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***Abstract*—A global finite-time observer is designed for nonlinear systems which are uniformly observable and globally Lipschitz. This result is based on a high-gain approach combined with recent advances on finite-time stability using Lyapunov function and homogeneity concepts.**

I. Introduction

Nonlinear observer design has a long standing history for more than twenty years (see [3]). The main stream being to use linear observer ideas. As a result, linearization of nonlinear system with algebraic methods have been investigated in [6], [19] and [13]. Another way to tackle such design is to use high-gain. The resulting observer, which is closely related to a triangular structure, has been developed by Gauthier et al. (see [11], [12]) and is derived from the uniform observability of nonlinear systems. Let us just mention few other ones: Kazantzis and Kravaris observer which uses the Lyapunov auxiliary theorem and a direct coordinate transformation in [17] ; backstepping design in [20] ; adaptive observer in [33] ; and many other ones ... All these approaches result in asymptotic convergence of the observer error dynamics whereas in some applications, finite-time convergence is needed: for instance like in secure communication where synchronization of chaotic signal is of major importance or for walking robots (see for instance [1], [28], [29]), for which each step has obviously to be completed in finite time. Less attention was paid to finite-time observer design except using some non-smooth techniques (see for example the sliding mode observers [7], [15] especially the step by step observer [9], [10]). Another approach based on the moving horizon observer was developed in [22]. Recently, **finite-time stability (FTS)** and stabilization (in the continuous time domain) using Lyapunov theory and homogeneity concept, has attracted a lot of attention: Bhat and Bernstein in [5], [4], Moulay and Perruquetti in [23], [25]. Continuous finite time observers are considered here. Such an observer has been designed for linear systems in [8] and extended to linear time-varying system in [21] and [30]. Let us mention [16] by Hong et al. dealing with output finite-time stabilization of fully actuated manipulators for which a **finite-time observer (FTO)** is designed for this special class of nonlinear systems. More recently, a global **FTO** for a linearizable system via input output injection has been designed in [28] and extended to **uniformly observable (UO)** systems in [31], [32] in a semi-global way. Semi-global means that the gains of the observer depend on a compact set (which can be chosen arbitrarily large) leading to finite-time convergence of the observer for any initial conditions within this compact set. This paper provides a **global** observer for uniformly observable systems which means that the parameters of the observer can be set once and then will provide finite time convergence whatever the initial conditions. The observer design is based on the observability normal form, Lyapunov theory and homogeneity.

The paper is organized as follows. The class of considered systems, the definitions and the properties of finite time stable systems are given in section II. Section III presents a global finite-time observer followed by the proof of its convergence. Section IV gives a convincing illustrative simulation of the obtained results.

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II. Preliminaries

Notations:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$, where $\mathbb{R}$ is the set of real number.
- For f a continuous vector field, $t \mapsto x(t, x_{t_0})$ denotes a solution starting from x_{t_0} at t_0 for system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0. \tag{1}$$

- $\lceil x \rfloor^{\alpha} = \mathrm{sign}(x).|x|^{\alpha}$, with $\alpha > 0$ and $x \in \mathbb{R}$,
- $\|.\|_{i,k}$ denotes the i-norm on $\mathbb{R}^k$,
- if $x \in \mathbb{R}^n$, $\overline{x}_i$ denotes the vector in $\mathbb{R}^i$ with the i^{th} first components of x $(1 \leq i \leq n)$,
- $\mathcal{B}_{\|.\|}(\varepsilon)$ is the ball centered at the origin and of radius ε, w.r.t. (with respect to) the norm $\|.\|$.

Context: Let us consider the following analytic system:

$$\dot{z} = F(z) + \sum_{i=1}^{m} G_i(z) u_i, \quad z \in \Omega, \quad y = h(z), \tag{2}$$

where Ω is an open subset of $\mathbb{R}^n$, $u = (u_1, \dots, u_m) \in \mathbb{R}^m$, $y \in \mathbb{R}$ (the measured output). If system (2) is **UO** for any bounded input (see [11]), then, a coordinate change can be found to transform system (2) into the form (see [14]):

$$\begin{cases} \dot{x}_1 = x_2 + \sum_{j=1}^{m} g_{1,j}(x_1) u_j \\ \dot{x}_2 = x_3 + \sum_{j=1}^{m} g_{2,j}(x_1, x_2) u_j \\ \vdots \\ \dot{x}_{n-1} = x_n + \sum_{j=1}^{m} g_{n-1,j}(x_1, \dots, x_{n-1}) u_j \\ \dot{x}_n = \varphi(x) + \sum_{j=1}^{m} g_{n,j}(x) u_j \\ y = x_1 = Cx \end{cases} \tag{3}$$

where $C = (1 \;\; 0 \cdots 0)$, φ and $g_{i,j}$ $(i = 1, \dots, n, j = 1, \dots, m)$ are analytic functions with $\varphi(0) = 0, g_{ij}(0, \dots, 0) = 0$. We assume furthermore that the functions $g_{i,j}$ and φ are globally Lipschitz with constant l and u is bounded by $u_0 \in \mathbb{R}_+$, that is $\|u\|_{\infty} \leq u_0$. Thus we concentrate here on systems of form (3).

