \cap S'$ is a block of $\pi \wedge \pi'$. If $S \cap S' \neq \emptyset$, then this is the case. If not, then $\perp$ is the only lower bound. This proves that $(\mathfrak{C}(n)_\perp, \leq)$ is a lattice, and (i), (ii) hold (the assertion on complemented elements is clear).

(iii) Clear from (ii) in Section 2.

(iv) Clear from (iii).

(v) From (iii), $S\pi$ covers $S'\pi'$ implies that if π is a k-partition, then π' is a $(k+1)$-partition. Hence, a maximal chain from $S\pi$ to the bottom element has length $n - k + 1$, which proves the Jordan-Dedekind chain condition. Now, the height function is $h(S\pi) = n - k + 1$.

(vi) The lattice is not (upper or lower locally) distributive since it contains diamonds. For example, with $n = 3$, the following 5 elements form a diamond (see Fig. 1):

$$(1, \{1, 2, 3\}), (12, \{12, 3\}), (1, \{1, 23\}), (13, \{13, 2\}), (123, \{123\}).$$

For atomisticity see (iv). Let us prove it is upper semimodular. Since the lattice is ranked, if x covers $x \wedge y$ and $x \wedge y$, then both x and y are one level above $x \wedge y$. Hence using (iii), if $x \wedge y := (S, \{S, S_2, \ldots, S_k\})$, then x has either the form $(S, \{S, S_i \cup S_j, \ldots\})$ or $(S \cup S_i, \{S \cup S_i, \ldots\})$, and similarly $y = (S, \{S, S_k \cup S_l, \ldots\})$ or $(S \cup S_j, \{S \cup S_j, \ldots\})$. To compute $x \vee y$, we have three cases:

- $x = (S, \{S, S_i \cup S_j, \ldots\})$ and $y = (S, \{S, S_k \cup S_l, \ldots\})$: then $x \vee y = (S, \{S, S_i \cup S_j, S_k \cup S_l, \ldots\})$ if $k, l \neq i, j$. If, e.g., $k = i$, $x \vee y = (S, \{S, S_i \cup S_j \cup S_l, \ldots\})$.
- $x = (S \cup S_i, \{S \cup S_i, \ldots\})$ and $y = (S \cup S_j, \{S \cup S_j, \ldots\})$: then $x \vee y = (S \cup S_i \cup S_j, \{S \cup S_i \cup S_j, \ldots\})$.
- $x = (S, \{S, S_i \cup S_j, \ldots\})$ and $y = (S \cup S_k, \{S \cup S_k, \ldots\})$: then $x \vee y = (S \cup S_k, \{S \cup S_k, S_i \cup S_j, \ldots\})$ if $k \neq i, j$. If $k = i$, $x \vee y = (S \cup S_i \cup S_j, \{S \cup S_i \cup S_j, \ldots\})$.

In all cases, we get a $(k-2)$-partition, so upper modularity holds. Lower semimodularity does not hold. Taking the example of $\mathfrak{C}(3)_\perp$, $123\{123\}$ covers $12\{12, 3\}$ and $3\{3, 12\}$, but these elements do not cover $12\{12, 3\} \wedge 3\{12, 3\} = \perp$.

(vii) Clear from the results on $\Pi(n)$.

(viii) Consider the element $(i, \pi^\perp)$ in $\mathfrak{C}(n)_\perp$, $i \in N$. Then $[(i, \pi^\perp), N\{N\}]$ is a sublattice isomorphic to $\Pi(n)$, since by (iii) the number of elements covering $(i, \pi^\perp)$ is the same as the number of elements covering $\pi^\perp$ in $\Pi(n)$, and that this property remain true for all elements above $(i, \pi^\perp)$.

The other assertions are clear.

