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ABOUT THE CHARACTERIZATION OF SOME RESIDUE CURRENTS

Pierre Dolbeault

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**1. Introduction.**

**1.1. Residue current in dimension 1.** Let  $\omega = g(z)dz$  be a meromorphic 1-form on a small enough open set  $0 \in U \subset \mathbb{C}$  having 0 as unique pole, with multiplicity  $k$ :

$$g = \sum_{l=1}^k \frac{a_{-l}}{z^l} + \text{holomorphic function}$$

Note that  $\omega$  is  $d$ -closed.

Let  $\psi = \psi_0 d\bar{z} \in \mathcal{D}^1(U)$  be a 1-test form. In general  $g\psi$  is not integrable, but the principal value

$$Vp[\omega](\psi) = \lim_{\epsilon \rightarrow 0} \int_{|z| \geq \epsilon} \omega \wedge \psi$$

exists, and  $dVp[\omega] = d''Vp[\omega] = \text{Res}[\omega]$  is the residue current of  $\omega$ . For any test function  $\varphi$  on  $U$ ,

$$\text{Res}[\omega](\varphi) = \lim_{\epsilon \rightarrow 0} \int_{|z|=\epsilon} \omega \wedge \varphi$$

Then  $\text{Res}[\omega] = 2\pi i \text{res}_0(\omega)\delta_0 + dB = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j} \delta_0$  where  $\text{res}_0(\omega) = a_{-1}$  is the Cauchy residue. We remark

that  $\delta_0$  is the integration current on the subvariety  $\{0\}$  of  $U$ , that  $D = \sum_{j=0}^{k-1} b_j \frac{\partial^j}{\partial z^j}$  and that  $b_j = \lambda_j a_{-j}$  where the  $\lambda_j$  are universal constants.

Conversely, given the subvariety  $\{0\}$  and the differential operator  $D$ , then the meromorphic differential form  $\omega$  is equal to  $g dz$ , up to holomorphic form; hence the residue current  $\text{Res}[\omega] = D\delta_0$ , can be constructed.

**1.2. Characterization of holomorphic chains.** P. Lelong (1957) proved that a complex analytic subvariety  $V$  in a complex analytic manifold  $X$  defines an integration current  $\varphi \mapsto [V](\varphi) = \int_{\text{Reg}V} \varphi$  on  $X$ . More

generally, a *holomorphic  $p$ -chain* is a current  $\sum_{l \in L} n_l [V_l]$  where  $n_l \in \mathbb{Z}$ ,  $[V_l]$  is the integration current defined

by an irreducible  $p$ -dimensional complex analytic subvariety  $V_l$ , the family  $(V_l)_{l \in L}$  being locally finite.

During more than twenty years, J. King [K 71], Harvey-Shiffman [HS 74], Shiffman [S 83], H. Alexander [A 97] succeeded in proving the following structure theorem: *Holomorphic  $p$ -chains on a complex manifold  $X$  are exactly the rectifiable  $d$ -closed currents of bidimension  $(p, p)$  on  $X$ .*

In the case of section 1.1,  $\text{Res}[\omega]$  is the holomorphic chain with complex coefficients  $2\pi i \text{res}_0(\omega)\delta_0$  if and only if 0 is a simple pole of  $\omega$ .

**1.3.** Our aim is to characterize residue currents using rectifiable currents with coefficients that are principal values of meromorphic differential forms and holomorphic differential operators acting on them.

We present a few results in this direction.

The structure theorem of section 1.2 concerns complex analytic varieties and closed currents. So, after generalities on residue currents of semi-meromorphic differential forms, we will concentrate on residue currents of closed meromorphic forms.

## 2. Preliminaries: local description of a residue current ([D 93], section 6)

**2.1.** We will consider a finite number of holomorphic functions defined on a small enough open neighborhood  $U$  of the origin 0 of  $\mathbb{C}^n$ , with coordinates  $(z_1, \dots, z_n)$ . For convenient coordinates, any semi-meromorphic differential form, for  $U$  small enough, can be written  $\frac{\alpha}{f}$ , where  $\alpha \in \mathcal{E}(U)$ ,  $f \in \mathcal{O}(U)$  and

$$f = u_j \prod_k {}_j\rho_k^{r_k},$$

where the  ${}_j\rho_k$  are irreducible distinct Weierstrass polynomials in  $z_j$  and the  $r_k \in \mathbb{N}$  are independent of  $j$ , moreover  $u_j$  is a unit at 0, i.e., for  $U$  small enough,  $u_j$  does not vanish on  $U$ . Let  $B_j$  be the discriminant of the polynomial  ${}_j\rho = \prod_k {}_j\rho_k$  and let  $Y_k = Z({}_j\rho_k)$ ; it is clear that  $Y_k$  is independent of  $j$ . Let  $Y = \cup_k Y_k$  and  $Z = \text{Sing } Y$ .