Finite-time stability: Since the main concern is **finite-time observer (FTO)**, the main definitions and properties for FTS are recalled now. In system (1), f is a continuous but not necessarily a Lipschitzian function, so it may happen that any solution of the system converges to zero in finite time (for example, the solutions of $\dot{x} = -\mathrm{sign}(x)\,|x|^{\frac{1}{3}}$, for $x \in \mathbb{R}$). It is aimed here to exploit this property of such dynamical nonlinear systems to design a **FTO**. Due to the non Lipschitz condition on the right hand side of (1) backward uniqueness may be lost, and thus we only consider forward uniqueness (see [28]). We recall the definition of finite-time stability.

Definition 1. *The origin of system (1) is said to be* **finite time stable (FTS)** *(at the origin, on an open neighborhood of the origin $\mathcal{V} \subset \mathbb{R}^n$) if:*

1) *there exists a function $T : \mathcal{V} \setminus \{0\} \to \mathbb{R}_+$, such that for all $x_0 \in \mathcal{V} \setminus \{0\}$, $x(t, x_0)$ is defined (and unique) on $[0, T(x_0))$, $x(t, x_0) \in \mathcal{V} \setminus \{0\}$ for all $t \in [0, T(x_0))$ and $\lim_{t \to T(x_0)} x(t, x_0) = 0$. T is called the* settling-time function *of the system (1).*
2) *for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for every $x_0 \in \left(\mathcal{B}_{\|.\|_{2,n}}(\delta(\epsilon)) \setminus \{0\}\right) \cap \mathcal{V}$, $x(t, x_0) \in \mathcal{B}_{\|.\|_{2,n}}(\epsilon)$ for all $t \in [0, T(x_0))$.*

Furthermore, if only 1) is fulfilled then the origin of system (1) is said to be finite-time attractive.

The following result gives a sufficient condition for system (1) to be finite time stable (see [26], [27] for ordinary differential equations, and [24] for differential inclusions):

Lemma 1. *[32, lemma 1] Suppose there exists a Lyapunov function $V(x)$ defined on a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin of system (1) and some constants $\tau, \gamma > 0$ and $0 < \beta < 1$ such that*

$$\frac{d}{dt}V(x)_{|(1)} \leq -\tau V(x)^{\beta} + \gamma V(x), \quad \forall x \in \mathcal{U}\backslash\{0\}.$$

Then the origin of system (1) is FTS. The set $\Omega = \left\{x \in \mathcal{U} : V(x)^{1-\beta} < \frac{\tau}{\gamma}\right\}$ is contained in the domain of attraction of the origin. The settling time satisfies $T(x) \leq \frac{\ln\left(1 - \frac{\gamma}{\tau}V(x)^{1-\beta}\right)}{\gamma(\beta - 1)}$, $x \in \Omega$.

To circumvent the standard design of Lyapunov functions, one can use homogeneity conditions recalled hereafter.

Homogeneity:

Definition 2. *A function $V : \mathbb{R}^n \to \mathbb{R}$ is* homogeneous of degree *d w.r.t. the weights $(r_1, \dots, r_n) \in \mathbb{R}^n_+$ if $V(\lambda^{r_1}x_1, \dots, \lambda^{r_n}x_n) = \lambda^d V(x_1, \dots, x_n)$, $\forall \lambda > 0$. A vector field f is* homogeneous of degree *d w.r.t. the weights $(r_1, \dots, r_n) \in \mathbb{R}^n_+$ if for all $1 \leq i \leq n$, the i-th component f_i is a homogeneous function of degree $r_i + d$. The system (1) is homogeneous of degree d if the vector field f is homogeneous of degree d.*

Previous observers: Our observer is directly based on the observer introduced by Shen and Xia in [32]. Let us recall this semi-global result.

Theorem 1. *[32, Theorem 1] System (3) admits a semi-global observer of the form:*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + k_1\lceil y - \hat{x}_1\rfloor^{\alpha_1} + \sum_{j=1}^{m} g_{1,j}(\hat{x}_1)u_j \\ \dot{\hat{x}}_2 = \hat{x}_3 + k_2\lceil y - \hat{x}_1\rfloor^{\alpha_2} + \sum_{j=1}^{m} g_{2j}(\hat{x}_1, \hat{x}_2)u_j \\ \vdots \\ \dot{\hat{x}}_n = \varphi(\hat{x}) + k_n\lceil y - \hat{x}_1\rfloor^{\alpha_n} + \sum_{j=1}^{m} g_{n,j}(\hat{x})u_j \end{cases} \tag{4}$$

where the α_i are defined by

$$\alpha_i = i\alpha - (i-1), \quad i = 1, \dots, n, \quad \alpha \in \left]1 - \frac{1}{n}, 1\right[. \tag{5}$$

The gains are given by

$$K = [k_1, \dots, k_n]^T = S_\infty^{-1}(\theta)C^T, \tag{6}$$

where $S_\infty(\theta)$ is the unique solution of the matrix equation:

$$\begin{cases} \theta S_\infty(\theta) + A^T S_\infty(\theta) + S_\infty(\theta)A - C^T C = 0 \\ S_\infty(\theta) = S_\infty^T(\theta) \end{cases} \tag{7}$$

where $(A)_{i,j} = \delta_{i,j-1}$, $1 \leq i, j \leq n$, and $C = (1 \quad 0 \dots 0)$.

