Geometric ergodicity for families of homogeneous Markov chains

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Abstract

In this paper we find nonasymptotic exponential upper bounds for the deviation in the ergodic theorem for families of homogeneous Markov processes. We find some sufficient conditions for geometric ergodicity uniformly over a parametric family. We apply this property to the nonasymptotic nonparametric estimation problem for ergodic diffusion processes.

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1 Introduction

In this paper we consider a family of homogeneous Markov chains

$$(\Phi^{\vartheta})_{\vartheta\in\Theta}\,, \tag{1.1}$$

where Θ is a parametric set for this family and for each $\vartheta \in \Theta$ the sequence $\Phi^{\vartheta} = (\Phi^{\vartheta}_n)_{n\ge 0}$ is a homogeneous Markov chain defined on some measurable space $(\mathcal{X}\,,\,\mathcal{B}(\mathcal{X}))$ with a transition probability $\mathbf{P}^{\vartheta}$.

Our main goal is to study the geometric ergodicity property for this family uniformly over the parameter $\vartheta \in \Theta$.

The geometric ergodicity property is studied in a number of papers (see, for example, [12]-[14]) for the case of single Markov chain. An important contribution is given by Meyn and Tweedie. The principal Meyn–Tweedie result concerning the geometric ergodicity is the following one (see, for example, [12]).

Let $(\Phi_n)_{n\ge 1}$ be an ergodic homogeneous Markov chain on the space $(\mathcal{X},\mathcal{B}(\mathcal{X}))$ with an invariant measure π. If there exists $\mathbb{R} \to [1,\infty[$ function $V(x)$ for which the chain $(\Phi_n)_{n\ge 1}$ satisfies the drift condition, then there exist some constants $R > 0$ and $\kappa > 0$ such that, for any $n \ge 1$,

$$\sup_{x\in\mathcal{X}}\sup_{1\le g\le V}\frac{1}{V(x)}|\mathbf{E}_x\,g(\Phi_n)-\pi(g)| \le R\,e^{-\kappa n}\,. \tag{1.2}$$

This property is called *geometric ergodicity*. It is useful in applied problems related to identification of stochastic systems, described by stochastic processes with dependent values, in particular, governed by stochastic difference or stochastic differential equations. As we will see later (see, Definition 5.1 below) the function V, providing the drift condition, is given by the Lyapunov functions (see [9]) in the case of diffusion processes and (see [4], [7], [8]) for Markov chains. For this reason, in the sequel, we will call such functions by *Lyapunov functions*.

Necessity of the uniform geometric ergodicity appears in statistics, when one studies nonasymptotic risk and has to evaluate the maximum of expected losses over the family of distributions related to a statistical experiment. In particular, in this paper we will apply the geometric ergodicity property to the nonasymptotic nonparametric estimation problem (see [1]) for the ergodic diffusion process governed by the stochastic differential equation

$$\mathrm{d}y_t = S(y_t)\,\mathrm{d}t + \sigma(y_t)\mathrm{d}W_t\,, \quad 0\le t\le T\,, \tag{1.3}$$

where S is a unknown function from some functional class, $\sigma(y)$ is supposed to be known. To construct an optimal estimator of S we need to apply the geometric ergodicity property to the homogeneous Markov process uniformly over S from some functional class (see (2.4) below). Note that the function S is the family parameter ϑ, in this case.

In order to explain the novelty of the introduced in the paper method, we give the scheme of proving the property (1.2) in the case of a single chain. The first step consists in passing to splitting chains, which yields a chain with an atom. Then, one makes use of the *Regenerative Decomposition* for splitting chains to evaluate the convergence rate (see [6], [12]). Let us remember that the principal term in this decomposition gives a deviation in the renewal theorem, which may be evaluated thanks to the Kendall renewal theorem that provides a geometric convergence rate.

Unfortunately, in the case of a family of Markov chains this method is inapplicable because the Kendall renewal theorem does not provide the explicit constant in the upper bound. The Kendall theorem claims only the existence of some finite constant that depends, in our case, on the parameter ϑ and there is no method to make obvious this dependence on the family parameter. As consequence, the constant R will depend on the parameter ϑ as well.

Therefore, the problem is to find a nonasymptotic exponential upper bound of type (1.2) with explicit constants involved in.

To that end in the paper we apply the coupling approach (see [10]) instead of the Kendall theorem. Note that in their book Meyn and Tweedie apply this approach for obtaining a polynomial convergence rate. We obtain an exponential convergence rate thanks to making use of Lyapunov functions for coupling renewal process which turn out to be independent of the parameter (see Theorem 3.1 in Section 3).

This upper bound enables us to find the explicit nonasymptotic exponential upper bound in the ergodic theorem for which we can find the supremum over all Markov processes family in (1.1).

In this paper we find some sufficient conditions which provide the geometric ergodicity for the homogeneous Markov chain family uniformly over this family. We check these conditions for the diffusion model (1.3).

The paper is organized as follows. In the next Section the main results are formulated. In Section 3 we give the coupling renewal methods. In Section 4 the geometric ergodicity is proved for a family of homogeneous Markov chains. In Section 5 we apply this property to nonasymptotic nonparametric estimation in stochastic differential equations. In the Appendix some basic results are given on homogeneous Markov chains.

2 Main results

Assume that the transition probability family $(\mathbf{P}^{\vartheta})_{\vartheta\in\Theta}$ satisfies the properties

$\mathbf{H}_1$) *There exist* $\mathcal{X} \to [1,\infty)$ *function* V, *some constants* ρ, D, $0 < \rho < 1$,

$D > 0$, *and a set* C *from* $\mathcal{B}(\mathcal{X})$ *such that*

$$V^* = \sup_{x \in C} V(x) \, < \infty$$

and, for any $x \in \mathcal{X}$,

$$\sup_{\vartheta \in \Theta} \mathbf{E}_x^{\vartheta} \left(V(\Phi_1) \right) \, \leq \, (1-\rho) V(x) \, + \, D \mathbf{1}_C(x) \,. \tag{2.1}$$

$\mathbf{H}_2$) *There exist* $\delta, 0 < \delta < 1/2$, *some set* $C \in \mathcal{B}(\mathcal{X})$ *and some probability measure* ν *on* $\mathcal{B}(\mathcal{X})$ *with* $\nu(C) = 1$ *such that, for any* $x \in \mathcal{X}$ *and any* $A \in \mathcal{B}(\mathcal{X})$,

$$\inf_{\vartheta \in \Theta} \, \mathbf{P}^{\vartheta}(x, A) \, \geq \, 2 \, \delta \, \mathbf{1}_C(x) \, \nu(A) \,. \tag{2.2}$$

Here $\mathbf{E}_x^{\vartheta}$ means the expectation with respect to the transition probability $\mathbf{P}^{\vartheta}(x, \cdot)$.

Remark 2.1. *Condition* $\mathbf{H}_1$) *is called the uniform drift condition and that of* $\mathbf{H}_2$) *is the uniform minorization condition.*

Theorem 2.1. *Assume that the family* (1.1) *satisfies the conditions* $\mathbf{H}_1$)–$\mathbf{H}_2$) *with same set* $C \in \mathcal{B}(\mathcal{X})$.

Then, for each $\theta \in \Theta$, *the chain* Φ^{ϑ} *admits an invariant distribution* π^{ϑ} *on* $\mathcal{B}(\mathcal{X})$. *Moreover, there exists* $\kappa > 0$ *such that*

$$\sup_{n \geq 0} e^{\kappa n} \sup_{\vartheta \in \Theta} \sup_{x \in \mathcal{X}} \sup_{1 \leq f \leq V} \frac{1}{V(x)} \left| \mathbf{E}_x^{\vartheta} f(\Phi_n) - \int_{\mathcal{X}} f(z) \, \pi^{\vartheta}(\mathrm{d}z) \right| \, < \, \infty \,. \tag{2.3}$$

Apply now to the process (1.3). We assume that the function S belongs to the functional class $\Sigma_{M,a,L}$ introduced in [2], i.e.

$$\Sigma_{M,a,L} = \{ S \in \mathcal{B}_{M,a} \, : \quad \inf_{|x| \geq a} \dot{S}(x) \geq -L \,, \quad \sup_{|x| \geq a} \dot{S}(x) \leq -1/L \}$$

with

$$\mathcal{B}_{M,a} = \{ S \in \mathbf{C}^1(\mathbb{R}) \, : \, \sup_{|x| \leq a} |S(x)| \leq M \} \,,$$

where $M > 0$, $L \geq 1$ and $a > 0$ are some fixed real numbers.

As concerning the diffusion coefficient, we suppose that $\sigma(x)$ is twice continuously differentiable and $0 < \sigma_0 \leq \sigma(x) \leq \sigma_1 < \infty$.

Note that (see, for example, [3]), for any function S from $\Sigma_{M,a,L}$, the equation (1.3) admits a unique strong solution, which is an ergodic process with an invariant density π_S having the following form

$$\pi_S(x) \, = \, \pi_{S,\sigma}(x) \, = \, \frac{(1/\sigma^2(x)) \exp\{2 \int_0^x (S(v)/\sigma^2(v)) \mathrm{d}v\}}{\int_{-\infty}^{+\infty} (1/\sigma^2(z)) \exp\{2 \int_0^z (S(v)/\sigma^2(v)) \mathrm{d}v\} \mathrm{d}z} \,.$$

Theorem 2.2. *For any $\epsilon > 0$, there exist constants $R = R(\epsilon) > 0$ and $\kappa = \kappa(\epsilon) > 0$ such that*

$$\sup_{t\geq 0} e^{\kappa t} \sup_{\|g\|_* \leq 1} \sup_{x\in\mathbb{R}} \sup_{S\in\Sigma_{M,a,L}} \frac{|\mathbf{E}_S \left(g(y_t)|y_0 = x\right) - \pi_S(g)|}{(1+x^2)^\epsilon} \leq R\,, \tag{2.4}$$

where

$$\|g\|_* = \sup_{x\in\mathbb{R}} |g(x)|\,.$$

Remark 2.2. *Note that the property* (2.4) *is called geometric ergodicity. As is shown in Section 5 the function* $(1+x^2)^\epsilon$ *is the Lyapunov function.*

3 Coupling Renewal Methods

In this Section we will obtain a nonasymptotic upper bound with explicit constants in the renewal theorem by making use of the coupling method. The used here notions can be found in ([5], [7], [10]).

