Multi-soliton solutions for the supercritical gKdV equations

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Abstract

For the L^2 subcritical and critical (gKdV) equations, Martel [11] proved the existence and uniqueness of multi-solitons. Recall that for any N given solitons, we call multi-soliton a solution of (gKdV) which behaves as the sum of these N solitons asymptotically as $t \to +\infty$. More recently, for the L^2 supercritical case, Côte, Martel and Merle [4] proved the existence of at least one multi-soliton. In the present paper, as suggested by a previous work concerning the one soliton case [3], we first construct an N-parameter family of multi-solitons for the supercritical (gKdV) equation, for N arbitrarily given solitons, and then prove that any multi-soliton belongs to this family. In other words, we obtain a complete classification of multi-solitons for (gKdV).

1 Introduction

1.1 The generalized Korteweg-de Vries equation

We consider the generalized Korteweg-de Vries equation:

$$\begin{cases} \partial_t u + \partial_x^3 u + \partial_x(u^p) = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases} \tag{gKdV}$$

where $(t,x) \in \mathbb{R}^2$ and $p \geqslant 2$ is integer. The following quantities are formally conserved for solutions of (gKdV):

$$\int u^2(t) = \int u^2(0) \quad \text{(mass)},$$
$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \quad \text{(energy)}.$$

Kenig, Ponce and Vega [10] have shown that the local Cauchy problem for (gKdV) is well posed in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ of (gKdV) satisfying $u(0) = u_0$ which is unique in some class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. Moreover, if $T^* \geqslant T$ is the maximal time of existence of u, then either $T^* = +\infty$ which means that $u(t)$ is a global solution, or $T^* < +\infty$ and then $\|u(t)\|_{H^1} \to +\infty$ as $t \uparrow T^*$ ($u(t)$ is a finite time blow up solution). Throughout this paper, when referring to an H^1 solution of (gKdV), we mean a solution in the above sense. Finally, if $u_0 \in H^s(\mathbb{R})$ for some $s \geqslant 1$, then $u(t) \in H^s(\mathbb{R})$ for all $t \in [0, T^*)$.

In the case where $2 \leqslant p < 5$, it is standard that all solutions in H^1 are global and uniformly bounded by the energy and mass conservations and the Gagliardo-Nirenberg inequality. In the case $p = 5$, the existence of finite time blow up solutions was proved by Merle [17] and Martel and Merle [12]. Therefore $p = 5$ is the critical exponent for the long time behavior of solutions of (gKdV). For $p > 5$, the existence of blow up solutions is an open problem.

We recall that a fundamental property of (gKdV) equations is the existence of a family of explicit traveling wave solutions. Let Q be the only solution (up to translations) of

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q'' + Q^p = Q, \quad \text{i.e. } Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)} \right)^{\frac{1}{p-1}}.$$

For all $c_0 > 0$ and $x_0 \in \mathbb{R}$,

$$R_{c_0,x_0}(t,x) = Q_{c_0}(x - c_0 t - x_0)$$

is a solution of (gKdV), where $Q_{c_0}(x) = c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x)$. We call solitons these solutions though they are known to be solitons only for $p = 2, 3$ (in the sense that they are stable by interaction).

It is well-known that the stability properties of a soliton solution depend on the sign of $\frac{d}{dc}\int Q_{c|c=c_0}^2$. Since $\int Q_c^2 = c^{\frac{5-p}{2(p-1)}} \int Q^2$, we distinguish the following three cases:

- For $p < 5$ (L^2 subcritical case), solitons are stable and asymptotically stable in H^1 in some suitable sense: see Cazenave and Lions [2], Weinstein [22], Grillakis, Shatah and Strauss [7] for orbital stability; and Pego and Weinstein [19], Martel and Merle [13] for asymptotic stability.
- For $p = 5$ (L^2 critical case), solitons are unstable, and blow up occurs for a large class of solutions initially arbitrarily close to a soliton, see [12, 17]. Moreover, for both critical and subcritical cases, previous works imply the following asymptotic classification result: if u is a solution of (gKdV) such that $\lim_{t\to+\infty} \|u(t) - Q(\cdot - t)\|_{H^1} = 0$, then $u(t) = Q(\cdot - t)$ for t large enough.
- For $p > 5$ (L^2 supercritical case), solitons are unstable (see Grillakis, Shatah and Strauss [7] and Bona, Souganidis and Strauss [1]). In particular, the previous asymptotic classification result does not hold in this case. More precisely, we have:

Theorem 1.1 ([3]). *Let $p > 5$.*

(i) There exists a one-parameter family $(U^A)_{A\in\mathbb{R}}$ of solutions of (gKdV) *such that, for all $A \in \mathbb{R}$,*

$$\lim_{t\to+\infty} \left\| U^A(t,\cdot+t) - Q \right\|_{H^1} = 0,$$

and if $A' \in \mathbb{R}$ satisfies $A' \neq A$, then $U^{A'} \neq U^A$.

(ii) Conversely, if u is a solution of (gKdV) *such that $\lim_{t\to+\infty} \inf_{y\in\mathbb{R}} \|u(t) - Q(\cdot - y)\|_{H^1} = 0$, then there exist $A \in \mathbb{R}$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $u(t) = U^A(t, \cdot - x_0)$ for $t \geqslant t_0$.*

We recall that this result was an adaptation to (gKdV) of previous works, concerning the nonlinear Schrödinger equation, of Duyckaerts and Merle [5] and Duyckaerts and Roudenko [6]. The purpose of this paper is to extend Theorem 1.1 to multi-solitons.

1.2 Multi-solitons

Now, we focus on multi-soliton solutions. Given $2N$ parameters defining $N \geqslant 2$ solitons with different speeds,

$$0 < c_1 < \cdots < c_N, \quad x_1, \ldots, x_N \in \mathbb{R}, \tag{1.1}$$

we set

$$R_j(t) = R_{c_j,x_j}(t) \quad \text{and} \quad R(t) = \sum_{j=1}^{N} R_j(t),$$

and we call multi-soliton a solution $u(t)$ of (gKdV) such that

$$\|u(t) - R(t)\|_{H^1} \longrightarrow 0 \quad \text{as} \quad t \to +\infty. \tag{1.2}$$

Let us recall known results on multi-solitons:

- For $p = 2$ and 3 (KdV and mKdV), multi-solitons (in a stronger sense) are well-known to exist for any set of parameters (1.1), as a consequence of the inverse scattering method.
- In the L^2 subcritical and critical cases, *i.e.* for (gKdV) with $p \leqslant 5$, Martel [11] constructed multi-solitons for any set of parameters (1.1). The proof in [11] follows the strategy of Merle [16] (compactness argument) and relies on monotonicity properties developed in [13] (see also [15]). Recall that Martel, Merle and Tsai [15] proved stability and asymptotic stability of a sum of N solitons for large time for the subcritical case. A refined version of the stability result of [15] shows that, for a given set of parameters, there exists a *unique* multi-soliton solution satisfying (1.2), see Theorem 1 in [11].
- In the L^2 supercritical case, *i.e.* in a situation where solitons are known to be unstable, Côte, Martel and Merle [4] have recently proved the existence of at least *one* multi-soliton solution for (gKdV):

Theorem 1.2 ([4])**.** *Let $p > 5$ and $N \geqslant 2$. Let $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $\varphi \in C([T_0, +\infty), H^1)$ of* (gKdV) *such that*

$$\forall t \in [T_0, +\infty), \quad \|\varphi(t) - R(t)\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2} t}.$$

Recall that, with respect to [11, 15], the proof of Theorem 1.2 relies on an additional topological argument to control the unstable nature of the solitons. Moreover, note that no uniqueness result is proved in [4], contrary to the subcritical and critical cases [11]. In fact, the objective of this paper is to prove uniqueness up to N parameters, as suggested by Theorem 1.1.

1.3 Main result and outline of the paper

The whole paper is devoted to prove the following theorem of existence and uniqueness of a family of multi-solitons for the supercritical (gKdV) equation.

Theorem 1.3. *Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R = \sum_{j=1}^{N} R_{c_j, x_j}$.*

1. *There exists an N-parameter family $(\varphi_{A_1,\ldots,A_N})_{(A_1,\ldots,A_N)\in\mathbb{R}^N}$ of solutions of* (gKdV) *such that, for all $(A_1, \ldots, A_N) \in \mathbb{R}^N$,*

$$\lim_{t\to+\infty} \|\varphi_{A_1,\ldots,A_N}(t) - R(t)\|_{H^1} = 0,$$

   and if $(A_1', \ldots, A_N') \neq (A_1, \ldots, A_N)$, then $\varphi_{A_1',\ldots,A_N'} \neq \varphi_{A_1,\ldots,A_N}$.

2. *Conversely, if u is a solution of* (gKdV) *such that $\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1} = 0$, then there exists $(A_1, \ldots, A_N) \in \mathbb{R}^N$ such that $u = \varphi_{A_1,\ldots,A_N}$.*

Remark 1.4. The convergence of $\varphi_{A_1,\ldots,A_N}$ to R in Theorem 1.3 is actually exponential in time, as in Theorem 1.2. See the proof of Theorem 1.3 at the beginning of Section 3 for more details.

Remark 1.5. For the nonlinear Schrödinger equation, the question of the classification of multi-solitons as in Theorem 1.3 is open. In fact, even for subcritical and critical cases, no general uniqueness result has been proved yet (see general existence results in [16, 20, 21, 14, 4]).

The paper is organized as follows. In the next section, we briefly recall some well-known results on solitons, multi-solitons, and on the linearized equation. One of the most important facts about the linearized equation, also strongly used in [4, 3], is the determination by Pego and Weinstein [18] of the spectrum of the linearized operator $\mathcal{L}$ around the soliton $Q(x-t)$: $\sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, +e_0\}$ with $e_0 > 0$, and moreover e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and Y^-. Indeed, $Y^\pm$ allow to control the negative directions of the linearized energy around a soliton (see

Lemma 2.5). Moreover, by a simple scaling argument, we determine eigenvalues of the linearized operator around Q_{c_j}: $\pm e_j = \pm c_j^{3/2} e_0$ are eigenvalues with eigenfunctions $Y_j^{\pm}$ (see Notation 2.6 for precise definitions).

In Section 3, we construct the family $(\varphi_{A_1,\dots,A_N})$ described in Theorem 1.3. To do this, we first claim Proposition 3.1, which is the new key point of the proof of the multi-existence result, and can be summarized as follows. *Let φ be a multi-soliton given by Theorem 1.2, $j \in [\![1, N]\!]$ and $A_j \in \mathbb{R}$. Then there exists a solution $u(t)$ of* (gKdV) *such that*

$$\left\| u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t) \right\|_{H^1} \leqslant e^{-(e_j+\gamma)t},$$

for t large and for some small $\gamma > 0$. This means that, similarly as in [3] for one soliton, we can perturb the multi-soliton φ locally around *one* given soliton at the order $e^{-e_j t}$. Since $e_1 < \cdots < e_N$, $\varphi_{A_1,\dots,A_N}$ has to be constructed by iteration, from $j = 1$ to $j = N$. Indeed, it is not significant to perturb φ at order e_j before order e_{j-1}, since $e_j > e_{j-1} + \gamma$. Similarly, it seems that there exists no simple way to compare $\varphi_{A_1,\dots,A_N}$ to φ. Finally, to prove Proposition 3.1, we rely on refinements of arguments developed in [4], in particular the topological argument to control the unstable directions.

In Section 4, we classify all multi-solitons in terms of the family previously constructed. Once again, it appears that the identification of the solution has to be done step by step (after an improvement of the convergence rate, as in [3]), from order e_1 to order e_N. In this section, we strongly use special monotonicity properties of (gKdV), in particular to prove that any multi-soliton converges exponentially (Section 4.1). Such arguments are not known for the nonlinear Schrödinger equations.

Finally, recall that in the one soliton case for (gKdV) [3], a construction of a family of approximate solutions of the linearized equation and fixed point arguments were used (among other things), as in the one soliton case for the nonlinear Schrödinger equation [6]. For multi-solitons, since the construction of approximate solutions is not natural (because of the interactions between solitons), we propose in this paper an alternate approach based only on compactness and energy methods.

2 Preliminary results

2.1 Notation and first properties of the solitons

Notation 2.1. They are available in the whole paper.

(a) $(\cdot,\cdot)$ denotes the $L^2(\mathbb{R})$ scalar product.

(b) The Sobolev space H^s is defined by $H^s(\mathbb{R}) = \{u \in \mathcal{D}'(\mathbb{R}) \mid (1+\xi^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R})\}$, and in particular $H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) \mid \|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|u'\|_{L^2}^2 < +\infty\} \hookrightarrow L^\infty(\mathbb{R})$.

(c) We denote $\partial_x v = v_x$ the partial derivative of v with respect to x.

(d) All numbers C, K appearing in inequalities are real constants (with respect to the context) strictly positive, which may change in each step of an inequality.

Claim 2.2. *For all $c > 0$, one has:*

(i) $Q_c > 0$, Q_c is even, Q_c is C^∞, and $Q_c'(x) < 0$ for all $x > 0$.

(ii) For all $j \geqslant 0$, there exists $C_j > 0$ such that $Q_c^{(j)}(x) \sim C_j e^{-\sqrt{c}|x|}$ as $|x| \to +\infty$.
In particular, for all $j \geqslant 0$, there exists $C_j' > 0$ such that $|Q_c^{(j)}(x)| \leqslant C_j' e^{-\sqrt{c}|x|}$ for all $x \in \mathbb{R}$.

(iii) $Q_c'' + Q_c^p = cQ_c$.

2.2 Linearized equation

Let $c > 0$.

2.2.1 Linearized operator around Q_c

The linearized equation appears if one considers a solution of (gKdV) close to the soliton $Q_c(x-ct)$. More precisely, if $u_c(t,x) = Q_c(x-ct) + h_c(t, x-ct)$ satisfies (gKdV), then h_c satisfies

$$\partial_t h_c + \mathcal{L}_c h_c = O(h_c^2)$$

where

$$\mathcal{L}_c a = -\partial_x(L_c a) \quad \text{and} \quad L_c a = -\partial_x^2 a + ca - pQ_c^{p-1}a.$$

The spectrum of $\mathcal{L}_c$ has been calculated by Pego and Weinstein for $c = 1$ in [18]. Their results are summed up in the following proposition for the reader's convenience.

Proposition 2.3 ([18])**.** *Let $\sigma(\mathcal{L})$ be the spectrum of the operator $\mathcal{L}$ defined on $L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L})$ be its essential spectrum. Then*

$$\sigma_{\text{ess}}(\mathcal{L}) = i\mathbb{R} \quad \textit{and} \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\} \textit{ with } e_0 > 0.$$

Furthermore, e_0 and $-e_0$ are simple eigenvalues of $\mathcal{L}$ with eigenfunctions Y^+ and $Y^- = \check{Y}^+$ which have an exponential decay at infinity, and the null space of $\mathcal{L}$ is spanned by Q'.

This result is extended to $\mathcal{L}_c$ in Corollary 2.4 by a simple scaling argument. Indeed, we recall that if u is a solution of (gKdV), then for all $\lambda > 0$, $u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^3 t, \lambda x)$ is also a solution. Moreover, we have $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$.

Corollary 2.4. *Let $\sigma(\mathcal{L}_c)$ be the spectrum of the operator $\mathcal{L}_c$ defined on $L^2(\mathbb{R})$ and let $\sigma_{\text{ess}}(\mathcal{L}_c)$ be its essential spectrum. Then*

$$\sigma_{\text{ess}}(\mathcal{L}_c) = i\mathbb{R} \quad \textit{and} \quad \sigma(\mathcal{L}_c) \cap \mathbb{R} = \{-e_c, 0, e_c\} \quad \textit{where} \quad e_c = c^{3/2} e_0 > 0.$$

Furthermore, e_c and $-e_c$ are simple eigenvalues of $\mathcal{L}_c$ with eigenfunctions Y_c^+ and $Y_c^- = \check{Y_c^+}$, where

$$Y_c^\pm(x) = c^{-1/2} Y^\pm(\sqrt{c}x),$$

and the null space of $\mathcal{L}_c$ is spanned by Q_c'.

2.2.2 Adjoint of $\mathcal{L}_c$

We recall that Lemma 4.9 in [3], under a suitable normalization of $Y^\pm$, shows important properties of the adjoint of $\mathcal{L}$. With the same normalization and by Corollary 2.4, we obtain the following lemma by a simple scaling argument. Recall that assertion (v) is proved in [4] for $c = 1$.

Lemma 2.5. *Let $Z_c^\pm = L_c Y_c^\pm$. Then the following properties hold:*

(i) *$Z_c^\pm$ are two eigenfunctions of $L_c\partial_x$: $L_c(\partial_x Z_c^\pm) = \mp e_c Z_c^\pm$.*

(ii) *There exists $\eta_0 > 0$ such that, for all $x \in \mathbb{R}$,*

$$|Y_c^\pm(x)| + |\partial_x Y_c^\pm(x)| + |Z_c^\pm(x)| + |\partial_x Z_c^\pm(x)| \leqslant Ce^{-\eta_0\sqrt{c}|x|}.$$

(iii) $(Y_c^+, Z_c^+) = (Y_c^-, Z_c^-) = 0$ *and* $(Z_c^+, Q_c') = (Z_c^-, Q_c') = 0$.

(iv) $(Y_c^+, Z_c^-) = (Y_c^-, Z_c^+) = 1$ *and* $(Q_c', \partial_x Y_c^+) > 0$.

(v) *There exists $\widetilde{\sigma_c} > 0$ such that, for all $v_c \in H^1$ such that $(v_c, Z_c^+) = (v_c, Z_c^-) = (v_c, Q_c') = 0$, $(L_c v_c, v_c) \geqslant \widetilde{\sigma_c}\|v_c\|_{H^1}^2$.*

(vi) *There exist $\sigma_c > 0$ and $C > 0$ such that, for all $v_c \in H^1$,*

$$(L_c v_c, v_c) \geqslant \sigma_c\|v_c\|_{H^1}^2 - C(v_c, Z_c^+)^2 - C(v_c, Z_c^-)^2 - C(v_c, Q_c')^2.$$

2.3 Multi-solitons results

A set of parameters (1.1) being given, we adopt the following notation.