The special case $g_{i,j} = 0$ and $\varphi = 0$, yields the observer by Perruquetti et al. (see [28]) which is based on homogeneity property (specifically on Theorem 5.8 in [2]).

III. Global Observer

In this section, Theorem 2 provides a global finite-time observer for system (3) based on the semi-global finite-time observer (4) designed by Shen and Xia in [32] and rooted in [28] :

Theorem 2. *Let us consider system (3) with a bounded input u. Then there exists $0 < \theta^* < \infty$ and $\varepsilon > 0$ such that for all $\theta > \theta^*$ and $\alpha \in]1 - \varepsilon, 1[$, system (3) admits the following* ***global*** *finite-time high-gain observer:*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + k_1(\lceil e_1\rfloor^{\alpha_1} + \rho e_1) + \sum_{j=1}^{m} g_{1,j}(\hat{x}_1)u_j \\ \dot{\hat{x}}_2 = \hat{x}_3 + k_2(\lceil e_1\rfloor^{\alpha_2} + \rho e_1) + \sum_{j=1}^{m} g_{2,j}(\hat{x}_1, \hat{x}_2)u_j \\ \vdots \\ \dot{\hat{x}}_n = k_n(\lceil e_1\rfloor^{\alpha_n} + \rho e_1) + \varphi(\hat{x}) + \sum_{j=1}^{m} g_{n,j}(\hat{x})u_j \end{cases}$$

where $e_1 = x_1 - \hat{x}_1$, the powers α_i are defined by (5), the gains k_i by (6), and $\rho = \left(\frac{n^2\theta^{\frac{2}{3}}S_1 + 1}{2}\right)$, where

$$S_1 = \max_{1 \leq i,j \leq n} |S_\infty(1)_{i,j}|.|S_\infty^{-1}(1)_{j,1}|. \tag{8}$$

In addition, the settling time $T(e_0)$ (where $e_0 = x_0 - \hat{x}_0$) of the error dynamics is bounded by $\frac{\ln\left(\frac{4r^2}{V(e_0)}\right)}{\kappa(\theta)} + \frac{\ln\left(1 - \frac{b_1}{b_2}\left(4r^2\right)^{1-\overline{\alpha}}\right)}{b_2(\overline{\alpha} - 1)}$ (where all the parameters and the Lyapunov function V are given in the proof).

To prove our result, we need the following technical lemmas:

Lemma 2. *[16, Remark 1] Assume that (2) is globally asymptotically stable and finite-time attractive on a neighborhood of the origin. Then system (2) is globally finite-time stable.*

Lemma 3. *The matrix $S_\infty(\theta)$ and $S_\infty^{-1}(\theta)$ verify the following properties:*

$$S_\infty(\theta)_{i,j} = S_\infty(1)_{i,j}\frac{1}{\theta^{i+j-1}} \tag{9}$$

$$S_\infty^{-1}(\theta)_{i,j} = S_\infty^{-1}(1)_{i,j}\theta^{i+j-1} \tag{10}$$

for any $\theta > 0$ and $1 \leq i, j \leq n$.

The proof of lemma 3 is not given here, but an explicit computation (straightforward but lengthy) gives the first equality from which the second easily follows.

Lemma 4. *[18, Lemma 2.5 p. 85] Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a smooth function such that*

$$\dot{\sigma}(t) \leq k\sigma(t), \; a \leq t \leq b,$$

for some constant $k \in \mathbb{R}$. Then $\sigma(t) \leq \sigma(a)e^{-k(a-t)}$, for $a \leq t \leq b$.

Proof of Theorem 2: Denote $e = x - \hat{x}$. By using

$$D(x, \hat{x}, u) = \Phi(x) - \Phi(\hat{x}) + \sum_{j=1}^{m}(g_j(x) - g_j(\hat{x}))u_j(t),$$

where $\Phi(x) = (0, \ldots, 0, \varphi(x))$, $g_j = (g_{1,j}, \ldots, g_{n,j})$, and

$$F(K, e) = \left(k_1\lceil e_1\rfloor^{\alpha_1}, \ldots, k_n\lceil e_1\rfloor^{\alpha_n}\right)^T,$$

the error dynamics is given by:

$$\dot{e} = Ae - F(K, e) - \rho S_\infty^{-1}(\theta)C^T Ce + D(x, \hat{x}, u). \tag{11}$$

The proof of the global finite-time convergence of the observer is split into two parts. Part 1 proves the existence of a "Lyapunov function" V for (11) which is positive definite on $\mathbb{R}^n$, radially unbounded and whose derivative is negative definite on $P^r = \mathbb{R}^n - \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(r)$ (for some $r > 1$). Then part 2 proves that (11) is **FTS** at the origin on $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$. Since $\dot{V}$ is negative on P^r and the **FTS** on $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ yield that (11) is globally asymptotic stable and locally **FTS** at the origin. We apply then Lemma 2 to complete the proof.