Let $(Y_j)_{j\geq 0}$ and $(Y'_j)_{j\geq 0}$ be two independent sequences of random variables taking values in $\mathbb{N}$. Assume that the initial random variables Y_0 and Y'_0 have distributions $a = (a(k))_{k\geq 0}$ and $b = (b(k))_{k\geq 0}$, respectively, i.e. for any $k \geq 0$,

$$\mathbf{P}(Y_0 = k) = a(k) \quad \text{and} \quad \mathbf{P}(Y'_0 = k) = b(k)\,.$$

The sequences $(Y_j)_{j\geq 1}$ and $(Y'_j)_{j\geq 1}$ are supposed to be the i.i.d. sequences with the same distribution $p = (p(k))_{k\geq 0}$, i.e. for any $k \geq 0$,

$$\mathbf{P}(Y_1 = k) = \mathbf{P}(Y'_1 = k) = p(k)\,.$$

We assume also that $a(0) > 0$ and $p(0) = \mathbf{P}(Y_1 = 0) = 0$, i.e. the sequences $(Y_j)_{j\geq 1}$ and $(Y'_j)_{j\geq 1}$ take values in $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Moreover, we suppose that the distributions a, b and p satisfy the following conditions

$\mathbf{C}_1$) *For any $k \geq 1$,*

$$p(k) > 0\,.$$

$\mathbf{C}_2$) *There exists a real number $r > 0$ such that*

$$\ln\left(\mathbf{E}\, e^{rY_0} + \mathbf{E}\, e^{rY'_0} + \mathbf{E}\, e^{rY_1}\right) \leq \upsilon^*(r) < \infty\,. \tag{3.1}$$

For any $n \geq 0$, we define the following stopping times

$$t_n = \inf\{k \geq 0 \,:\, \sum_{i=0}^{k} Y_i > n\}\,,$$

$$t'_n = \inf\{k \geq 0 \,:\, \sum_{i=0}^{k} Y'_i > n\}$$

and we set

$$W_n = \sum_{j=0}^{t_n} Y_j - n \quad \text{and} \quad W'_n = \sum_{j=0}^{t'_n} Y'_j - n\,. \tag{3.2}$$

Further we introduce the sequence of stopping times $(\sigma_k)_{k\geq 0}$:

$$\sigma_k = \inf\{l \geq \sigma_{k-1} + 1 \,:\, W_l = 1\}\,, \sigma_0 = 0\,. \tag{3.3}$$

Proposition 3.1. *Assume that the conditions* $\mathbf{C}_1) - \mathbf{C}_2)$ *hold. Then*

$$\mathbf{E}\, e^{r\sigma_1} \leq 3\, e^{\upsilon^*(r)}\,.$$

Proof. First of all, note that

$$\mathbf{P}(\sigma_1 = l | W_0 = 1) = p(l)$$

and, for $k \geq 2$,

$$\mathbf{P}(\sigma_1 = l | W_0 = k) = \mathbf{1}_{\{l=k-1\}}\,.$$

Furthermore,

$$\mathbf{P}(W_0 = k) = a(0)p(k) + a(k)\,.$$

This implies that, for any $l \geq 1$,

$$\mathbf{P}(\sigma_1 = l) = a(0)p(l)p(1) + a(1)p(l) + a(0)p(l+1) + a(l+1)\,. \tag{3.4}$$

Thus we calculate

$$\begin{aligned}\mathbf{E}\, e^{r\sigma_1} &= \mathbf{P}(W_0 = 1)\, \mathbf{E}\, e^{rY_1} + \sum_{k=2}^{\infty} \mathbf{E}\, e^{r(k-1)}\, \mathbf{P}(W_0 = k) \\ &\leq \mathbf{E}\, e^{rY_1} + a(0)\mathbf{E}\, e^{rY_1} + \mathbf{E}\, e^{rY_0}\,.\end{aligned}$$

This implies the desired inequality, due to the condition $\mathbf{C}_2$).

□

Now, we introduce the embedded Markov chain $(C_k)_{k\geq 0}$ by

$$C_k = W'_{\sigma_k} \tag{3.5}$$

and the corresponding entrance time

$$\varpi = \inf\{k \geq 1 \,:\, Z_k = 1\}\,. \tag{3.6}$$

In order to study the property of this stopping time, we need the following notations

$$r_1 = \frac{r^2}{2\upsilon^*(r)}\,, \quad \rho^* = \frac{1-e^{-r_1}}{2}\,,$$

$$A^*(r) \quad = \frac{3e^{\upsilon^*(r)+r/2}}{1-e^{-r/2}} \qquad \text{and} \qquad l_* = \left[\frac{2}{r}\ln\left(\frac{2A^*(r)}{1-e^{-r_1}}\right)\right] + 1\,. \tag{3.7}$$

Moreover, we set

$$A_1^*(r) = \frac{\sqrt{1-\rho^*} + (1+A^*(r))e^{r_1 l_*}}{1-\sqrt{1-\rho^*}}\,. \tag{3.8}$$

Proposition 3.2. *Assume that the conditions* $\mathbf{C}_1$) *and* $\mathbf{C}_2$) *hold. Then*

$$\mathbf{E}\, e^{\gamma_1 \varpi} \le A_2^*(r) e^{\upsilon^*(r)}\,, \tag{3.9}$$

where $\gamma_1 = \min(\gamma_*, \iota_*)$, $\gamma^* = -\dfrac{1}{2}\ln(1-\rho^*)$,

$$\iota_* = -\frac{r\ln(1-\varsigma_*)}{2(\ln A^*(r) + r_1 l_*)}\,, \quad \varsigma_* = a(0)p^2(1)p_{min}(l_*)$$

and

$$A_2^*(r) = A_1^*(r)\left(1 + \frac{A_1^*(r)e^{r_1 l_*}}{1-(1-\varsigma_*)^{1/4}}\right)\,.$$

Proof. First of all, note that the sequence $(W_n)_{n\ge0}$ is a homogeneous Markov chain taking values in $\mathbb{N}^*$, i.e. for any n, k and l from $\mathbb{N}^*$

$$\mathbf{P}(W_n = k | W_{n-1} = l) = p(k)\mathbf{1}_{\{l=1\}} + \mathbf{1}_{\{k=l-1\}}\mathbf{1}_{\{l\ge2\}}\,.$$

Hence, the sequence σ_k can be represented as

$$\sigma_k = \sigma_1 + \sum_{j=2}^{k} \xi_j\,, \tag{3.10}$$

where $(\xi_j)_{j\ge2}$ are i.i.d. random variables such that, for any $l \in \mathbb{N}^*$,

$$\mathbf{P}(\xi_2 = l) = \mathbf{P}(\sigma_1 = l | W_0 = 1) = p(l)\,.$$

Therefore, the process (Z_k) defined in (3.5) is a homogeneous Markov chain.

For any positive function V, one can calculate directly

$$\mathbf{E}\,[V(Z_1)|Z_0 = l] = \mathbf{E}\,V(l - \sigma_1)\,\mathbf{1}_{\{\sigma_1 < l\}} + \mathbf{E}\,V(\eta_1 - \sigma_1 + l)\,\mathbf{1}_{\{\sigma_1 \ge l\}}\,,$$

where

$$\eta_1 = \sum_{j=t'_{\sigma_0}+1}^{t'_{\sigma_1}} Y'_j\,.$$

Choosing now $V(x) = e^{r_1 x}$ with r_1 defined in (3.7) yields

$$\frac{\mathbf{E}\,[V(Z_1)|Z_0 = l]}{V(l)} = \mathbf{E}\,e^{-r_1\sigma_1}\,\mathbf{1}_{\{\sigma_1 < l\}} + \mathbf{E}\,e^{r_1(\eta_1 - \sigma_1)}\,\mathbf{1}_{\{\sigma_1 \ge l\}}\,.$$

Taking into account that $t'_n \le n + 1$, we estimate the exponential moment for η_1 as

$$\mathbf{E}\,e^{r_1\eta_1}\,\mathbf{1}_{\{\sigma_1 \ge l\}} \le \sum_{n=l}^{\infty} \left(\mathbf{E}\,e^{r_1 Y_1}\right)^{n+1}\,\mathbf{P}(\sigma_1 = n)\,.$$

By the Hölder inequality (with $p = r/r_1$), one has

$$\mathbf{E}\,e^{r_1 Y_1} \le e^{r/2}\,.$$

Thus

$$\mathbf{E}\,e^{r_1\eta_1}\,\mathbf{1}_{\{\sigma_1 \ge l\}} \le \sum_{n=l}^{\infty} e^{r(n+1)/2}\mathbf{P}(\sigma_1 = n)\,.$$

By making use of the upper bound from Proposition 3.1, we find

$$\mathbf{P}(\sigma_1 = n) \le 3e^{v^*(r)}e^{-rn}\,.$$

Therefore, for any $l \ge 1$,

$$\mathbf{E}\,e^{r_1\eta_1}\,\mathbf{1}_{\{\sigma_1 \ge l\}} \le A^*(r)\,e^{-rl/2}\,,$$

where $A^*(r)$ is defined in (3.7). Moreover, taking into account the definition of l_* in (3.7), we obtain

$$\sup_{l \ge l_*} \frac{\mathbf{E}\,[V(Z_1)|Z_0 = l]}{V(l)} \le e^{-r_1} + \frac{1 - e^{-r_1}}{2} = 1 - \rho^* < 1\,,$$

i.e. the chain $(Z_k)_{k\ge1}$ satisfies the condition (A.2) in the Appendix with

$$C = \{1, \ldots, l_*\} \quad \text{and} \quad D = (1 + A^*(r))e^{r_1 l_*}\,.$$

Therefore, by using Proposition A.2 with γ^* defined in (3.7), one gets

$$\sup_{l\ge1}\frac{U_C(l,\gamma^*,V)}{V(l)}\le A_1^*(r)\,,$$

where $A_1^*(r)$ is given in (3.8). Moreover, note that, for any $l\ge1$,

$$\begin{aligned}\mathbf{P}(Z_1=1|Z_0=l)&=\mathbf{P}(\sigma_1=l-1)+\sum_{j=l}^{\infty}\mathbf{P}(\sigma_1=j)\mathbf{P}(\eta_1=1+j-l)\\&\ge\mathbf{P}(\sigma_1=l)p(1)\,.\end{aligned}$$

Therefore, from (3.4) we estimate from below as

$$\mathbf{P}(Z_1=1|Z_0=l)\ge a(0)p(l)p^2(1)\,,$$

i.e.

$$\min_{1\le l\le l_*}\mathbf{P}(Z_1=1|Z_0=l)\ge a(0)p^2(1)p_{min}(l_*):=\varsigma_*\,,$$

where $p_{min}(k)=\min_{1\le l\le k}p(l)$. Now the condition $\mathbf{C}_1$) implies the inequality (A.5) for the set $B=\{1\}$. Then Proposition A.3 implies directly the inequality (3.9). □

Let us define the renewal sequence $(u(n))_{n\ge0}$ as follows

$$u(n)=\sum_{j=0}^{\infty}p^{*j}(n)\,,\tag{3.11}$$

where p^{*j} denotes the jth convolution power. We remind that for $j=0$, we set $p^0(n)=1$ for $n=0$ and $p^0(n)=0$ for $n\ge1$.