(ix) Since by Prop. 1, $\mathcal{C}(\Pi(n)) = \frac{n((n-1)!)^2}{2^{n-1}}$, and using (viii) and the fact that there are n mutually incomparable elements $(i, \pi^\perp)$ in $\mathfrak{C}(n)_\perp$, the result follows.

(x) The proof follows the same technique as for Prop. 1 (ii). Using the second assertion of (viii) and noting that deleting the bottom element does not change the number of maximal (longest) chains, we can write immediately

$$|\mathcal{C}([\bot, S\pi])| = \prod_{i=1}^{k} |\mathcal{C}(\Pi(s_i))||\mathcal{C}(\mathfrak{C}(s))| \frac{\left(\sum_{i=1}^{k} s_i - 1\right)!}{\prod_{i=1}^{k}(s_i - 1)!}.$$

The result follows by using Prop. 1 (i) and (ix).

The second assertion is clear since $[S\pi, N\{N\}]$ is isomorphic to $\Pi(k)$ by (viii).

(xi) Same as for Proposition 1 (iv). □

4 Functions on $\mathfrak{C}(n)_\bot$

We investigate properties of some classes of real-valued functions over $\mathfrak{C}(n)_\bot$. As our motivation comes from game theory, we will focus on games, that is, functions vanishing at the bottom element, and on valuations, which are related to additive games, another fundamental notion in game theory.

Definition 1. A *game in partition function form* on N (called here for short simply *game on* $\mathfrak{C}(N)_\bot$) is a mapping $v : \mathfrak{C}(N)_\bot \to \mathbb{R}$, such that $v(\bot) = 0$. The set of all games in partition function form on N is denoted by $\mathcal{PG}(N)$.

Definition 2. Let $v \in \mathcal{PG}(N)$.

(i) v is *monotone* if $S\pi \sqsubseteq S'\pi'$ implies $v(S\pi) \leq v(S'\pi')$. A monotone game on $\mathfrak{C}(N)_\bot$ is called a *capacity on* $\mathfrak{C}(N)_\bot$. A capacity on $\mathfrak{C}(N)_\bot$ v is normalized if $v(N\{N\}) = 1$.

(ii) v is *supermodular* if for every $S\pi, S'\pi'$ we have

$$v(S\pi \vee S'\pi') + v(S\pi \wedge S'\pi') \geq v(S\pi) + v(S'\pi').$$

It is *submodular* if the reverse inequality holds.

(iii) A game is *additive* if it is both supermodular and submodular.

(iv) More generally, for a given $k \geq 2$, a game is *k-monotone* if for all families of k elements $S_1\pi_1, \dots, S_k\pi_k$ (not necessarily different), we have

$$v\big(\bigvee_{i \in K} S_i\pi_i\big) \geq \sum_{J \subseteq K, J \neq \emptyset} (-1)^{|J+1|} v\big(\bigwedge_{i \in J} S_i\pi_i\big)$$

putting $K := \{1, \dots, k\}$. A game is *∞-monotone* if it is k-monotone for every $k \geq 2$. Note that k-monotonicity implies k'-monotonicity for any $2 \leq k' \leq k$.

(v) A game is a *belief function* if it is a normalized ∞-monotone capacity.

The following result is due to Barthélemy [4].

Proposition 3. Let L be a lattice. Then f is monotone and ∞-monotone on L if and only if it is monotone and $(|L| - 2)$-monotone.

In lattice theory, a *valuation* (or *2-valuation*) on a lattice L is a real-valued function on L being both super- and submodular (i.e., it is additive in our terminology). More generally, for a given $k \geq 2$, a *k-valuation* satisfies

$$v(\bigvee_{i\in K} x_i) = \sum_{J\subseteq K, J\neq\emptyset} (-1)^{|J+1|} v(\bigwedge_{i\in J} x_i)$$

for every family of k elements. An ∞-valuation is a function f which is a k-valuation for every $k \geq 2$. The following well-known results clarify the existence of valuations (see [5, Ch. X], and also [4]).