Part 1: Follow [11], and consider:

$$V(e) = e^T S_\infty(\theta)e.$$

For all $\theta > 0$, the function V is positive definite positive and radially unbounded, since, according to [11], there exists δ_θ>0, such that:

$$S_\infty(\theta) \geq \delta_\theta I_n,$$

where I_n is the identity matrix of dimension n. By using (7) and (11), the derivative of V along the solutions of (11) is given by:

$$\begin{aligned}\tfrac{d}{dt}(e^T S_\infty(\theta)e) = -\theta e^T S_\infty(\theta)e - (2\rho - 1)(Ce)^2 \\ -2e^T S_\infty(\theta)F(K, e) + 2e^T S_\infty(\theta)D(x, \hat{x}, u).\end{aligned}$$

It leads to:

$$\begin{aligned}\tfrac{d}{dt}(e^T S_\infty(\theta)e) \leq -\theta\|e\|^2_{S_\infty(\theta)} - (2\rho - 1)(Ce)^2 \\ -2e^T S_\infty(\theta)F(K, e) + 2\|e\|_{S_\infty(\theta)}\|D(x, \hat{x}, u)\|_{S_\infty(\theta)}.\end{aligned}$$

Since φ and g_{ij} $(i = 1, \ldots, n, j = 1, \ldots, m)$ are globally Lipschitzian functions with a constant l and $\|u\|_\infty$ is bounded by u_0, by using (9) and following the same computations as in [11], we obtain:

$$\|D(x, \hat{x}, u)\|_{S_\infty(\theta)} \leq nl(u_0 + 1)mC_1\sqrt{S}\|e\|_{S_\infty(\theta)},$$

where $S = \max_{1\leq i,j\leq n}|S_\infty(1)_{i,j}|$ and by norm equivalence, there exists $C_1 > 0$ such that:

$$\|x\|_{1,n} \leq C_1\|x\|_{S_\infty(1)}, \; \forall x \in \mathbb{R}^n. \tag{12}$$

Hence,

$$\begin{aligned}\tfrac{d}{dt}V(e) \;\leq\; & (-\theta + M)V(e) - (2\rho - 1)(Ce)^2 \\ & -2e^T S_\infty(\theta)F(K, e),\end{aligned} \tag{13}$$

where $M = 2nl(u_0 + 1)mC_1\sqrt{S}$.
According to (13), to prove that $\dot{V}$ is negative definite on $P^r = \mathbb{R}^n - \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(r)$, use an overvaluation of $e^T S_\infty(\theta)F(K, e)$. According to Lemma 3, the following equalities hold:

$$\begin{aligned}& e^T S_\infty(\theta)F(K, e) \\ &= \sum_{1\leq i,j\leq n} e_i \frac{(S_\infty(1))_{i,j}}{\theta^{i+j-1}} \left(S_\infty^{-1}(1)\right)_{j,1} \theta^j \lceil e_1\rfloor^{\alpha_j}, \\ &= \sum_{j=1}^{n} \left(S_\infty^{-1}(1)\right)_{j,1} \lceil e_1\rfloor^{\alpha_j} \sum_{i=1}^{n} \frac{e_i}{\theta^{i-1}} \left(S_\infty(1)\right)_{i,j}.\end{aligned}$$

Overvalue $e^T S_\infty(\theta)F(K, e)$ in two steps. For this, the set P^r is partitioned in two complementary parts:

$$P^r_{<1} = \{e \in P^r : |e_1| < 1\}, \quad P^r_{\geq 1} = \{e \in P^r : |e_1| \geq 1\}.$$

On $P^r_{<1}$, one have $|e_1|^{\alpha_i} < 1$, $i = 1, \ldots, n$. Hence $|e^T S_\infty(\theta)F(K, e)| \leq nS_1\theta \sum_{i=1}^{n} \left|\frac{e_i}{\theta^i}\right|$, where S_1 is defined by (8). Let $\xi_i = \frac{e_i}{\theta^i}$ for $i = 1, \ldots, n$, it follows:

$$|e^T S_\infty(\theta)F(K, e)| \leq nS_1\theta\|\xi\|_{1,n}.$$

Now, using (12) and $\|\xi\|^2_{S_\infty(1)} = \frac{1}{\theta}\|e\|^2_{S_\infty(\theta)}$, one gets $|e^T S_\infty(\theta)F(K, e)| \leq nS_1C_1\sqrt{\theta}\|e\|_{S_\infty(\theta)}$. Let $C_2 = nSC_1$. Taking $r > 1$, then $\|e\|_{S_\infty(\theta)} \leq \|e\|^2_{S_\infty(\theta)}$ for $e \in P^r$, thus:

$$|e^T S_\infty(\theta)F(K, e)| \leq C_2\sqrt{\theta}\|e\|^2_{S_\infty(\theta)}.$$