Notation 2.6. For all $j \in [\![1, N]\!]$, define:

(i) $R_j(t,x) = Q_{c_j}(x - c_j t - x_j)$, where $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$.

(ii) $Y_j^{\pm}(t,x) = Y_{c_j}^{\pm}(x - c_j t - x_j)$, where $Y_c^{\pm}(x) = c^{-1/2} Y^{\pm}(\sqrt{c}x)$ is defined in Corollary 2.4.

(iii) $Z_j^{\pm}(t,x) = Z_{c_j}^{\pm}(x - c_j t - x_j)$, where $Z_c^{\pm} = L_c Y_c^{\pm}$.

(iv) $e_j = e_{c_j}$, where $e_c = c^{3/2} e_0$.

Now, to estimate interactions between solitons, we denote the small parameters

$$\sigma_0 = \min\{\eta_0^{2/3} c_1, e_0^{2/3} c_1, c_1, c_2 - c_1, \dots, c_N - c_{N-1}\} \quad \text{and} \quad \gamma = \frac{\sigma_0^{3/2}}{10^6}. \tag{2.1}$$

From [11], it appears that γ is a suitable parameter to quantify interactions between solitons in large time. For instance, we have, for $j \neq k$ and all $t \geqslant 0$,

$$\int R_j(t) R_k(t) + |(R_j)_x(t)||(R_k)_x(t)| \leqslant C e^{-10\gamma t}. \tag{2.2}$$

From the definition of σ_0 and Lemma 2.5, such an inequality is also true for $Y_j^{\pm}$ and $Z_j^{\pm}$.

Moreover, since σ_0 has the same definition as in [4], then from their Remark 1, Theorem 1.2 can be rewritten as follows. There exist $T_0 \in \mathbb{R}$ and $\varphi \in C([T_0, +\infty), H^1)$ such that, for all $s \geqslant 1$, there exists $A_s > 0$ such that

$$\|\varphi(t) - R(t)\|_{H^s} \leqslant A_s e^{-4\gamma t}. \tag{2.3}$$

3 Construction of a family of multi-solitons

In this section, we prove the first point of Theorem 1.3 as a consequence of the following crucial Proposition 3.1. Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ and $x_1, \dots, x_N \in \mathbb{R}$. Denote $R = \sum_{k=1}^{N} R_k$ and φ a multi-soliton solution satisfying (2.3), as defined in Theorem 1.2 for example.

Proposition 3.1. *Let $j \in [\![1, N]\!]$ and $A_j \in \mathbb{R}$. Then there exist $t_0 > 0$ and $u \in C([t_0, +\infty), H^1)$ a solution of* (gKdV) *such that*

$$\forall t \geqslant t_0, \quad \left\| u(t) - \varphi(t) - A_j e^{-e_j t} Y_j^{+}(t) \right\|_{H^1} \leqslant e^{-(e_j + \gamma)t}. \tag{3.1}$$

Before proving this proposition, let us show how this proposition implies the first point of Theorem 1.3.

Proof of 1. of Theorem 1.3. Let $(A_1, \dots, A_N) \in \mathbb{R}^N$.

(i) Consider φ_{A_1} the solution of (gKdV) given by Proposition 3.1 applied with φ given by Theorem 1.2. Thus there exists $t_0 > 0$ such that

$$\forall t \geqslant t_0, \quad \left\| \varphi_{A_1}(t) - \varphi(t) - A_1 e^{-e_1 t} Y_1^{+}(t) \right\|_{H^1} \leqslant e^{-(e_1 + \gamma)t}.$$

Now remark that φ_{A_1} is also a multi-soliton, which satisfies (2.3) by the definition of γ and the same techniques used in [11, Section 3.4] to improve the estimate in higher order Sobolev norms. Hence we can apply Proposition 3.1 with φ_{A_1} instead of φ, so that we obtain φ_{A_1,A_2} such that

$$\forall t \geqslant t_0', \quad \left\| \varphi_{A_1,A_2}(t) - \varphi_{A_1}(t) - A_2 e^{-e_2 t} Y_2^{+}(t) \right\|_{H^1} \leqslant e^{-(e_2 + \gamma)t}.$$

Similarly, for all $j \in [\![1, N]\!]$, we construct by induction a solution $\varphi_{A_1,\dots,A_j}$ such that

$$\forall t \geqslant t_0, \quad \left\| \varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t} Y_j^{+}(t) \right\|_{H^1} \leqslant e^{-(e_j + \gamma)t}. \tag{3.2}$$

Observe finally that $\varphi_{A_1,\dots,A_N}$ constructed by this way satisfies (2.3).

(ii) Let $(A'_1,\dots,A'_N)\in\mathbb{R}^N$ be such that $(A'_1,\dots,A'_N)\neq(A_1,\dots,A_N)$, and suppose in the sake of contradiction that $\varphi_{A'_1,\dots,A'_N}=\varphi_{A_1,\dots,A_N}$. Denote $i_0=\min\{i\in[\![1,N]\!]\mid A'_i\neq A_i\}$. Hence we have $A'_i=A_i$ for $i\in[\![1,i_0-1]\!]$, $A'_{i_0}\neq A_{i_0}$ and from the construction of $\varphi_{A_1,\dots,A_N}$,

$$\begin{aligned}\varphi_{A_1,\dots,A_N}(t)&=\varphi_{A_1,\dots,A_{N-1}}(t)+A_Ne^{-e_Nt}Y_N^+(t)+z_N(t)\\&=\varphi_{A_1,\dots,A_{N-2}}(t)+A_{N-1}e^{-e_{N-1}t}Y_{N-1}^+(t)+A_Ne^{-e_Nt}Y_N^+(t)+z_{N-1}(t)+z_N(t)\\&=\cdots=\varphi_{A_1,\dots,A_{i_0-1}}(t)+A_{i_0}e^{-e_{i_0}t}Y_{i_0}^+(t)+\sum_{k>i_0}A_ke^{-e_kt}Y_k^+(t)+\sum_{k\geqslant i_0}z_k(t)\end{aligned}$$

where z_k satisfies $\|z_k(t)\|_{H^1}\leqslant e^{-(e_k+\gamma)t}$ for $t\geqslant t_0$ and each $k\geqslant i_0$. Similarly, we get

$$\varphi_{A'_1,\dots,A'_N}(t)=\varphi_{A'_1,\dots,A'_{i_0-1}}(t)+A'_{i_0}e^{-e_{i_0}t}Y_{i_0}^+(t)+\sum_{k>i_0}A'_ke^{-e_kt}Y_k^+(t)+\sum_{k\geqslant i_0}\widetilde{z_k}(t),$$

and so using $\varphi_{A'_1,\dots,A'_N}=\varphi_{A_1,\dots,A_N}$ and $\varphi_{A'_1,\dots,A'_{i_0-1}}=\varphi_{A_1,\dots,A_{i_0-1}}$, we obtain

$$e^{-e_{i_0}t}|A_{i_0}-A'_{i_0}|\leqslant Ce^{-(e_{i_0}+\gamma)t}$$

for $t\geqslant t_0$, thus $|A_{i_0}-A'_{i_0}|\leqslant Ce^{-\gamma t}$, and so $A'_{i_0}=A_{i_0}$ by letting $t\to+\infty$, which is a contradiction and concludes the proof. □

Now, the only purpose of the rest of this section is to prove Proposition 3.1. Let $j\in[\![1,N]\!]$ and $A_j\in\mathbb{R}$. We want to construct a solution u of (gKdV) such that

$$z(t,x)=u(t,x)-\varphi(t,x)-A_je^{-e_jt}Y_j^+(t,x)$$

satisfies $\|z(t)\|_{H^1}\leqslant e^{-(e_j+\gamma)t}$ for $t\geqslant t_0$ with t_0 large enough.

3.1 Equation of z

Since u is a solution of (gKdV) and also φ is (and this fact is crucial for the whole proof), we get

$$\partial_tz+\partial_x^3z+\partial_x[(\varphi+A_je^{-e_jt}Y_j^++z)^p-\varphi^p]+A_je^{-e_jt}[\partial_x^3Y_j^+-c_j\partial_xY_j^+-e_jY_j^+]=0.$$

But from Corollary 2.4, we have

$$\mathcal{L}_{c_j}Y_{c_j}^+=e_jY_{c_j}^+=\partial_x^3Y_{c_j}^+-c_j\partial_xY_{c_j}^++p\partial_x(Q_{c_j}^{p-1}Y_{c_j}^+)$$

and so following Notation 2.6, we get the following equation for z:

$$\partial_tz+\partial_x^3z+\partial_x[(\varphi+A_je^{-e_jt}Y_j^++z)^p-\varphi^p-pA_je^{-e_jt}R_j^{p-1}Y_j^+]=0. \tag{3.3}$$

This can also be written

$$\begin{aligned}&\partial_tz+\partial_x\big[\partial_x^2z+p\varphi^{p-1}z\big]+\partial_x\big[(\varphi+A_je^{-e_jt}Y_j^++z)^p-(\varphi+A_je^{-e_jt}Y_j^+)^p-p(\varphi+A_je^{-e_jt}Y_j^+)^{p-1}z\big]\\&+p\partial_x\big[((\varphi+A_je^{-e_jt}Y_j^+)^{p-1}-\varphi^{p-1})\cdot z\big]=-\partial_x\big[(\varphi+A_je^{-e_jt}Y_j^+)^p-\varphi^p-pA_je^{-e_jt}Y_j^+R_j^{p-1}\big].\end{aligned}$$

Finally, if we denote

$$\begin{cases}\omega_1=p[(\varphi+A_je^{-e_jt}Y_j^+)^{p-1}-\varphi^{p-1}],\\ \omega(z)=(\varphi+A_je^{-e_jt}Y_j^++z)^p-(\varphi+A_je^{-e_jt}Y_j^+)^p-p(\varphi+A_je^{-e_jt}Y_j^+)^{p-1}z,\\ \Omega=(\varphi+A_je^{-e_jt}Y_j^+)^p-\varphi^p-pA_je^{-e_jt}Y_j^+R_j^{p-1},\end{cases}$$

we obtain the shorter form of the equation of z:

$$\partial_tz+\partial_x\big[\partial_x^2z+p\varphi^{p-1}z\big]+\partial_x[\omega_1\cdot z]+\partial_x[\omega(z)]=-\partial_x\Omega. \tag{3.4}$$

Note that the term $\omega(z)$ is the nonlinear term in z, and that ω_1 satisfies, for all $s \geqslant 0$, $\|\omega_1(t)\|_{H^s} \leqslant C_s e^{-e_j t}$ for all $t \geqslant 0$. Moreover, the source term Ω satisfies

$$\forall s \geqslant 1, \exists C_s > 0, \forall t \geqslant 0, \quad \|\Omega(t)\|_{H^s} \leqslant C_s e^{-(e_j+4\gamma)t}. \tag{3.5}$$

Indeed, if we write Ω under the form

$$\Omega = \left[(\varphi + A_j e^{-e_j t} Y_j^+)^p - \varphi^p - p\varphi^{p-1} A_j e^{-e_j t} Y_j^+\right] + pA_j e^{-e_j t} Y_j^+ (\varphi^{p-1} - R^{p-1}) + pA_j e^{-e_j t} Y_j^+ (R^{p-1} - R_j^{p-1}),$$

we deduce from (2.3), (2.2) and the definition of γ (2.1) that

$$\|\Omega(t)\|_{H^s} \leqslant Ce^{-2e_j t} + Ce^{-e_j t}\|\varphi(t) - R(t)\|_{H^s} + Ce^{-e_j t} \cdot e^{-4\gamma t} \leqslant Ce^{-(e_j+4\gamma)t}.$$

3.2 Compactness argument assuming uniform estimate

To prove Proposition 3.1, we follow the strategy of [11, 4]. Let $S_n \to +\infty$ be an increasing sequence of time, $\mathfrak{b}_n = (b_{n,k})_{j<k\leqslant N} \in \mathbb{R}^{N-j}$ be a sequence of parameters to be determined, and let u_n be the solution of

$$\begin{cases} \partial_t u_n + \partial_x[\partial_x^2 u_n + u_n^p] = 0, \\ u_n(S_n) = \varphi(S_n) + A_j e^{-e_j S_n} Y_j^+(S_n) + \displaystyle\sum_{k>j} b_{n,k} Y_k^+(S_n). \end{cases} \tag{3.6}$$

Notation 3.2. (i) $\mathbb{R}^N$ is equipped with the ℓ^2 norm, simply denoted $\|\cdot\|$.

(ii) $B_{\mathcal{B}}(P, r)$ is the closed ball of the Banach space $\mathcal{B}$, centered at P and of radius $r \geqslant 0$. If $P = 0$, we simply write $B_{\mathcal{B}}(r)$.

(iii) $\mathbb{S}_{\mathbb{R}^N}(r)$ denotes the sphere of radius r in $\mathbb{R}^N$.

Proposition 3.3. *There exist $n_0 \geqslant 0$ and $t_0 > 0$ (independent of n) such that the following holds. For each $n \geqslant n_0$, there exists $\mathfrak{b}_n \in \mathbb{R}^{N-j}$ with $\|\mathfrak{b}_n\| \leqslant 2e^{-(e_j+2\gamma)S_n}$, and such that the solution u_n of (3.6) is defined on the interval $[t_0, S_n]$, and satisfies*

$$\forall t \in [t_0, S_n], \quad \left\|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\right\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

Assuming this proposition and the following lemma of weak continuity of the flow, we can deduce the proof of Proposition 3.1. The proof of Proposition 3.3 is postponed to the next section, whereas the proof of Lemma 3.4 is postponed to Appendix A.

Lemma 3.4. *Suppose that $z_{0,n} \rightharpoonup z_0$ in H^1, and that there exits $T > 0$ such that the solution $z_n(t)$ corresponding to initial data $z_{0,n}$ exists for $t \in [0, T]$ and $\sup_{t\in[0,T]} \|z_n(t)\|_{H^1} \leqslant K$. Then for all $t \in [0, T]$, the solution $z(t)$ corresponding to initial data z_0 exists, and $z_n(T) \rightharpoonup z(T)$ in H^1.*

Remark 3.5. Note that the proof of Lemma 3.4 strongly relies on the Cauchy theory in H^s with $s < 1$, developed in [10]. Thus this argument is quite similar to the compactness argument developed in [4] or [11].

Proof of Proposition 3.1 assuming Proposition 3.3. We may assume $n_0 = 0$ in Proposition 3.3 without loss of generality. It follows from this proposition that there exists a sequence $u_n(t)$ of solutions to (gKdV), defined on $[t_0, S_n]$, such that the following uniform estimates hold:

$$\forall n \geqslant 0, \forall t \in [t_0, S_n], \quad \left\|u_n(t) - \varphi(t) - A_j e^{-e_j t} Y_j^+(t)\right\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

In particular, there exists $C_0 > 0$ such that $\|u_n(t_0)\|_{H^1} \leqslant C_0$ for all $n \geqslant 0$. Thus there exists $u_0 \in H^1(\mathbb{R})$ such that $u_n(t_0) \rightharpoonup u_0$ in H^1 weak (after passing to a subsequence). Now consider u solution of

$$\begin{cases} \partial_t u + \partial_x[\partial_x^2 u + u^p] = 0, \\ u(t_0) = u_0. \end{cases}$$

Let $T \geqslant t_0$. For n such that $S_n > T$, $u_n(t)$ is well defined for all $t \in [t_0, T]$, and moreover $\|u_n(t)\|_{H^1} \leqslant C$. By Lemma 3.4, we have $u_n(T) \rightharpoonup u(T)$ in H^1. As

$$\left\|u_n(T) - \varphi(T) - A_j e^{-e_j T} Y_j^+(T)\right\|_{H^1} \leqslant e^{-(e_j+\gamma)T},$$

we finally obtain, by weak convergence, $\left\|u(T) - \varphi(T) - A_j e^{-e_j T} Y_j^+(T)\right\|_{H^1} \leqslant e^{-(e_j+\gamma)T}$. Thus u is a solution of (gKdV) which satisfies (3.1). □

3.3 Proof of Proposition 3.3

The proof proceeds in several steps. For the sake of simplicity, we will drop the index n for the rest of this section (except for S_n). As Proposition 3.3 is proved for given n, this should not be a source of confusion. Hence we will write u for u_n, z for z_n, $\mathfrak{b}$ for $\mathfrak{b}_n$, etc. We possibly drop the first terms of the sequence S_n, so that, for all n, S_n is large enough for our purposes.

From (3.4), the equation satisfied by z is

$$\begin{cases} \partial_t z + \partial_x[\partial_x^2 z + p\varphi^{p-1} z] + \partial_x[\omega_1 \cdot z] + \partial_x[\omega(z)] = -\partial_x \Omega, \\ z(S_n) = \sum_{k>j} b_k Y_k^+(S_n). \end{cases} \tag{3.7}$$

Moreover, for all $k \in [\![1, N]\!]$, we denote

$$\alpha_k^{\pm}(t) = \int z(t) \cdot Z_k^{\pm}(t).$$

In particular, we have

$$\alpha_k^{\pm}(S_n) = \sum_{l>j} b_l \int Y_l^+(S_n) \cdot Z_k^{\pm}(S_n).$$

Finally, we denote $\boldsymbol{\alpha}^-(t) = (\alpha_k^-(t))_{j<k\leqslant N}$.