Proposition 4. Let L be a lattice.

(i) L is modular if and only if it admits a strictly monotone valuation.

(ii) L is distributive if and only if it admits a strictly monotone 3-valuation.

(iii) L is distributive if and only if it is modular and every strictly monotone valuation is a k-valuation for any $k \geq 2$.

(iv) Any lattice admits an ∞-valuation.

The consequence of (i) is that no strictly monotone additive game exists since $\mathfrak{C}(n)_\perp$ is not modular when $n > 2$. The question is: does it exist an additive game? We shall prove that the answer is no (except for the trivial game $v = 0$) as soon as $n > 2$.

Proposition 5. $\mathfrak{C}(2)_\perp$ admits a strictly monotone 2-valuation (hence by Prop. 3, a strictly monotone ∞-monotone valuation). If $n > 2$, the only possible valuations on $\mathfrak{C}(n)_\perp$ are constant valuations.

Proof. For $n = 2$, the result is clear from Proposition 4 since $\mathfrak{C}(2)_\perp = 2^2$. For $n > 2$, we shall proceed by induction on n. Let us show the result for $n = 3$. Let $f : \mathfrak{C}(3)_\perp \to \mathbb{R}$. To check whether f is a valuation amounts to verify that

$$f(x) + f(y) = f(x \vee y) + f(x \wedge y), \quad \forall x, y \in \mathfrak{C}(3),\ x \text{ and } y \text{ not comparable.}$$

This leads to a linear system, for which the constant function is an obvious solution. Let us show that it is the only one. We extract the following subsystem, naming for brevity elements $S\pi$ by $a, b, c, \ldots$ as on Figure 2:

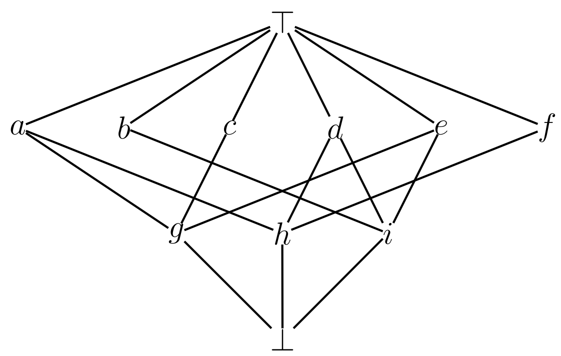

Figure 2: Hasse diagram of $(\mathfrak{C}(3)_\perp, \sqsubseteq)$

$$
\begin{align}
f(a)+f(b) &= f(\top)+f(\perp) \tag{1}\\
f(a)+f(c) &= f(\top)+f(g) \tag{2}\\
f(a)+f(e) &= f(\top)+f(g) \tag{3}\\
f(a)+f(i) &= f(\top)+f(\perp) \tag{4}\\
f(b)+f(c) &= f(\top)+f(\perp) \tag{5}\\
f(b)+f(f) &= f(\top)+f(\perp) \tag{6}\\
f(b)+f(g) &= f(\top)+f(\perp) \tag{7}\\
f(b)+f(h) &= f(\top)+f(\perp) \tag{8}\\
f(c)+f(d) &= f(\top)+f(\perp) \tag{9}\\
f(c)+f(f) &= f(\top)+f(\perp) \tag{10}\\
f(c)+f(h) &= f(\top)+f(\perp) \tag{11}\\
f(f)+f(g) &= f(\top)+f(\perp) \tag{12}\\
f(f)+f(i) &= f(\top)+f(\perp) \tag{13}
\end{align}
$$

$f(b) = f(i)$ from (1) and (4), $f(c) = f(f) = f(g) = f(h)$ from (5), (6), (7) and (8), $f(d) = f(f) = f(h)$ from (9), (10) and (11), $f(g) = f(i)$ from (12) and (13). Then $f(g) = f(\perp)$ from (1) and (2), and hence $f(a) = f(\top)$ from (1), and $f(e) = f(g)$ from (3). Finally $f(\top) = f(\perp)$ from (5). This proves the assertion for $n = 3$.