After shrinkage of  $(0 \in) U$ , the following expressions of  $\frac{1}{f}$  are valid on  $U$ : for every  $j \in [1, \dots, n]$ ,

$$\frac{1}{f} = u_j^{-1} \sum_k \sum_{\mu=1}^{r_k} {}_j c_\mu^k \frac{1}{{}_j\rho_k^\mu}$$

where  ${}_j c_\mu^k$  is a meromorphic function whose polar set, in  $Y_k$ , is contained in  $Z(B_j)$ . Notice that  $B_j$  is a holomorphic function of  $(z_1, \dots, \widehat{z}_j, \dots, z_n)$ . In the following, for simplicity, we omit the unit  $u_j^{-1}$ .

**2.2.** Let  $\omega = \frac{1}{f}$ ,  $Vp[\omega](\psi) = \lim_{\epsilon \rightarrow 0} \int_{|f| \geq \epsilon} \omega \wedge \psi$ ;  $\psi \in \mathcal{D}^{n,n}(U)$ . The residue of  $\omega$  is

$$\text{Res}[\omega] = (dVp - Vpd)[\omega] = (d^n Vp - Vpd^n)[\omega]$$

For every  $\varphi \in \mathcal{D}^{n,n-1}(U)$ , let  $\varphi = \sum_{j=1}^n \varphi_j$  with

$$\varphi_j = \psi_j dz_1 \wedge \dots \wedge d\bar{z}_1 \wedge \dots \wedge \widehat{d\bar{z}_j} \wedge \dots$$

Then, from Herrera-Lieberman [HL 71], and the next lemma about  $B_j$ , we have:

$$\text{Res}[\omega](\varphi) = \sum_{j=1}^n \sum_k \sum_{\mu=1}^{r_k} \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta |{}_j\rho_k| = \epsilon} {}_j c_\mu^k \frac{1}{{}_j\rho_k^\mu} \varphi_j.$$

The lemma we have used here is the following:

**Lemma 2.1.** ([D 93], Lemma 6.2.2).

$$\text{Res}[\omega](\varphi_j) = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta |f| = \epsilon} \omega \varphi_j.$$

Outside  $Z(B_j)$ , for  $|{}_j\rho_k|$  small enough (since  $\frac{\partial_j\rho_k}{\partial z_j} \neq 0$ ), we take  $(z_1, \dots, z_{j-1}, {}_j\rho_k, z_{j+1}, \dots, z_n)$  as local coordinates.

**2.3. Notations.** For the sake of simplicity, until the end of this section, we assume  $j = 1$  and write  $\rho_k, c_\mu^k$  instead of  ${}_1\rho_k, {}_1c_\mu^k$ . Outside  $Z(B_1)$ , we take  $(\rho_k, z_2, \dots, z_n)$  as local coordinates; then, for every  $C^\infty$  function  $h$  and every  $s \in \mathbb{N}$ , we have

$$\frac{\partial^s h}{\partial \rho_k^s} = \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1}\right)^{2s-1}} D_s h, \quad \text{for } s \geq 1,$$

where  $D_s = \sum_{\alpha=1}^s \beta_\alpha^s \frac{\partial^\alpha}{\partial z_1^\alpha}$ ,  $\beta_\alpha^s$  is a holomorphic function determined by  $\rho_k$  and  $D_0 = \left(\frac{\partial \rho_k}{\partial z_1}\right)^{-1}$ .

Let

$$g_l^\mu = \binom{\mu-1}{l} \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1}\right)^{2\mu-4}} D_l \left(\frac{c_\mu^k}{\frac{\partial \rho_k}{\partial z_1}}\right), \quad (0 \leq l \leq \mu-2);$$

$$g_{\mu-1}^\mu = \frac{1}{\left(\frac{\partial \rho_k}{\partial z_1}\right)^{2\mu-3}} D_{\mu-1} \left(\frac{c_\mu^k}{\frac{\partial \rho_k}{\partial z_1}}\right)$$

Let  $Vp_{Y_k, B_1}^1[g_l^\mu]$  also denote the direct image, by the inclusion  $Y_k \rightarrow U$ , of the Cauchy principal value  $Vp_{Y_k, B_1}[g_l^\mu]$  of  $g_l^\mu|_{Y_k}$ ;

$$D_{1,k}^{\mu,l} = \sum_{\alpha=1}^{\mu-1-l} (-1)^\alpha \beta_\alpha^{\mu-1-l} \frac{\partial^\alpha}{\partial z_1^\alpha}, \quad \text{and } D_{1,k}^{\mu,\mu-1} = \text{id}.$$

