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Functional Ito calculus and stochastic integral representation of martingales

Rama Cont

David-Antoine Fournié

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Abstract

We develop a non-anticipative calculus for functionals of a continuous semimartingale, using an extension of the Ito formula to path-dependent functionals which possess certain directional derivatives. The construction is based on a pathwise derivative, introduced by B Dupire, for functionals on the space of right-continuous functions with left limits. We show that this functional derivative admits a suitable extension to the space of square-integrable martingales. This extension defines a weak derivative which is shown to be the inverse of the Ito integral and which may be viewed as a non-anticipative “lifting” of the Malliavin derivative.

These results lead to a constructive martingale representation formula for Ito processes. By contrast with the Clark-Haussmann-Ocone formula, this representation only involves non-anticipative quantities which may be computed pathwise.

Keywords: stochastic calculus, functional calculus, Ito formula, integration by parts, Malliavin derivative, martingale representation, semimartingale, Wiener functionals, Clark-Ocone formula.

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1 Introduction

In the analysis of phenomena with stochastic dynamics, Ito's stochastic calculus [15, 16, 8, 23, 19, 28, 29] has proven to be a powerful and useful tool. A central ingredient of this calculus is the *Ito formula* [15, 16, 23], a change of variable formula for functions $f(X_t)$ of a *semimartingale* X which allows to represent such quantities in terms of a stochastic integral. Given that in many applications such as statistics of processes, physics or mathematical finance, one is led to consider path-dependent functionals of a semimartingale X and its quadratic variation process $[X]$ such as:

$$\int_0^t g(t, X_t) d[X](t), \quad G(t, X_t, [X]_t), \quad \text{or} \quad E[G(T, X(T), [X](T)) | \mathcal{F}_t] \quad (1)$$

(where $X(t)$ denotes the value at time t and $X_t = (X(u), u \in [0, t])$ the path up to time t) there has been a sustained interest in extending the framework of stochastic calculus to such path-dependent functionals.

In this context, the Malliavin calculus [3, 24, 22, 25, 30, 31, 32] has proven to be a powerful tool for investigating various properties of Brownian functionals. Since the construction of Malliavin derivative does not refer to an underlying filtration \mathcal{F}_t , it naturally leads to representations of functionals in terms of *anticipative* processes [4, 14, 25]. However, in most applications it is more natural to consider non-anticipative versions of such representations.

In a recent insightful work, B. Dupire [9] has proposed a method to extend the Ito formula to a functional setting in a *non-anticipative* manner, using a pathwise functional derivative which

quantifies the sensitivity of a functional $F_t : D([0, t], \mathbb{R}) \rightarrow \mathbb{R}$ to a variation in the endpoint of a path $\omega \in D([0, t], \mathbb{R})$:

$$\nabla_\omega F_t(\omega) = \lim_{\epsilon \rightarrow 0} \frac{F_t(\omega + \epsilon 1_t) - F_t(\omega)}{\epsilon}$$

Building on this insight, we develop hereafter a non-anticipative calculus [6] for a class of processes –including the above examples– which may be represented as

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}, \{A(u), 0 \leq u \leq t\}) = F_t(X_t, A_t) \quad (2)$$

where A is the local quadratic variation defined by $[X](t) = \int_0^t A(u) du$ and the functional

$$F_t : D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+) \rightarrow \mathbb{R}$$

represents the dependence of Y on the path $X_t = \{X(u), 0 \leq u \leq t\}$ of X and its quadratic variation.

Our first result (Theorem 4.1) is a change of variable formula for path-dependent functionals of the form (2). Introducing A_t as additional variable allows us to control the dependence of Y with respect to the ”quadratic variation” $[X]$ by requiring smoothness properties of F_t with respect to the variable A_t in the supremum norm, without resorting to p -variation norms as in “rough path” theory [20]. This allows our result to cover a wide range of functionals, including the examples in (1).