Assume the assertion holds till n, and let us prove it for $n+1$. The idea is to cover entirely $\mathfrak{C}(n+1)_\perp$ by overlapping copies of $\mathfrak{C}(n)_\perp$. Then since on each copy the valuation has to be constant by assumption, it will be constant everywhere. Let us consider the set $\mathfrak{C}[ij]$ of those embedded subsets in $\mathfrak{C}(n+1)_\perp$ where elements $i, j \in N$ belong to the same block (i.e., as if $[ij]$ were a single element). Then this set plus the bottom element $\perp$ of $\mathfrak{C}(n+1)_\perp$, which we denote by $\mathfrak{C}[ij]_\perp$, is isomorphic to $\mathfrak{C}(n)_\perp$. Moreover, $\mathfrak{C}[ij]_\perp$ is a sublattice of $\mathfrak{C}(n+1)_\perp$, since the supremum and infimum in $\mathfrak{C}(n+1)_\perp$ of embedded subsets in $\mathfrak{C}[ij]_\perp$ remains in $\mathfrak{C}[ij]_\perp$ (because $[ij]$ is never splitted when taking the infimum and supremum over partitions). This proves that the system for equations of valuation in $\mathfrak{C}[ij]_\perp$ is the same than the set of equations in $\mathfrak{C}(n+1)_\perp$ restricted to $\mathfrak{C}[ij]_\perp$. Taking all possibilities for i, j cover all elements of $\mathfrak{C}(n+1)_\perp$, except atoms. Also, since $\top$ and $\perp$ belongs to each $\mathfrak{C}[ij]_\perp$, an overlap exists. It remains to cover the set of atoms. For this we consider $\mathfrak{C}(n)_\perp$, and to each embedded subset $S\pi$ we add the block $\{n+1\}$ in π.

Again, this is a sublattice of $\mathfrak{C}(n+1)_\perp$ isomorphic to $\mathfrak{C}(n)_\perp$, covering all atoms except $\{n+1\}\pi^\perp$. For this last one, it suffices to do the same on the set $\{2,\dots,n+1\}$ and to add the block $\{1\}$ to each partition. □

Since any game v satisfies $v(\perp) = 0$, we have:

Corollary 1. The only additive game is the constant game $v = 0$.

We comment on this surprising result. Our definition of an additive game follows tradition in the theory of posets: additivity means both supermodularity and submodularity, and as a consequence, it converts supremum (similar as union) into addition, provided the elements do not cover a common element different from the bottom element (similar to elements with an empty intersection). But it does not match with the traditional view in game theory, where additive games are assimilated to imputations, hence to values. An imputation is a vector defined on the set of players, often indicating how the total worth of the game is shared among players. Incidentally, in the classical setting, both notions coincide. In this more complex structure, this is no more the case. The nonexistence of additive games does not imply therefore the absence of imputation or value. It simply says that it is not possible to have an additivity property (converting supremum into a sum) for such a game. The consequence is that games in partition function form cannot be written in a simpler form, like an additive game which is equivalent to a n-dimensional vector in the classical setting.

5 The Möbius function on $\mathfrak{C}(n)_\perp$

The Möbius function is a central notion in combinatorics and posets (see [18]). It is also very useful in cooperative game theory, since it leads to the Möbius transform (known in this domain as the Harsanyi dividends of a game), which are the coordinates of a game in the basis of unanimity games (see end of this section).

First we give the results for the lattice of partitions $\Pi(n)$.