Proposition 3.3. *Assume that the conditions $\mathbf{C}_1$) and $\mathbf{C}_2$) hold. Then, for any $n\ge2$,*

$$|a*u(n)-b*u(n)|\le M^*\,e^{-\kappa n}\,,\tag{3.12}$$

where $a*u(n)=\sum_{j=0}^{n}a(j)u(n-j)$,

$$M^*(r)=\frac{\sqrt{3A_2^*(r)e^{\upsilon^*(r)}}e^{\gamma_1/4}}{e^{\gamma_1/4}-1}\quad\text{and}\quad\kappa=\frac{\gamma_1 r}{2\upsilon^*(r)}\,.$$

Proof. Obviously, that for $n\ge1$,

$$a*u(n)=\mathbf{P}\left(\cup_{j=0}\left\{\sum_{i=0}^{j}Y_i=n\right\}\right)=\mathbf{P}(W_{n-1}=1)$$

and

$$b * u(n) = \mathbf{P}\left(\cup_{j=0}\left\{\sum_{i=0}^{j} Y'_i = n\right\}\right) = \mathbf{P}(W'_{n-1} = 1)\,.$$

Thus,

$$\Delta(n) = a * u(n) - b * u(n) = \mathbf{P}(W_{n-1} = 1\,,\, W'_{n-1} \geq 2) - \mathbf{P}(W'_{n-1} = 1\,,\, W_{n-1} \geq 2)\,.$$

Now, we introduce the "coupling" stopping time τ as

$$\tau = \inf\{k \geq 1\,:\, (W_k, W'_k) = (1,1)\}\,.$$

Note that, for any $n \geq 2$, by strong Markov property, one has

$$\begin{aligned}\mathbf{P}(W_n = 1\,,\, W'_n \geq 2, \tau \leq n-1) &= \sum_{k=1}^{n-1} \mathbf{P}(W_n = 1\,,\, W'_n \geq 2, \tau = k)\\ &= \sum_{k=1}^{n-1} \mathbf{P}(\tau = k)\mathbf{P}^2(W_{n-k} = 1 \,|\, W_0 = 1)\,.\end{aligned}$$

Similarly, one gets

$$\mathbf{P}(W'_n = 1\,,\, W_n \geq 2, \tau \leq n-1) = \sum_{k=1}^{n-1} \mathbf{P}(\tau = k)\mathbf{P}^2(W_{n-k} = 1 \,|\, W_0 = 1)\,.$$

Therefore, putting

$$\alpha_1(n) = \mathbf{P}(W_n = 1\,,\, W'_n \geq 2\,, \tau \geq n)$$

and

$$\alpha_2(n) = \mathbf{P}(W'_n = 1\,,\, W_n \geq 2\,, \tau \geq n)$$

yields, for any $n \geq 2$,

$$\begin{aligned}|\Delta(n)| &= |\alpha_1(n-1) - \alpha_2(n-1)|\\ &\leq \max\left(\alpha_1(n-1)\,,\, \alpha_2(n-1)\right) \leq \mathbf{P}(\tau > n)\,.\end{aligned}$$

Taking into account that $\tau \leq \sigma_{\varpi}$ a.s., we obtain

$$|\Delta(n)| \leq \mathbf{P}(\sigma_{\varpi} > n) \leq e^{-\kappa n}\, \mathbf{E} e^{\kappa \sigma_{\varpi}}\,.$$

Note now that

$$\mathbf{E} e^{\kappa\sigma_{\varpi}} = \sum_{k=1}^{\infty} \mathbf{E}\, e^{\kappa\sigma_k}\, \mathbf{1}_{\{\varpi = k\}} \leq \sum_{k=1}^{\infty} \sqrt{\mathbf{E}\, e^{2\kappa\sigma_k}}\, \sqrt{\mathbf{P}(\varpi \geq k)}\,.$$

Moreover, by Proposition 3.1 and by the Hölder inequality, one has

$$\mathbf{E}\, e^{\kappa\sigma_k} = \mathbf{E}\, e^{\kappa\sigma_1} \left(\mathbf{E}\, e^{\kappa Y_1}\right)^{k-1} \le 3\, e^{k\gamma_1/2}\,.$$

Now, the inequality (3.9) implies the upper bound (3.12). $\square$

Theorem 3.1. *Assume that the distribution of Y_1 satisfies the condition $\mathbf{C}_1$) and, for some $r > 0$,*

$$\mathbf{E}\, e^{rY_1} < \infty\,. \tag{3.13}$$

Then, for any $n \ge 2$,

$$\left| u(n) - \frac{1}{\mathbf{E}\, Y_1} \right| \le M^*(r)\, e^{-\kappa n}\,, \tag{3.14}$$

where the coefficient $M^*(r)$ is given in (3.12) *with*

$$\upsilon^*(r) = \ln\left(1 + \frac{e^r}{e^r - 1}\, \mathbf{E}\, e^{rY_1}\right)\,. \tag{3.15}$$

Proof. We choose the distribution $a(0) = 1$, i.e. $Y_0 = 0$ a.s. and

$$b(j) = \frac{1}{\mathbf{E}Y_1} \sum_{i=j+1}^{\infty} p(i)\,.$$

It is easy to see directly that, for any $j \ge 1$,

$$b * u(j) = \frac{1}{\mathbf{E}Y_1}\,.$$

Moreover,

$$\sum_{j=1}^{\infty} e^{rj}\, b(j) = \frac{1}{\mathbf{E}Y_1} \sum_{j=1}^{\infty} p(j) \sum_{i=0}^{j-1} e^{ri} \le \frac{1}{e^r - 1}\, \mathbf{E}\, e^{rY_1}\,,$$

i.e. the distributions a, b and p satisfy the condition $\mathbf{C}_2$) with $\upsilon^*(r)$ given by (3.15).
$\square$

4 Proof of Theorem 2.1

First of all note that, the condition $\mathbf{H}_1$) and Proposition A.2 with

$$r = -\frac{1}{2}\ln(1-\rho)$$

imply immediately that

$$\mathcal{D}_1(r) = \sup_{\vartheta\in\Theta} \sup_{x\in\mathcal{X}} \frac{U_C^\vartheta(x,r,V)}{V(x)} \, < \, \infty \,, \tag{4.1}$$

where

$$U_C^\vartheta(x,r,V) = \mathbf{E}_x^\vartheta \sum_{j=1}^{\tau_C} e^{rj}\, V(\Phi_j) \quad \text{and} \quad \tau_C = \inf\{n\ge 1\,:\,\Phi_n\in C\}\,.$$

Now, we introduce a splitting chain family as in [12], p. 108 (see also [13]). We set $\check{\mathcal{X}} = \mathcal{X}\times\{0,1\}$, $\mathcal{X}_0 = \mathcal{X}\times\{0\}$ and $\mathcal{X}_1 = \mathcal{X}\times\{1\}$. Let $\mathcal{B}(\mathcal{X}_i)$ be the $\sigma-$fields generated by the set $A_i = A\times\{i\}$ with $A\in\mathcal{B}(\mathcal{X})$, $i=0,1$. Further we define the $\sigma-$field $\mathcal{B}(\check{\mathcal{X}})$ as a $\sigma-$field generated by $\mathcal{B}(\mathcal{X}_0)\cup\mathcal{B}(\mathcal{X}_1)$. Moreover, for any measure λ on $\mathcal{B}(\mathcal{X})$ we relate the measure λ^* on $\mathcal{B}(\check{\mathcal{X}})$ as

$$\lambda^*(A_0) = (1-\delta)\lambda(A\cap C) + \lambda(A\cap C^c)$$

and

$$\lambda^*(A_1) = \delta\lambda(A\cap C)\,.$$

Now, for each $\vartheta\in\Theta$, we define a homogeneous Markov chain $(\check{\Phi}_n^\vartheta)_{n\ge0}$ by the following transition probabilities $\check{\mathbf{P}}^\vartheta(\check{x},\cdot) = Q^\vartheta(\check{x},\cdot)^*$ with

$$Q^\vartheta(\check{x},\cdot) = \begin{cases} \mathbf{P}^\vartheta(x,\cdot)\,, & \text{if} \quad \check{x}\in\mathcal{X}_0\setminus C_0\,;\\ \dfrac{\mathbf{P}^\vartheta(x,\cdot)-\delta\nu(\cdot)}{1-\delta}\,, & \text{if} \quad \check{x}\in C_0\,;\\ \nu(\cdot)\,, & \text{if} \quad \check{x}\in\mathcal{X}_1\,. \end{cases} \tag{4.2}$$

Obviously, that the set $\alpha = C_1$ is an accessible atom for the chain $(\check{\Phi}_n^\vartheta)_{n\ge1}$, i.e. for any positive $\check{\mathcal{X}}\to\mathbb{R}$ function g

$$\check{\mathbf{E}}_{\check{x}}^\vartheta\, g(\check{\Phi}_1^\vartheta) = \check{\mathbf{E}}_{\check{y}}^\vartheta\, g(\check{\Phi}_1^\vartheta)\,, \quad \text{for any} \quad \check{x}\,,\,\check{y}\in\alpha\,.$$

Moreover, the chain $(\check{\Phi}_n^\vartheta)_{n\ge1}$ is ν^*-irreducible, thus $\psi-$irreducible. Indeed, let $\nu^*\check{\Gamma}>0$ for some set $\check{\Gamma}$. Then one can see directly that, for any $\check{x}\in\mathcal{X}_1\cup C_0$,

$$\check{\mathbf{P}}(\check{x},\check{\Gamma}) > 0\,.$$

Moreover, for $\check{x}\in\mathcal{X}_0\setminus C_0$,

$$\check{\mathbf{P}}^2(x_0,\check{\Gamma}) = \int_{\check{\mathcal{X}}} \check{\mathbf{P}}(\check{x},\mathrm{d}\check{z})\,\check{\mathbf{P}}(\check{z},\check{\Gamma}) \ge \,\mathbf{P}^*(x,\mathcal{X}_1)\,\check{\mathbf{P}}(\check{z},\check{\Gamma}) \ge \delta\nu^*(\check{\Gamma})\,.$$

This implies directly that, for any nonnegative random variable ξ measurable with respect to the $\sigma-$field generated by the chain $(\check{\Phi}^{\vartheta}_n)_{n\ge 1}$, one has

$$\check{\mathbf{E}}^{\vartheta}_{\check{x}}\,\xi = \check{\mathbf{E}}^{\vartheta}_{\check{y}}\,\xi \quad \text{for any} \quad \check{x}\,,\,\check{y}\in\alpha\,.$$

In the sequel denote by $\check{\mathbf{E}}^{\vartheta}_{\alpha}(\cdot)$ the such expectations.