3.3.1 Modulated final data

Lemma 3.6. *For $n \geqslant n_0$ large enough, the following holds. For all $\mathfrak{a}^- \in \mathbb{R}^{N-j}$, there exists a unique $\mathfrak{b} \in \mathbb{R}^{N-j}$ such that $\|\mathfrak{b}\| \leqslant 2\|\mathfrak{a}^-\|$ and $\boldsymbol{\alpha}^-(S_n) = \mathfrak{a}^-$.*

Proof. Consider the linear application

$$\begin{array}{rccc} \Phi : & \mathbb{R}^{N-j} & \to & \mathbb{R}^{N-j} \\ & \mathfrak{b} = (b_l)_{j<l\leqslant N} & \mapsto & \left(\sum_{l>j} b_l \int Y_l^+(S_n) Z_k^-(S_n)\right)_{j<k\leqslant N}. \end{array}$$

From the normalization of Lemma 2.5, its matrix in the canonical basis is

$$\operatorname{Mat}\Phi = \begin{pmatrix} 1 & \int Y_{j+2}^+ Z_{j+1}^-(S_n) & \cdots & \int Y_{j+N}^+ Z_{j+1}^-(S_n) \\ \int Y_{j+1}^+ Z_{j+2}^-(S_n) & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \int Y_{j+1}^+ Z_{j+N}^-(S_n) & \cdots & \cdots & 1 \end{pmatrix}.$$

But from (2.2), we have, for $k \neq l$,

$$\left|\int Y_l^{\pm} Z_k^{\pm}(S_n)\right| \leqslant C_0 e^{-\gamma S_n}$$

with C_0 independent of n, and so by taking n_0 large enough, we have $\Phi = \mathrm{Id} + A_n$ where $\|A_n\| \leqslant \frac{1}{2}$. Thus Φ is invertible and $\|\Phi^{-1}\| \leqslant 2$. Finally, for a given $\mathfrak{a}^- \in \mathbb{R}^{N-j}$, it is enough to define $\mathfrak{b}$ by $\mathfrak{b} = \Phi^{-1}(\mathfrak{a}^-)$ to conclude the proof of Lemma 3.6. □

Claim 3.7. *The following estimates at S_n hold:*

- $|\alpha_k^+(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\|$ *for all* $k \in [\![1, N]\!]$,
- $|\alpha_k^-(S_n)| \leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\|$ *for all* $k \in [\![1, j]\!]$,
- $\|z(S_n)\|_{H^1} \leqslant C\|\mathfrak{b}\|$.

3.3.2 Equations on $\alpha_k^\pm$

Let $t_0 > 0$ independent of n to be determined later in the proof, $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ to be chosen, $\mathfrak{b}$ be given by Lemma 3.6 and u be the corresponding solution of (3.6). We now define the maximal time interval $[T(\mathfrak{a}^-), S_n]$ on which suitable exponential estimates hold.

Definition 3.8. Let $T(\mathfrak{a}^-)$ be the infimum of $T \geqslant t_0$ such that for all $t \in [T, S_n]$, both following properties hold:

$$e^{(e_j+\gamma)t}z(t) \in B_{H^1}(1) \quad \text{and} \quad e^{(e_j+2\gamma)t}\boldsymbol{\alpha}^-(t) \in B_{\mathbb{R}^{N-j}}(1). \tag{3.8}$$

Observe that Proposition 3.3 is proved if for all n, we can find $\mathfrak{a}^-$ such that $T(\mathfrak{a}^-) = t_0$. The rest of the proof is devoted to prove the existence of such a value of $\mathfrak{a}^-$.

First, we prove the following estimate on $\alpha_k^\pm$.

Claim 3.9. *For all $k \in [\![1, N]\!]$ and all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\left|\frac{d}{dt}\alpha_k^\pm(t) \mp e_k\alpha_k^\pm(t)\right| \leqslant C_0e^{-4\gamma t}\|z(t)\|_{H^1} + C_1\|z(t)\|_{H^1}^2 + C_2e^{-(e_j+4\gamma)t}. \tag{3.9}$$

Proof. Using the equation of z (3.7), we first compute

$$\begin{aligned}
\frac{d}{dt}\alpha_k^\pm(t) &= \int z_t Z_k^\pm + \int zZ_{kt}^\pm \\
&= \int (z_{xx} + p\varphi^{p-1}z)Z_{kx}^\pm + \int \omega_1 zZ_{kx}^\pm + \int \omega(z)Z_{kx}^\pm + \int \Omega Z_{kx}^\pm - c_k\int zZ_{kx}^\pm \\
&= \int (z_{xx} - c_kz + pR_k^{p-1}z)Z_{kx}^\pm + p\int(\varphi^{p-1} - R_k^{p-1})zZ_{kx}^\pm + \int(\omega_1 z + \omega(z) + \Omega)Z_{kx}^\pm.
\end{aligned}$$

But from (i) of Lemma 2.5, we have

$$\begin{aligned}
\int (z_{xx} - c_kz + pR_k^{p-1}z)Z_{kx}^\pm &= (-L_{c_k}z(t, \cdot + c_kt), \partial_x Z_{c_k}^\pm) \\
&= (z(t, \cdot + c_kt), -L_{c_k}(\partial_x Z_{c_k}^\pm)) = \pm e_k(z(t, \cdot + c_kt), Z_{c_k}^\pm) = \pm e_k\alpha_k^\pm,
\end{aligned}$$

and from (2.3) and (3.5), we have the following estimates:

- $|\int(\varphi^{p-1} - R_k^{p-1})zZ_{kx}^\pm| \leqslant C\|\varphi - R\|_{L^\infty}\|z\|_{L^\infty} + Ce^{-4\gamma t}\|z\|_{L^2} \leqslant Ce^{-4\gamma t}\|z\|_{H^1}$,
- $|\int \omega_1 zZ_{kx}^\pm| \leqslant \|\omega_1\|_{L^\infty}\|z\|_{L^\infty}\|Z_{kx}^\pm\|_{L^1} \leqslant Ce^{-e_jt}\|z\|_{H^1} \leqslant Ce^{-4\gamma t}\|z\|_{H^1}$,
- $|\int \omega(z)Z_{kx}^\pm| \leqslant C\|z\|_{L^2}^2 \leqslant C\|z\|_{H^1}^2$,
- $|\int \Omega Z_{kx}^\pm| \leqslant C\|\Omega\|_{L^\infty} \leqslant Ce^{-(e_j+4\gamma)t}$,

which conclude the proof of the claim. □

3.3.3 Control of the stable directions

We estimate here $\alpha_k^+(t)$ for all $k \in [\![1, N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$. From (3.9) and (3.8), we have

$$\left|\frac{d}{dt}\alpha_k^+(t) - e_k\alpha_k^+(t)\right| \leqslant C_0 e^{-(e_j+5\gamma)t} + C_1 e^{-2(e_j+\gamma)t} + C_2 e^{-(e_j+4\gamma)t} \leqslant K_2 e^{-(e_j+4\gamma)t}.$$

Thus $|(e^{-e_k s}\alpha_k^+(s))'| \leqslant K_2 e^{-(e_j+e_k+4\gamma)s}$, and so by integration on $[t, S_n]$ we get $|e^{-e_k S_n}\alpha_k^+(S_n) - e^{-e_k t}\alpha_k^+(t)| \leqslant K_2 e^{-(e_j+e_k+4\gamma)t}$ and so

$$|\alpha_k^+(t)| \leqslant e^{e_k(t-S_n)}|\alpha_k^+(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But from Claim 3.7 and Lemma 3.6, we have

$$\begin{aligned} e^{e_k(t-S_n)}|\alpha_k^+(S_n)| \leqslant |\alpha_k^+(S_n)| &\leqslant Ce^{-2\gamma S_n}\|\mathfrak{b}\| \\ &\leqslant Ce^{-2\gamma S_n}e^{-(e_j+2\gamma)S_n} \leqslant K_2 e^{-(e_j+4\gamma)S_n} \leqslant K_2 e^{-(e_j+4\gamma)t}, \end{aligned}$$

and so finally

$$\forall k \in [\![1, N]\!], \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^+(t)| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.10}$$

3.3.4 Control of the unstable directions for $k \leqslant j$

We estimate here $\alpha_k^-(t)$ for all $k \in [\![1, j]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$. Note first that, as in the previous paragraph, we get for all $k \in [\![1, N]\!]$ and $t \in [T(\mathfrak{a}^-), S_n]$,

$$\left|\frac{d}{dt}\alpha_k^-(t) + e_k\alpha_k^-(t)\right| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.11}$$

Now suppose $k \leqslant j$, which implies $e_k \leqslant e_j$. Since $|(e^{e_k s}\alpha_k^-(s))'| \leqslant K_2 e^{(e_k-e_j-4\gamma)s}$, we obtain, by integration on $[t, S_n]$,

$$|\alpha_k^-(t)| \leqslant e^{e_k(S_n-t)}|\alpha_k^-(S_n)| + K_2 e^{-(e_j+4\gamma)t}.$$

But again from Claim 3.7 and Lemma 3.6, we have

$$\begin{aligned} e^{e_k(S_n-t)}|\alpha_k^-(S_n)| &\leqslant K_2 e^{e_k(S_n-t)}e^{-2\gamma S_n}e^{-(e_j+2\gamma)S_n} = K_2 e^{e_k(S_n-t)}e^{-(e_j+4\gamma)S_n} \\ &\leqslant K_2 e^{(S_n-t)(e_k-e_j)}e^{-e_j t}e^{-4\gamma S_n} \leqslant K_2 e^{-(e_j+4\gamma)t}, \end{aligned}$$

and so finally

$$\forall k \in [\![1, j]\!], \forall t \in [T(\mathfrak{a}^-), S_n], \quad |\alpha_k^-(t)| \leqslant K_2 e^{-(e_j+4\gamma)t}. \tag{3.12}$$

3.3.5 Monotonicity property of the energy

We follow here the same strategy as in [11, Section 4] to estimate the energy backwards. Since calculations are long and technical, we refer to [11] for more details.

We define the following function

$$\psi(x) = \frac{2}{\pi}\arctan(\exp(-\sqrt{\sigma_0}x/2))$$

so that $\lim_{+\infty}\psi = 0$, $\lim_{-\infty}\psi = 1$, and for all $x \in \mathbb{R}$, $\psi(-x) = 1 - \psi(x)$. Note that by a direct calculation, we have $|\psi'''(x)| \leqslant \frac{\sigma_0}{4}|\psi'(x)|$. Moreover, we set

$$h(t, x) = \frac{1}{c_N} + \sum_{k=1}^{N-1}\left(\frac{1}{c_k} - \frac{1}{c_{k+1}}\right)\psi\left(x - \frac{c_k + c_{k+1}}{2}t - \frac{x_k + x_{k+1}}{2}\right).$$

Observe that the function h takes values close to $\frac{1}{c_k}$ for x close to $c_k t + x_k$, and has large variations only in regions far away from the solitons (for instance we have, for all $k \in [\![1, N]\!]$ and $t \geqslant 0$, $\|R_k(t)h_x(t)\|_{L^\infty} \leqslant Ce^{-4\gamma t}$). We also define a quantity related to the energy for z:

$$H(t) = \int \Big\{ \Big(z_x^2(t,x) - F(t, z(t,x)) \Big) h(t,x) + z^2(t,x) \Big\} \, dx$$

where

$$F(t,z) = 2\left[\frac{(\varphi + v_j + z)^{p+1}}{p+1} - \frac{(\varphi + v_j)^{p+1}}{p+1} - (\varphi + v_j)^p z \right],$$

and $v_j(t,x) = A_j e^{-e_j t} Y_j^+(t,x)$.

Lemma 3.10. *For all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\frac{dH}{dt}(t) \geqslant -C_0 \|z(t)\|_{H^1}^3 - C_1 e^{-2\gamma t} \|z(t)\|_{H^1}^2 - C_2 e^{-(e_j + 3\gamma)t} \|z(t)\|_{H^1}.$$

Proof. Since $\frac{\partial F}{\partial z} = 2[(\varphi + v_j + z)^p - (\varphi + v_j)^p]$, we can first compute

$$\frac{dH}{dt} = \int (z_x^2 - F(z)) h_t - 2\int z_t \big[(\varphi + v_j + z)^p - (\varphi + v_j)^p \big] h + 2\int z_{xt} z_x h + 2\int z_t z$$
$$- 2\int (\varphi + v_j)_t \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1} z \Big] h.$$

Moreover $2\int z_{xt} z_x h = -2\int z_t (z_{xx} h + z_x h_x)$, thus

$$\frac{dH}{dt} = \int (z_x^2 - F(z)) h_t - 2\int z_t \big[z_{xx} + (\varphi + v_j + z)^p - (\varphi + v_j)^p \big] h + 2\int z_t (z - z_x h_x)$$
$$- 2\int (\varphi + v_j)_t \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1} z \Big] h.$$

Now we replace z_t thanks to the equation that it satisfies, which can be written, from (3.3),

$$z_t + \Big[z_{xx} + (\varphi + v_j + z)^p - (\varphi + v_j)^p \Big]_x = -\Omega_x.$$

Using multiple integrations by parts, we finally obtain

$$\frac{dH}{dt} = \int (z_x^2 - F(z)) h_t + \int z_x^2 h_{xxx} \tag{3.13}$$
$$+ 2\int z_x h_x \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p \Big]_x \tag{3.14}$$
$$- 2\int z \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p \Big]_x - 2\int \varphi_t h \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1} z \Big] \tag{3.15}$$
$$- 2\int z\Omega_x + 2\int z h \Omega_{xxx} + 2\int z h_x \Omega_{xx} + 2\int h\Omega_x \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p \Big] \tag{3.16}$$
$$- 2\int h v_{jt} \Big[(\varphi + v_j + z)^p - (\varphi + v_j)^p - p(\varphi + v_j)^{p-1} z \Big] \tag{3.17}$$
$$- \int \Big[z_{xx} + (\varphi + v_j + z)^p - (\varphi + v_j)^p \Big]^2 h_x - 2\int z_{xx}^2 h_x. \tag{3.18}$$

To conclude, we estimate each term of this equality:

- First note that (3.18) $\geqslant 0$ since $h_x < 0$.

- (3.13): By the expression of h and $|\psi'''| \leqslant \frac{\sigma_0}{4}|\psi'|$, we see after direct calculation that $h_t \geqslant \sigma_0|h_x| \geqslant 4|h_{xxx}|$, thus

$$(3.13) \geqslant \frac{3}{4}\int z_x^2 h_t - \int F(z)h_t \geqslant -\int |F(z)|h_t.$$

Moreover, since $\|Rh_t\|_{L^\infty} \leqslant Ce^{-4\gamma t}$, and

$$\begin{aligned}|F(z)| &\leqslant C|z|^{p+1} + Cz^2|\varphi + v_j|^{p-1} \leqslant C\|z\|_{L^\infty}^{p-1} z^2 + Cz^2(|\varphi|^{p-1} + |v_j|^{p-1}) \\ &\leqslant C\|z\|_{L^\infty} z^2 + Cz^2|\varphi - R|^{p-1} + Cz^2|R|^{p-1} + Cz^2\|v_j\|_{L^\infty},\end{aligned}$$

then $\int |F(z)|h_t \leqslant C_0\|z\|_{H^1}^3 + C_1 e^{-2\gamma t}\|z\|_{H^1}^2$.

- For (3.17), first note that $\|v_{jt}\|_{L^\infty} \leqslant Ce^{-e_j t}$, and so

$$|(3.17)| \leqslant C\|v_{jt}\|_{L^\infty}\|z\|_{L^2}^2 \leqslant C_1 e^{-2\gamma t}\|z\|_{H^1}^2.$$

- $|(3.16)| \leqslant C\|\Omega\|_{H^3}\|z\|_{L^2} \leqslant C_2 e^{-(e_j+4\gamma)t}\|z\|_{H^1}$ by (3.5).

- To estimate (3.14), we develop it as

$$\begin{aligned}\frac{1}{2}(3.14) &= \int z_x h_x \sum_{k=1}^{p}\binom{p}{k}\Big[(\varphi+v_j)^{p-k}z^k\Big]_x = \sum_{k=1}^{p-1}\binom{p}{k}k\int z_x^2 z^{k-1}(\varphi+v_j)^{p-k}h_x \\ &\quad + \sum_{k=1}^{p-1}\binom{p}{k}(p-k)\int (\varphi+v_j)_x(\varphi+v_j)^{p-k-1}h_x z_x z^k + p\int z_x^2 z^{p-1}h_x.\end{aligned}$$

Since $|\varphi_x h_x| + |\varphi h_x| \leqslant Ce^{-2\gamma t}$ and $|v_{jx}| + |v_j| \leqslant Ce^{-e_j t}$, then

$$|(3.14)| \leqslant C_1 e^{-2\gamma t}\|z\|_{H^1}^2 + C_0\|z\|_{H^1}^3.$$

- We finally estimate (3.15) to conclude. The key point to control it is that locally around $x = c_k t + x_k$, φ behaves as a solitary wave of speed c_k. More precisely, we strongly use the estimate $\|\varphi_t h + \varphi_x\|_{L^\infty} \leqslant Ce^{-2\gamma t}$, proved in [11]. Note that the proof uses the H^4 norm of the difference $\varphi - R$, *i.e.* (2.3). Now, we compute

$$\begin{aligned}-\frac{1}{2}(3.15) &= \int z\Big[(\varphi+v_j+z)^p - (\varphi+v_j)^p - p(\varphi+v_j)^{p-1}z\Big]_x \\ &\quad + \int \varphi_t h\Big[(\varphi+v_j+z)^p - (\varphi+v_j)^p - p(\varphi+v_j)^{p-1}z - \frac{p(p-1)}{2}(\varphi+v_j)^{p-2}z^2\Big] \\ &\qquad - p\int (\varphi+v_j)^{p-1}z_x z + \frac{p(p-1)}{2}\int \varphi_t h(\varphi+v_j)^{p-2}z^2 = \mathbf{I}+\mathbf{II}+\mathbf{III}+\mathbf{IV}.\end{aligned}$$

First notice that $|\mathbf{I}| + |\mathbf{II}| \leqslant C_0\|z\|_{H^1}^3$. Moreover, an integration by parts gives

$$\begin{aligned}\mathbf{III}+\mathbf{IV} &= \frac{p}{2}\int z^2(p-1)(\varphi_x+v_{jx})(\varphi+v_j)^{p-2} + \frac{p(p-1)}{2}\int \varphi_t h(\varphi+v_j)^{p-2}z^2 \\ &= \frac{p(p-1)}{2}\int z^2(\varphi+v_j)^{p-2}(\varphi_x+\varphi_t h) + \frac{p(p-1)}{2}\int z^2 v_{jx}(\varphi+v_j)^{p-2},\end{aligned}$$

thus

$$|\mathbf{III}+\mathbf{IV}| \leqslant C\|\varphi_x+\varphi_t h\|_{L^\infty}\|z\|_{L^2}^2 + C\|v_{jx}\|_{L^\infty}\|z\|_{L^2}^2 \leqslant Ce^{-2\gamma t}\|z\|_{H^1}^2 + Ce^{-e_j t}\|z\|_{H^1}^2,$$

and so finally $|(3.15)| \leqslant C_0\|z\|_{H^1}^3 + C_1 e^{-2\gamma t}\|z\|_{H^1}^2$. □

We can now prove that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\int \Big(z_x^2(t) - pR^{p-1}(t)z^2(t)\Big)h(t) + z^2(t) \leqslant K_1 e^{-2(e_j+2\gamma)t}. \tag{3.19}$$

Indeed, from Lemma 3.10 and estimates (3.8), we deduce that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\frac{dH}{dt}(t) \geqslant -C_0 e^{-3(e_j+\gamma)t} - C_1 e^{-2\gamma t} e^{-2(e_j+\gamma)t} - C_2 e^{-(e_j+3\gamma)t} e^{-(e_j+\gamma)t} \geqslant -K_1 e^{-2(e_j+2\gamma)t}.$$

Thus by integration on $[t, S_n]$, we obtain $H(S_n) - H(t) \geqslant -K_1 e^{-2(e_j+2\gamma)t}$, and so

$$H(t) \leqslant H(S_n) + K_1 e^{-2(e_j+2\gamma)t}.$$

But from Claim 3.7 and Lemma 3.6, we have

$$\begin{aligned} H(S_n) \leqslant |H(S_n)| &\leqslant C\|z(S_n)\|_{H^1}^2 \leqslant C\|\mathfrak{b}\|^2 \leqslant C\|\mathfrak{a}^-\|^2 \\ &\leqslant Ce^{-2(e_j+2\gamma)S_n} \leqslant Ce^{-2(e_j+2\gamma)t}, \end{aligned}$$

and so

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad H(t) \leqslant K_1 e^{-2(e_j+2\gamma)t}.$$

Finally, since

$$\begin{aligned} |F(z) - pR^{p-1}z^2| &\leqslant |F(z) - p(\varphi+v_j)^{p-1}z^2| + p|((\varphi+v_j)^{p-1} - \varphi^{p-1})z^2| + p|(\varphi^{p-1} - R^{p-1})z^2| \\ &\leqslant C_0|z|^3 + C_1 e^{-2\gamma t}|z|^2, \end{aligned}$$

we easily obtain (3.19) from the definition of H.