We then extend this notion of functional derivative to *processes*: we show that for Y of the form (2) where F satisfies some regularity conditions, the process $\nabla_X Y = \nabla_\omega F(X_t, A_t)$ may be defined intrinsically, independently of the choice of F in (2). The operator ∇_X is shown to admit an extension to the space of square-integrable martingales, which is the inverse of the Ito integral with respect to X : for $\phi \in \mathcal{L}^2(X)$, $\nabla_X (\int \phi . dX) = \phi$ (Theorem 5.8). In particular, we obtain a constructive version of the martingale representation theorem (Theorem 5.9), which states that for any square-integrable \mathcal{F}_t^X -martingale Y ,

$$Y(T) = Y(0) + \int_0^T \nabla_X Y . dX \quad \mathbb{P} - a.s.$$

This formula can be seen as a non-anticipative counterpart of the Clark-Haussmann-Ocone formula [4, 13, 14, 18, 25]. The integrand $\nabla_X Y$ is an adapted process which may be computed pathwise, so this formula is more amenable to numerical computations than those based on Malliavin calculus.

Finally, we show that this functional derivative ∇_X may be viewed as a non-anticipative “lifting” of the Malliavin derivative (Theorem 6.1): for square-integrable martingales Y whose terminal values is differentiable in the sense of Malliavin $Y(T) \in \mathbf{D}^{1,2}$, we show that $\nabla_X Y(t) = E[\mathbb{D}_t H | \mathcal{F}_t]$.

These results provide a rigorous mathematical framework for developing and extending the ideas proposed by B. Dupire [9] for a large class of functionals which notably includes stochastic integrals and allows for dependence on the quadratic variation along a path.

2 Functional representation of non-anticipative processes

Let $X : [0, T] \times \Omega \mapsto \mathbb{R}^d$ be a continuous, \mathbb{R}^d -valued semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ assumed to satisfy the usual hypotheses [8]. Denote by \mathcal{P} (resp. \mathcal{O}) the associated *predictable* (resp. *optional*) sigma-algebra on $[0, T]$. \mathcal{F}_t^X denotes the (\mathbb{P} -completed) natural filtration

of X . The paths of X then lie in $C_0([0, T], \mathbb{R}^d)$, which we will view as a subspace of $D([0, t], \mathbb{R}^d)$ the space of cadlag functions with values in \mathbb{R}^d . For a path $x \in D([0, T], \mathbb{R}^d)$, denote by $x(t)$ the value of x at t and by $x_t = (x(u), 0 \leq u \leq t)$ the restriction of x to $[0, t]$. Thus $x_t \in D([0, t], \mathbb{R}^d)$. For a process X we shall similarly denote $X(t)$ its value at t and $X_t = (X(u), 0 \leq u \leq t)$ its path on $[0, t]$.

We denote by $[X] = ([X^i, X^j], i, j = 1..d)$ the quadratic (co-)variation process associated to X , taking values in the set S_d^+ of positive $d \times d$ matrices. We assume that

$$[X](t) = \int_0^t A(s) ds \quad (3)$$

for some cadlag process A with values in S_d^+ . Note that A need not be a semimartingale. The paths of A lie in $\mathcal{S}_t = D([0, t], S_d^+)$, the space of cadlag functions with values S_d^+ .

A process $Y : [0, T] \times \Omega \mapsto \mathbb{R}^d$ adapted to \mathcal{F}_t^X may be represented as

$$Y(t) = F_t(\{X(u), 0 \leq u \leq t\}, \{A(u), 0 \leq u \leq t\}) = F_t(X_t, A_t) \quad (4)$$

where $F = (F_t)_{t \in [0, T]}$ is a family of functionals

$$F_t : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \rightarrow \mathbb{R}$$

representing the dependence of $Y(t)$ on the underlying path of X and its quadratic variation.