Proposition 6. Let π, σ be partitions in $\Pi(n)$ such that $\pi < \sigma$. Let us denote by $b(\pi)$ the number of blocks of π, with $n_1, \dots, n_{b(\pi)}$ the sizes of the blocks. Then the Möbius function on $\Pi(n)$ is given by:

(i) $\mu_{\Pi(n)}(\pi^\perp, \{N\}) = (-1)^{n-1}(n-1)!$

(ii) $\mu_{\Pi(n)}(\pi, \{N\}) = (-1)^{b(\pi)-1}(b(\pi)-1)!$

(iii) $\mu_{\Pi(n)}(\pi^\perp, \pi) = (-1)^{n-b(\pi)}(n_1-1)!\cdots(n_{b(\pi)-1})!$

(iv) $\mu_{\Pi(n)}(\pi, \sigma) = (-1)^{b(\pi)-b(\sigma)}(m_1-1)!\cdots(m_{b(\sigma)}-1)!$, with $m_1, \dots, m_{b(\pi)}$ integers such that $\sum_{i=1}^{b(\sigma)} m_i = b(\pi)$.

Proof. (i) is proved in Aigner [1, p. 154]. The rest is deduced from the isomorphisms given in Section 2. □

We recall also the following fundamental result [12].

Proposition 7. If P is a lattice with bottom element 0 and set of atoms $\mathcal{A}$, for every $x \in P$, the Möbius function reads

$$\mu(0,x) = \sum_{S\subseteq\mathcal{A} | \bigvee S = x} (-1)^{|S|}.$$

In $\mathfrak{C}(N)_\perp$, there are only a few elements representable by atoms. These are $S\pi_S^\perp$, where $\pi_S^\perp$ is the partition formed by S and singletons. The unique way to write $S\pi_S^\perp$ is $\bigvee_{i\in S}(i\pi^\perp)$. Hence, the Möbius function over $\mathfrak{C}(N)_\perp$, denoted simply by μ if no confusion occurs, otherwise by $\mu_{\mathfrak{C}(N)_\perp}$, is known for $(\perp, S\pi)$:

$$\mu(\perp, S\pi) = \begin{cases} (-1)^{|S|}, & \text{if } \pi = \pi_S^\perp \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Let us find $\mu(S\pi, N\{N\})$. Since $[S\{S, S_2, \dots, S_k\}, N\{N\}] \cong [i\{i, i_2, \dots, i_k\}, K\{K\}]$ (see Proposition 2 (viii)), we have $\mu_{\mathfrak{C}(n)_\perp}(S\{S, S_2, \dots, S_k\}, N\{N\}) = \mu_{\mathfrak{C}(k)_\perp}(i\pi^\perp, K\{K\})$. We know also that $[i\pi^\perp, N\{N\}] \cong \Pi(N)$, from which we deduce by Proposition 6 that $\mu(i\pi^\perp, N\{N\}) = \mu_{\Pi(n)}(\pi^\perp, \{N\}) = (-1)^{n-1}(n-1)!$. Hence

$$\mu(S\{S, S_2, \dots, S_k\}, N\{N\}) = (-1)^{k-1}(k-1)!. \tag{2}$$

It remains to compute the general expression for $(S'\pi', S\pi)$. We put $S\pi := S\{S, S_2, \dots, S_k\}$ and $S'\pi' := S'\{S', S_{12}, \dots, S_{1l_1}, S_{21}, \dots, S_{2l_2}, \dots, S_{k1}, \dots, S_{kl_k}\}$, and $k' := \sum_{i=1}^k l_i$. Then $[S'\pi', S\pi] \cong [i\pi^\perp, S\pi]_{\mathfrak{C}(k')_\perp}$, where the subscript $\mathfrak{C}(k')_\perp$ means that elements in the brackets are understood to belong to $\mathfrak{C}(k')_\perp$. Hence we deduce:

$$\mu(S'\pi', S\pi) = \mu_{\Pi(k')}(\pi^\perp, \pi) = (-1)^{k'-k}(l_1-1)!\cdots(l_k-1)!.$$

In summary, we have proved:

Proposition 8. The Möbius function on $\mathfrak{C}(n)_\perp$ is given by (with the above notation):

$$\mu(\perp, S\pi) = \begin{cases} (-1)^{|S|}, & \text{if } \pi = \pi_S^\perp \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

$$\mu(S'\pi', S\pi) = (-1)^{k'-k}(l_1-1)!\cdots(l_k-1)!, \text{ for } S'\pi' \sqsubseteq S\pi. \tag{4}$$

In particular, $\mu(i\pi^\perp, S\pi) = (-1)^{n-k}(s-1)!(s_2-1)!\cdots(s_k-1)!$.