**2.4. Final expression of the residue.** All what has been done for  $j = 1$  is valid for any  $j \in \{1, \dots, n\}$ : the principal value  $Vp^j(k, \mu, l) = Vp_{Y_k, B_j}^j[g_l^\mu]$  defined on  $Y_k$  and the holomorphic differential operator  $D_{j,k}^{\mu,l}$ . We also denote  $Vp^j(k, \mu, l)$  the direct image of the principal value by the canonical injection  $Y \hookrightarrow U$ . Then, denoting  $L$  the inner product, we have:

$$(*) \quad \text{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^n \left[ \sum_k \sum_{\mu=1}^{r_k} \frac{1}{(\mu-1)!} \sum_{l=0}^{\mu-1} D_{j,k}^{\mu,l} Vp^j(k, \mu, l) \right] \left( \frac{\partial}{\partial z_j} L\varphi_j \right)$$

### 3. The case of simple poles.

#### 3.1. The case $\omega = \frac{1}{f}$ .

**Lemma 3.1.** *For a simple pole and for every  $k$ ,  ${}^j c_1^k$  is holomorphic.*

*Proof.* Let  $w = z_j$  and  $y = (z_1, \dots, \hat{z}_j, \dots, z_n)$ . At points  $z \in U$  where  $B_j(z) \neq 0$ , for given  $y$ , let  $w_{ks}, s = 1, \dots, s_k$ , be the zeros of  $\rho_k$ . For given  $y$ ,  $\rho_k = \prod_{s=1}^{s_k} (w - w_{ks})$ ,

$$\frac{1}{f} = u_j \sum_k \sum_{s=1}^{s_k} {}^j c_1^{k,s} (w - w_{ks})^{-1}$$

where  ${}^j c_1^{k,s} = \frac{1}{\frac{\partial}{\partial w} f(w_{ks}, y)}$ ; let  $\prod_\sigma^s$  denote the product for all  $\sigma \neq s$ ,

$$\sum_{s=1}^{s_k} {}^j c_1^{k,s} (w - w_{ks})^{-1} = \sum_{s=1}^{s_k} {}^j c_1^{k,s} \frac{\prod_\sigma^s (w - w_{k\sigma})}{\prod_\sigma (w - w_{k\sigma})} = {}^j c_1^k(w, y) \rho_k^{-1},$$

with

$${}^j c_1^k(w, y) = \sum_{s=1}^{s_k} \frac{\prod_{\sigma}^s (w - w_{k\sigma})}{\frac{\partial}{\partial w} f(w_{ks}, y)} \quad ([D 57], \text{IV.B.3 et C.1}).$$

Here  ${}^j c_1^k(w, y)$  holomorphically extends to points of  $U$  where the  $w_s$  are not all distinct because: if  $w_s$  appears  $m$  times in  $\prod_{\sigma} (w - w_{k\sigma})$ , it appears  $(m-1)$  times in the numerator and the denominator of  $\frac{\prod_{\sigma}^s (w - w_{k\sigma})}{\frac{\partial}{\partial w} f(w_{ks}, y)}$ .

□

All the poles of  $\omega$  are simple, i.e. for every  $k$ ,  $r_k = 1$ ; then  $\mu = 1$ ,  $l = 0$ .

$$\text{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^n \left[ \sum_k D_{j,k}^{1,0} V p^j(k, 1, 0) \right] \left( \frac{\partial}{\partial z_j} L\varphi_j \right)$$

$$\begin{aligned} \text{But } D_{1,k}^{1,0} &= \text{id}; \quad D_0 = \left( \frac{\partial \rho_k}{\partial z_1} \right)^{-1}; \quad g_{\mu-1}^{\mu} = \frac{1}{\left( \frac{\partial \rho_k}{\partial z_1} \right)^{2\mu-3}} D_{\mu-1} \left( \frac{c_{\mu}^k}{\frac{\partial \rho_k}{\partial z_1}} \right); \quad g_0^1 = \frac{1}{\left( \frac{\partial \rho_k}{\partial z_1} \right)^{-1}} D_0 \left( \frac{c_1^k}{\frac{\partial \rho_k}{\partial z_1}} \right) \\ &= \frac{1}{\left( \frac{\partial \rho_k}{\partial z_1} \right)^{-1}} \left( \frac{\partial \rho_k}{\partial z_1} \right)^{-1} \left( \frac{c_1^k}{\frac{\partial \rho_k}{\partial z_1}} \right) = \left( \frac{\partial \rho_k}{\partial z_1} \right)^{-1} c_1^k; \end{aligned}$$

$$V p^j(k, 1, 0) = V p_{Y_k, B_j}^j [g_0^1] = V p_{Y_k, B_j}^j \left[ \left( \frac{\partial \rho_k}{\partial z_j} \right)^{-1} {}^j c_1^k \right],$$

hence

$$\text{Res}[\omega](\varphi) = 2\pi i \sum_{j=1}^n \left[ \sum_k V p_{Y_k, B_j}^j \left[ \left( \frac{\partial \rho_k}{\partial z_j} \right)^{-1} {}^j c_1^k \right] \right] \left( \frac{\partial}{\partial z_j} L\varphi_j \right)$$

where  ${}^j c_1^k$  is holomorphic.