3.3.6 Control of the R_{kx} directions

Define $\widetilde{z}(t) = z(t) + \sum_{k=1}^{N} a_k(t)R_{kx}(t)$, where $a_k(t) = -\frac{\int z(t)R_{kx}(t)}{\int (Q'_{c_k})^2}$, so that by (2.2)

$$\left|\int \widetilde{z}R_{kx}\right| \leqslant Ce^{-\gamma t}\|z\|_{H^1} \tag{3.20}$$

and there exist $C_1, C_2 > 0$ such that

$$C_1\|z\|_{H^1} \leqslant \|\widetilde{z}\|_{H^1} + \sum_{k=1}^{N}|a_k| \leqslant C_2\|z\|_{H^1}. \tag{3.21}$$

As in [11, Section 4], we find

$$\int \left[(\widetilde{z}_x^2 - pR^{p-1}\widetilde{z}^2)h + \widetilde{z}^2\right] \leqslant \int \left[(z_x^2 - pR^{p-1}z^2)h + z^2\right] + Ce^{-2\gamma t}\|z\|_{H^1}^2.$$

Using (3.19), we deduce that

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \int \Big(\widetilde{z}_x^2(t) - pR^{p-1}(t)\widetilde{z}^2(t)\Big)h(t) + \widetilde{z}^2(t) \leqslant K_1 e^{-2(e_j+2\gamma)t}. \tag{3.22}$$

Now, from the property of coercivity (vi) in Lemma 2.5, and since h takes values close to $\frac{1}{c_k}$ for x close to $c_k t + x_k$, we obtain, by simple localization arguments (see [15, Lemma 4] for details), that there exists $\lambda_2 > 0$ such that

$$\int (\widetilde{z}_x^2 - pR^{p-1}\widetilde{z}^2)h + \widetilde{z}^2 \geqslant \lambda_2\|\widetilde{z}\|_{H^1}^2 - \frac{1}{\lambda_2}\sum_{k=1}^{N}\left[\left(\int \widetilde{z}R_{kx}\right)^2 + \left(\int \widetilde{z}Z_k^+\right)^2 + \left(\int \widetilde{z}Z_k^-\right)^2\right].$$

Moreover, gathering all previous estimates, we have for all $t \in [T(\mathfrak{a}^-), S_n]$:

(a) For all $k \in [\![1, N]\!]$, $\left(\int \widetilde{z} R_{kx}\right)^2 \leqslant Ce^{-2\gamma t}\|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (3.20).

(b) For all $k \in [\![1, N]\!]$, $\left(\int \widetilde{z} Z_k^+\right)^2 \leqslant 2(\alpha_k^+)^2 + Ce^{-2\gamma t}\|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (iii) of Lemma 2.5, (3.10) and (2.2).

(c) For all $k \in [\![1, j]\!]$, $\left(\int \widetilde{z} Z_k^-\right)^2 \leqslant 2(\alpha_k^-)^2 + Ce^{-2\gamma t}\|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (iii) of Lemma 2.5, (3.12) and (2.2).

(d) For all $k \in [\![j+1, N]\!]$, $\left(\int \widetilde{z} Z_k^-\right)^2 \leqslant 2(\alpha_k^-)^2 + Ce^{-2\gamma t}\|z\|_{H^1}^2 \leqslant Ce^{-2(e_j+2\gamma)t}$ by (3.8).

Finally, we have proved that there exists $K > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,

$$\|\widetilde{z}(t)\|_{H^1} \leqslant Ke^{-(e_j+2\gamma)t}.$$

We want now to prove the same estimate for z.

Lemma 3.11. *There exists $K_0 > 0$ such that, for all $t \in [T(\mathfrak{a}^-), S_n]$,*

$$\|z(t)\|_{H^1} \leqslant K_0 e^{-(e_j+2\gamma)t}.$$

Proof. By (3.21), it is enough to prove this estimate for $|a_k(t)|$ with $k \in [\![1, N]\!]$ fixed. To do this, write first the equation of $\widetilde{z}$ from the equation of z (3.4):

$$\begin{aligned}
&\widetilde{z}_t + (\widetilde{z}_{xx} + p\varphi^{p-1}\widetilde{z})_x \\
&= z_t + \sum_{l=1}^N a_l R_{lxt} + \sum_{l=1}^N a_l' R_{lx} + z_{xxx} + \sum_{l=1}^N a_l R_{lxxx} + p\sum_{l=1}^N a_l (R_{lx}\varphi^{p-1})_x + p(\varphi^{p-1} z)_x \\
&= -(\omega_1 \cdot z)_x - (\omega(z))_x - \Omega_x + \sum_{l=1}^N a_l' R_{lx} + \sum_{l=1}^N a_l \Big[-c_l R_{lx} + R_{lxxx} + p\varphi^{p-1} R_{lx} \Big]_x.
\end{aligned}$$

Then multiply this equation by R_{kx} and integrate, so that we obtain

$$\begin{aligned}
\int \widetilde{z}_t R_{kx} - \int (\widetilde{z}_{xx} + p\varphi^{p-1}\widetilde{z}) R_{kxx} &= a_k' \int R_{kx}^2 + \sum_{l\neq k} a_l' \int R_{lx} R_{kx} \\
&+ \sum_{l=1}^N a_l \int \Big[R_{lxxx} - c_l R_{lx} + p\varphi^{p-1} R_{lx} \Big]_x R_{kx} + \int \omega_1 z R_{kxx} + \int \omega(z) R_{kxx} + \int \Omega R_{kxx}.
\end{aligned}$$

But from (2.3) and (iii) of Claim 2.2, we have

$$\begin{aligned}
\big\|(R_{lxxx} - c_l R_{lx} + p\varphi^{p-1} R_{lx})_x\big\|_{L^\infty} &\leqslant p\|R_{lx}(\varphi^{p-1} - R_l^{p-1})\|_{H^2} \\
&\leqslant C\|\varphi - R\|_{H^2} + p\|R_{lx}(R^{p-1} - R_l^{p-1})\|_{H^2} \leqslant Ce^{-2\gamma t},
\end{aligned}$$

and consequently

$$\begin{aligned}
|a_k'| \leqslant C\left|\int \widetilde{z} R_{kx}\right| + C\|\widetilde{z}\|_{L^2} + Ce^{-\gamma t}\sum_{l\neq k}|a_l'| + Ce^{-2\gamma t}\sum_{l=1}^N |a_l| \\
+ Ce^{-e_j t}\|z\|_{L^2} + C\|z\|_{L^2}^2 + C\|\Omega\|_{L^2}.
\end{aligned}$$

Moreover, from $\int \widetilde{z} R_{kx} = \sum_{l\neq k} a_l \int R_{lx} R_{kx}$, we deduce that

$$\begin{aligned}
\frac{d}{dt}\int \widetilde{z} R_{kx} &= \sum_{l\neq k} a_l' \int R_{kx} R_{lx} + \sum_{l\neq k} a_l \int (-c_l R_{lxx} R_{kx} - c_k R_{lx} R_{kxx}) \\
&= \int \widetilde{z}_t R_{kx} + \int \widetilde{z}(-c_k R_{kxx}),
\end{aligned}$$

and so

$$\left|\int \widetilde{z}_t R_{kx}\right| \leqslant C\|\widetilde{z}\|_{H^1} + Ce^{-\gamma t}\sum_{l\neq k}|a_l'| + Ce^{-2\gamma t}\sum_{l=1}^{N}|a_l|.$$

Gathering previous estimates, we have from (3.21) and (3.5),

$$\begin{aligned}|a_k'| &\leqslant C\|\widetilde{z}\|_{H^1} + C_4e^{-\gamma t}\sum_{l\neq k}|a_l'| + Ce^{-2\gamma t}\|z\|_{H^1} + C\|z\|_{H^1}^2 + C\|\Omega\|_{L^2}\\ &\leqslant Ke^{-(e_j+2\gamma)t} + C_4e^{-\gamma t}\sum_{l\neq k}|a_l'| + Ce^{-2\gamma t}e^{-(e_j+\gamma)t} + Ce^{-2(e_j+\gamma)t} + Ce^{-(e_j+4\gamma)t}.\end{aligned}$$

Finally, if we choose t_0 large enough so that $C_4e^{-\gamma t_0} \leqslant \frac{1}{N}$, we obtain for all $s \in [T(\mathfrak{a}^-), S_n]$,

$$|a_k'(s)| \leqslant Ke^{-(e_j+2\gamma)s}.$$

By integration on $[t, S_n]$ with $t \in [T(\mathfrak{a}^-), S_n]$, we get $|a_k(t)| \leqslant |a_k(S_n)| + Ke^{-(e_j+2\gamma)t}$. But from Claim 3.7 and Lemma 3.6, we have

$$|a_k(S_n)| \leqslant C\|z(S_n)\|_{H^1} \leqslant C\|\mathfrak{b}\| \leqslant C\|\mathfrak{a}^-\| \leqslant Ce^{-(e_j+2\gamma)S_n} \leqslant Ce^{-(e_j+2\gamma)t},$$

and so finally,

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad |a_k(t)| \leqslant Ke^{-(e_j+2\gamma)t}. \qquad \square$$

3.3.7 Control of the unstable directions for $k > j$ by a topological argument

Lemma 3.11 being proved, we choose t_0 large enough so that $K_0e^{-\gamma t_0} \leqslant \frac{1}{2}$. Therefore, we have

$$\forall t \in [T(\mathfrak{a}^-), S_n], \quad \|z(t)\|_{H^1} \leqslant \frac{1}{2}e^{-(e_j+\gamma)t}.$$

We can now prove the following final lemma, which concludes the proof of Proposition 3.3.

Lemma 3.12. *For t_0 large enough, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$.*

Proof. For the sake of contradiction, suppose that for all $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$, $T(\mathfrak{a}^-) > t_0$. As $e^{(e_j+\gamma)T(\mathfrak{a}^-)}z(T(\mathfrak{a}^-)) \in B_{H^1}(1/2)$, then by definition of $T(\mathfrak{a}^-)$ and continuity of the flow, we have

$$e^{(e_j+2\gamma)T(\mathfrak{a}^-)}\boldsymbol{\alpha}^-(T(\mathfrak{a}^-)) \in \mathbb{S}_{\mathbb{R}^{N-j}}(1). \tag{3.23}$$

Now let $T \in [t_0, T(\mathfrak{a}^-)]$ be close enough to $T(\mathfrak{a}^-)$ such that z is defined on $[T, S_n]$, and by continuity,

$$\forall t \in [T, S_n], \quad \|z(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

We can now consider, for $t \in [T, S_n]$,

$$\mathcal{N}(t) = \mathcal{N}(\boldsymbol{\alpha}^-(t)) = \|e^{(e_j+2\gamma)t}\boldsymbol{\alpha}^-(t)\|^2.$$

To calculate $\mathcal{N}'$, we start from estimate (3.11):

$$\forall k \in [\![j+1, N]\!], \forall t \in [T, S_n], \quad \left|\frac{d}{dt}\alpha_k^-(t) + e_k\alpha_k^-(t)\right| \leqslant K_2'e^{-(e_j+4\gamma)t}.$$

Multiplying by $|\alpha_k^-(t)|$, we obtain

$$\left|\alpha_k^-(t)\frac{d}{dt}\alpha_k^-(t) + e_k\alpha_k^-(t)^2\right| \leqslant K_2'e^{-(e_j+4\gamma)t}|\alpha_k^-(t)|,$$

and thus

$$2\alpha_k^-(t)\frac{d}{dt}\alpha_k^-(t) + 2e_{j+1}\alpha_k^-(t)^2 \leqslant 2\alpha_k^-(t)\frac{d}{dt}\alpha_k^-(t) + 2e_k\alpha_k^-(t)^2 \leqslant K_2e^{-(e_j+4\gamma)t}|\alpha_k^-(t)|.$$

By summing on $k \in [\![j+1, N]\!]$, we get

$$(\|\boldsymbol{\alpha}^-(t)\|^2)' + 2e_{j+1}\|\boldsymbol{\alpha}^-(t)\|^2 \leqslant K_2 e^{-(e_j+4\gamma)t}\|\boldsymbol{\alpha}^-(t)\|.$$

Therefore we can estimate

$$\begin{aligned}\mathcal{N}'(t) &= (e^{2(e_j+2\gamma)t}\|\boldsymbol{\alpha}^-(t)\|^2)' = e^{2(e_j+2\gamma)t}\left[2(e_j+2\gamma)\|\boldsymbol{\alpha}^-(t)\|^2 + (\|\boldsymbol{\alpha}^-(t)\|^2)'\right] \\ &\leqslant e^{2(e_j+2\gamma)t}\left[2(e_j+2\gamma)\|\boldsymbol{\alpha}^-(t)\|^2 - 2e_{j+1}\|\boldsymbol{\alpha}^-(t)\|^2 + K_2 e^{-(e_j+4\gamma)t}\|\boldsymbol{\alpha}^-(t)\|\right].\end{aligned}$$

Hence we have, for all $t \in [T, S_n]$,

$$\mathcal{N}'(t) \leqslant -\theta \cdot \mathcal{N}(t) + K_2 e^{e_j t}\|\boldsymbol{\alpha}^-(t)\|,$$

where $\theta = 2(e_{j+1} - e_j - 2\gamma) > 0$ by definition of γ (2.1). In particular, for all $\tau \in [T, S_n]$ satisfying $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leqslant -\theta + K_2 e^{e_j\tau}\|\boldsymbol{\alpha}^-(\tau)\| = -\theta + K_2 e^{e_j\tau}e^{-(e_j+2\gamma)\tau} = -\theta + K_2 e^{-2\gamma\tau} \leqslant -\theta + K_2 e^{-2\gamma t_0}.$$

Now we fix t_0 large enough so that $K_2 e^{-2\gamma t_0} \leqslant \frac{\theta}{2}$, and so for all $\tau \in [T, S_n]$ such that $\mathcal{N}(\tau) = 1$, we have

$$\mathcal{N}'(\tau) \leqslant -\frac{\theta}{2}. \tag{3.24}$$

In particular, by (3.23), we have $\mathcal{N}'(T(\mathfrak{a}^-)) \leqslant -\frac{\theta}{2}$.

First consequence: $\mathfrak{a}^- \mapsto T(\mathfrak{a}^-)$ is continuous. Indeed, let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $\mathcal{N}(T(\mathfrak{a}^-) - \varepsilon) > 1 + \delta$ and $\mathcal{N}(T(\mathfrak{a}^-) + \varepsilon) < 1 - \delta$. Moreover, by definition of $T(\mathfrak{a}^-)$ and (3.24), there can not exist $\tau \in [T(\mathfrak{a}^-) + \varepsilon, S_n]$ such that $\mathcal{N}(\tau) = 1$, and so by choosing δ small enough, we have for all $t \in [T(\mathfrak{a}^-) + \varepsilon, S_n]$, $\mathcal{N}(t) < 1 - \delta$. But from continuity of the flow, there exists $\eta > 0$ such that, for all $\widetilde{\mathfrak{a}}^-$ satisfying $\|\widetilde{\mathfrak{a}}^- - \mathfrak{a}^-\| \leqslant \eta$, we have

$$\forall t \in [T(\mathfrak{a}^-) - \varepsilon, S_n], \quad |\mathcal{N}(\widetilde{\boldsymbol{\alpha}}^-(t)) - \mathcal{N}(\boldsymbol{\alpha}^-(t))| \leqslant \delta/2.$$

We finally deduce that $T(\mathfrak{a}^-) - \varepsilon \leqslant T(\widetilde{\mathfrak{a}}^-) \leqslant T(\mathfrak{a}^-) + \varepsilon$, as expected.