It leads to:

$$\frac{d}{dt}V(e) \leq (-\theta + M + C_2\sqrt{\theta})V(e). \tag{14}$$

On $P^r_{\geq 1}$, one has $|e_1| \geq 1$ so $|e_1|^{\alpha_i} \leq |e_1|$ for $i = 1, \ldots, n$. Hence

$$\begin{aligned}|e^T S_\infty(\theta)F(K, e)| \;&\leq\; nS_1\theta \sum_{i=1}^{n} \left|\frac{e_i}{\theta^i}\right| .|e_1|, \\ &=\; nS_1 \sum_{i=1}^{n} \left(\theta^{\frac{2}{3}} \left|\frac{e_i}{\theta^i}\right|\right) \left(\theta^{\frac{1}{3}}|e_1|\right), \\ &\leq\; \frac{nS_1\theta^{\frac{4}{3}}}{2}\|\xi\|^2_{2,n} + \frac{n^2\theta^{\frac{2}{3}}S_1}{2}|e_1|^2.\end{aligned}$$

But $\|\xi\|^2_{2,n} \leq C_3\|\xi\|^2_{S_\infty(1)}$ and $\|\xi\|^2_{S_\infty(1)} = \frac{1}{\theta}\|e\|^2_{S_\infty(\theta)}$, hence

$$|e^T S_\infty(\theta)F(K, e)| \leq C_4\theta^{\frac{1}{3}}\|e\|^2_{S_\infty(\theta)} + \frac{n^2\theta^{\frac{2}{3}}S_1}{2}|e_1|^2, \tag{15}$$

where $C_4 = \frac{nS_1C_3}{2}$. Combining (13) and (15), we have:

$$\frac{d}{dt}V(e) \leq \left(-\theta + M + 2C_4\theta^{\frac{1}{3}}\right)V(e). \tag{16}$$

Combining the two inequalities (14) and (16), with $r > 1$, there exists $\theta_1 > 0$ such that for all $\theta \geq \theta_1$, $\frac{d}{dt}V(e) < 0, \forall e \in P^r$ and more precisely:

$$\frac{d}{dt}V(e) \leq \kappa(\theta)V(e), \tag{17}$$

where $\kappa(\theta) = \max\{(-\theta + M + 2C_4\theta^{\frac{1}{3}}), (-\theta + M + C_2\sqrt{\theta})\}$. Thus applying Lemma 4 to inequality (17), one gets $V(e(t)) \leq V(e_0)e^{\kappa(\theta)t}$. Since we look for trajectories entering into $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$, it is sufficient to have $V(e_0)e^{\kappa(\theta)t} \leq 4r^2$ or equivalently $t \geq \frac{\ln\left(\frac{4r^2}{V(e_0)}\right)}{\kappa(\theta)}$. Which is an overvaluation of $T_1(e_0)$ the time for a trajectory starting at e_0 to enter into $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$:

$$T_1(e_0) \leq \frac{\ln\left(\frac{4r^2}{V(e_0)}\right)}{\kappa(\theta)}. \tag{18}$$

Part 2: The proof of FTS of the error dynamics (11) on $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ is broken into two steps: firstly, prove that the linear part contributes to the convergence of the error and secondly, finite-time stability on this compact is obtained following similar lines as in the semi-global case (see the proof of the main result in [32]). Consider the following Lyapunov function:

$$\tilde{V}_\alpha(e) = \tilde{e}^T S_\infty(\theta)\tilde{e},$$

where $\tilde{e} = \left(\lceil e_1 \rfloor^{\frac{1}{q}} \; \lceil e_2 \rfloor^{\frac{1}{\alpha_1 q}} \dots \lceil e_n \rfloor^{\frac{1}{\alpha_{n-1} q}}\right)$, $q = \prod_{i=1}^{n-1}[(i-1)\alpha - (i-2)]$ is the product of the weights. It is obvious that $\tilde{V}_\alpha$ is homogeneous of degree $\frac{2}{q}$ with respect to the weights $\{(i-1)\alpha - (i-2)\}_{1 \leq i \leq n}$. The function $\tilde{V}_\alpha$ is positive definite and radially unbounded, since according to [11], for all $\theta > 0$, there exists δ_θ such that for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$:

$$\tilde{V}_\alpha(x) = \tilde{x}^T S_\infty(\theta)\tilde{x} \geq \delta_\theta \tilde{x}^T \tilde{x} = \delta_\theta \sum_{i=1}^n |x_i|^{\frac{2}{\alpha_{i-1} q}},$$

and $\frac{2}{\alpha_{i-1} q} > 0$, for $i = 1, \dots, n$. We have:

$$\frac{d}{dt}\tilde{V}_\alpha(e) = W_1 + W_2 + W_3$$

where

$$W_1 = 2\tilde{e}^T S_\infty(\theta) \begin{pmatrix} \frac{1}{q}|e_1|^{\frac{1}{q}-1}(\frac{1}{2}e_2 - k_1\lceil e_1 \rfloor^{\alpha_1}) \\ \vdots \\ \frac{1}{\alpha_{n-1} q}|e_{n-1}|^{\frac{1}{\alpha_{n-2} q}-1}(\frac{1}{2}e_n - k_{n-1}\lceil e_1 \rfloor^{\alpha_{n-1}}) \\ \frac{1}{\alpha_n q}|e_n|^{\frac{1}{\alpha_{n-1} q}-1}(-k_n\lceil e_1 \rfloor^{\alpha_n}) \end{pmatrix},$$