**3.2. The case of any degree.** Let  $\omega = \frac{\alpha}{f}$ . Then  $\text{Res}[\omega] = \alpha \wedge \text{Res}\left(\frac{1}{f}\right)$ . Moreover,  $d \text{Res}[\omega] = \pm \text{Res}[d\omega]$ , then  $\text{Res}[\omega]$  is  $d$ -closed if  $\omega$  is  $d$ -closed.

#### 4. Expression of the residue current of a closed meromorphic differential form.

In this section and a part of the following one, we give statements on residue currents according to the general hypotheses and proofs of sections 2 and 3. Proofs in a particular case where the polar set is equisingular and the singularity of the polar set is a 2-codimensional smooth submanifold are given in ([D 57], IV.D).

##### 4.1. Closed meromorphic differential forms.

**4.1.1.** Let  $\omega = \frac{\alpha}{f}$  be a  $d$ -closed meromorphic differential  $p$ -form on a small enough open neighborhood  $U$  of the origin 0 of  $\mathbb{C}^n$ . From section 2.1, we get  $\omega = \sum \omega_k$  with  $\omega_k = \sum_{\mu=1}^{r_k} {}^j c_{\mu}^k \frac{\alpha}{j \rho_k^{\mu}}$  for every  $j = 1, \dots, n$ . We have

$${}^j c_{\mu}^k = \frac{{}^j a_{\mu}^k(z_1, \dots, z_n)}{{}^j b_{\mu}^k(z_1, \dots, \widehat{z}_j, \dots, z_n)},$$

where  $a$  and  $b$  are holomorphic. Then  $d\omega = \sum d\omega_k$  and  $d\omega_k$  is the quotient of a holomorphic form by a product of  ${}^j b_{\mu}^k(z_1, \dots, \widehat{z}_j, \dots, z_n)$  and  ${}^j \rho_k^{r_k+1}$  (see [D 57], IV.D.1).

As at the end of section 2.2, using the local coordinates

$$(z_1, \dots, z_{j-1}, \rho_k, z_{j+1}, \dots, z_n),$$

we have

$$(4.1) \quad \omega_k = \sum_{\mu=1}^{r_k} [{}^j A_{\mu}^k \wedge {}^j \rho_k^{-\mu} d_j \rho_k + {}^j \rho_k^{-\mu} B_{\mu}^k],$$

where the coefficients are meromorphic.

Let  $\mathcal{R}_j$  be the ring of meromorphic forms on  $U$  whose coefficients are quotients of holomorphic forms on  $U$  by products of powers of  $\frac{\partial_j \rho_k}{\partial z_j}$  and  $j b_\mu^k$ .

**Lemma 4.1** ([D 57], Lemme 4.10). *Assume that  $d\omega_k \in \mathcal{R}_j$ . Then*

$$\omega_k = {}_j \rho_k^{-1} d_j \rho_k \wedge a_j^k + \beta_j^k + dR_j^k$$

with

$$R_j^k = \sum_{\nu=1}^{r_k-1} {}_j e_{\nu,j}^k \rho_k^{-\nu} \quad \text{and} \quad da_j^k = d_j \rho_k \wedge {}^k a_j' + C_j^k \rho_k,$$

where  $a_j^k, \beta_j^k, {}_j e_{\nu,j}^k, {}^k a_j', C_j \in \mathcal{R}_j$  and are independent of  $dz_j$ .

**4.1.2.** Let  $\varphi$  be of type  $(n-p, n-1)$ . Then

$$\varphi = \sum \varphi_j, \quad \text{with} \quad \varphi_j = \sum \psi_{l_1, \dots, l_{n-p}} dz_{l_1} \wedge \dots \wedge dz_{l_{n-p}} \wedge \dots \wedge \widehat{dz_j} \wedge \dots$$

**Proposition 4.2.** *Let  $\omega = \frac{\alpha}{f}$  be a  $d$ -closed meromorphic  $p$ -form on  $U$ . Given a coordinate system on  $U$ , and with notations of section 2.1, there exists a current  $S_j^{p-1,1}$  such that  $d'' S_j|_{U \setminus Z} = 0$ ,  $\text{supp} S_j = Y$  and, for every  $k, j$ , a  $d$ -closed meromorphic  $(p-1)$ -form  $A_j^k$  on  $Y_k$  with polar set  $Z$  such that*

$$\text{Res}[\omega](\varphi) = \sum_{j=1}^n \left( 2\pi i \sum_k V p_{Y_k, B_j} A_j^k + d' S_j \right) \left( \frac{\partial}{\partial z_j} \mathbf{L} \varphi_j \right).$$

When the coordinate system is changed, the first term of the parenthesis is modified by addition of  $2\pi i \sum_k d' V p_{Y_k, B_j} [F_j^k]$  where  $F_j^k$  is a meromorphic  $(p-2)$ -form on  $Y_k$  with polar set  $Z$ .