Second consequence: We can define the map

$$\begin{array}{rccl}\mathcal{M} \ : & B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n}) & \to & \mathbb{S}_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n}) \\ & \mathfrak{a}^- & \mapsto & e^{-(e_j+2\gamma)(S_n - T(\mathfrak{a}^-))}\boldsymbol{\alpha}^-(T(\mathfrak{a}^-)).\end{array}$$

Note that $\mathcal{M}$ is continuous by the previous point. Moreover, let $\mathfrak{a}^- \in \mathbb{S}_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$. As $\mathcal{N}'(S_n) \leqslant -\frac{\theta}{2}$ by (3.24), we deduce by definition of $T(\mathfrak{a}^-)$ that $T(\mathfrak{a}^-) = S_n$, and so $\mathcal{M}(\mathfrak{a}^-) = \mathfrak{a}^-$. In other words, $\mathcal{M}$ restricted to $\mathbb{S}_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ is the identity. But the existence of such a map $\mathcal{M}$ contradicts Brouwer's fixed point theorem.

In conclusion, there exists $\mathfrak{a}^- \in B_{\mathbb{R}^{N-j}}(e^{-(e_j+2\gamma)S_n})$ such that $T(\mathfrak{a}^-) = t_0$. □

4 Classification of multi-solitons

This section is devoted to prove the second assertion of Theorem 1.3. Let $p > 5$, $N \geqslant 2$, $0 < c_1 < \cdots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R = \sum_{j=1}^N R_{c_j, x_j}$ and φ the multi-soliton given by Theorem 1.2. Let u be a solution of (gKdV), defined on $[t_1, +\infty)$ with $t_1 > 0$ large, satisfying

$$\lim_{t \to +\infty} \|u(t) - R(t)\|_{H^1} = 0. \tag{4.1}$$

4.1 Convergence at exponential rate γ

We first improve condition (4.1) into an exponential convergence, with a small rate $\gamma > 0$, where γ is defined by (2.1).

Lemma 4.1. *Let $\varepsilon = u - \varphi$. Then there exist $C, t_0 > 0$ such that, for all $t \geqslant t_0$, $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma t}$.*

Proof. Step 1: Modulation. Denote $v = u - R$, so that $\|v(t)\|_{H^1} \to 0$ as $t \to +\infty$ by (4.1). Therefore, by a standard lemma of modulation (see for example [11, Lemma 2]), for t_0 large enough, there exist N functions $y_j : [t_0, +\infty) \to \mathbb{R}$ of class C^1 such that $w = u - \widetilde{R}$, where $\widetilde{R} = \sum \widetilde{R}_j$ and $\widetilde{R}_j(t) = R_j(t, \cdot - y_j(t))$, satisfies

$$\begin{cases} \forall j \in [\![1, N]\!], \quad \int w(t)(\widetilde{R}_j)_x(t) = 0, \\ \|w(t)\|_{H^1} + \sum_{j=1}^N |y_j(t)| \leqslant C\|v(t)\|_{H^1}, \\ \forall j \in [\![1, N]\!], \quad |y_j'(t)| \leqslant C\|w(t)\|_{H^1} + Ce^{-\gamma t}. \end{cases}$$

Note that the first two facts are a simple consequence of the implicit function theorem, while the last estimate comes from the equation satisfied by w,

$$\partial_t w + \partial_x^3 w = \sum_{k=1}^N y_k' \partial_x(\widetilde{R}_k) - \partial_x \left((w + \widetilde{R})^p - \sum_{k=1}^N \widetilde{R}_k^p \right),$$

multiplied by $(\widetilde{R}_j)_x$ and integrated. Similarly, if we denote $\widetilde{Z}_j^{\pm}(t) = Z_j^{\pm}(t, \cdot - y_j(t))$ and $\widetilde{\alpha}_j^{\pm}(t) = \int w(t)\widetilde{Z}_j^{\pm}(t)$, the equation of w multiplied by $\widetilde{Z}_j^{\pm}$ leads to

$$\forall t \geqslant t_0, \quad \left| \frac{d}{dt}\widetilde{\alpha}_j^{\pm}(t) \mp e_j \widetilde{\alpha}_j^{\pm}(t) \right| \leqslant C\|w(t)\|_{H^1}^2 + Ce^{-2\gamma t}. \tag{4.2}$$

Step 2: Monotonicity. We use again the function ψ introduced in Section 3.3.5. Following [11], we introduce moreover $\psi_N \equiv 1$ and for $j \in [\![1, N-1]\!]$,

$$m_j(t) = \frac{c_j + c_{j+1}}{2} t + \frac{x_j + x_{j+1}}{2}, \quad \psi_j(t) = \psi(x - m_j(t)),$$

and

$$\phi_1 \equiv \psi_1, \quad \phi_N \equiv 1 - \psi_{N-1}, \quad \phi_j \equiv \psi_j - \psi_{j-1} \quad \text{for } j \in [\![2, N-1]\!].$$

We also define some local quantities related to L^2 mass and energy:

$$M_j(t) = \int u^2(t)\phi_j(t), \quad E_j(t) = \int \left(\frac{1}{2}u_x^2(t) - \frac{1}{p+1}u^{p+1}(t) \right) \phi_j(t), \quad \widetilde{E}_j(t) = E_j(t) + \frac{\sigma_0}{100} M_j(t).$$

Then, by (4.1) and monotonicity results on the quantities $t \mapsto \sum_{k=1}^j M_k(t)$ and $t \mapsto \sum_{k=1}^j E_k(t)$, we have, for all $t \geqslant t_0$ and all $j \in [\![1, N]\!]$, following Lemmas 1 and 3 of [11],

$$\begin{cases} \displaystyle\sum_{k=1}^j \left(\int Q_{c_k}^2 - M_k(t) \right) \geqslant -K_2 e^{-2\gamma t}, & (4.3) \\ \displaystyle\sum_{k=1}^j \left(E(Q_{c_k}) + \frac{\sigma_0}{100} \int Q_{c_k}^2 - \widetilde{E}_k(t) \right) \geqslant -K_2 e^{-2\gamma t}, & (4.4) \end{cases}$$

and

$$\left| \left(E_j(t) + \frac{c_j}{2} M_j(t) \right) - \left(E(Q_{c_j}) + \frac{c_j}{2} \int Q_{c_j}^2 \right) - \frac{1}{2} H_j(t) \right| \leqslant K_4 e^{-2\gamma t} + K_4 \|w(t)\|_{H^1} \int w^2 \phi_j, \tag{4.5}$$

where $H_j(t) = \int \left(w_x^2(t) + c_j w^2(t) - p\widetilde{R}_j^{p-1}(t) w^2(t) \right) \phi_j(t)$. But if we write

$$\sum_{j=1}^{N} \frac{1}{c_j^2} \left(E_j + \frac{c_j}{2} M_j \right) = \sum_{j=1}^{N-1} \left[\left(\frac{1}{c_j^2} - \frac{1}{c_{j+1}^2} \right) \sum_{k=1}^{j} \widetilde{E}_k \right] + \frac{1}{c_N^2} \sum_{k=1}^{N} \widetilde{E}_k + \frac{1}{2c_N} \left(1 - \frac{\sigma_0}{50 c_N} \right) \sum_{k=1}^{N} M_k + \sum_{j=1}^{N-1} \left[\frac{1}{2} \left(\frac{1}{c_j} - \frac{1}{c_{j+1}} \right) \left(1 - \frac{\sigma_0}{50} \left(\frac{1}{c_j} + \frac{1}{c_{j+1}} \right) \right) \sum_{k=1}^{j} M_k \right],$$

and similarly

$$\sum_{j=1}^{N} \frac{1}{c_j^2} \left(E(Q_{c_j}) + \frac{c_j}{2} \int Q_{c_j}^2 \right) = \sum_{j=1}^{N-1} \left[\left(\frac{1}{c_j^2} - \frac{1}{c_{j+1}^2} \right) \sum_{k=1}^{j} \left(E(Q_{c_k}) + \frac{\sigma_0}{100} \int Q_{c_k}^2 \right) \right] + \sum_{j=1}^{N-1} \left[\frac{1}{2} \left(\frac{1}{c_j} - \frac{1}{c_{j+1}} \right) \left(1 - \frac{\sigma_0}{50} \left(\frac{1}{c_j} + \frac{1}{c_{j+1}} \right) \right) \sum_{k=1}^{j} \int Q_{c_k}^2 \right] + \frac{1}{c_N^2} \sum_{k=1}^{N} \left(E(Q_{c_k}) + \frac{\sigma_0}{100} \int Q_{c_k}^2 \right) + \frac{1}{2c_N} \left(1 - \frac{\sigma_0}{50 c_N} \right) \sum_{k=1}^{N} \int Q_{c_k}^2,$$

and we remark that all coefficients in these decompositions are positive, we obtain by (4.3) and (4.4),

$$\sum_{j=1}^{N} \frac{1}{c_j^2} \left(E_j(t) + \frac{c_j}{2} M_j(t) \right) - \sum_{j=1}^{N} \frac{1}{c_j^2} \left(E(Q_{c_j}) + \frac{c_j}{2} \int Q_{c_j}^2 \right) \leqslant C e^{-2\gamma t}.$$

Therefore, we have by (4.5),

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{N} \frac{1}{c_j^2} H_j(t) &\leqslant \sum_{j=1}^{N} \frac{1}{c_j^2} \left(E_j(t) + \frac{c_j}{2} M_j(t) \right) - \sum_{j=1}^{N} \frac{1}{c_j^2} \left(E(Q_{c_j}) + \frac{c_j}{2} \int Q_{c_j}^2 \right) \\ &\quad + K_4 \sum_{j=1}^{N} \frac{1}{c_j^2} e^{-2\gamma t} + K_4 \|w(t)\|_{H^1} \sum_{j=1}^{N} \frac{1}{c_j^2} \int w^2 \phi_j \\ &\leqslant C_1 e^{-2\gamma t} + \frac{K_4}{\sigma_0^2} \|w(t)\|_{H^1} \int w^2 \sum_{j=1}^{N} \phi_j \end{aligned}$$

since $\phi_j \geqslant 0$. Finally, as $\sum_{j=1}^{N} \phi_j \equiv 1$, we obtain

$$\sum_{j=1}^{N} \frac{1}{c_j^2} H_j(t) \leqslant C_1 e^{-2\gamma t} + C_2 \|w(t)\|_{H^1}^3. \tag{4.6}$$

Step 3: Coercivity. Now, from the property of coercivity (vi) in Lemma 2.5 and by standard localization arguments (as in Section 3), we have

$$\sum_{j=1}^{N} \frac{1}{c_j^2} H_j(t) \geqslant \lambda_c \|w(t)\|_{H^1}^2 - \frac{1}{\lambda_c} \sum_{j=1}^{N} \left(\int w(t) (\widetilde{R}_j)_x(t) \right)^2 - \frac{1}{\lambda_c} \sum_{j,\pm} \left(\int w(t) \widetilde{Z}_j^{\pm}(t) \right)^2.$$

As $\int w(t) (\widetilde{R}_j)_x(t) = 0$ and $\widetilde{\alpha}_j^{\pm}(t) = \int w(t) \widetilde{Z}_j^{\pm}(t)$, we obtain by (4.6),

$$\lambda_c \|w(t)\|_{H^1}^2 \leqslant C_1 e^{-2\gamma t} + C_2 \|w(t)\|_{H^1}^3 + C_3 \|\widetilde{\boldsymbol{\alpha}}(t)\|^2,$$

where $\widetilde{\boldsymbol{\alpha}}(t) = (\widetilde{\alpha}_j^{\pm}(t))_{j,\pm}$. For t_0 large enough so that $C_2 \|w(t)\|_{H^1} \leqslant \frac{\lambda_c}{2}$, we obtain

$$\forall t \geqslant t_0, \quad \|w(t)\|_{H^1}^2 \leqslant C_1 \|\widetilde{\boldsymbol{\alpha}}(t)\|^2 + C_2 e^{-2\gamma t}. \tag{4.7}$$

Step 4: Exponential decay of $\widetilde{\boldsymbol{\alpha}}$. From (4.2) and (4.7), we have for all $j \in [\![1, N]\!]$ and all $t \geqslant t_0$,

$$\left| \frac{d}{dt}\widetilde{\alpha}_j^{\pm}(t) \mp e_j \widetilde{\alpha}_j^{\pm}(t) \right| \leqslant C_1 \|\widetilde{\boldsymbol{\alpha}}(t)\|^2 + C_2 e^{-2\gamma t}.$$

We follow here the strategy of [3, Section 4.4.2]. Define $A(t) = \sum_{j=1}^N \widetilde{\alpha}_j^+(t)^2$ and $B(t) = \sum_{j=1}^N \widetilde{\alpha}_j^-(t)^2$, and let us prove that $A(t) \leqslant B(t) + Le^{-2\gamma t}$ for L large enough. First, we have, by multiplying the previous estimate by $|\widetilde{\alpha}_j^+(t)|$ (that we can of course suppose less than 1),

$$\widetilde{\alpha}_j^+(t)\frac{d}{dt}\widetilde{\alpha}_j^+(t) \geqslant e_j \widetilde{\alpha}_j^+(t)^2 - C_1|\widetilde{\alpha}_j^+(t)| \cdot \|\widetilde{\boldsymbol{\alpha}}(t)\|^2 - C_2 e^{-2\gamma t},$$

and so by summing,

$$A'(t) \geqslant 2e_1 A(t) - C_1\|\widetilde{\boldsymbol{\alpha}}(t)\|^3 - C_2 e^{-2\gamma t}.$$

Similarly, we obtain

$$B'(t) \leqslant -2e_1 B(t) + C_1\|\widetilde{\boldsymbol{\alpha}}(t)\|^3 + C_2 e^{-2\gamma t}. \tag{4.8}$$

Now let $h(t) = A(t) - B(t) - Le^{-2\gamma t}$ with L to be determined later. We have of course $h(t) \to 0$ as $t \to +\infty$, and by the previous estimates, we can calculate

$$\begin{aligned} h'(t) &= A'(t) - B'(t) + 2L\gamma e^{-2\gamma t} \\ &\geqslant 2e_1 A(t) + 2e_1 B(t) - C_1\|\widetilde{\boldsymbol{\alpha}}(t)\|^3 - C_2 e^{-2\gamma t} \\ &\geqslant 2e_1 h(t) + 4e_1 B(t) - C_1\|\widetilde{\boldsymbol{\alpha}}(t)\|^3 - C_2 e^{-2\gamma t} + 2Le_1 e^{-2\gamma t}. \end{aligned}$$

Since $\|\widetilde{\boldsymbol{\alpha}}(t)\|^2 = A(t) + B(t) = h(t) + 2B(t) + Le^{-2\gamma t}$, we get

$$h'(t) \geqslant h(t)(2e_1 - C_1\|\widetilde{\boldsymbol{\alpha}}(t)\|) + B(t)(4e_1 - 2C_1\|\widetilde{\boldsymbol{\alpha}}(t)\|) + e^{-2\gamma t}(2Le_1 - C_2 - C_1 L\|\widetilde{\boldsymbol{\alpha}}(t)\|).$$

Now choose t_0 large enough so that $C_1\|\widetilde{\boldsymbol{\alpha}}(t)\| \leqslant \frac{e_1}{2}$ for $t \geqslant t_0$, and fix $L = \frac{C_2}{e_1}$. Therefore, we have, for all $t \geqslant t_0$ such that $h(t) \geqslant 0$, $h'(t) \geqslant e_1 h(t)$. Hence, if there exists $T \geqslant t_0$ such that $h(T) \geqslant 0$, then $h(t) \geqslant 0$ for all $t \geqslant T$, and thus $h(t) \geqslant Ce^{e_1 t}$, which would be in contradiction with $\lim_{t\to+\infty} h(t) = 0$. So we have proved that $h(t) \leqslant 0$ for all $t \geqslant t_0$, as expected.

Now, from (4.8) and the choice of t_0 to have $C_1\|\widetilde{\boldsymbol{\alpha}}(t)\| \leqslant \frac{e_1}{2}$ for all $t \geqslant t_0$, it comes

$$B'(t) + 2e_1 B(t) \leqslant e_1 B(t) + \left(\frac{Le_1}{2} + C_2\right) e^{-2\gamma t},$$

and so $B'(t) + e_1 B(t) \leqslant Ke^{-2\gamma t}$. Therefore, $(e^{e_1 s}B(s))' \leqslant Ke^{(e_1-2\gamma)s}$ for $s \geqslant t_0$, and so by integration on $[t_0, t]$,

$$e^{e_1 t}B(t) - e^{e_1 t_0}B(t_0) \leqslant Ke^{(e_1-2\gamma)t},$$

since $e_1 - 2\gamma > 0$. We deduce that

$$B(t) \leqslant Ke^{-2\gamma t} + K'e^{-e_1 t} \leqslant Ke^{-2\gamma t}.$$

Finally, we also have by the previous point $A(t) \leqslant K'e^{-2\gamma t}$, and so

$$\forall t \geqslant t_0, \quad \|\widetilde{\boldsymbol{\alpha}}(t)\|^2 \leqslant Ce^{-2\gamma t}. \tag{4.9}$$

Step 5: Conclusion. By (4.7), we deduce that $\|w(t)\|_{H^1} \leqslant Ce^{-\gamma t}$, and from the estimate on $|y_j'|$, we have for all $j \in [\![1, N]\!]$ and all $t \geqslant t_0$, $|y_j(t)| \leqslant Ce^{-\gamma t}$, by integration and the fact that $y_j(t) \to 0$ as $t \to +\infty$. To conclude, write

$$\varepsilon = u - \varphi = w + \widetilde{R} - \varphi = w - (\varphi - R) + (\widetilde{R} - R),$$

so that

$$\|\varepsilon(t)\|_{H^1} \leqslant \|w(t)\|_{H^1} + \|(\varphi - R)(t)\|_{H^1} + \|(\widetilde{R} - R)(t)\|_{H^1} \leqslant Ce^{-\gamma t} + \|(\widetilde{R} - R)(t)\|_{H^1}.$$

But we have

$$\begin{aligned}\|(\widetilde{R}-R)(t)\|_{H^1} &\leqslant \sum_{j=1}^{N}\|R_j(t,\cdot-y_j(t))-R_j(t)\|_{H^1}\\ &\leqslant C\sum_{j=1}^{N}\Big\|Q(\sqrt{c_j}x-\sqrt{c_j}y_j(t)-c_j^{3/2}t-\sqrt{c_j}x_j)-Q(\sqrt{c_j}x-c_j^{3/2}t-\sqrt{c_j}x_j)\Big\|_{H^1}\\ &\leqslant C\sum_{j=1}^{N}|y_j(t)|\leqslant Ce^{-\gamma t},\end{aligned}$$

and so finally, for all $t\geqslant t_0$, $\|\varepsilon(t)\|_{H^1}\leqslant Ce^{-\gamma t}$. □

4.2 Convergence at exponential rate e_1

Now, we improve the convergence of the previous lemma, with an exponential rate $e_1\gg\gamma$. The proof will mainly use arguments developed in [11, Section 4].