For any set $\check{C}$ from $\mathcal{B}(\check{\mathcal{X}})$, we denote

$$\check{\tau}_{\check{C}} = \inf\left\{n\ge 1\,:\,\check{\Phi}^{\vartheta}_n\in\check{C}\right\}\,. \tag{4.3}$$

Now, we define the $\check{\mathcal{X}}\to[1,\infty)$ function $\check{V}$ as

$$\check{V}(\check{x}) = V(<\check{x}>_1)\,, \tag{4.4}$$

where $<\check{x}>_1$ is the first component of $\check{x}\in\check{\mathcal{X}}$ which belongs to $\mathcal{X}$. We set

$$\check{U}^{\vartheta}_{\check{C}}(\check{x},r,\check{V}) = \check{\mathbf{E}}^{\vartheta}_{\check{x}}\sum_{j=1}^{\check{\tau}_{\check{C}}} e^{rj}\,\check{V}(\check{\Phi}^{\vartheta}_j)\,. \tag{4.5}$$

By Proposition A.6 we obtain that, for any $x\in\mathcal{X}$,

$$\begin{aligned}U^{\vartheta}_C(x,r,V) &= (1-\delta)\check{U}^{\vartheta}_{C_0\cup C_1}(x_0,r,\check{V})\,\mathbf{1}_{\{x\in C\}} + \check{U}^{\vartheta}_{C_0\cup C_1}(x_0,r,\check{V})\,\mathbf{1}_{\{x\in C^c\}}\\ &\quad+\delta\check{U}^{\vartheta}_{C_0\cup C_1}(x_1,r,\check{V})\,\mathbf{1}_{\{x\in C\}}\,,\end{aligned}$$

where $x_i=(x,i)$ for $i=0,1$. Taking into account that, for any x, y from $\check{\mathcal{X}}$ and any set $\check{C}$ from $\mathcal{B}(\check{\mathcal{X}})$,

$$\check{U}^{\vartheta}_{\check{C}}(x_1,r,\check{V}) = \check{U}^{\vartheta}_{\check{C}}(y_1,r,\check{V})\,,$$

we obtain from (4.1)

$$\sup_{\vartheta\in\Theta}\sup_{\check{x}\in\check{\mathcal{X}}}\frac{\check{U}^{\vartheta}_{C_0\cup C_1}(\check{x},r,\check{V})}{\check{V}(\check{x})}\le\check{\mathcal{D}}(r)\,<\,\infty\,,$$

where

$$\check{\mathcal{D}}(r) = \frac{1}{\delta(1-\delta)}\,V^*\mathcal{D}(r)\,.$$

Note now that, for $\check{x}\in C_0$ by the definition (4.2) and the condition $\mathbf{H}_2$),

$$\check{\mathbf{P}}^{\vartheta}(\check{x},\alpha) = \check{\mathbf{P}}^{\vartheta}(\check{x},C_1) = \delta\frac{\mathbf{P}^{\vartheta}(x,C)-\delta\nu(C)}{1-\delta}\ge\frac{\delta^2}{1-\delta}\,.$$

Similarly, for $\check{x}\in C_1$,

$$\check{\mathbf{P}}^{\vartheta}(\check{x},\alpha) = \check{\mathbf{P}}^{\vartheta}(\check{x},C_1) = \delta\nu(C) = \delta\,.$$

Hence, taking into account that $0 < \delta < 1/2$, one gets

$$\inf_{\check{x} \in C_0 \cup C_1} \inf_{\vartheta \in \Theta} \check{\mathbf{P}}^{\vartheta}(\check{x}, \alpha) \geq \frac{\delta^2}{1-\delta} \,. \tag{4.6}$$

Thus, by Proposition A.4, one obtains that

$$\mathcal{D}_2(r) = \sup_{\vartheta \in \Theta} \sup_{\check{x} \in \check{\mathcal{X}}} \frac{\check{U}^{\vartheta}_{\alpha}(\check{x}, r, \check{V})}{\check{V}(\check{x})} < \infty \,. \tag{4.7}$$

Therefore, by Proposition A.1, the chain $(\check{\Phi}^{\vartheta}_n)_{n \geq 0}$ is ergodic for each $\vartheta \in \Theta$ with the invariant measure given as

$$\check{\pi}^{\vartheta}(\check{\Gamma}) = \mu_{\vartheta} \, \check{\mathbf{E}}^{\vartheta}_{\alpha} \sum_{j=1}^{\check{\tau}_{\alpha}} \mathbf{1}_{\{\check{\Phi}^{\vartheta}_j \in \check{\Gamma}\}} \,, \quad \text{where} \quad \mu_{\vartheta} = \frac{1}{\check{\mathbf{E}}^{\vartheta}_{\alpha} \check{\tau}_{\alpha}} \,. \tag{4.8}$$

Now, for any $n \geq 1$, we define

$$\iota^* = \max\{1 \leq j \leq n-1 \, : \, \check{\Phi}^{\vartheta}_j \in \alpha\}$$

and we put $\iota^* = 0$ if $\check{\tau}_{\alpha} \geq n$. Moreover, note that, for any $\check{\mathcal{X}} \to \mathbb{R}$ function f and any $n \geq 2$,

$$\begin{aligned}
\check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}^{\vartheta}_n) \, \mathbf{1}_{\{\check{\tau}_{\alpha} < n\}} &= \sum_{j=1}^{n-1} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}^{\vartheta}_n) \, \mathbf{1}_{\{\check{\tau}_{\alpha} \leq j\}} \mathbf{1}_{\{\iota^* = j\}} \\
&= \sum_{j=1}^{n-1} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, \mathbf{1}_{\{\check{\tau}_{\alpha} \leq j\}} \, \check{\mathbf{E}}^{\vartheta}_{\check{x}} \left(f(\check{\Phi}^{\vartheta}_n) \, \mathbf{1}_{\{\iota^* = j\}} | \check{\Phi}^{\vartheta}_1, \ldots, \check{\Phi}^{\vartheta}_j \right) \,.
\end{aligned}$$

Note that, for $j \leq n-2$,

$$\{\iota^* = j\} = \left\{ \check{\Phi}^{\vartheta}_j \in \alpha \,, \, \check{\Phi}^{\vartheta}_{j+1} \notin \alpha \,, \ldots , \check{\Phi}^{\vartheta}_{n-1} \notin \alpha \right\}$$

and

$$\{\iota^* = n-1\} = \left\{ \check{\Phi}^{\vartheta}_{n-1} \in \alpha \right\} \,.$$

Now, taking into account that $(\check{\Phi}^{\vartheta}_n)_{n \geq 1}$ is a homogeneous Markov chain, we can calculate the last conditional expectation as follows

$$\begin{aligned}
\check{\mathbf{E}}^{\vartheta}_{\check{x}} \left(f(\check{\Phi}^{\vartheta}_n) \, \mathbf{1}_{\{\iota^* = j\}} | \check{\Phi}^{\vartheta}_1, \ldots, \check{\Phi}^{\vartheta}_j \right) &= \mathbf{1}_{\{\check{\Phi}^{\vartheta}_j \in \alpha\}} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \left(f(\check{\Phi}^{\vartheta}_n) \, \mathbf{1}_{\{\check{\Phi}^{\vartheta}_{j+1} \notin \alpha, \ldots, \check{\Phi}^{\vartheta}_{n-1} \notin \alpha\}} | \check{\Phi}^{\vartheta}_j \right) \\
&= \mathbf{1}_{\{\check{\Phi}^{\vartheta}_j \in \alpha\}} \check{\mathbf{E}}^{\vartheta}_{\alpha} \left(f(\check{\Phi}^{\vartheta}_{n-j}) \, \mathbf{1}_{\{\check{\Phi}^{\vartheta}_1 \notin \alpha, \ldots, \check{\Phi}^{\vartheta}_{n-j-1} \notin \alpha\}} \right) \\
&= \mathbf{1}_{\{\check{\Phi}^{\vartheta}_j \in \alpha\}} t_{f,\vartheta}(n-j) \,,
\end{aligned}$$

where, for $k \geq 1$,

$$t_{f,\vartheta}(k) = \check{\mathbf{E}}_{\alpha}^{\vartheta}\, f(\check{\Phi}_k^{\vartheta})\, \mathbf{1}_{\{\check{\tau}_\alpha \geq k\}}\,. \tag{4.9}$$

By convention, we set $t_{f,\vartheta}(0) = 0$. Therefore,

$$\check{\mathbf{E}}_{\check{x}}^{\vartheta}\, f(\check{\Phi}_n^{\vartheta})\, \mathbf{1}_{\{\check{\tau}_\alpha < n\}} = \sum_{j=1}^{n} \check{\mathbf{P}}_{\check{x}}^{\vartheta}\left(\check{\tau}_\alpha \leq j\right)\, t_{f,\vartheta}(n-j) = \upsilon_{\check{x},\vartheta} * t_{f,\vartheta}(n)\,,$$

where $\upsilon_{\check{x},\vartheta}(0) = 0$ and, for $j \geq 1$,

$$\upsilon_{\check{x},\vartheta}(j) = \check{\mathbf{P}}_{\check{x}}^{\vartheta}\left(\check{\tau}_\alpha \leq j\right) = \check{\mathbf{P}}_{\check{x}}^{\vartheta}\left(\check{\Phi}_j^{\vartheta} \in \alpha\right)\,.$$

Moreover, for $j \geq 1$,

$$\begin{aligned}\upsilon_{\check{x},\vartheta}(j) &= \sum_{l=1}^{j} \check{\mathbf{P}}_{\check{x}}^{\vartheta}\left(\check{\tau}_\alpha = l\,,\, \check{\Phi}_j^{\vartheta} \in \alpha\right)\\ &= \sum_{l=1}^{j} \gamma_{\check{x},\vartheta}(l)\, u_{\vartheta}(l-j) = \gamma_{\check{x},\vartheta} * u_{\vartheta}(j)\,,\end{aligned}$$

where

$$\gamma_{\check{x},\vartheta}(l) = \check{\mathbf{P}}_{\check{x}}^{\vartheta}\left(\check{\tau}_\alpha = l\right) \quad \text{and} \quad u_{\vartheta}(l) = \check{\mathbf{P}}_{\alpha}^{\vartheta}\left(\check{\Phi}_l^{\vartheta} \in \alpha\right)\,. \tag{4.10}$$

It is clear that $\gamma_{\check{x},\vartheta}(0) = 0$, i.e. $\gamma_{\check{x},\vartheta} * u_{\vartheta}(0) = 0$. This implies that

$$\upsilon_{\check{x},\vartheta}(j) = \gamma_{\check{x},\vartheta} * u_{\vartheta}(j)\,,$$

for all $j \geq 0$. Finally, for any $n \geq 2$,

$$\check{\mathbf{E}}_{\check{x}}^{\vartheta}\, f(\check{\Phi}_n^{\vartheta})\, \mathbf{1}_{\{\check{\tau}_\alpha < n\}} = \gamma_{\check{x},\vartheta} * u_{\vartheta} * t_{f,\vartheta}(n)\,. \tag{4.11}$$

Note that the sequence $(u_{\vartheta}(n))_{n\geq 0}$ is a renewal sequence, i.e.

$$u_{\vartheta}(n) = \sum_{j=0}^{\infty} p_{\vartheta}^{*j}(n)\,,$$

where $p_{\vartheta}(k) = \check{\mathbf{P}}_{\alpha}^{\vartheta}(\check{\tau}_\alpha = k)$. Now, we set

$$\Delta_{\vartheta}(n) = \left|u_{\vartheta}(n) - \mu_{\vartheta}\right|\,.$$

We estimate this term by Theorem 3.1. First we have to check the condition $\mathbf{C}_1$) uniformly over the parameter $\vartheta \in \Theta$, i.e. to show that, for any $k \geq 1$,

$$\inf_{\vartheta \in \Theta}\, \min_{1 \leq j \leq k} \check{\mathbf{P}}_{\alpha}^{\vartheta}(\check{\tau}_\alpha = j) > 0\,. \tag{4.12}$$