$$W_2 = 2\tilde{e}^T S_\infty(\theta) \begin{pmatrix} \frac{1}{q}|e_1|^{\frac{1}{q}-1}(\frac{1}{2}e_2 - \rho k_1 e_1) \\ \vdots \\ \frac{1}{\alpha_{n-1} q}|e_{n-1}|^{\frac{1}{\alpha_{n-2} q}-1}(\frac{1}{2}e_n - \rho k_{n-1} e_1) \\ \frac{1}{\alpha_n q}|e_n|^{\frac{1}{\alpha_{n-1} q}-1}(-\rho k_n e_1) \end{pmatrix},$$

$$W_3 = 2\tilde{e}^T S_\infty(\theta) \begin{pmatrix} \frac{1}{q}|e_1|^{\frac{1}{q}-1}D_1 \\ \vdots \\ \frac{1}{\alpha_{n-2} q}|e_{n-1}|^{\frac{1}{\alpha_{n-2} q}-1}D_{n-1} \\ \frac{1}{\alpha_{n-1} q}|e_n|^{\frac{1}{\alpha_{n-1} q}-1}D_n \end{pmatrix}.$$

Overvaluation of W_1 : this term is homogeneous, Lemma 4.2 in [5] leads to:

$$W_1 \leq -b_1(\alpha, \theta)\left(\tilde{V}_\alpha(e)\right)^{\frac{\frac{2}{q}+\alpha-1}{\frac{2}{q}}},$$

where b_1 verifies $\lim_{\alpha \to 1} b_1(\alpha, \theta) = \frac{\theta}{2}$ (see Lemma 4 in [32]).
Overvaluation of W_2 : use the tube lemma as done in [28]. Since V is proper, $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ is a compact set of $\mathbb{R}^n$. Define the function $\varphi : \mathbb{R}_+ \times \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r) \to \mathbb{R}$ by

$$\varphi(\alpha, e) = W_2$$

By using the same technique as in [11], it is easily proved that $\varphi(1, e) < 0$ for $e \in \mathbb{R}^n$. Since φ is continuous, $\varphi^{-1}(\mathbb{R}_-)$ is an open subset of $\mathbb{R}_+ \times \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ containing the slice $\{1\} \times \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$. Since $\mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ is compact, it follows from the tube lemma that $\varphi^{-1}(\mathbb{R}_-)$ contains some tube $(1-\mu_1, 1+\mu_2) \times \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ about $\{1\} \times \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$. For all $(\alpha, e) \in (1-\mu_1, 1+\mu_2) \times \mathcal{B}_{\|.\|_{S_\infty(\theta)}}(2r)$ one has $\varphi(\alpha, e) < 0$. Thus there exists $\varepsilon_1 > 0$ such that for all $\alpha \in (1-\varepsilon_1, 1) : W_2 \leq 0$.
Overvaluation of W_3 : noting that $\frac{d}{dt}\lceil e_i \rfloor^{\alpha_i} = \alpha_i |e_i|^{\alpha_i - 1}$, one obtains

$$W_3 \leq 2l(u_0+1)m\left(\tilde{e}^T S_\infty(\theta)\tilde{e}\right)^{\frac{1}{2}} \times \left(\sum_{1 \leq i,j \leq n} \frac{|S_\infty(1)_{i,j}|}{\theta^{i+j-1}} \frac{|e_i|^{\frac{1}{\alpha_{i-1} q}-1}\sum_{k=1}^{i}|e_k|}{\alpha_{i-1} q} \times \frac{|e_j|^{\frac{1}{\alpha_{j-1} q}-1}\sum_{k=1}^{j}|e_k|}{\alpha_{j-1} q}\right)^{\frac{1}{2}}.$$

By using Young's inequality, for any reals x, y and $p > 0$ one has $|x||y|^{p-1} \leq \frac{1}{p}|x|^p + \frac{p-1}{p}|y|^p$. This leads to $|e_k||e_i|^{\frac{1}{\alpha_{i-1} q}-1} \leq \alpha_{i-1} q |e_k|^{\frac{1}{\alpha_{i-1} q}} + (1 - \alpha_{i-1} q)|e_i|^{\frac{1}{\alpha_{i-1} q}}$, thus

$$\begin{aligned} &\sum_{k=1}^{i} |e_i|^{\frac{1}{\alpha_{i-1} q}-1}|e_k| \\ &\leq \sum_{k=1}^{i}\left((1-\alpha_{i-1} q)|e_i|^{\frac{1}{\alpha_{i-1} q}} + \alpha_{i-1} q|e_k|^{\frac{1}{\alpha_{i-1} q}}\right), \\ &\triangleq \sum_{k=1}^{i} b_{i,k}|e_k|^{\frac{1}{\alpha_{i-1} q}}, \end{aligned}$$

where $b_{i,k} > 0$. Let $b = \max_{i,k} b_{i,k}$. Thus

$$W_3 \leq \frac{2bl(u_0+1)mS^{\frac{1}{2}}\theta^{\frac{1}{2}}}{\alpha_{n-1} q} \times \left(\tilde{V}_\alpha(e)\right)^{\frac{1}{2}} \left(\sum_{1 \leq i,j \leq n} \left(\sum_{k=1}^{i} \frac{e_k^{\frac{2}{\alpha_{k-1} q}}}{\theta^{2k}}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{j} \frac{e_k^{\frac{2}{\alpha_{k-1} q}}}{\theta^{2k}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