Lemma 4.2. *There exist $C,t_0>0$ such that, for all $t\geqslant t_0$, $\|\varepsilon(t)\|_{H^1}\leqslant Ce^{-e_1t}$.*

Proof. Step 1: Estimates. We follow the same strategy as in Section 3.3. First, from the equation of ε,

$$\varepsilon_t+(\varepsilon_{xx}+(\varphi+\varepsilon)^p-\varphi^p)_x=0,$$

we can estimate $\alpha_j^{\pm}(t)=\int\varepsilon(t)Z_j^{\pm}(t)$ for $j\in[\![1,N]\!]$ and $t\geqslant t_0$. Indeed, we have

$$\begin{aligned}\frac{d}{dt}\alpha_j^{\pm}(t)&=\int\varepsilon_tZ_j^{\pm}+\int\varepsilon Z_{jt}^{\pm}=\int\Big(\varepsilon_{xx}+(\varphi+\varepsilon)^p-\varphi^p\Big)Z_{jx}^{\pm}-c_j\int\varepsilon Z_{jx}^{\pm}\\ &=\int\left[\varepsilon_{xx}-c_j\varepsilon+\sum_{k=1}^{p}\binom{p}{k}\varphi^{p-k}\varepsilon^k\right]Z_{jx}^{\pm}\\ &=\int\Big[\varepsilon_{xx}-c_j\varepsilon+pR_j^{p-1}\varepsilon\Big]Z_{jx}^{\pm}+p\int(\varphi^{p-1}-R_j^{p-1})\varepsilon Z_{jx}^{\pm}+\sum_{k=2}^{p}\binom{p}{k}\int\varphi^{p-k}\varepsilon^kZ_{jx}^{\pm}\\ &=\mathbf{I}+\mathbf{II}+\mathbf{III}.\end{aligned}$$

But we have $\mathbf{I}=\pm e_j\alpha_j^{\pm}(t)$ (see proof of (3.9)), $|\mathbf{II}|\leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1}$ and $|\mathbf{III}|\leqslant C\|\varepsilon(t)\|_{H^1}^2$, and so, for all $t\geqslant t_0$ and all $j\in[\![1,N]\!]$,

$$\left|\frac{d}{dt}\alpha_j^{\pm}(t)\mp e_j\alpha_j(t)\right|\leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1}. \tag{4.10}$$

To control the R_{jx} directions, we proceed exactly as in Section 3.3.6. Define $\widetilde{\varepsilon}(t)=\varepsilon(t)+\sum_{j=1}^{N}a_j(t)R_{jx}(t)$, where $a_j(t)=-\frac{\int\varepsilon(t)R_{jx}(t)}{\int(Q'_{c_j})^2}$, so that $|\int\widetilde{\varepsilon}(t)R_{jx}(t)|\leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1}$ and

$$C_1\|\varepsilon\|_{H^1}\leqslant\|\widetilde{\varepsilon}\|_{H^1}+\sum_{j=1}^{N}|a_j|\leqslant C_2\|\varepsilon\|_{H^1}. \tag{4.11}$$

As $\|\varepsilon(t)\|_{H^1}\leqslant Ce^{-\gamma t}$, we have exactly as in [11], for all $t\geqslant t_0$, by monotonicity arguments,

$$\int\Big[\varepsilon_x^2(t)-pR^{p-1}(t)\varepsilon^2(t)\Big]h(t)+\varepsilon^2(t)\leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2,$$

where h is defined in Section 3.3.5. We also have from [11],

$$\int(\widetilde{\varepsilon}_x^2-pR^{p-1}\widetilde{\varepsilon}^2)h+\widetilde{\varepsilon}^2\leqslant\int\left[(\varepsilon_x^2-pR^{p-1}\varepsilon^2)h+\varepsilon^2\right]+Ce^{-2\gamma t}\sum_{j=1}^{N}a_j^2+Ce^{-2\gamma t}\|\varepsilon\|_{H^1}^2,$$

and thus

$$\int \Big[\widetilde{\varepsilon}_x^2(t) - pR^{p-1}(t)\widetilde{\varepsilon}^2(t)\Big]h(t) + \widetilde{\varepsilon}^2(t) \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2.$$

But as in Section 3.3.5, a localization argument of the property of coercivity (vi) in Lemma 2.5 leads to

$$\int (\widetilde{\varepsilon}_x^2 - pR^{p-1}\widetilde{\varepsilon}^2)h + \widetilde{\varepsilon}^2 \geqslant \lambda_2\|\widetilde{\varepsilon}\|_{H^1}^2 - \frac{1}{\lambda_2}\sum_{j=1}^N\left[\left(\int \widetilde{\varepsilon}R_{jx}\right)^2 + \left(\int \widetilde{\varepsilon}Z_j^+\right)^2 + \left(\int \widetilde{\varepsilon}Z_j^-\right)^2\right].$$

Since $\left(\int \widetilde{\varepsilon}R_{jx}\right)^2 \leqslant Ce^{-2\gamma t}\|\varepsilon(t)\|_{H^1}^2$ and $\left(\int \widetilde{\varepsilon}Z_j^{\pm}\right)^2 \leqslant 2(\alpha_j^{\pm})^2 + Ce^{-2\gamma t}\|\varepsilon(t)\|_{H^1}^2$, then

$$\lambda_2\|\widetilde{\varepsilon}\|_{H^1}^2 \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2 + Ce^{-2\gamma t}\|\varepsilon(t)\|_{H^1}^2 + C\sum_{j=1}^N(\alpha_j^+)^2 + C\sum_{j=1}^N(\alpha_j^-)^2.$$

By denoting $\boldsymbol{\alpha}(t) = (\alpha_j^{\pm}(t))_{j,\pm}$, we thus have

$$\|\widetilde{\varepsilon}(t)\|_{H^1}^2 \leqslant Ce^{-2\gamma t}\sup_{t'\geqslant t}\|\varepsilon(t')\|_{H^1}^2 + C\|\boldsymbol{\alpha}(t)\|^2. \tag{4.12}$$

Finally, to estimate $|a_j(t)|$ for all $j \in [\![1, N]\!]$, we follow the strategy and some calculation from the proof of Lemma 3.11. First write the equation satisfied by $\widetilde{\varepsilon}$:

$$\begin{aligned}
&\widetilde{\varepsilon}_t + (\widetilde{\varepsilon}_{xx} + p\varphi^{p-1}\widetilde{\varepsilon})_x \\
&= \varepsilon_t + \varepsilon_{xxx} + p(\varphi^{p-1}\varepsilon)_x + \sum_{k=1}^N a_k R_{kxt} + \sum_{k=1}^N a_k' R_{kx} + \sum_{k=1}^N a_k R_{kxxx} + p\sum_{k=1}^N a_k(R_{kx}\varphi^{p-1})_x \\
&= -[(\varphi+\varepsilon)^p - \varphi^p]_x + p(\varphi^{p-1}\varepsilon)_x + \sum_{k=1}^N a_k' R_{kx} + \sum_{k=1}^N a_k\big[-c_k R_{kx} + R_{kxxx} + p\varphi^{p-1}R_{kx}\big]_x \\
&= \sum_{k=1}^N a_k' R_{kx} + \sum_{k=1}^N a_k\big[R_{kxxx} - c_k R_{kx} + p\varphi^{p-1}R_{kx}\big]_x - \big[(\varphi+\varepsilon)^p - \varphi^p - p\varphi^{p-1}\varepsilon\big]_x.
\end{aligned}$$

Then multiply by R_{jx} and integrate, so

$$\begin{aligned}
\int \widetilde{\varepsilon}_t R_{jx} - \int (\widetilde{\varepsilon}_{xx} + p\varphi^{p-1}\widetilde{\varepsilon})R_{jxx} &= a_j'\int R_{jx}^2 + \sum_{k\neq j} a_k'\int R_{kx}R_{jx} \\
&+ \sum_{k=1}^N a_k\int \big[R_{kxxx} - c_k R_{kx} + p\varphi^{p-1}R_{kx}\big]_x R_{jx} + \int \big[(\varphi^p+\varepsilon)^p - \varphi^p - p\varphi^{p-1}\varepsilon\big]R_{jxx}.
\end{aligned}$$

As $\big\|(R_{kxxx} - c_k R_{kx} + p\varphi^{p-1}R_{kx})_x\big\|_{L^\infty} \leqslant Ce^{-\gamma t}$, we obtain

$$|a_j'(t)| \leqslant C\left|\int \widetilde{\varepsilon}_t(t)R_{jx}(t)\right| + Ce^{-\gamma t}\sum_{k\neq j}|a_k'(t)| + Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\varepsilon(t)\|_{H^1}^2 + C\|\widetilde{\varepsilon}(t)\|_{H^1}.$$

Moreover, we still have

$$\left|\int \widetilde{\varepsilon}_t(t)R_{jx}(t)\right| \leqslant C\|\widetilde{\varepsilon}(t)\|_{H^1} + Ce^{-\gamma t}\sum_{k\neq j}|a_k'(t)| + Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1},$$

and so

$$|a_j'(t)| \leqslant C_1 e^{-\gamma t}\sum_{k\neq j}|a_k'(t)| + Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\varepsilon(t)\|_{H^1}^2 + C\|\widetilde{\varepsilon}(t)\|_{H^1}.$$

Finally choose t_0 large enough such that $C_1 e^{-\gamma t_0} \leqslant \frac{1}{N}$, so that we obtain, for all $j \in [\![1, N]\!]$ and all $t \geqslant t_0$,

$$|a_j'(t)| \leqslant Ce^{-\gamma t}\|\varepsilon(t)\|_{H^1} + C\|\widetilde{\varepsilon}(t)\|_{H^1}. \tag{4.13}$$

Step 2: Induction. With estimates (4.10) to (4.13), we can now improve exponential convergence of ε by a bootstrap argument. We recall that we already have $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma_0 = \gamma$. Now, we prove that if $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma \leqslant \gamma_0 < e_1 - \gamma$, then $\|\varepsilon(t)\|_{H^1} \leqslant C'e^{-(\gamma_0+\gamma)t}$. So, suppose that $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma \leqslant \gamma_0 < e_1 - \gamma$.

(a) From (4.10), we get for all $j \in [\![1, N]\!]$, $|(e^{-e_j t}\alpha_j^+(t))'| \leqslant Ce^{-(e_j+\gamma_0+\gamma)t}$, and so by integration on $[t, +\infty)$, $|\alpha_j^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$, since $\alpha_j^+(t) \to 0$ as $t \to +\infty$.

(b) Still from (4.10), we get for all $j \in [\![1, N]\!]$, $|(e^{e_j t}\alpha_j^-(t))'| \leqslant Ce^{(e_j-\gamma-\gamma_0)t}$. As $e_j - \gamma - \gamma_0 \geqslant e_1 - \gamma - \gamma_0 > 0$, we obtain by integration on $[t_0, t]$, $|e^{e_j t}\alpha_j^-(t) - e^{e_j t_0}\alpha_j^-(t_0)| \leqslant Ce^{(e_j-\gamma-\gamma_0)t}$, and so
$$|\alpha_j^-(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} + Ce^{-e_j t} \leqslant Ce^{-(\gamma_0+\gamma)t}.$$

(c) Therefore we have $\|\boldsymbol{\alpha}(t)\|^2 \leqslant Ce^{-2(\gamma_0+\gamma)t}$, and so by (4.12), we obtain $\|\widetilde{\varepsilon}(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$.

(d) From (4.13), we deduce that for all $j \in [\![1, N]\!]$, $|a_j'(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$, and so, by integration on $[t, +\infty)$, $|a_j(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$, since $a_j(t) \to 0$ as $t \to +\infty$.

(e) Finally, from (4.11), we have $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$, as expected.

Step 3: Conclusion. We apply the previous induction until to have $e_1 - \gamma < \gamma_0 < e_1$. Note that if $\gamma_0 = e_1 - \gamma$, then the estimate is still true for $\gamma_0 = e_1 - \frac{3}{2}\gamma < e_1 - \gamma$, and so for $\gamma_0 = e_1 - \frac{1}{2}\gamma > e_1 - \gamma$ by the previous step. Now we follow the scheme of step 2. We still have, for all $j \in [\![1, N]\!]$, $|\alpha_j^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_1 t}$, and $|(e^{e_j t}\alpha_j^-(t))'| \leqslant Ce^{(e_j-\gamma-\gamma_0)t}$. In particular, for $j = 1$, we have
$$|(e^{e_1 t}\alpha_1^-(t))'| \leqslant Ce^{(e_1-\gamma-\gamma_0)t} \in L^1([t_0, +\infty)),$$
since $e_1 - \gamma - \gamma_0 < 0$. Hence there exists $A_1 \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} e^{e_1 t}\alpha_1^-(t) = A_1, \tag{4.14}$$

and $|e^{e_1 t}\alpha_1^-(t) - A_1| \leqslant Ce^{(e_1-\gamma-\gamma_0)t}$, and so $|\alpha_1^-(t)| \leqslant Ce^{-e_1 t}$. For $j \geqslant 2$, since $e_j - \gamma - \gamma_0 > e_2 - \gamma - e_1 > 0$ by definition of γ, we still obtain by integration on $[t_0, t]$, $|\alpha_j^-(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_1 t}$. As in step 2, it follows $\|\boldsymbol{\alpha}(t)\|^2 \leqslant Ce^{-2e_1 t}$, then $\|\widetilde{\varepsilon}(t)\|_{H^1} \leqslant Ce^{-e_1 t}$ by (4.12), $|a_j(t)| \leqslant Ce^{-e_1 t}$ for all $j \in [\![1, N]\!]$ by (4.13), and finally $\|\varepsilon(t)\|_{H^1} \leqslant Ce^{-e_1 t}$ by (4.11), as expected. □

4.3 Identification of the solution

We now prove the following proposition by induction, following the strategy of the previous section. We identify u among the family $(\varphi_{A_1,\dots,A_N})$ constructed in Section 3. We recall that this family was constructed thanks to the subfamilies $(\varphi_{A_1,\dots,A_j})$, satisfying (3.2) for all $j \in [\![1, N]\!]$:

$$\forall t \geqslant t_0, \quad \left\|\varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t}Y_j^+(t)\right\|_{H^1} \leqslant e^{-(e_j+\gamma)t}.$$

Proposition 4.3. *For all $j \in [\![1, N]\!]$, there exist $t_0, C > 0$ and $(A_1, \dots, A_j) \in \mathbb{R}^j$ such that, defining $\varepsilon_j(t) = u(t) - \varphi_{A_1,\dots,A_j}(t)$, one has*

$$\forall t \geqslant t_0, \quad \|\varepsilon_j(t)\|_{H^1} \leqslant Ce^{-e_j t}.$$

Moreover, defining $\alpha_{j,k}^{\pm}(t) = \int \varepsilon_j(t)Z_k^{\pm}(t)$ for all $k \in [\![1, N]\!]$, one has

$$\forall k \in [\![1, j]\!], \quad \lim_{t\to+\infty} e^{e_k t}\alpha_{j,k}^-(t) = 0.$$

Remark 4.4. As $\varepsilon_1 = u - \varphi_{A_1} = \varepsilon + (\varphi - \varphi_{A_1})$, we have

$$\|\varepsilon_1(t)\|_{H^1} \leqslant \|\varepsilon(t)\|_{H^1} + \|\varphi(t) - \varphi_{A_1}(t)\|_{H^1} \leqslant Ce^{-e_1 t}$$

by Lemma 4.2 and (3.2). Moreover, if we define z_1 by $z_1(t) = \varphi_{A_1}(t) - \varphi(t) - A_1 e^{-e_1 t} Y_1^+(t)$, we have

$$\begin{aligned}\alpha_{1,1}^-(t) &= \int \varepsilon_1(t) Z_1^-(t) = \int \varepsilon(t) Z_1^-(t) - A_1 e^{-e_1 t} \int Y_1^+(t) Z_1^-(t) - \int z_1(t) Z_1^-(t) \\ &= \alpha_1^-(t) - A_1 e^{-e_1 t} - \int z_1(t) Z_1^-(t)\end{aligned}$$

by definition of α_1^- in the previous section and by normalization (iv) of Lemma 2.5. As $\|z_1(t)\|_{H^1} \leqslant e^{-(e_1+\gamma)t}$, we finally deduce, by (4.14),

$$|e^{e_1 t}\alpha_{1,1}^-(t)| \leqslant |e^{e_1 t}\alpha_1^-(t) - A_1| + Ce^{-\gamma t} \xrightarrow[t\to+\infty]{} 0.$$

Therefore, Proposition 4.3 is proved for $j = 1$.

Proof of Proposition 4.3. By remark 4.4, it is enough to prove the inductive step: we suppose the assertion true for $j-1$ with $j \geqslant 2$, and we prove it for j. So, suppose that there exist $t_0, C > 0$ and $(A_1, \dots, A_{j-1}) \in \mathbb{R}^{j-1}$ such that $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-e_{j-1}t}$ for all $t \geqslant t_0$, and moreover, for all $k \in [\![1, j-1]\!]$, $e^{e_k t}\alpha_{j-1,k}^-(t) \to 0$ as $t \to +\infty$.