Let us check this property for $k = 1$. We remind that, by the condition $\mathbf{H}_2$), one has $\nu(C) = 1$. Thus, the definition (4.2) implies

$$\check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_\alpha = 1) = \nu^*(C_1) = \delta > 0\,.$$

Moreover, for $\check{z} \in C_0$, i.e. $\check{z} = (z, 0)$ with $z \in C$, one has

$$\check{\mathbf{P}}^{\vartheta}(\check{z}, C_0) = \mathbf{P}^{\vartheta}(z, C) - \delta\nu(C) \geq \delta\nu(C) = \delta\,.$$

By induction, one can show that, for any $j \geq 1$,

$$\check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \in C_0\,, \ldots, \check{\Phi}_j \in C_0\right) \geq (1-\delta)\,\delta^{j-1}\,.$$

Therefore, taking into account (4.6) yields, for $j \geq 2$,

$$\begin{aligned}\check{\mathbf{P}}^{\vartheta}_{\alpha}(\check{\tau}_\alpha = j) &= \check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \notin \alpha\,, \ldots, \check{\Phi}_{j-1} \notin \alpha\,, \check{\Phi}_j \in \alpha\right)\\ &\geq \check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \in C_0\,, \ldots, \check{\Phi}_{j-1} \in C_0\,, \check{\Phi}_j \in C_1\right)\\ &\geq \frac{\delta\,\check{\mathbf{P}}^{\vartheta}_{\alpha}\left(\check{\Phi}_1 \in C_0\,, \ldots, \check{\Phi}_{j-1} \in C_0\right)}{1-\delta} \geq \delta^{j-1}\,.\end{aligned}$$

Hence, by Theorem 3.1, there exists a constant κ, $0 < \kappa \leq r/2$ such that

$$\Delta^* = \sup_{n\geq 1} e^{2\kappa n} \sup_{\vartheta\in\Theta} \Delta_\vartheta(n) \,<\, \infty\,. \tag{4.13}$$

Moreover, noting that

$$\check{\pi}^{\vartheta}(f) = \mu_\vartheta \sum_{j=0}^{+\infty} t_{f,\vartheta}(j)\,,$$

and that $\mu_\vartheta \leq 1$ one obtains, for any $n \geq 2$,

$$|\check{\mathbf{E}}^{\vartheta}_{\check{x}}\, f(\check{\Phi}^{\vartheta}_n)\,\mathbf{1}_{\{\check{\tau}_\alpha < n\}} - \check{\pi}^{\vartheta}(f)| \,\leq\, \gamma_{\check{x},\vartheta} * \Delta_\vartheta * t_{f,\vartheta}(n) + q_{\check{x},\vartheta} * t_{f,\vartheta}(n) + s_{f,\vartheta}(n)\,,$$

where

$$q_{\check{x},\vartheta}(n) = \check{\mathbf{P}}^{\vartheta}_{\check{x}}\,(\tau_\alpha > n) \quad \text{and} \quad s_{f,\vartheta}(n) = \sum_{j=n+1} t_{f,\vartheta}(j)\,.$$

Therefore, for any $n \geq 2$,

$$\begin{aligned}|\check{\mathbf{E}}^{\vartheta}_{\check{x}}\, f(\check{\Phi}^{\vartheta}_n) - \check{\pi}^{\vartheta}(f)| &\leq \gamma_{\check{x},\vartheta} * \Delta_\vartheta * t_{f,\vartheta}(n) + q_{\check{x},\vartheta} * t_{f,\vartheta}(n)\\ &\quad + \check{\mathbf{E}}^{\vartheta}_{\check{x}}\, f(\check{\Phi}^{\vartheta}_n)\,\mathbf{1}_{\{\check{\tau}_\alpha \geq n\}} + s_{f,\vartheta}(n)\,.\end{aligned} \tag{4.14}$$

Note now, that from (4.7) we obtain, for $n \geq 2$ and for any function $1 \leq f \leq \check{V}$,

$$e^{rn}\,\check{\mathbf{E}}^{\vartheta}_{\check{x}}\, f(\check{\Phi}^{\vartheta}_n)\,\mathbf{1}_{\{\check{\tau}_\alpha \geq n\}} \,\leq\, \check{U}^{\vartheta}_{\alpha}(\check{x}, r, \check{V}) \leq \mathcal{D}_2(r)\check{V}(\check{x})\,.$$

Similarly,

$$e^{rn} s_{f,\vartheta}(n) \le \check{U}^{\vartheta}_{\alpha}(\alpha, r, \check{V}) \le \mathcal{D}_2(r) V^* \le \mathcal{D}_2(r) V^* \, \check{V}(\check{x}) \,.$$

Putting

$$\varrho(\check{x}, \vartheta) = \sum_{n\ge 2} e^{\kappa n} \, |\check{\mathbf{E}}^{\vartheta}_{\check{x}} \, f(\check{\Phi}^{\vartheta}_n) \, - \check{\pi}^{\vartheta}(f)| \tag{4.15}$$

yields

$$\varrho_{\check{x}}(\kappa, \vartheta) \le (\widehat{\gamma}_{\check{x},\vartheta} \widehat{\Delta}_{\vartheta} + \widehat{q}_{\check{x},\vartheta}) \widehat{t}_{f,\vartheta} + \mathcal{D}_3(r) \, \check{V}(\check{x}) \,,$$

where

$$\widehat{\gamma}_{\check{x},\vartheta} = \sum_{n\ge 0} e^{\kappa n} \, \gamma_{\check{x},\vartheta}(n) \,, \quad \widehat{\Delta}_{\vartheta} = \sum_{n\ge 0} e^{\kappa n} \, \Delta_{\vartheta}(n) \,, \quad \widehat{q}_{\check{x},\vartheta} = \sum_{n\ge 0} e^{\kappa n} \, q_{\check{x},\vartheta}(\kappa)(n)$$

and

$$\mathcal{D}_3(r) = 2 \frac{\mathcal{D}_2(r) V^* e^{r/2}}{e^{r/2} - 1} \,.$$

Now, one obtains the following estimates

$$\widehat{\gamma}_{\check{x},\vartheta} = \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, e^{\kappa \check{\tau}_{\alpha}} \le \check{U}^{\vartheta}_{\alpha}(\check{x}, r, \check{V}) \le \mathcal{D}_2(r) \check{V}(\check{x}) \,.$$

Similarly,

$$\widehat{q}_{\check{x},\vartheta} \le 1 + \check{U}^{\vartheta}_{\alpha}(\check{x}, r, \check{V}) \le (1 + \mathcal{D}_2(r)) \check{V}(\check{x}) \,.$$

The inequality (4.13) implies

$$\widehat{\Delta}_{\vartheta} \le \frac{e^{\kappa}}{e^{\kappa} - 1} \, \Delta^* \,.$$

Thus,

$$\varrho^* = \sup_{\check{x} \in \check{\mathcal{X}}} \sup_{\vartheta \in \Theta} \frac{\varrho(\check{x}, \vartheta)}{\check{V}(\check{x})} \, < \, \infty \,.$$

Note that the chain $(\Phi_n)_{n\ge 1}$ is ergodic with the invariant measure π^{ϑ} defined in (4.8) and (A.14). By applying Proposition A.6 with λ equals to the Dirac measure of x, we obtain that, for any function $0 < f \le V$,

$$\begin{aligned} \mathbf{E}^{\vartheta}_{x} f(\Phi_n) - \pi^{\vartheta}(f) = (1 - \delta) \left(\check{\mathbf{E}}^{\vartheta}_{x_0} \, \check{f}(\check{\Phi}_n) - \check{\pi}^{\vartheta}(\check{f}) \right) \mathbf{1}_{\{x \in C\}} \\ + \delta \left(\check{\mathbf{E}}^{\vartheta}_{x_1} \, \check{f}(\check{\Phi}_n) - \check{\pi}^{\vartheta}(\check{f}) \right) \mathbf{1}_{\{x \in C\}} \\ + \left(\check{\mathbf{E}}^{\vartheta}_{x_0} \, \check{f}(\check{\Phi}_n) - \check{\pi}^{\vartheta}(\check{f}) \right) \mathbf{1}_{\{x \in C^c\}} \,, \end{aligned}$$

where $\check{f}(\check{x}) = f(< \check{x} >_1)$. Thus, for any $x \in \mathcal{X}$, one gets

$$|\mathbf{E}^{\vartheta}_{x} f(\Phi_n) - \pi^{\vartheta}(f)| \le |\check{\mathbf{E}}^{\vartheta}_{x_0} \, \check{f}(\check{\Phi}_n) - \check{\pi}^{\vartheta}(\check{f})| + |\check{\mathbf{E}}^{\vartheta}_{x_1} \, \check{f}(\check{\Phi}_n) - \check{\pi}^{\vartheta}(\check{f})| \,.$$

Taking into account that, for any $x \in \mathcal{X}$,

$$V(x) = \check{V}(\check{x}) = \check{V}(x_0) = \check{V}(x_1)$$

yields the inequality

$$\sum_{n\geq 2} e^{\kappa n} \, |\mathbf{E}_x^{\vartheta} f(\Phi_n) - \pi^{\vartheta}(f)| \leq 2\varrho^* V(x) \, .$$

Using here the condition $\mathbf{H}_1$) we obtain, for any $\mathcal{X} \to \mathbb{R}$ function $0 < f \leq V$,

$$\sum_{n\geq 0} e^{\kappa n} \, |\mathbf{E}_x^{\vartheta} f(\Phi_n) - \pi^{\vartheta}(f)| \leq (1 + e^{\kappa}(1+D) + 2\varrho^*) \, V(x) \, .$$

$\square$

5 Application to diffusion processes

In order to study the geometric ergodicity for the process (1.3) we start with the chain $(\Phi_n^y)_{n\geq 0}$, where $\Phi_n^y = y_n$.