Using this, the Möbius transform of any game v on $\mathfrak{C}(N)_\perp$ is defined by

$$m(S\pi) = \sum_{S'\pi'\sqsubseteq S\pi} \mu(S'\pi', S\pi)v(S'\pi'), \quad \forall S\pi \in \mathfrak{C}(N)_\perp.$$

Now, m gives the coordinates of v in the basis of unanimity games defined by:

$$u_{S\pi}(S'\pi') = \begin{cases} 1, & \text{if } S'\pi' \sqsupseteq S\pi \\ 0, & \text{otherwise.} \end{cases}$$

Example 1. Application for $n = 3$. Let us compute the Möbius transform of a game v. We have, for all distinct $i, j, k \in \{1, 2, 3\}$:

$$\begin{aligned}
m(\perp) &= 0\\
m(i\pi^\perp) &= v(i\pi^\perp)\\
m(ij\{ij,k\}) &= v(ij\{ij,k\}) - v(i\pi^\perp) - v(j\pi^\perp)\\
m(i\{i,jk\}) &= v(i\{i,jk\}) - v(i\pi^\perp)\\
m(123\{123\}) &= v(123\{123\}) - \sum_{i,j} v(ij\{ij,k\}) - \sum_i v(i\{i,jk\}) + 2\sum_i v(i\pi^\perp).
\end{aligned}$$

Consider now the following game: $v(123\{123\}) = 3$, $v(12\{12,3\}) = 2$, $v(3\{12,3\}) = 0$, $v(1\{1,23\}) = 1$, $v(23\{1,23\}) = 2$, $v(13\{13,2\}) = 1$, $v(2\{13,2\}) = 1$, $v(2\{13,2\}) = 1$, $v(1\{1,2,3\}) = 1$, $v(2\{1,2,3\}) = 1$, and $v(3\{1,2,3\}) = 0$. Let us find its coordinates in the basis of unanimity games. It suffices to compute $m(S\pi)$ for all $S\pi \in \mathfrak{C}(3)$ from the above formulas. We obtain:

$$m(1\{1,2,3\}) = m(2\{1,2,3\}) = m(23\{1,23\}) = 1$$

and $m(S\pi) = 0$ otherwise. Therefore,

$$v = u_{1\{1,2,3\}} + u_{2\{1,2,3\}} + u_{23\{1,23\}}.$$

This decomposition can be checked on the figure of $\mathfrak{C}(3)$ (Figure 1).

6 Belief functions and minitive functions on $\mathfrak{C}(n)_\perp$

Belief functions and minitive functions (i.e., inf-preserving mappings, also called necessity measures) are well-known in artificial intelligence and decision making, where the underlying lattice is the Boolean lattice. In the case of an arbitrary lattice, they have interesting properties, investigated by Barthlemy [4] and the author [13].

Barthélemy proved the following.

Proposition 9. Let L be any lattice, and $f : L \to \mathbb{R}$. If the Möbius transform of f, denoted by m, satisfies $m(\perp) = 0$, $m(x) \geq 0$ for all $x \in L$, and $\sum_{x \in L} m(x) = 1$ (normalization), then f is a belief function.

The converse of this proposition does not hold in general (it holds for the Boolean lattice 2^N). A belief function is *invertible* if its Möbius transform is nonnegative, normalized and vanishes at $\perp$. The following counterexample shows that for $\mathfrak{C}(n)_\perp$, there exist belief functions which are not invertible.