According to Lemma 6 in [32], there exists θ_2 such that for $\theta \geq \theta_2 \geq 1$, for $i = 1, \dots, n, k = 1, \dots, i$, one has $\frac{|e_k(t)|^{\frac{1}{\alpha_{i-1} q}}}{\theta^i} \leq \frac{|e_k(t)|^{\frac{1}{\alpha_{k-1} q}}}{\theta^k}$. Thus, using $\xi_k = \frac{\lceil e_k \rfloor^{\frac{1}{\alpha_{k-1} q}}}{\theta^k}$, one obtains

$$\sum_{1 \leq i,j \leq n} \left(\sum_{k=1}^{i} \frac{e_k^{\frac{2}{\alpha_{k-1} q}}}{\theta^{2k}}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{j} \frac{e_k^{\frac{2}{\alpha_{k-1} q}}}{\theta^{2k}}\right)^{\frac{1}{2}} \leq n^2 \sum_{k=1}^{n} \xi_k^2.$$

On the other hand, according to [11], there exists $\delta_1 > 0$ such that:

$$S_\infty(1) \geq \delta_1 I$$

and using $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$, we have

$$\begin{aligned}\sum_{k=1}^{n}\xi_k^2 &\leq \frac{1}{\delta_1}\xi^T S_\infty(1)\xi \\ &\leq \frac{1}{\theta\delta_1}\sum_{1\leq i,j\leq n}\left(\lceil e_i\rfloor^{\frac{1}{\alpha_{i-1}q}}\frac{S(1)_{i,j}}{\theta^{i+j-1}}\lceil e_j\rfloor^{\frac{1}{\alpha_{j-1}q}}\right) \\ &\leq \frac{1}{\theta\delta_1}\tilde{V}_\alpha(e).\end{aligned}$$

Thus $W_3 \leq \frac{2bl(u_0+1)mn^2 S^{\frac{1}{2}}}{\alpha_{n-1}q\delta_1^{\frac{1}{2}}}\tilde{V}_\alpha(e)$. Finally, one obtains:

$$\frac{d}{dt}\tilde{V}_\alpha(e)_{(11)} \leq -b_1(\alpha,\theta)\left(\tilde{V}_\alpha\right)^{\frac{\frac{2}{q}+\alpha-1}{\frac{2}{q}}} + b_2(\alpha)\tilde{V}_\alpha(e), \quad (19)$$

where $b_2(\alpha) = \frac{2bl(u_0+1)mn^2 S^{\frac{1}{2}}}{\alpha_{n-1}q\delta_1^{\frac{1}{2}}}$. By (19) and Theorem 1, the domain of attraction of the observer is given by:

$$\Omega = \left\{e : \tilde{V}_\alpha(e) < \left(\frac{b_1}{b_2}\right)^{\frac{2}{q(1-\alpha)}}\right\}. \quad (20)$$

From (20) and the inequality $\tilde{V}_\alpha(e) \leq e_0^T S_\infty(\theta)e_0, \forall t > 0$ (see Lemma 6 in [32]), one has

$$\mathcal{U} = \left\{e \,:\, V(e) = e^T S_\infty(\theta)e < \left(\frac{b_1}{b_2}\right)^{\frac{2}{q(1-\alpha)}}\right\} \subset \Omega, \quad (21)$$

since $\lim_{\alpha\to 1} b_1(\alpha,\theta) = \dfrac{\theta}{2}$ there exists $\varepsilon_2 > 0$ such that

$$b_1(\alpha,\theta) \geq \frac{\theta}{4}, \quad \text{for } \alpha \in]1-\varepsilon_2, 1[,$$

thus for $\alpha \in (1-\varepsilon_2, 1)$, we have:

$$\frac{b_1}{b_2} \to +\infty, \quad \theta \to \infty, \quad \text{for } \alpha \in]1-\varepsilon_2, 1[. \quad (22)$$

Considering (21) and (22), there exists $\theta_3 > 0$ such that for all $\theta \geq \theta_3$:

$$\mathcal{B}_{\|\cdot\|_{S_\infty(\theta)}}(2r) \subset \Omega.$$

Finally, take $\theta^* = \max\{\theta_1, \theta_2, \theta_3\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. According to equation (19) and Lemma 1, for a trajectory starting in Ω at e_0, the following inequality is obtained for the settling time $T_2(e_0) \leq \frac{\ln\left(1-\frac{b_1}{b_2}\tilde{V}_\alpha^{1-\overline{\alpha}}\right)(e_0)}{b_2(\overline{\alpha}-1)}$, where $\overline{\alpha} = \frac{\frac{2}{q}+\alpha-1}{\frac{2}{q}}$. According to Lemma 6 in [32], $\tilde{V}_\alpha(e_0) \leq e_0^T S_\infty(\theta)e_0$. Hence a straightforward computation yields:

$$T_2(e_0) \leq \frac{\ln\left(1-\frac{b_1}{b_2}\left(4r^2\right)^{1-\overline{\alpha}}\right)}{b_2(\overline{\alpha}-1)} \quad (23)$$