Here  $2\pi i \sum_{j=1}^n \sum_k V p_{Y_k, B_j} A_j^k(\cdot_j)$  will be called the *reduced residue* of  $\omega$ .

*Proof.* Apply the proof of (\*) (section 2) to the meromorphic form of Lemma 4.1.

We shall use the expression of  $\text{Res}[\omega](\varphi)$  of section 2.2, for  $\omega$  closed.

For  $k$  and  $j$  fixed, we consider

$$J_{kj} = \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |{}_j \rho_k| = \epsilon} \omega_k(\varphi_j).$$

Then  $\text{Res}[\omega](\varphi) = \sum_{k,j} J_{kj}$ .

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |{}_j \rho_k| = \epsilon} dR_j^k \wedge \varphi_j = (-1)^p \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |{}_j \rho_k| = \epsilon} R_j^k \wedge d\varphi_j.$$

Let  $S_j^k$  be the current defined by

$$S_j^k(\psi_j) = - \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |{}_j \rho_k| = \epsilon} R_j^k \wedge \psi_j.$$

By Lemme 4.1.  $R_j^k$  is independent of  $dz_j$ .

Let  $\psi_j = dz_j \wedge \eta^j + \xi^j$ , where  $\xi^j$  is independent of  $dz_j$ , then  $\eta^j = \frac{\partial}{\partial z_j} \mathbf{L} \psi_j$ .

After change of coordinates:

$$(4.2) \quad S_j^k(\psi_j) = - \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |{}_j \rho_k| = \epsilon} \left( \frac{\partial_j \rho_k}{\partial z_j} \right)^{-1} R_j^k \wedge d_j \rho_k \wedge \eta^j$$

$$= (-1)^p 2\pi i \lim_{\delta \rightarrow 0} \sum_{\nu} \int_{Y_k |_{B_j} \geq \delta} (\nu - 1)!^{-1} \left( \frac{\partial^{\nu-1} (j e_{\nu}^k \wedge \eta^j \left( \frac{\partial_j \rho_k}{\partial z_j} \right)^{-1})}{\partial_j \rho_k^{\nu-1}} \right)_{j \rho_k = 0}$$

We have  $S_j(\psi_j) = \sum_k S_j^k$ .

$$\lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \int_{|B_j| \geq \delta, |j \rho_k| = \epsilon} j \rho_k^{-1} d_j \rho_k \wedge a_j^k + \beta_j^k = 2\pi i \lim_{\delta \rightarrow 0} \int_{|B_j| \geq \delta} a_j^k |_{Y_k} = 2\pi i V p_{Y_k, B_j} A_j^k, \text{ with } A_j^k = a_j^k |_{Y_k}$$

The last ainea is proved as in ([D 57], IV.D.4).  $\square$

**Corollary 4.3.** *The current  $S_j$  is obtained by application of holomorphic differential operators to currents principal values of meromorphic forms supported by the irreducible components of  $Y$ .*

*Proof.* The corollary follows from the above expression for  $S_j$  and the computations in section 2.  $\square$

We remark that  $d'$  itself is a holomorphic differential operator.

#### 4.2. Particular cases.

**4.2.1. The case  $p = 1$ .** With the notations of Proposition 4.2, the forms  $A^k$  are of degree 0 and are  $d$ -closed, hence constant and unique: the reduced residue is a *divisor with complex coefficients*.

**4.2.2.** With the hypotheses and the notations of section 2.1, if all the multiplicities  $r_k$  are equal to 1, the reduced residue is uniquely determined and the current  $S = 0$ .

#### 4.3. Comparison with the expression of $Res[\omega]$ in section 2, when $\omega$ is $d$ -closed.

The reduced residue is equal to

$$2\pi i \sum_{j=1}^n \left[ \sum_k V p_{Y_k, B_j}^j \left( \frac{\partial \rho_k}{\partial z_j} \right)^{-1} j c_1^k \left( \frac{\partial}{\partial z_j} L(\alpha \wedge \cdot)_j \right) \right].$$

It is well defined if all the poles of  $\omega$  are simple.