Proposition 5.1. *For any $S \in \Sigma_{M,a,L}$, the sequence $(\Phi_n^y)_{n\geq 0}$ is a homogeneous Markov chain aperiodic and ψ-irreducible, where ψ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.*

Proof. Taking into account (see, for example, [3]) that the solution of the equation (1.3) is a homogeneous Markov process, we obtain immediately that $(\Phi_n^y)_{n\geq 0}$ is a homogeneous Markov chain. In this case (see [3]), the transition density of the process (y_t) has the following form :

$$\mathbf{P}_S(y_t \, = \, y | y_0 \, = \, x) = g(t,x,y) = \frac{G(t,x,y)}{\sqrt{2\pi t}\sigma(y)} e^{\int_{f(x)}^{f(y)} \tilde{S}(u)\mathrm{d}u - \frac{(f(y)-f(x))^2}{2t}} \, , \qquad (5.1)$$

where

$$G(t,x,y) \, = \, \mathbf{E}_S \, \exp\{-\frac{1}{2}\int_0^t \mathbf{B}(S)(\hat{w}_u)\mathrm{d}u\} \, , \quad \mathbf{B}(S)(x) \, = \, \dot{\tilde{S}}(x) \, + \, \tilde{S}^2(x) \, ,$$

$$\tilde{S}(z) = \frac{S(g(z))}{\sigma(g(z))} \, - \, \frac{1}{2}\dot{\sigma}(g(z)) \, , f(x) = \int_0^x \frac{\mathrm{d}u}{\sigma(u)} \, ,$$

$$\hat{w}_u \, = \, f(x) \, + \, \frac{u}{t}(f(y) - f(x)) \, + \, w_u \, - \, \frac{u}{t} w_t \, ,$$

$g(z)$ is the inverse function of f, i.e. $g(z)$ is the solution of the equation $z \, = \, \int_0^g \frac{\mathrm{d}u}{\sigma(u)}$. The solution exists since σ does not change the sign.

This means that, for any $n \geq 1$, for any $A \in \mathcal{B}(\mathbb{R})$ and for any $x \in \mathbb{R}$,

$$\mathbf{P}_S(\Phi^y_n \in A | \Phi^y_0 = x) = \int_A g(n,x,z)\mathrm{d}z\,. \tag{5.2}$$

Thus, the chain $(\Phi^y_n)_{n\geq 0}$ is ψ-irreducible, where ψ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Moreover, in this case the chain is aperiodic, i.e. $d = 1$.

□

Now, we check the minorization condition $\mathbf{H}_2$) for the chain $(\Phi^y_n)_{n\geq 0}$.

Proposition 5.2. *For any $K > 0$, the chain $(\Phi^y_n)_{n\geq 0}$ satisfies the minorization condition with $C = [-f(K), f(K)]$ uniformly over S, i.e. for any $K > 0$, there exist $\delta = \delta_K > 0$ and some probability measure ν_K on $\mathcal{B}(\mathbb{R})$ with $\nu_K(C) = 1$ such that, for any set $A \in \mathcal{B}(\mathbb{R})$,*

$$\inf_{x\in C}\ \inf_{S\in\Sigma_{M,a,L}} \mathbf{P}_S(\Phi^y_1 \in A|\Phi^y_0 = x) \geq \delta\nu_K(A)\,. \tag{5.3}$$

Proof. From (5.2) it follows that

$$\mathbf{P}_S(\Phi^y_1 \in A|\Phi^y_0 = x) = \int_A g(1,x,z)\mathrm{d}z\,.$$

Now, setting $v_{1,K} = \mathbf{P}_S(\max_{0\leq u\leq 1}|w_u| \leq K)$ and

$$v_{2,K} = \sup_{S\in\Sigma_{M,a,L}}\ \sup_{|z|\leq 3K}\left(|\mathbf{B}(S)(z)| + 2\int_0^K |\tilde{S}(u)|\mathrm{d}u\right)\,,$$

one finds that, for $x \in C$ and $y \in C$,

$$g(1,x,y) \geq \frac{v_{1,K}\,e^{-v_{2,K}}}{\sqrt{2\pi}\sigma_1}e^{-f^2(y)/2} := \frac{v_{1,K}\,v_{3,K}\,e^{-v_{2,K}}}{\sqrt{2\pi}\sigma_1}\,\varrho_K(y)$$

with

$$\varrho_K(y) = \frac{1}{v_{3,K}}e^{-f^2(y)}\,\mathbf{1}_{\{y\in C\}} \quad \text{and} \quad v_{3,K} = 2\int_C e^{-f^2(t)}\mathrm{d}t\,.$$

This implies the inequality (5.3) with

$$\delta_K = \frac{v_{1,K}v_{3,K}e^{-v_{2,K}}}{\sqrt{2\pi}\sigma_1} \quad \text{and} \quad \nu_K(A) = \int_A \varrho_K(y)\mathrm{d}y\,.$$

□

Definition 5.1. *A $\mathbb{R} \to [1,\infty)$ function V is called uniform over S Lyapunov function for the equation* (1.3) *if it is twice continuously differentiable and such that, for some constants $\gamma > 0$ and $\beta > 0$ and for any $x \in \mathbb{R}$,*

$$\sup_{S\in\Sigma_{M,a,L}}\left(\dot{V}(x)S(x) + \frac{\sigma^2(x)}{2}\ddot{V}(x)\right) \leq -\gamma V(x) + \beta\,. \tag{5.4}$$

Moreover, $\lim_{x\to\infty} V(x) = \infty$ *and there exists* $m > 0$ *such that*

$$\sup_{x\in\mathbb{R}} \frac{V(x)}{1+|x|^m} < \infty\,. \tag{5.5}$$

Remark 5.1. *For any* $c > 0$ *and* $0 < \epsilon \le 1$*, the function*

$$V(x) = (1+x^2)^{\epsilon} \tag{5.6}$$

satisfies the inequality (5.4) *with* $\gamma = \epsilon/(2L)$ *and*

$$\beta = \sup_{|x|\le x^*} \left(\dot V(x)(M\,+\,L\,x^*)\,+\,\frac{\sigma^2(x)}{2}\ddot V(x)\,+\,\gamma V(x)\right)\,,$$

where $x^* > a$*. Indeed, one has*

$$\frac{S(x)}{x} = \frac{S(x)}{x} + \frac{\dot S(\theta)(x-a)}{x} \le \frac{M}{x} - \frac{1}{L} + \frac{L\,a}{x} \le -\frac{1}{2L}\,, a\le\theta\le x\,,$$

provided $|x| > a^* = 2L(M + a\,L)$*. Therefore*

$$\dot V(x)S(x)\,+\,\frac{\sigma^2(x)}{2}\ddot V(x)\;=$$

$$2\epsilon V(x)\frac{x^2}{1+x^2}\frac{S(x)}{x}\,+\,\epsilon V(x)\frac{\sigma^2(x)}{1+x^2}\,+\,2\epsilon(\epsilon-1)V(x)\frac{x^2\sigma^2(x)}{(1+x^2)^2}$$
$$\le \epsilon V(x)\left(-\frac{1}{L}\,+\,\frac{L^{-1}+\sigma_1^2}{1+x^2}\right) \le -\frac{\epsilon}{2L}V(x) = -\gamma V(x)$$

provided $(L^{-1}+\sigma_1^2)/(1+x^2) \le 1/(2L)$*, i.e.* $|x| > \sqrt{1+2L\sigma_1^2}$*.*

Choosing $x^* = \max[2L(M+aL)\,,\sqrt{1+2L\sigma_1^2}]$ *yields*

$$\sup_{S\in\Sigma_{M,a,L}}\ \sup_{x\in\mathbb{R}}[\dot V(x)S(x)\,+\,\frac{\sigma^2(x)}{2}\ddot V(x)\,+\,\gamma V(x)] \le \beta\,.$$

Proposition 5.3. *Assume that*

$$\sup_{x\in\mathbb{R}} \frac{|S(x)|}{1+|x|} < \infty \tag{5.7}$$

and there exists a Lyapunov function for the equation (1.3). Then the chain $(\Phi_n^y)_{n\ge0}$ *satisfies the drift condition uniformly over* S*, i.e. there exist constants* $K > 0$*,* $D = D_K > 0$ *and* $0 < \rho = \rho_K < 1$ *such that, for any* $x \in \mathbb{R}$*, one has*

$$\sup_{S\in\Sigma_{M,a,L}} \mathbf{E}_S\,\left(V(\Phi_1^y)|\Phi_0 = x\right) \le\,(1-\rho)V(x) + D\mathbf{1}_{\{|x|\le K\}}\,. \tag{5.8}$$

Proof. By the Ito formula, one gets

$$V(y_t) = V(y_0) + \int_0^t \left(\dot{V}(y_s) S(y_s) + \frac{\sigma^2(y_s)}{2} \ddot{V}(y_s) \right) \mathrm{d}s + \int_0^t \dot{V}(y_s) \sigma(y_s) \mathrm{d}w_s \,.$$

In Proposition 4.1 from [2], we have proved that the moments of the solution of equation (1.3) are bounded, when $\sigma(x) = 1$, i.e. for any $d > 0$,

$$\sup_{t \geq 0} \sup_{|x| \leq K} \sup_{S \in \Sigma_{M,a,L}} \mathbf{E}_S \left(|y_t|^d | y_0 = x \right) \leq c^* (d+1)^{d/2} \varrho^d \,, \tag{5.9}$$

where c^* and $\varrho = \varrho_K$ are some positive constants independent of d. The same kind bounds are true when $0 < \sigma_0 \leq \sigma(x) \leq \sigma_1 < \infty$. This implies that the stochastic integral is a martingale in the above Ito formula.

Therefore, by setting

$$g(t) = \mathbf{E}_S \left(V(y_t) | y_0 = x \right)$$

one has

$$g(t) = \int_0^t \mathbf{E}_S \left(\dot{V}(y_u) S(y_u) + \frac{\sigma^2(y_u)}{2} \ddot{V}(y_u) | y_0 = x \right) \mathrm{d}u \,.$$

This relationship and the definition 5.1 give

$$\dot{g}(t) \leq -\gamma g(t) + \beta \,.$$

By the Gronwall inequality, it follows

$$g(t) \leq g(0) e^{-\gamma t} + \beta/\gamma \,.$$

This implies

$$\sup_{S \in \Sigma_{M,a,L}} \mathbf{E}_S \left(V(\Phi^y_1) | y_0 = x \right) \leq V(x) e^{-\gamma} + \beta/\gamma \,.$$

It is clear now that there exists $K > 0$ for which the inequality (5.8) holds with $\rho = 1 - e^{-\gamma/2}$ and $D = \beta/\gamma$.

□

5.1 Proof of Theorem 2.2

For any $t \geq 1$ and any $\mathbb{R} \to]0,1]$ function g, we set

$$\widetilde{g}(x) = \mathbf{E}_S \left(g(y_t) | y_{[t]} = x \right) = \mathbf{E}_S \left(g(y_{\{t\}}) | y_0 = x \right) \,.$$

Moreover, taking into account that $\pi(g) = \pi(\tilde{g})$, one has

$$\mathbf{E}_S(g(y_t)|y_0 = x) - \pi(g) = \mathbf{E}_S \left(\widetilde{g}(\Phi^y_{[t]}) | y_0 = x \right) - \pi(\tilde{g}) \,.$$

Therefore by applying Theorem 2.1 to the chain $(\Phi^y_n)_{n \geq 0}$, we come the upper bound (2.4). Hence Theorem 2.2. □

A Appendix

A.1 Homogeneous Markov chains with atoms

We follow the Meyn-Tweedie approach (see [12]). We remind some definitions from [12] for a homogeneous Markov chains $(\Phi_n)_{n\ge 0}$ defined on a measurable state space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. Denote by $P(x,\cdot)\,, x\in\mathcal{X}$, the transition probability of this chain, i.e. for any $A\in\mathcal{B}(\mathcal{X}), x\in\mathcal{X}$,

$$P(x,A) \,=\, \mathbf{P}_x(\Phi_1\in A) = \mathbf{P}(\Phi_1\in A|\Phi_0 \,=\, x)\,.$$

Therefore, the nth convolution power of this distribution is

$$P^n(x,A) \,=\, \mathbf{P}_x(\Phi_n\in A)\,.$$

We remind that a measure π on $\mathcal{B}(\mathcal{X}))$ is called *invariant* for this chain if, for any $A\in\mathcal{B}(\mathcal{X})$,

$$\pi(A) \,=\, \int_{\mathcal{X}} P(x,A)\pi(\mathrm{d}x)\,.$$

If there exists an invariant positive measure π with $\pi(\mathcal{X}) \,=\, 1$ then the chain is called *positive*.