Example 2. Let us take $\mathfrak{C}(3)_\perp$, and consider a function f whose values are given on the figure below. Monotonicity implies that $1 \geq \beta \geq \alpha \geq 0$. In order to check ∞-monotonicity, from Proposition 3 we know that it suffices to check till 7-monotonicity. We write below the most constraining inequalities only, keeping in mind that $1 \geq \beta \geq \alpha \geq 0$.

2-monotonicity is equivalent to $\beta \geq 2\alpha$ and $1 \geq 2\beta - \alpha$. 3-monotonicity is equivalent to $1 \geq 3\beta - \alpha$. 4-monotonicity is equivalent to $1 \geq 4\beta - 3\alpha$. 5-monotonicity is equivalent

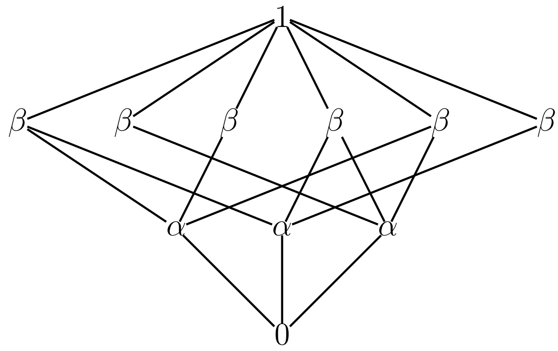

Figure 3: Hasse diagram of $(\mathfrak{C}(3)_\perp, \sqsubseteq)$

to $1 \geq 5\beta - 4\alpha$, while 6-monotonicity is equivalent to $1 \geq 6\beta - 9\alpha$. 7-monotonicity does not add further constraints.

From Example 1, nonnegativity of the Möbius transform implies $\alpha \geq 0$ (atoms), $\beta \geq 2\alpha$ (2nd level), and $m(\top) = 6\alpha - 6\beta + 1 \geq 0$. Then taking $\alpha = 0.1$, $\beta = 0.28$ make that f is a belief function, but $m(\top) = -0.08$.

The last point concerns minitive functions. A *minitive function* on a lattice L is a real-valued function f on L such that $f(\perp) = 0$, $f(\top) = 1$, and

$$f(x \wedge y) = \min(f(x), f(y)).$$

Hence a minitive function is a capacity. The following proposition by Barthélemy shows that minitive functions always exist on $\mathfrak{C}(n)_\perp$.

Proposition 10. Let L be a lattice and $f : L \to \mathbb{R}$ such that $f(\perp) = 0$, $f(\top) = 1$. Then f is a minitive function if and only if it is an invertible belief function whose Möbius transform is nonzero on a chain of $\mathfrak{C}(n)_\perp$.

In other words, taking any chain C in $\mathfrak{C}(n)$ and assigning nonnegative numbers on elements of C such that their sum is 1 generates (by Proposition 9) a belief function which is a minitive function.

7 Concluding remarks

The paper has given a natural structure to embedded subsets, and hence a better understanding to games in partition function form. Specifically, the main results from a game theoretic viewpoint are:

- The set of embedded subsets forms a lattice, whose structure is much more complicated and less easy to handle than the Boolean lattice of coalitions in the classical setting. In particular, since the lattice is not distributive, no simple decomposition of elements is possible, a fortiori no decompositions in atoms.

- The number of maximal chains between any two elements is known. This allows the definition of many notions in game theory, like the Shapley value.

- The decomposition of games into the basis of unanimity games, and hence its Harsanyi dividends, is known.
- There is no additivity property for such games, hence there is few hope to express them in a simpler form.
- Infinite monotonicity is no more equivalent to the nonnegativity of the Möbius transform (Harsanyi dividends).

We hope that this work can help in the clarification and further investigation on games in partition function form.

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