### 5. Generalization of a theorem of Picard. Structure of residue currents of closed meromorphic forms.

**5.1.** The theorem of Picard [P 01] characterizes the divisor with complex coefficients associated to a  $d$ -closed differential form, of degree 1 of the third kind, on a complex projective algebraic surface; this result has been generalized by S. Lefschetz (1924): "the divisor has to be homologous to 0", then by A. Weil (1947). Locally, one of its assertions is a particular case of the theorem of Dickenstein-Sessa ([DS 85], Theorem 7.1): *Analytic cycles are locally residual currents* (see section 5.5), with a variant by D. Boudiaf ([B 92], Ch.1, sect.3).

#### 5.2. Main results.

**Theorem 5.1.** *Let  $X$  be a complex manifold which is compact Kähler or Stein, and  $Y$  be a complex hypersurface of  $X$ , then  $Y = \cup_{\nu} Y_{\nu}$  is a locally finite union of irreducible hypersurfaces. Let  $Z = \text{Sing } Y$ , and let  $A_{\nu}$  be a  $d$ -closed meromorphic  $(p-1)$ -form on  $Y_{\nu}$  with polar set  $Y_{\nu} \cap Z$  such that the current  $t = 2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu}$  is  $d$ -closed.*

*Then the following two conditions are equivalent:*

(i)  *$t$  is the residue current of a  $d$ -closed meromorphic  $p$ -form on  $X$  having  $Y$  as polar set with multiplicity one.*

(ii)  *$t = dv$  on  $X$ , where  $v$  is a current, i.e., is cohomologous to 0 on  $X$ .*

*Proof.* From section 4 locally, and a sheaf cohomology machinery globally; detailed proof will be given later for the more general theorem 5.5.  $\square$

For  $p = 1$ , the  $A_{\nu}$  are complex constants, then  $t$  is the divisor with complex coefficients  $2\pi i \sum_{\nu} A_{\nu} Y_{\nu}$ .

**Corollary 5.1.1.** *Under the hypotheses of Theorem 5.1, every residue current of a closed meromorphic  $p$ -form appears as a divisor, homologous to 0, whose coefficients are principal values of meromorphic  $(p-1)$ -forms on the irreducible components of the support of the divisor and conversely.*

Let  $\mathcal{R}_{q,q}^{loc}(X)$  be the vector space of locally rectifiable currents of bidimension  $(q, q)$  on the complex manifold  $X$  and

$$\mathcal{R}_{q,q}^{loc\mathbb{C}}(X) = \mathcal{R}_{q,q}^{loc} \otimes_{\mathbb{Z}} \mathbb{C}(X)$$

**Theorem 5.2.** *Let  $T \in \mathcal{R}_{q,q}^{loc\mathbb{C}}(X)$ ,  $dT = 0$ . Then  $T$  is a holomorphic  $q$ -chain with complex coefficients.*

This is the structure theorem of holomorphic chains of Harvey-Shiffman-Alexander for complex coefficients; thanks to it, divisors will be translated into rectifiable currents.

**Theorem 5.3.** *Let  $X$  be a Stein manifold or a compact Kähler manifold. Then the following conditions are equivalent:*

(i)  *$T$  is the residue current of a  $d$ -closed meromorphic 1-form on  $X$  having  $\text{supp } T$  as polar set with multiplicity 1;*

(ii)  *$T \in \mathcal{R}_{n-1,n-1}^{loc\mathbb{C}}(X)$ ,  $T = dV$ .*

In the same way, we can reformulate the Theorem 5.1 with rectifiable currents:

**Theorem 5.4.** *Let  $X$  be a Stein manifold or a compact Kähler manifold. Then the following conditions are equivalent:*

(i)  *$T = \sum_{\nu} a_{\nu} T_{\nu}$ , with  $T_{\nu} \in \mathcal{R}_{n-1,n-1}^{loc\mathbb{C}}(X)$ ,  $d$ -closed, and  $a_{\nu}$  the principal value of a  $d$ -closed meromorphic  $(p-1)$ -form on  $\text{supp } T_{\nu}$ , such that  $T = dV$ ;*

(ii)  *$T$  is the residue current of a  $d$ -closed meromorphic  $p$ -form on  $X$  having  $\cup_l T_l$  as polar set with multiplicity 1.*

**5.3. Remark.** The global Theorem 5.1 gives also local results since any open ball centered at 0 in  $\mathbb{C}^n$  is a Stein manifold.

#### 5.4. Generalization.

**5.4.1.** With the notations of section 4.1, what has been done with the current  $2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu}$  is also possible in the general case. The current  $S$  is defined as follows: let  $\psi = \sum_j \psi_j$ , then  $S(\psi) = \sum_j \sum_k S_j^k(\psi_j)$ .