Step 1: Another induction. Following the proof of Lemma 4.2, we prove that if $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $e_{j-1} \leqslant \gamma_0 < e_j - \gamma$, then $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant C'e^{-(\gamma_0+\gamma)t}$. But, as φ_{A_1} is a soliton like φ, estimates (4.10) to (4.13) of the previous section hold. In other words, we have, with obvious notation, for all $t \geqslant t_0$,

$$\begin{cases} \forall k \in [\![1, N]\!], \quad \left|\frac{d}{dt}\alpha_{j-1,k}^{\pm}(t) \mp e_k \alpha_{j-1,k}^{\pm}(t)\right| \leqslant Ce^{-\gamma t}\|\varepsilon_{j-1}(t)\|_{H^1}, \\ \|\widetilde{\varepsilon}_{j-1}(t)\|_{H^1}^2 \leqslant Ce^{-2\gamma t}\sup_{t' \geqslant t}\|\varepsilon_{j-1}(t')\|_{H^1}^2 + C\|\boldsymbol{\alpha}_{j-1}(t)\|^2, \\ \forall k \in [\![1, N]\!], \quad |a'_{j-1,k}(t)| \leqslant Ce^{-\gamma t}\|\varepsilon_{j-1}(t)\|_{H^1} + C\|\widetilde{\varepsilon}_{j-1}(t)\|_{H^1}, \\ \|\varepsilon_{j-1}(t)\|_{H^1} \leqslant C\|\widetilde{\varepsilon}_{j-1}(t)\|_{H^1} + C\sum_{k=1}^N |a_{j-1,k}(t)|. \end{cases}$$

From these estimates, we deduce the following steps as in the previous section.

(a) For all $k \in [\![1, N]\!]$, $|\alpha_{j-1,k}^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$.

(b) For all $k \in [\![1, j-1]\!]$, we have $|(e^{e_k t}\alpha_{j-1,k}^-(t))'| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$. As $e_k - \gamma_0 - \gamma \leqslant e_{j-1} - \gamma_0 - \gamma \leqslant -\gamma < 0$ and $e^{e_k t}\alpha_{j-1,k}^-(t) \to 0$ as $t \to +\infty$ by hypothesis, we deduce by integration on $[t, +\infty)$ that $|e^{e_k t}\alpha_{j-1,k}^-(t)| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$, and so $|\alpha_{j-1,k}^-(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$.

(c) For all $k \in [\![j, N]\!]$, we still have $|(e^{e_k t}\alpha_{j-1,k}^-(t))'| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$. As $e_k - \gamma_0 - \gamma \geqslant e_j - \gamma_0 - \gamma > 0$, we deduce, by integration on $[t_0, t]$, $|e^{e_k t}\alpha_{j-1,k}^-(t) - e^{e_k t_0}\alpha_{j-1,k}^-(t_0)| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}$, and so

$$|\alpha_{j-1,k}^-(t)| \leqslant Ce^{-e_k t} + Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-(\gamma_0+\gamma)t}.$$

(d) Hence we have $\|\boldsymbol{\alpha}_{j-1}(t)\|^2 \leqslant Ce^{-2(\gamma_0+\gamma)t}$. It follows $\|\widetilde{\varepsilon}_{j-1}(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$, $|a_{j-1,k}(t)| \leqslant Ce^{-(\gamma_0+\gamma)t}$ by integration, and finally $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-(\gamma_0+\gamma)t}$ as expected.

Step 2: Identification of A_j. We apply the previous induction until to have $e_j - \gamma < \gamma_0 < e_j$. Moreover, following the same scheme, we obtain the following estimates.

(a) For all $k \in [\![1, N]\!]$, $|\alpha_{j-1,k}^+(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_j t}$, and we still have

$$|(e^{e_k t}\alpha_{j-1,k}^-(t))'| \leqslant Ce^{(e_k-\gamma_0-\gamma)t}.$$

(b) For all $k \in [\![1, j-1]\!]$, we still have $|\alpha^-_{j-1,k}(t)| \leqslant Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_j t}$.

(c) For $k = j$, we have $|(e^{e_j t}\alpha^-_{j-1,j}(t))'| \leqslant Ce^{(e_j-\gamma_0-\gamma)t} \in L^1([t_0, +\infty))$ since $e_j - \gamma_0 - \gamma < 0$. Thus there exists $A_j \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} e^{e_j t}\alpha^-_{j-1,j}(t) = A_j,$$

and moreover $|e^{e_j t}\alpha^-_{j-1,j}(t) - A_j| \leqslant Ce^{(e_j-\gamma_0-\gamma)t}$. Hence we have $|\alpha^-_{j-1,j}(t)| \leqslant Ce^{-e_j t}$.

(d) For all $k \in [\![j+1, N]\!]$, we have $e_k - \gamma_0 - \gamma > e_{j+1} - e_j - \gamma > 0$, thus by integration on $[t_0, t]$, we get $|\alpha^-_{j-1,k}(t)| \leqslant Ce^{-e_k t} + Ce^{-(\gamma_0+\gamma)t} \leqslant Ce^{-e_j t}$.

(e) We now have $\|\boldsymbol{\alpha}_{j-1}(t)\|^2 \leqslant Ce^{-2e_j t}$, and so as in the first step, we conclude that $\|\varepsilon_{j-1}(t)\|_{H^1} \leqslant Ce^{-e_j t}$.

Step 3: Conclusion. To conclude the induction, we write

$$\begin{aligned}\varepsilon_j(t) &= u(t) - \varphi_{A_1,\dots,A_j}(t) = \varepsilon_{j-1}(t) + [\varphi_{A_1,\dots,A_{j-1}}(t) - \varphi_{A_1,\dots,A_j}(t)]\\ &= \varepsilon_{j-1}(t) - A_j e^{-e_j t}Y^+_j(t) - z_j(t),\end{aligned}$$

where z_j, defined by $z_j(t) = \varphi_{A_1,\dots,A_j}(t) - \varphi_{A_1,\dots,A_{j-1}}(t) - A_j e^{-e_j t}Y^+_j(t)$, satisfies $\|z_j(t)\|_{H^1} \leqslant e^{-(e_j+\gamma)t}$ by (3.2). Thus, we first have

$$\|\varepsilon_j(t)\|_{H^1} \leqslant \|\varepsilon_{j-1}(t)\|_{H^1} + Ce^{-e_j t} + \|z_j(t)\|_{H^1} \leqslant Ce^{-e_j t}.$$

Moreover, we find

$$\alpha^-_{j,k}(t) = \int \varepsilon_j(t)Z^-_k(t) = \alpha^-_{j-1,k}(t) - A_j e^{-e_j t}\int Y^+_j(t)Z^-_k(t) - \int z_j(t)Z^-_k(t).$$

Therefore, for all $k \in [\![1, j-1]\!]$, we have $|\alpha^-_{j,k}(t)| \leqslant |\alpha^-_{j-1,k}(t)| + Ce^{-e_j t} + Ce^{-(e_j+\gamma)t}$, and so

$$e^{e_k t}|\alpha^-_{j,k}(t)| \leqslant e^{e_k t}|\alpha^-_{j-1,k}(t)| + Ce^{-(e_j-e_k)t} \xrightarrow[t\to+\infty]{} 0.$$

Finally, for $k = j$, we have by (iv) of Lemma 2.5, $\alpha^-_{j,j}(t) = \alpha^-_{j-1,j}(t) - A_j e^{-e_j t} - \int z_j(t)Z^-_j(t)$, and so

$$e^{e_j t}|\alpha^-_{j,j}(t)| \leqslant |e^{e_j t}\alpha^-_{j-1,j}(t) - A_j| + Ce^{-\gamma t} \xrightarrow[t\to+\infty]{} 0,$$

which achieves the proof of Proposition 4.3. □

Corollary 4.5. *There exist $(A_1, \dots, A_N) \in \mathbb{R}^N$ and $C, t_0 > 0$ such that, defining $z(t) = u(t) - \varphi_{A_1,\dots,A_N}(t)$, we have $\|z(t)\|_{H^1} \leqslant Ce^{-2e_N t}$ for all $t \geqslant t_0$.*

Proof. Applying Proposition 4.3 with $j = N$, we obtain $(A_1, \dots, A_N) \in \mathbb{R}^N$ and $C, t_0 > 0$ such that $\|z(t)\|_{H^1} \leqslant Ce^{-e_N t}$ for all $t \geqslant t_0$. Moreover, if we set

$$\alpha^\pm_k(t) = \int z(t)Z^\pm_k(t)$$

for all $k \in [\![1, N]\!]$, we have $e^{e_k t}\alpha^-_k(t) \to 0$ as $t \to +\infty$. But, as in the previous proof, it easily follows that if $\|z(t)\|_{H^1} \leqslant Ce^{-\gamma_0 t}$ with $\gamma_0 \geqslant e_N$, then $\|z(t)\|_{H^1} \leqslant C'e^{-(\gamma_0+\gamma)t}$, and we apply this induction until to have $\gamma_0 = 2e_N$. □

4.4 Uniqueness

Finally, we prove the following proposition, which achieves the proof of Theorem 1.3. Note that its proof is based on the schemes developed above, and on arguments developed in [11, Section 4].

Proposition 4.6. *There exists $t_0 > 0$ such that, for all $t \geqslant t_0$, $z(t) = 0$.*

Proof. We start from the conclusion of Corollary 4.5, we set

$$\theta(t) = \sup_{t' \geqslant t} e^{e_N t'} \|z(t')\|_{H^1},$$

well defined and decreasing, and we prove that $\theta = 0$. Indeed, with obvious notation, we still have the following estimates, for all $t \geqslant t_0$,

$$\begin{cases} \forall k \in [\![1, N]\!], \quad \left|\frac{d}{dt}\alpha_k^{\pm}(t) \mp e_k \alpha_k^{\pm}(t)\right| \leqslant Ce^{-\gamma t}\|z(t)\|_{H^1}, \\ \forall k \in [\![1, N]\!], \quad |a_k'(t)| \leqslant Ce^{-\gamma t}\|z(t)\|_{H^1} + C\|\widetilde{z}(t)\|_{H^1}, \\ \|z(t)\|_{H^1} \leqslant C\|\widetilde{z}(t)\|_{H^1} + C\sum_{k=1}^{N} |a_k(t)|. \end{cases}$$

Moreover, if we define H_0 as in [11] by

$$H_0(t) = \int \left\{ \Big(z_x^2(t,x) - F_0(t, z(t,x)) \Big) h(t,x) + z^2(t,x) \right\} dx,$$

where

$$F_0(t,z) = 2\left[\frac{(\varphi_{A_1,\dots,A_N}(t) + z)^{p+1}}{p+1} - \frac{\varphi^{p+1}_{A_1,\dots,A_N}(t)}{p+1} - \varphi^{p}_{A_1,\dots,A_N}(t) z \right]$$

and h is defined in Section 3.3.5, we also have $\frac{dH_0}{dt}(t) \geqslant -Ce^{-2\gamma t}\|z(t)\|^2_{H^1}$. Now, we want to prove that $\theta(t) = 0$, for $t \geqslant t_0$ with t_0 large enough. Let $t \geqslant t_0$.

First, we have for all $k \in [\![1, N]\!]$, $\left|\frac{d}{dt}\alpha_k^{\pm}(t) \mp e_k \alpha_k^{\pm}(t)\right| \leqslant Ce^{-\gamma t}e^{-e_N t}\theta(t)$, and thus, for all $s \geqslant t$,

$$\left| \frac{d}{dt}\alpha_k^{\pm}(s) \mp e_k \alpha_k^{\pm}(s) \right| \leqslant Ce^{-(e_N+\gamma)s}\theta(t).$$

Hence, we have $|(e^{-e_k s}\alpha_k^{+}(s))'| \leqslant Ce^{-(e_N+e_k+\gamma)s}\theta(t)$, and so by integration on $[t, +\infty)$,

$$|\alpha_k^{+}(t)| \leqslant Ce^{-(e_N+\gamma)t}\theta(t).$$

Similarly, we have $|(e^{e_k s}\alpha_k^{-}(s))'| \leqslant Ce^{-(e_N-e_k+\gamma)s}\theta(t)$, and since $e_N - e_k + \gamma \geqslant \gamma > 0$ and $e^{e_k t}\alpha_k^{-}(t) \to 0$ as $t \to +\infty$, we also get by integration on $[t, +\infty)$,

$$|\alpha_k^{-}(t)| \leqslant Ce^{-(e_N+\gamma)t}\theta(t).$$

We thus have $\|\boldsymbol{\alpha}(t)\|^2 \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t)$. But we also have, for $s \geqslant t$,

$$\frac{dH_0}{dt}(s) \geqslant -Ce^{-2\gamma s}\|z(s)\|^2_{H^1} = -Ce^{-2(e_N+\gamma)s}(e^{e_N s}\|z(s)\|_{H^1})^2 \geqslant -Ce^{-2(e_N+\gamma)s}\theta^2(t),$$

and so by integration on $[t, +\infty)$, $H_0(t) \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t)$. As in the proof of Lemma 4.2, we deduce that

$$\|\widetilde{z}(t)\|^2_{H^1} \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t) + C\|\boldsymbol{\alpha}(t)\|^2 \leqslant Ce^{-2(e_N+\gamma)t}\theta^2(t),$$

and so $\|\widetilde{z}(t)\|_{H^1} \leqslant Ce^{-(e_N+\gamma)t}\theta(t)$. But, for all $k \in [\![1, N]\!]$ and all $s \geqslant t$, we have

$$|a_k'(s)| \leqslant Ce^{-\gamma s}\|z(s)\|_{H^1} + C\|\widetilde{z}(s)\|_{H^1} \leqslant Ce^{-(e_N+\gamma)s}\theta(s) \leqslant Ce^{-(e_N+\gamma)s}\theta(t),$$

and so by integration on $[t, +\infty)$, $|a_k(t)| \leqslant Ce^{-(e_N+\gamma)t}\theta(t)$.

Finally, we have shown that there exists $C^* > 0$ such that, for all $t \geqslant t_0$, $\|z(t)\|_{H^1} \leqslant C^* e^{-(e_N+\gamma)t}\theta(t)$. Now fix $t \geqslant t_0$. We have, for all $t' \geqslant t$,

$$e^{e_N t'}\|z(t')\|_{H^1} \leqslant C^* e^{-\gamma t'}\theta(t') \leqslant C^* e^{-\gamma t_0}\theta(t),$$

and thus $\theta(t) \leqslant C^* e^{-\gamma t_0}\theta(t)$. Choosing t_0 large enough so that $C^* e^{-\gamma t_0} \leqslant \frac{1}{2}$, we obtain $\theta(t) \leqslant \frac{1}{2}\theta(t)$, so $\theta(t) \leqslant 0$, and so finally $\theta(t) = 0$, as expected. □

A Appendix

Proof of Lemma 3.4. The scheme of the proof is quite similar to the proof of [9, Theorem 5], and uses moreover some arguments developed in [11, section 3.4]. Let $T^* = T^*(\|z_0\|_{H^{\frac{3}{4}}}) > 0$ be the maximum time of existence of the solution $z(t)$ associated to z_0. We distinguish two cases, whether $T < T^*$ or not, and we show that this last case is in fact impossible.

First case. Suppose that $T < T^*$, and let us show that $z_n(T) \rightharpoonup z(T)$ in H^1. Since C_0^∞ is dense in H^{-1} and $\|z_n(T) - z(T)\|_{H^1} \leqslant \|z_n(T)\|_{H^1} + \|z(T)\|_{H^1} \leqslant K'$, it is enough to show that $z_n(T) \to z(T)$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$. So let $g \in C_0^\infty(\mathbb{R})$ and $\varepsilon > 0$, and let us show the lemma in three steps, using a H^3 regularization.

Step 1. For $N \gg 1$ to fix later, we define $z_{0,n}^N$ and z_0^N by

$$\begin{cases} \widehat{z_{0,n}^N}(\xi) = \mathbb{1}_{[-N,N]}(\xi)\widehat{z_{0,n}}(\xi), \\ \widehat{z_0^N}(\xi) = \mathbb{1}_{[-N,N]}(\xi)\widehat{z_0}(\xi). \end{cases}$$

In particular, $z_{0,n}^N$ and z_0^N belong to H^3, and $z_{0,n}^N \to z_0^N$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$, since Fourier transform is continuous in $\mathcal{D}'(\mathbb{R})$. Moreover, since $(z_{0,n})$ is uniformly bounded in H^1 by Banach-Steinhaus' theorem, we have $\|z_{0,n}^N\|_{H^3} \leqslant C(N)\|z_{0,n}\|_{H^1} \leqslant C(N)$, and

$$\begin{aligned} \big\|z_{0,n}^N - z_{0,n}\big\|_{H^{\frac{3}{4}}}^2 &= \int_{|\xi|\geqslant N} \big(1+\xi^2\big)^{3/4} |\widehat{z_{0,n}}(\xi)|^2 \, d\xi \leqslant 2^{3/4} \int_{|\xi|\geqslant N} |\xi|^{3/2} \cdot |\widehat{z_{0,n}}(\xi)|^2 \, d\xi \\ &\leqslant \frac{2^{3/4}}{\sqrt{N}} \int_{|\xi|\geqslant N} \xi^2 |\widehat{z_{0,n}}(\xi)|^2 \, d\xi \leqslant \frac{2^{3/4}}{\sqrt{N}} \|z_{0,n}\|_{H^1}^2 \leqslant \frac{C}{\sqrt{N}}, \end{aligned}$$

so $z_{0,n}^N \to z_{0,n}$ as $N \to +\infty$ in $H^{\frac{3}{4}}$ uniformly in n. If we call $z_n^N(t)$ the solution corresponding to initial data $z_{0,n}^N$, and since $\|z_n(t)\|_{H^{\frac{3}{4}}} \leqslant \|z_n(t)\|_{H^1} \leqslant K$, we deduce that

$$\sup_{t\in[0,T]} \big\|z_n^N(t) - z_n(t)\big\|_{H^{\frac{3}{4}}} \leqslant C\big\|z_{0,n}^N - z_{0,n}\big\|_{H^{\frac{3}{4}}}$$

for N large enough, by applying [10, Corollary 2.18] with $s = \frac{3}{4} > \frac{p-5}{2(p-1)}$ and $T = T_K = T(\|z_n(t)\|_{H^{\frac{3}{4}}})$. As a consequence, we have

$$\begin{aligned} \sup_{t\in[0,T]} \big\|z_n^N(t)\big\|_{H^{\frac{3}{4}}} &\leqslant \sup_{t\in[0,T]} \|z_n(t)\|_{H^{\frac{3}{4}}} + C\big\|z_{0,n}^N\big\|_{H^{\frac{3}{4}}} + C\|z_{0,n}\|_{H^{\frac{3}{4}}} \\ &\leqslant \sup_{t\in[0,T]} \|z_n(t)\|_{H^1} + 2C\|z_{0,n}\|_{H^1} \leqslant C. \end{aligned}$$

Similarly, since $\sup_{t\in[0,T]} \|z(t)\|_{H^1} \leqslant K'$ by hypothesis, we also obtain, for N large enough,

$$\sup_{t\in[0,T]} \big\|z^N(t) - z(t)\big\|_{H^{\frac{3}{4}}} \leqslant C'\big\|z_0^N - z_0\big\|_{H^{\frac{3}{4}}},$$

where $z^N(t)$ is the solution corresponding to initial data z_0^N. Notice that C and C' are independent of n, and that by propagation of the regularity, we have $z_n^N(t), z^N(t) \in H^3$ for all $t \in [0,T]$. Finally, we have by the Cauchy-Schwarz inequality

$$\begin{aligned} \left|\int (z_n(T) - z(T))g - \int (z_n^N(T) - z^N(T))g\right| &\leqslant \left|\int (z_n(T) - z_n^N(T))g\right| + \left|\int (z(T) - z^N(T))g\right| \\ &\leqslant (\big\|z_n(T) - z_n^N(T)\big\|_{L^2} + \big\|z(T) - z^N(T)\big\|_{L^2})\|g\|_{L^2} \leqslant \frac{C}{\sqrt[4]{N}} \leqslant \frac{\varepsilon}{2} \end{aligned}$$

for N large enough, and we now fix it to this value.