Definition A.1. *The chain $(\Phi_n)_{n\ge 0}$ is φ-irreducible if there exists a nontrivial measure φ on $\mathcal{B}(\mathcal{X})$ such that, whenever $\varphi(A) \,>\, 0$, one has*

$$L(x,A) = \mathbf{P}_x(\cup_{n=1}^{\infty}\{\Phi_n \,\in\, A\}) \,>\, 0 \quad \text{for any} \quad x\,\in\,\mathcal{X}\,.$$

One can show that, for any φ-irreducible chain, there exists a "maximal" irreducible measure which is noted as ψ and the chain is called ψ-irreducible. An irreducible measure ψ is maximal if and only if $\psi(A)=0$ implies

$$\psi(x\in\mathbb{R}\,:\,L(x,A)>0)=0\,.$$

In the sequel, we denote

$$\mathcal{B}_+(\mathcal{X}) \,=\, \{A\,\in\,\mathcal{B}(\mathcal{X}): \psi(A)\,>\,0\}\,.$$

Definition A.2. *The chain $(\Phi_n)_{n\ge 0}$ is Harris recurrent if it is ψ-irreducible and, for any $A\,\in\,\mathcal{B}_+(\mathcal{X})$, one has*

$$\mathbf{P}_x\left(\sum_{n=1}^{\infty}\mathbf{1}_{\{\Phi_n\in A\}}\right) = 1\,, \quad \text{for any} \quad x\,\in\,A\,.$$

Definition A.3. *The Markov ψ-irreducible chain $(\Phi_n)_{n\ge 0}$ is called periodic of period d if there exist disjoint sets $\Gamma_1, \ldots, \Gamma_d$ in $\mathcal{B}(\mathcal{X})$ with*

$$\psi\left(\cap_{j=1}^{d}\Gamma_j^c\right) = 0$$

such that, for $1 \le i \le d-1$ and for any $x \in \Gamma_i$, one has

$$\mathbf{P}_x(\Phi_1 \in \Gamma_{i+1}) = 1$$

and for $x \in \Gamma_d$ one has $\mathbf{P}_x(\Phi_1 \in \Gamma_1) = 1$. The chain is aperiodic if $d = 1$.

Definition A.4. *We will say that the chain $(\Phi_n)_{n\ge 0}$ satisfies the minorization condition if, for some $\delta > 0$, some set $C \in \mathcal{B}(\mathcal{X})$ and some probability measure ν with $\nu(C) = 1$, one has*

$$P(x, A) \ge 2\delta \mathbf{1}_C(x)\nu(A)\,, \quad A \in \mathcal{B}(\mathcal{X}) \quad \text{and} \quad x \in \mathcal{X}\,. \tag{A.1}$$

Obviously, that the minorization condition implies that the chain is ν - irreducible, therefore ψ- irreducible.

Definition A.5. *A set $\alpha \in \mathcal{B}_+(\mathcal{X})$ is called accessible atom if, for any x and y from α,*

$$\mathbf{P}(x, \Gamma) = \mathbf{P}(y, \Gamma)\,, \quad \forall\, \Gamma \in \mathcal{B}(\mathcal{X})\,.$$

In order to study the ergodicity property, we associate to any set $C \in \mathcal{B}(\mathcal{X})$ the stopping time

$$\tau_C = \inf\{k \ge 1\,:\, \Phi_k \in C\}\,.$$

Proposition A.1. *Suppose that the Markov chain Φ is ψ- irreducible and contains an accessible atom α such that*

$$\mathbf{E}_\alpha \tau_\alpha < \infty\,.$$

Then the chain is ergodic with the invariant probability measure π defined as

$$\pi(\Gamma) = \frac{1}{\mathbf{E}_\alpha\, \tau_\alpha}\, \mathbf{E}_\alpha \sum_{j=1}^{\tau_\alpha} \mathbf{1}_{\{\Phi_j \in \Gamma\}}\,.$$

Proof. Indeed, by the definition of π, for any set $\Gamma \in \mathcal{B}(\mathcal{X})$, one has

$$\int_{\mathcal{X}} \pi(\mathrm{d}z)\, \mathbf{P}(z, \Gamma) = \frac{1}{\mathbf{E}_\alpha\, \tau_\alpha}\, \mathbf{E}_\alpha \sum_{j=1}^{\infty} \mathbf{1}_{\{j \le \tau_\alpha\}}\, \mathbf{E}_\alpha \left(\mathbf{1}_{\{\Phi_{j+1} \in \Gamma\}} | \Phi_1, \ldots, \Phi_j\right)$$

$$= \frac{1}{\mathbf{E}_\alpha\, \tau_\alpha} \left(\mathbf{E}_\alpha \sum_{j=2}^{\tau_\alpha} \mathbf{1}_{\{\Phi_j \in \Gamma\}} + \mathbf{P}_\alpha \left(\Phi_{\tau_\alpha + 1} \in \Gamma\right)\right)\,.$$

Moreover, it is easy to see that

$$\mathbf{P}_\alpha\left(\Phi_{\tau_\alpha+1} \in \Gamma\right) = \mathbf{P}_\alpha\left(\Phi_1 \in \Gamma\right) .$$

This implies the relationship

$$\int_{\mathcal{X}} \pi(\mathrm{d}z)\, \mathbf{P}(z,\Gamma) = \pi(\Gamma)\,,$$

i.e. the measure π is invariant. Obviously, that $\pi(\mathcal{X}) = 1$, i.e. π is a probability measure. □

A.2 Lyapunov functions method for Markov chains

We start with the definition of a "Lyapunov function".

Definition A.6. *We will say that the chain $(\Phi_n)_{n\ge 0}$ satisfies the drift condition if there exists $\mathcal{X} \to [1,\infty)$ function V such that for some constants ρ, $0<\rho<1$, $D>0$ and a small set C from $\mathcal{B}(\mathcal{X})$ one has*

$$\mathbf{E}_x\left(V(\Phi_1)\right) \le (1-\rho)V(x) + D\mathbf{1}_C(x) \tag{A.2}$$

for all $x \in \mathcal{X}$. In this case the function V is called the Lyapunov function.

We remind that $\mathbf{E}_x$ denotes the expectation with respect to the measure $\mathbf{P}_x(\cdot)$.

Now, for any $\mathcal{X} \to [1,+\infty)$ function f and any set $A \in \mathcal{B}(\mathcal{X})$, we set

$$U_A(x,r,f) = \mathbf{E}_x \sum_{j=1}^{\tau_A} e^{rj}\, f(\Phi_j)\,. \tag{A.3}$$

Proposition A.2. *Assume that for the Markov chain $(\Phi_n)_{n\ge 1}$ the condition* (A.2) *holds. Then, for any r, $0<r<-\ln(1-\rho)$, one has*

$$\sup_{x\in\mathcal{X}} \frac{U_C(x,r,V)}{V(x)} \le D_1(r)\,, \tag{A.4}$$

where

$$D_1(r) = \frac{(1-\rho)e^r + D\,e^r}{1-(1-\rho)e^r}\,.$$

Proof. The condition (A.2) implies immediately

$$U_C(x,r,V) \le (1-\rho)e^r V(x) + (1-\rho)e^r U_C(x,r,V) + D\,e^r\,.$$

Taking into account that $V(x) \geq 1$, we obtain the inequality (A.4). $\square$

Further, for any set C from $\mathcal{B}(\mathcal{X})$, we introduce the sequence of stopping times $(\tau_C(n))_{n\geq 0}$ as follows : $\tau_C(0) = 0$ and, for $n \geq 1$,

$$\tau_C(n) = \inf\{k \geq \tau_C(n-1) + 1 \,:\, \Phi_k \in C\}\,.$$

Obviously, that $\tau_C(1) = \tau_C$. We use the following property.

Proposition A.3. *Let C and B be two sets from $\mathcal{B}(\mathcal{X})$ such that*

$$\inf_{x\in C} \mathbf{P}(x, B) \geq \varsigma_* > 0\,. \tag{A.5}$$

Then, for any $n \geq 1$,

$$\sup_{x\in\mathcal{X}} \mathbf{P}_x(\tau_C(n) < \tau_B) \leq (1 - \varsigma_*)^{n-1}\,.$$

Proof. First, note that, for any $x \in C$,

$$\mathbf{P}_x\left(\tau_C < \tau_B\right) \leq \mathbf{P}_x\left(1 < \tau_B\right) = 1 - \mathbf{P}(x, B) \leq 1 - \varsigma_*\,.$$

Indeed, using the strong Markov property and denoting

$$z_n = \Phi_{\tau_C(n)}\, \mathbf{1}_{\{\tau_C(n)<\infty\}}\,, \tag{A.6}$$

one gets, for $n \geq 2$ and for any $x \in \mathcal{X}$,

$$\begin{aligned}\mathbf{P}_x\left(\tau_C(n) < \tau_B\right) &= \mathbf{E}_x\left(\mathbf{1}_{\{\tau_C(n-1)<\tau_B\}}\mathbf{P}_{z_{n-1}}(\tau_C < \tau_B)\right)\\ &\leq (1 - \varsigma_*)\mathbf{P}_x\left(\tau_C(n-1) < \tau_B\right)\,.\end{aligned}$$

Obviously, that for $n = 2$,

$$\sup_{x\in\mathcal{X}} \mathbf{P}_x\left(\tau_C(2) < \tau_B\right) \leq 1 - \varsigma_*\,.$$