From (4.2), we have:

$$(5.3) \quad S_j^k(\psi_j) = 2\pi i \sum_{\mu=1}^{r_k} \sum_{l=0}^{\mu-1} \Delta_{j,k}^{\mu,l} V p_{Y_k, B_j}^j [\gamma_{k,l}^{\mu,j}] \left( \frac{\partial}{\partial z_j} L \psi_j \right)$$

where  $\gamma_{k,l}^{\mu,j}$  is a meromorphic form on  $Y_k$ , with polar set contained in  $Y_k \cap \{B_j = 0\}$ , and where  $\Delta_{j,k}^{\mu,l}$  is a holomorphic differential operator in the neighborhood of  $Y_k$ . In the global case, for  $Y = \cup_{\nu} Y_{\nu}$  locally finite, we take  $k = \nu$ , the sum  $\sum_{\nu} S_{\nu}^{\nu}$  being locally finite.

Then we will get generalizations of the results in sections 5.2 and 5.3 completing the programme of section 1.3.

**Lemma 5.1.** *Let  $m^p$  be the sheaf of closed meromorphic differential forms. Let  $\overline{m}^p$  be the image by  $V_p$  of  $m^p$  in the sheaf of germs of currents on  $X$ . Then, for  $X$  Stein or compact Kähler manifold, we have the commutative diagram*

$$\begin{array}{ccccccc} H^0(X, m^p) & \rightarrow & H^0(X, \overline{m}^p) & \rightarrow & H^0(X, \overline{m}^p/E^p) & \rightarrow & H^1(X, E^p) \\ & & \text{Res} \downarrow & & & & \downarrow \\ & & H^0(X, d''\overline{m}^p) & \rightarrow & H^{p+1}(X, \mathbb{C}) & & \end{array}$$

(from [D 57], IV.D.7)

**5.4.2.** The residue current of a  $d$ -closed meromorphic  $p$ -form is globally written  $t = 2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu} + d'S$ , where  $S = \sum_{\nu} \sum_j S_j^{\nu}$ , with  $dt = 0$ , from the local Proposition 4.2.

**Theorem 5.5.** *If  $X$  is a complex manifold which is compact Kähler, or Stein, and  $Y$  is a complex hypersurface of  $X$ , then  $Y = \cup_{\nu} Y_{\nu}$  is a locally finite union of irreducible hypersurfaces. Let  $Z = \text{Sing} Y$ ; for every  $\nu$ , let  $A_{\nu}$  be a  $d$ -closed meromorphic  $(p-1)$ -form on  $Y_{\nu}$ , and, in the notations of (5.3) with  $k = \nu$ ,  $\gamma_{\nu,l}^{\mu,j}$  be meromorphic  $(p-2)$ -forms on  $Y_{\nu}$ , with polar set  $Y_{\nu} \cap Z$  such that the current  $t = 2\pi i \sum_{\nu} V p_{Y_{\nu}} A_{\nu} + d'S$ , with  $S = \sum_{\nu} \sum_j S_j^{\nu}$ , be  $d$ -closed.*



Then the following two conditions are equivalent:

- (i)  $t$  is the residue current of a  $d$ -closed meromorphic  $p$ -form on  $X$  having  $Y$  as polar set.
- (ii)  $t = dv$  on  $X$ , where  $v$  is a current, i.e.  $t$  is cohomologous to 0 on  $X$ .

*Proof.*

(i)  $\Rightarrow$  (ii): From Lemma 5.1, the cohomology class of a residue current is 0; it is the case of  $t$ .

(ii)  $\Rightarrow$  (i):  $t = dv$  on  $X$ ;  $t$  of type  $(p, 1)$  implies:  $t = dv = d^{\#}v$ ;  $v$  of type  $(p, 0)$ ; the current  $v$  is closed on  $X \setminus Y$ , therefore it is a holomorphic  $p$ -form on  $X \setminus Y$ . Let  $m_Y^p$  be the sheaf of closed meromorphic  $p$ -forms with polar set  $Y$ ; the Lemma 5.1 is valid for  $m_Y^p$  instead of  $m^p$ . At a point  $O \in Y$ ,  $Y$  is defined by  $\Pi_k \rho_k = 0$  (omitting the index  $j$ ); the  $r_k$  being the integers in (5.3), then  $d(\Pi_k \rho_k^{r_k} v) = \Pi_k \rho_k^{r_k} d^{\#}v = \Pi_k \rho_k^{r_k} t = 0$  from Lemma 4.1; therefore  $\Pi_k \rho_k^{r_k} v$  is a germ of holomorphic form at  $O$  and  $v$  extends a closed meromorphic form  $G \in H^0(X, m_Y^p)$  on  $X$ .

We will show that  $t$  is the residue current of  $G$ .