Step 2. Now that N is fixed, we forget it and the situation amounts in: $z_n(t), z(t) \in H^3$ for all $t \in [0,T]$, $\sup_{t\in[0,T]} \|z_n(t)\|_{H^{\frac{3}{4}}} \leqslant C$, $\|z_{0,n}\|_{H^3} \leqslant C'$ (with C and C' independent of n) and $z_{0,n} \to z_0$ in $\mathcal{D}'(\mathbb{R})$ as $n \to +\infty$. The aim of this step is to show consecutively that $z_n(t)$ is uniformly bounded in H^1, H^2 and H^3, and finally z_n is uniformly bounded in $H^1([0,T]\times\mathbb{R})$.

Since $\sup_{t\in[0,T]} \|z_n(t)\|_{H^{\frac{3}{4}}} \leqslant C$ and $H^{\frac{3}{4}}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ continuously, then we have

$$\sup_{t\in[0,T]} \|z_n(t)\|_{L^\infty} \leqslant C \quad \text{and} \quad \sup_{t\in[0,T]} \|z_n(t)\|_{L^2} \leqslant C.$$

But energy conservation gives, for all $t \in [0,T]$,

$$\frac{1}{2}\int (\partial_x z_n(t))^2 - \frac{1}{p+1}\int z_n(t)^{p+1} = \frac{1}{2}\int (\partial_x z_{0,n})^2 - \frac{1}{p+1}\int z_{0,n}^{p+1}.$$

We deduce that:

$$\left|\int (\partial_x z_n(t))^2\right| \leqslant C\|z_n(t)\|_{L^\infty}^{p-1}\|z_n(t)\|_{L^2}^2 + C\|z_{0,n}\|_{H^1}^2 + C\|z_{0,n}\|_{H^1}^{p+1} \leqslant C,$$

and so $\sup_{t\in[0,T]} \|z_n(t)\|_{H^1} \leqslant C$.

To estimate $\|z_n(t)\|_{H^2}$, we use the "modified energy" as in [11, Section 3.4] (see also [8]). If we denote z_n by z for a short moment, and if we define $G_2(t) = \int \left(z_{xx}^2(t) - \frac{5p}{3} z_x^2(t) z^{p-1}(t)\right)$ for $t \in [0,T]$, we have the identity

$$G_2'(t) = \frac{1}{12}p(p-1)(p-2)(p-3)\int z_x^5(t) z^{p-4}(t) + \frac{5}{3}p^2(p-1)\int z_x^3(t) z^{2p-3}(t).$$

But Gagliardo-Nirenberg inequalities give, for all $k \geqslant 2$,

$$\int |u_x|^k \leqslant C\left(\int u_x^2\right)^{\frac{k+2}{4}} \left(\int u_{xx}^2\right)^{\frac{k-2}{4}},$$

and since $\sup_{t\in[0,T]} \|z(t)\|_{L^\infty} \leqslant C$, we have

$$\begin{aligned} G_2'(t) &\leqslant C\|z(t)\|_{L^\infty}^{p-4}\int |z_x(t)|^5 + C'\|z(t)\|_{L^\infty}^{2p-3}\int |z_x(t)|^3 \\ &\leqslant C\left(\int z_x^2(t)\right)^{7/4}\left(\int z_{xx}^2(t)\right)^{3/4} + C'\left(\int z_x^2(t)\right)^{5/4}\left(\int z_{xx}^2(t)\right)^{1/4} \\ &\leqslant C\left(\int z_{xx}^2(t)\right)^{3/4} + C'\left(\int z_{xx}^2(t)\right)^{1/4}. \end{aligned}$$

Since $a \leqslant a^{4/3} + 1$ and $a \leqslant a^4 + 1$ for $a \geqslant 0$, we deduce that for some $C, D > 0$ (still independent of n), we have, for all $s \in [0,T]$,

$$G_2'(s) \leqslant C\left(\int z_{xx}^2(s)\right) + D.$$

Now, for $t \in [0,T]$, we integrate between 0 and t, and we obtain

$$G_2(t) - G_2(0) \leqslant C\int_0^t \|z_{xx}(s)\|_{L^2}^2\, ds + Dt.$$

Moreover, by definition of G_2,

$$\begin{aligned}\|z_{xx}(t)\|_{L^2}^2 &\leqslant \frac{5p}{3}\left|\int z_x^2(t)z^{p-1}(t)\right| + \frac{5p}{3}\left|\int z_x^2(0)z^{p-1}(0)\right| \\ &\quad + \|z_{xx}(0)\|_{L^2}^2 + C\int_0^t \|z_{xx}(s)\|_{L^2}^2\,ds + DT \\ &\leqslant C\|z(t)\|_{H^1}^{p+1} + C\|z(0)\|_{H^1}^{p+1} + \|z(0)\|_{H^2} + DT + C\int_0^t \|z_{xx}(s)\|_{L^2}^2\,ds \\ &\leqslant B + C\int_0^t \|z_{xx}(s)\|_{L^2}^2\,ds.\end{aligned}$$

Finally, we obtain by Grönwall's lemma that, for all $t \in [0,T]$,

$$\|z_{xx}(t)\|_{L^2}^2 \leqslant Be^{Ct} \leqslant Be^{CT}.$$

We can conclude that $\sup_{t\in[0,T]} \|z_n(t)\|_{H^2} \leqslant C$ with $C > 0$ independent of n.

For a uniform bound in H^3, we use the same arguments as for H^2. In fact, it is easier, since we have, by straightforward calculation (we forget again n for a while),

$$\begin{aligned}\frac{d}{dt}\int z_{xxx}^2(t) &= -7p(p-1)\int z_{xxx}^2(t)z_x(t)z^{p-2}(t) \\ &\quad + 14p(p-1)(p-2)\int z_{xx}^3(t)z_x(t)z^{p-3}(t) \\ &\quad + 14p(p-1)(p-2)(p-3)\int z_{xx}^2(t)z_x^3(t)z^{p-4}(t) \\ &\quad + 2p(p-1)(p-2)(p-3)(p-4)\int z_{xx}(t)z_x^5(t)z^{p-5}(t).\end{aligned}$$

But we have now $\sup_{t\in[0,T]} \|z_x(t)\|_{L^\infty} \leqslant C\sup_{t\in[0,T]} \|z_x(t)\|_{H^1} \leqslant C\sup_{t\in[0,T]} \|z(t)\|_{H^2} \leqslant C$, and still $\sup_{t\in[0,T]} \|z(t)\|_{L^\infty} \leqslant C$, so

$$\frac{d}{dt}\int z_{xxx}^2(t) \leqslant A\int z_{xxx}^2(t) + B\int |z_{xx}(t)|^3 + C\int z_{xx}^2(t) + D\int |z_{xx}(t)||z_x(t)|.$$

Using a Gagliardo-Nirenberg inequality for the second term and the Cauchy-Schwarz one for the last term, we obtain

$$\begin{aligned}\frac{d}{dt}\int z_{xxx}^2(t) &\leqslant A\int z_{xxx}^2(t) + B'\left(\int z_{xx}^2(t)\right)^{5/4}\left(\int z_{xxx}^2(t)\right)^{1/4} \\ &\quad + C\|z(t)\|_{H^2}^2 + D\|z_{xx}(t)\|_{L^2}\|z_x(t)\|_{L^2} \\ &\leqslant A\int z_{xxx}^2(t) + B''\int z_{xxx}^2(t) + B'' + C' + D\|z(t)\|_{H^2}^2 \\ &\leqslant A'\int z_{xxx}^2(t) + D'.\end{aligned}$$

Now, if we integrate this inequality between 0 and $t \in [0,T]$, we get

$$\begin{aligned}\|z_{xxx}(t)\|_{L^2}^2 &\leqslant \|z_{xxx}(0)\|_{L^2}^2 + A'\int_0^t \|z_{xxx}(s)\|_{L^2}^2\,ds + D't \\ &\leqslant \|z(0)\|_{H^3}^2 + A'\int_0^t \|z_{xxx}(s)\|_{L^2}^2\,ds + D'T \\ &\leqslant A'\int_0^t \|z_{xxx}(s)\|_{L^2}^2\,ds + D'',\end{aligned}$$

and we conclude again by Grönwall's lemma that $\|z_{xxx}(t)\|_{L^2}^2 \leqslant D''e^{A't} \leqslant D''e^{A'T}$. Finally, we have the desired bound: $\sup_{t\in[0,T]} \|z_n(t)\|_{H^3} \leqslant C$.

As $z_{nt}(t) = -z_{nxxx}(t) - pz_{nx}(t)z_n^{p-1}(t)$, then we have, for all $t \in [0,T]$,

$$\|z_{nt}(t)\|_{L^2} \leqslant \|z_{nxxx}(t)\|_{L^2} + p\|z_n(t)\|_{L^\infty}^{p-1}\|z_{nx}\|_{L^2} \leqslant \|z_n(t)\|_{H^3} + C\|z_n(t)\|_{H^1}^p \leqslant C.$$

We deduce that (z_n) is uniformly bounded in $H^1([0,T]\times\mathbb{R})$, thus there exists $\tilde{z}$ such that $z_n \rightharpoonup \tilde{z}$ weakly in $H^1([0,T]\times\mathbb{R})$ (after passing to a subsequence), and in particular strongly on compacts in $L^2([0,T]\times\mathbb{R})$. Moreover, since $\sup_t \|z_n(t)\|_{H^3} \leqslant C$, we have $\sup_t \|\tilde{z}(t)\|_{H^3} \leqslant C$.

Step 3. This step is very similar to the first one of the proof of [9, Theorem 5]. We recall that we want to prove $\int (z_n(T) - z(T))g \to 0$ as $n \to +\infty$. Let $w_n = z_n - z$. The equation satisfied by w_n is $w_{nt} + w_{nxxx} + (z_n^p - z^p)_x = 0$, and moreover

$$\begin{aligned}(z_n^p - z^p)_x &= pz_{nx}z_n^{p-1} - pz_x z^{p-1} = p[(z_{nx} - z_x)z_n^{p-1} + z_x(z_n^{p-1} - z^{p-1})] \\ &= p\left[w_{nx}z_n^{p-1} + z_x(z_n - z)\sum_{k=0}^{p-2} z_n^k z^{p-2-k}\right].\end{aligned}$$

If we define $S(u,v) = \sum_{k=0}^{p-2} v^k u^{p-2-k}$, the equation satisfied by w_n can be written

$$\begin{cases} w_{nt} + w_{nxxx} + pz_n^{p-1}w_{nx} + pz_x S(z,z_n)w_n = 0, \\ w_n(0) = \psi_n = z_{0,n} - z_0. \end{cases}$$

Now consider $v(t)$ the solution of

$$\begin{cases} v_t + v_{xxx} + p(\tilde{z}^{p-1}v)_x + pz_x S(z,\tilde{z})v = 0, \\ v(T) = g. \end{cases}$$

First notice that $\sup_t \|v\|_{L^2} \leqslant C$ by an energy method. Indeed, we have by direct calculation

$$\frac{d}{dt}\int v^2 = -p\int v^2\left[(p-1)\tilde{z}_x\tilde{z}^{p-2} + 2z_x S(z,\tilde{z})\right].$$

But $\sup_t \|\tilde{z}_x(t)\|_{L^\infty} \leqslant \sup_t \|\tilde{z}(t)\|_{H^2} \leqslant C$, and similarly $\sup_t \|\tilde{z}(t)\|_{L^\infty} \leqslant C$, $\sup_t \|z_x(t)\|_{L^\infty} \leqslant C$ and $\sup_t \|S(z(t),\tilde{z}(t))\|_{L^\infty} \leqslant C$, and so

$$-\frac{d}{ds}\int v^2(s) \leqslant C\int v^2(s).$$

By integration between $t \in [0,T]$ and T, we obtain

$$\|v(t)\|_{L^2}^2 - \|v(T)\|_{L^2}^2 \leqslant C\int_t^T \|v(s)\|_{L^2}^2\,ds,$$

that is to say $\|v(t)\|_{L^2}^2 \leqslant \|g\|_{L^2}^2 + C\int_t^T \|v(s)\|_{L^2}^2\,ds$. We conclude, by Grönwall's lemma, that

$$\|v(t)\|_{L^2}^2 \leqslant \|g\|_{L^2}^2 e^{C(T-t)} \leqslant \|g\|_{L^2}^2 e^{CT} = K.$$

Now write

$$\int w_n(T,x)g(x)\,dx - \int \psi_n(x)v(0,x)\,dx = \int_0^T\int w_{nt}v + \int_0^T\int w_n v_t = \mathbf{I} + \mathbf{II}$$

with

$$\begin{cases} \mathbf{I} = \displaystyle\int_0^T\int w_n\left[v_{xxx} + p(vz_n^{p-1})_x - pz_x S(z,z_n)v\right], \\ \mathbf{II} = \displaystyle\int_0^T\int w_n\left[-v_{xxx} - p(v\tilde{z}^{p-1})_x + pz_x S(z,\tilde{z})v\right], \end{cases}$$

and so

$$\begin{aligned}\mathbf{I}+\mathbf{II} &= p\int_0^T\int w_n[v(z_n^{p-1}-\tilde{z}^{p-1})]_x + p\int_0^T\int w_n z_x v[S(z,\tilde{z})-S(z,z_n)]\\ &= -p\int_0^T\int w_{nx}v(z_n^{p-1}-\tilde{z}^{p-1}) - p\int_0^T\int w_n z_x v\sum_{k=1}^{p-2} z^{p-2-k}(z_n^k-\tilde{z}^k)\\ &= -p\int_0^T\int w_{nx}v(z_n-\tilde{z})S(\tilde{z},z_n) - p\int_0^T\int w_n z_x v(z_n-\tilde{z})S'(z,\tilde{z},z_n)\\ &= -p\int_0^T\int [w_{nx}S(\tilde{z},z_n)+w_n z_x S'(z,\tilde{z},z_n)]v(z_n-\tilde{z}),\end{aligned}$$

where $S(\tilde{z},z_n)=\sum_{k=0}^{p-2}\tilde{z}^{p-2-k}z_n^k$ and $S'(z,\tilde{z},z_n)=\sum_{k=1}^{p-2}\sum_{l=0}^{k-1} z^{p-2-k}\tilde{z}^{k-1-l}z_n^l$ both satisfy

$$\sup_{t\in[0,T]}\|S(\tilde{z},z_n)\|_{L^\infty}\leqslant C \quad\text{and}\quad \sup_{t\in[0,T]}\|S'(z,\tilde{z},z_n)\|_{L^\infty}\leqslant C.$$

Since $\psi_n\rightharpoonup 0$ in L^2 and $v(0)\in L^2$, then, for n large enough, $\left|\int\psi_n(x)v(0,x)\,dx\right|\leqslant\frac{\varepsilon}{4}$. Therefore, it is enough to conclude to show that, for n large enough, $|\mathbf{I}+\mathbf{II}|\leqslant\frac{\varepsilon}{4}$. But

$$\sup_t\|w_{nx}S(\tilde{z},z_n)+w_n z_x S'(z,\tilde{z},z_n)\|_{L^\infty}\leqslant C,$$

and $\sup_t\|z_n-\tilde{z}\|_{L^2}\leqslant C\|z_n-\tilde{z}\|_{H^1(]0,T[\times\mathbb{R})}\leqslant C$, $\sup_t\|v\|_{L^2}\leqslant C$. Hence, there exists $R>0$ such that

$$\left|-p\int_0^T\int_{|x|>R}[w_{nx}S(\tilde{z},z_n)+w_n z_x S'(z,\tilde{z},z_n)]v(z_n-\tilde{z})\right|\leqslant\frac{\varepsilon}{8}.$$

And finally, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|-p\int_0^T\int_{|x|\leqslant R}[w_{nx}S(\tilde{z},z_n)+w_n z_x S'(z,\tilde{z},z_n)]v(z_n-\tilde{z})\right|\leqslant C\int_0^T\int_{|x|\leqslant R}|z_n-\tilde{z}||v|\\ &\qquad\leqslant C\left(\int_0^T\int_{|x|\leqslant R}|z_n-\tilde{z}|^2\right)^{1/2}\left(\int_0^T\int_{|x|\leqslant R}v^2\right)^{1/2}\leqslant C\left(\int_0^T\int_{|x|\leqslant R}|z_n-\tilde{z}|^2\right)^{1/2}\leqslant\frac{\varepsilon}{8}\end{aligned}$$

for n large enough, which concludes the first case.

Second case. Suppose that $T^*\leqslant T$ and let us show that it implies a contradiction. Indeed, there exists $T'<T^*$ such that $\|z(T')\|_{H^{\frac{3}{4}}}\geqslant 2K$ (where K is the same constant as in the hypothesis of the lemma). But we can apply the first case with T replaced by T', so that $z_n(T')\rightharpoonup z(T')$ in H^1, and since $\|z_n(T')\|_{H^1}\leqslant K$, we obtain by weak convergence $\|z(T')\|_{H^{\frac{3}{4}}}\leqslant\|z(T')\|_{H^1}\leqslant K$, and so the desired contradiction and the end of the proof of the lemma. □

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