Therefore, by induction method, we obtain the desired inequality.
$\square$

Proposition A.4. *Assume that there exist a set $C \in \mathcal{B}(\mathcal{X})$, a real $r > 0$ and a $\mathbb{R} \to [1, +\infty)$ function V such that*

$$D^*(r) = 1 + \sup_{x\in\mathcal{X}} \frac{1}{V(x)}\, U_C(x, r, V) < \infty \tag{A.7}$$

and

$$V^* = \sup_{x\in C} V(x) < \infty\,. \tag{A.8}$$

Then, for any set $B\in\mathcal{B}(\mathcal{X})$ *satisfying the condition* (A.5)*, one has*

$$\sup_{x\in\mathcal{X}}\frac{1}{V(x)}\,U_B(x,\gamma,V)\le D_1^*\,, \tag{A.9}$$

where

$$\gamma=\min(r,\iota_0)\,,\quad \iota_0=-\frac{r}{2}\,\frac{\ln(1-\varsigma_*)}{\ln(D^*(r)V^*)}$$

and

$$D_1^*(r)=D^*(r)\left(1+\frac{D^*(r)V^*}{1-(1-\varsigma_*)^{1/4}}\right)\,.$$

Proof. Indeed, note that

$$U_B(x,\gamma,V)=\sum_{n=0}^{\infty}\mathbf{E}_x\sum_{j=\tau_C(n)+1}^{\tau_C(n+1)}e^{\gamma j}\,V(\Phi_j)\,\mathbf{1}_{\{\tau_B\ge j\}}$$

$$\le U_C(x,\gamma,V)+\sum_{n=1}^{\infty}\mathbf{E}_x\,\mathbf{1}_{\{\tau_B>\tau_C(n)\}}\,e^{\gamma\tau_C(n)}\,U_C(z_n,\gamma,V)\,,$$

where z_n is defined in (A.6). Taking into account that $0<\gamma\le r$ and the conditions (A.7) and (A.8), we get

$$\sup_{x\in C}U_C(x,\gamma,V)\le D^*(r)\,V^*\,.$$

Therefore, setting

$$\Upsilon(x)=\sum_{n=1}^{\infty}\mathbf{E}_x\,\mathbf{1}_{\{\tau_B>\tau_C(n)\}}\,e^{\gamma\tau_C(n)}\,, \tag{A.10}$$

we obtain

$$U_B(x,\gamma,V)\le D^*(r)V(x)+D^*(r)V^*\Upsilon(x)\,.$$

By the strong Markov property, we find, for any $\gamma>0$,

$$\mathbf{E}_x\,e^{\gamma\tau_C(n)}=\mathbf{E}_x\,e^{\gamma\tau_C(n-1)}\,\mathbf{E}_{z_{n-1}}\,e^{\gamma\tau_C}\le\rho(\gamma)\,\mathbf{E}_x\,e^{\gamma\tau_C(n-1)}$$

with

$$\rho(\gamma)=\sup_{z\in C}\mathbf{E}_z\,e^{\gamma\tau_C}\,.$$

By the induction method, we obtain

$$\mathbf{E}_x\,e^{\gamma\tau_C(n)}\le\rho^{n-1}(\gamma)\mathbf{E}_x\,e^{\gamma\tau_C}$$

and, for $0 < a \le r$, it follows that

$$\mathbf{E}_x\, e^{a\tau_C(n)} \le V(x) D^*(r) \rho^{n-1}(a)\,.$$

From here by the Cauchy - Schwartz inequality, we give

$$\begin{aligned}\mathbf{E}_x\, \mathbf{1}_{\{\tau_B > \tau_C(n)\}}\, e^{\kappa\tau_C(n)} &\le \sqrt{\mathbf{P}_x(\tau_B > \tau_C(n))}\,\sqrt{\mathbf{E}_x\, e^{2\kappa\tau_C(n)}}\\ &\le D^*(r)\, V(x)\, q^{n-1}\,,\end{aligned}$$

where

$$q = q(\kappa) = \sqrt{(1-\varsigma_*)\rho(2\kappa)}\,.$$

By the Hölder inequality and the definition of κ, we obtain

$$\rho(2\kappa) \le \left(D^*(r)V^*\right)^{2\kappa/r} \le \sqrt{1-\varsigma_*}\,,$$

i.e.

$$q(\kappa) \le (1-\varsigma_*)^{1/4}\,.$$

This implies the inequality (A.9). Hence, the Proposition A.4. □

A.3 Properties of splitting chains

Now, we study some property of the splitting chain $(\check{\Phi}^\vartheta_n)_{n\ge 1}$ constructed in Section 4 , which we represent as

$$\check{\Phi}^\vartheta_n = (\check{\phi}_n, \check{\iota}_n)\,, \qquad (A.11)$$

where $\check{\phi}_n \in \mathcal{X}$ and $\check{\iota}_n \in \{0,1\}$.

Proposition A.5. *For any measure λ on $\mathcal{B}(\mathcal{X})$ and any set $\check{\Gamma} \in \mathcal{B}(\check{\mathcal{X}})$,*

$$\int_{\check{\mathcal{X}}} \check{\mathbf{P}}(\check{x}, \check{\Gamma})\, \lambda^*(\mathrm{d}\check{x}) = \lambda^*_1(\check{\Gamma})\,, \qquad (A.12)$$

where

$$\lambda_1(\cdot) = \int_{\mathcal{X}} \mathbf{P}(x,\cdot)\, \lambda(\mathrm{d}x)\,.$$

Proof. Indeed, by the definition of the $*$ operation and of the transition probability $\check{\mathbf{P}}(\cdot,\cdot)$ we obtain

$$\int_{\check{\mathcal{X}}} \check{\mathbf{P}}(\check{x}, \check{\Gamma})\, \lambda^*(\mathrm{d}\check{x}) = \int_{\mathcal{X}} \mathbf{P}(x, \check{\Gamma})^*\, \lambda(\mathrm{d}x) = \lambda^*_1(\check{\Gamma})\,.$$

□

Proposition A.6. *For any $n \geq 1$, any measurable positive $\mathcal{X}^n \to \mathbb{R}$ function G and for any measure λ on $\mathcal{B}(\mathcal{X})$, one has*

$$\int_{\mathcal{X}} \mathbf{E}_x \, G_n(\Phi_1, \ldots, \Phi_n) \, \lambda(\mathrm{d}x) = \int_{\check{\mathcal{X}}} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, G_n(\check{\phi}_1, \ldots, \check{\phi}_n) \, \lambda^*(\mathrm{d}\check{x}) \,. \tag{A.13}$$

Proof. It is clear, that it suffices to check this equality for positive functions of the form

$$G_n(x_1, \ldots, x_n) = \prod_{j=1}^{n} g_j(x_j) \,.$$

First, we check this equality for $n = 1$. Note that, for any $\mathcal{X} \to \mathbb{R}$ function g and for any $x \in \mathcal{X}$, one has

$$\int_{\check{\mathcal{X}}} g(< \check{y} >_1) \, \mathbf{P}^*(x, \mathrm{d}\check{y}) = \int_{\mathcal{X}} g(y) \, \mathbf{P}(x, \mathrm{d}y) \,,$$

where $< \check{y} >_1$ denotes the first component of the $\check{y} \in \check{\mathcal{X}} = \mathcal{X} \times \{0, 1\}$. Making use of this equality implies easy (A.13) for $n = 1$. Assume now that the equality (A.13) is true until $n - 1$. We check it for n. Indeed, we have

$$\check{\mathbf{E}}^{\vartheta}_{\check{x}} G_n(\check{\phi}_1, \ldots, \check{\phi}_n) = \check{\mathbf{E}}^{\vartheta}_{\check{x}} \prod_{j=1}^{n} g_j(x_j) = \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, g_1(\check{\phi}_1) \, T(\check{\Phi}^{\vartheta}_1) \,,$$

where

$$T(\check{y}) = \check{\mathbf{E}}^{\vartheta}_{\check{y}} \prod_{j=1}^{n-1} g_{j+1}(\check{\phi}_j) \,.$$

Now, we set

$$\mu_{\vartheta}(\Gamma) = \int_{\Gamma} g_1(y) \, \lambda_1(\mathrm{d}y) \,,$$

where the measure $\lambda_1(\cdot)$ is defined in (A.12). Therefore, taking into account Proposition A.5 , we can represent the integral on the right hand side of the equality (A.13) as

$$\int_{\check{\mathcal{X}}} \check{\mathbf{E}}^{\vartheta}_{\check{x}} \, G_n(\check{\phi}_1, \ldots, \check{\phi}_n) \, \lambda^*(\mathrm{d}\check{x}) = \int_{\check{\mathcal{X}}} T(\check{y}) \mu^*(\mathrm{d}\check{y}) \,.$$

By the induction assumption, one has

$$\int_{\check{\mathcal{X}}} T(\check{y}) \mu^*(\mathrm{d}\check{y}) = \int_{\mathcal{X}} \mathbf{E}_y \prod_{j=1}^{n-1} g_{j+1}(\Phi_j) \; \mu_{\vartheta}(\mathrm{d}y) = \int_{\mathcal{X}} \mathbf{E}_x \, G_n(\Phi_1, \ldots, \Phi_n) \, \lambda(\mathrm{d}x) \,.$$

Hence, the Proposition A.6. $\square$

Proposition A.7. *Assume that the splitting chain* $(\check{\Phi}^{\vartheta}_n)_{n\ge 1}$ *has an invariant probability measure* $\check{\pi}$. *Then, the chain* $(\Phi_n)_{n\ge 1}$ *has the invariant probability measure* π *on* $\mathcal{B}(\mathcal{X})$ *which is given as*

$$\pi(\Gamma) = \check{\pi}(\Gamma_0) + \check{\pi}(\Gamma_1)\,. \tag{A.14}$$

Moreover, $\check{\pi} = \pi^*$.

Proof. Define the measure $\pi(\cdot)$ on $\mathcal{B}(\mathcal{X})$ as follows

$$\pi(\cdot) = \int_{\check{\mathcal{X}}} \check{\pi}(\mathrm{d}\check{z})\, Q(\check{z}, \cdot)\,,$$

where the kernel $Q(\cdot,\cdot)$ is defined in (4.2). It is clear, that $\pi(\cdot)$ is a probability measure on $\mathcal{B}(\mathcal{X})$ such that, for $\check{\Gamma} \in \mathcal{B}(\mathcal{X})$,

$$\check{\pi}(\check{\Gamma}) = \int_{\check{\mathcal{X}}} \check{\pi}(\mathrm{d}\check{z})\, \check{\mathbf{P}}(\check{z}, \check{\Gamma}) = \pi(\check{\Gamma})^*\,.$$

Thus, $\pi(\Gamma) = \check{\pi}(\Gamma_0) + \check{\pi}(\Gamma_1)$ for any set Γ from $\mathcal{B}(\mathcal{X})$. Moreover,

$$\begin{aligned}
\pi(\Gamma) &= \int_{\check{\mathcal{X}}} \check{\pi}(\mathrm{d}\check{z})\, Q(\check{z}, \Gamma) = \int_{\check{\mathcal{X}}} \pi^*(\mathrm{d}\check{z})\, Q(\check{z}, \Gamma) \\
&= (1-\delta) \int_C \pi(\mathrm{d}z)\, Q(z_0, \Gamma) + \int_{C^c} \pi(\mathrm{d}z)\, Q(z_0, \Gamma) \\
&\quad + \delta \int_C \pi(\mathrm{d}z)\, Q(z_1, \Gamma)\,.
\end{aligned}$$

Taking into account the definition (4.2) we obtain

$$\pi(\Gamma) = \int_{\mathcal{X}} \mathbf{P}(z, \Gamma)\, \pi(\mathrm{d}z)\,,$$

i.e. π is the invariant measure for the chain $(\Phi_n)_{n\ge 1}$. Hence Proposition A.7. $\square$

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