From Proposition 4.2,

$$\text{Res}[G] = d^{\#} \text{Vp } G = 2\pi i \sum_{\nu} \text{Vp}_{Y_{\nu}} B_{\nu} + d^{\#} T$$

where  $B_{\nu}$  and  $T$  are of the same nature as  $A_{\nu}$  and  $S$ .

**Lemma 5.2.**  $M = v - \text{Vp } G$  satisfies  $d^{\#} M = 0$ .

*Proof.* We have:

$$(5.4) \quad d^{\#} M = 2\pi i \sum_{\nu} \text{Vp}_{Y_{\nu}} (A_{\nu} - B_{\nu}) + d^{\#} (S - T)$$

Let  $O_1$  be a non singular point of  $Y$ ; there exists  $k$  such that:  $O_1 \in \{j \rho_k = 0\}$ , ( $j = 1, \dots, n$ ); in the neighborhood of  $O_1$ ,  $j \rho_k$  can be used as local coordinate. We have:  $M = M_j$  where  $M_j$  is written with the local coordinates  $(\dots, z_{j-1}, j \rho_k, z_{j+1}, \dots)$ ;  $d^{\#} M = d^{\#} M_j$ ; the support of  $d^{\#} M$  is  $Y$ , then, in the neighborhood of  $O_1$ ,  $d^{\#} M_j$  vanishes on the differential forms containing  $d_j \rho_k$  or  $d_j \bar{\rho}_k$ . Then

$$(5.5) \quad d^{\#} M_j = d_j \rho_k \wedge d_j \bar{\rho}_k \wedge N_j$$

$M_j$  is of type  $(p, 0)$ , therefore without term in  $d_j \bar{\rho}_k$  and in  $d \bar{z}_l$ ,  $l \neq j$ .

From (5.5),  $\frac{\partial M_j}{\partial \bar{z}_l} = 0$ , then

$$(5.6) \quad d^{\#} M_j = d_j \bar{\rho}_k \wedge \frac{\partial M_j}{\partial_j \bar{\rho}_k}$$

$d^{\#} M_j$  is a differential form with distribution coefficients supported by  $Y_k$ , therefore, outside  $Z$ , from the structure theorem of distributions supported by a submanifold ([Sc 50], ch. III, théorème XXXVII), and from (5.6), the coefficients of  $d^{\#} M_j$  being those of  $\frac{\partial M_j}{\partial_j \bar{\rho}_k}$ , then  $d^{\#} M_j$  contains transversal derivatives with respect  $j \rho_k$  or  $j \bar{\rho}_k$  of order at least equal to  $r_k + 1$ , what is incompatible with the initial expression (5.4) of  $d^{\#} M_j$ , except if  $d^{\#} M_j = 0$  outside  $Z$ . From (5.4) the  $\text{Vp}_{Y_k}(A_{\nu} - B_{\nu})$  and  $(S - T)$  being defined as limits of integrals of forms vanishing on  $Y \setminus Z$ , we have:  $d^{\#} M = 0$  on  $X$ .  $\square$

From Lemma 5.2,  $\text{Res}[G] = d^{\#} v = t$ .  $\square$

**Corollary 5.5.1.** Under the hypotheses of Theorem 5.5, the current  $S$  is a sum of currents obtained by application of holomorphic differential operators to principal values of meromorphic forms on the irreducible components  $Y_{\nu}$  of  $Y$ .

**Corollary 5.5.2.** Under the hypotheses of Theorem 5.5, the residue current of a  $d$ -closed meromorphic differential  $p$ -form is the sum, cohomologous to 0, of currents obtained by application of holomorphic differential operators to currents  $\sum_{\nu} a_{\nu} T_{\nu}$ , with  $T_{\nu} \in \mathcal{R}_{n-1, n-1}^{\text{loc}}(X)$ ,  $d$ -closed, and  $a_{\nu}$  the principal value of a meromorphic  $(p - 1)$ -form on  $\text{supp } T_{\nu}$ .

**5.5. Remarks.** The Theorems of the sections 5.2 and 5.4 and their Corollaries are valid for locally residue currents in the terminology of [DS 85]. Results are also valid for any complex analytic manifold, using less natural cohomology (cf [D 57], IV.D.7).

**6. Remarks about residual currents** [CH 78], [DS 85].

In the classical definition and notations, we consider residual currents  $R^p[\mu] = R^p P^0[\mu]$ , where  $\mu$  is a semi-meromorphic form  $\frac{\alpha}{f_1 \dots f_p}$ , and  $\alpha$  a differential  $(p, 0)$ -form. Then,  $R^p[\mu]$  satisfies a formula analogous to (\*) of section 2.4. ([D 93], section 8).

Locally, one of the assertions of the theorem of Picard is valid for any  $p$ , from the result of Dickenstein-Sessa quoted in section 5.1. So generalizations of theorems in sections 5.2 to 5.4, for residual currents, seem valid.

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