Combining (18) and (23), one obtains the following overvaluation for the settling time of the observer :

$$T(e_0) \leq \frac{\ln\left(\frac{4r^2}{V(e_0)}\right)}{\kappa(\theta)} + \frac{\ln\left(1-\frac{b_1}{b_2}\left(4r^2\right)^{1-\overline{\alpha}}\right)}{b_2(\overline{\alpha}-1)}$$

■

IV. EXAMPLE

Consider the following system (which is already in the form (3)):

$$\begin{cases}\dot{x}_1 = x_2, \\ \dot{x}_2 = x_3 + x_1\sin(x_2), \\ \dot{x}_3 = \sin(x_1+x_2+x_3).\end{cases}$$

Following the line of our result, the observer dynamics is chosen as:

$$\left\{\begin{array}{rcl}\dot{\hat{x}}_1 &=& \hat{x}_2 - k_1(\lceil e_1\rfloor^{\alpha} + \rho e_1), \\ \dot{\hat{x}}_2 &=& \hat{x}_3 + \hat{x}_1\sin(\hat{x}_2) - k_2(\lceil e_1\rfloor^{2\alpha-1} + \rho e_1), \\ \dot{\hat{x}}_3 &=& \sin(\hat{x}_1+\hat{x}_2+\hat{x}_3) - k_3(\lceil e_1\rfloor^{3\alpha-2} + \rho e_1),\end{array}\right.$$

with gains set as follows: $k_1 = 3\theta, k_2 = 3\theta, k_3 = \theta$ and $\rho = \frac{(81\theta^{\frac{2}{3}}+1)}{2}$. The simulations in Figure 1 show effectiveness of our algorithm even in the case of a noisy measurement (a Gaussian white noise with 0.01 correlation and 0.05 covariance) for different values of α and θ. As it can seen in Figure 1.b) and 1.d) for $\theta = 5$, the gain-selection is noise-sensitive as usual for such high-gain observers. Thus, a future research topic will be to design adaptive tuning gain using only local informations on the non linearities. On the contrary the parameter α seems not to be much sensitive w.r.t. the noise.

V. CONCLUSION

A global finite-time observer for uniformly observable systems with the global Lipschitzian properties has been introduced. This was achieved through an extension of a sufficient condition for local finite-time stability and Lyapunov theories.

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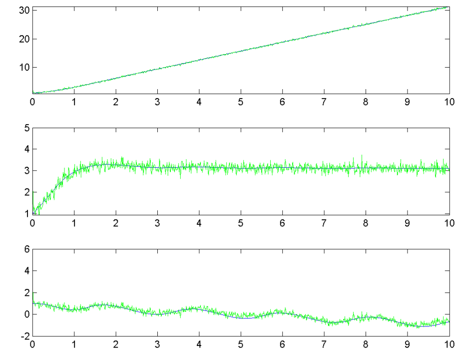
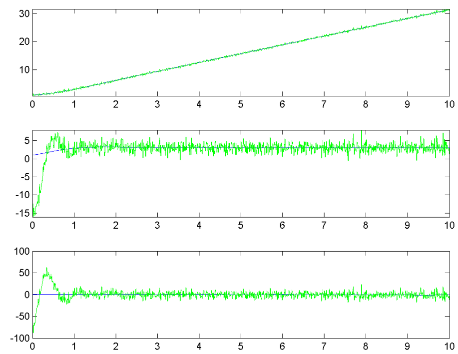
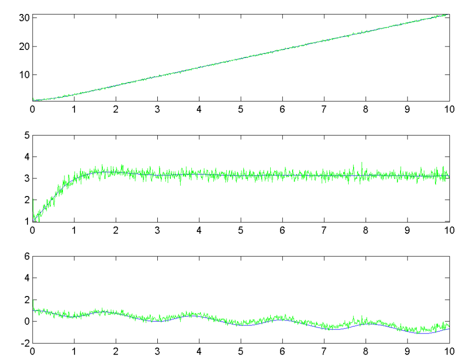


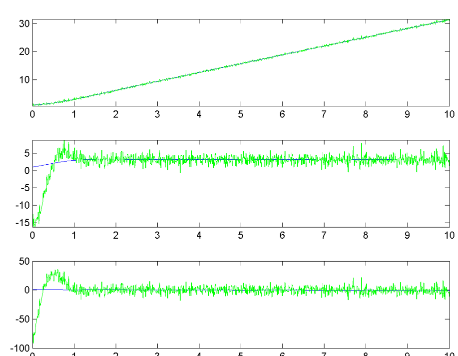
(a) $\alpha = 0.7$ and $\theta = 1$



(b) $\alpha = 0.7$ and $\theta = 5$



(c) $\alpha = 0.9$ and $\theta = 1$



(d) $\alpha = 0.9$ and $\theta = 5$

Fig. 1. States and its estimates