Introducing the process A as additional variable may seem redundant at this stage: indeed $A(t)$ is itself \mathcal{F}_t -measurable i.e. a functional of X_t . However, it is not a *continuous* functional with respect to the supremum norm or other usual topologies on $D([0, t], \mathbb{R}^d)$. Introducing A_t as a second argument in the functional will allow us to control the regularity of Y with respect to $[X]_t = \int_0^t A(u) du$ simply by requiring continuity of F_t in supremum or L^p norms with respect to the “lifted process” (X, A) (see Section 2.2). This idea is analogous in some ways to the approach of rough path theory [20], although here we do not resort to p-variation norms.

Since Y is non-anticipative, $Y(t, \omega)$ only depends on the restriction of ω on $[0, t]$. This motivates the following definition:

Definition 2.1 (Non-anticipative functional). A non-anticipative functional on Υ is a family of functionals $F = (F_t)_{t \in [0, T]}$ where

$$\begin{aligned} F_t : D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+) &\mapsto \mathbb{R} \\ (x, v) &\rightarrow F_t(x, v) \end{aligned}$$

is measurable with respect to \mathcal{B}_t , the canonical filtration on $D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+)$.

We can also $F = (F_t)_{t \in [0, T]}$ as a map defined on the vector bundle:

$$\Upsilon = \bigcup_{t \in [0, T]} D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+) \quad (5)$$

If Y is a \mathcal{B}_t -predictable process, then [8, Vol. I, Par. 97]

$$\forall t \in [0, T], \quad Y(t, \omega) = Y(t, \omega_{t-})$$

where ω_{t-} denotes the path defined on $[0, t]$ by

$$\omega_{t-}(u) = \omega(u) \quad u \in [0, t[\quad \omega_{t-}(t) = \omega(t-)$$

Note that ω_{t-} is cadlag and should *not* be confused with the caglad path $u \mapsto \omega(u-)$.

The functionals discussed in the introduction depend on the process A via $[X] = \int_0^\cdot A(t)dt$. In particular, they satisfy the condition $F_t(X_t, A_t) = F_t(X_t, A_{t-})$. Accordingly, we will assume throughout the paper that all functionals $F_t : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \rightarrow \mathbb{R}$ considered have “predictable” dependence with respect to the second argument:

$$\forall t \in [0, T], \quad \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad F_t(x_t, v_t) = F_t(x_t, v_{t-}) \quad (6)$$

2.1 Horizontal and vertical perturbation of a path

Consider a path $x \in D([0, T], \mathbb{R}^d)$ and denote by $x_t \in D([0, t], \mathbb{R}^d)$ its restriction to $[0, t]$ for $t < T$. For $h \geq 0$, the *horizontal* extension $x_{t,h} \in D([0, t+h], \mathbb{R}^d)$ of x_t to $[0, t+h]$ is defined as

$$x_{t,h}(u) = x(u) \quad u \in [0, t] ; \quad x_{t,h}(u) = x(t) \quad u \in]t, t+h] \quad (7)$$

For $h \in \mathbb{R}^d$, we define the *vertical* perturbation x_t^h of x_t as the cadlag path obtained by shifting the endpoint by h :

$$x_t^h(u) = x_t(u) \quad u \in [0, t[\quad x_t^h(t) = x(t) + h \quad (8)$$

or in other words $x_t^h(u) = x_t(u) + h1_{t=u}$.

We now define a distance between two paths, not necessarily defined on the same time interval. For $T \geq t' = t + h \geq t \geq 0$, $(x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t^+$ and $(x', v') \in D([0, t+h], \mathbb{R}^d) \times \mathcal{S}_{t+h}$ define

$$d_\infty((x, v), (x', v')) = \sup_{u \in [0, t+h]} |x_{t,h}(u) - x'(u)| + \sup_{u \in [0, t+h]} |v_{t,h}(u) - v'(u)| + h \quad (9)$$

If the paths $(x, v), (x', v')$ are defined on the same time interval, then $d_\infty((x, v), (x', v'))$ is simply the distance in supremum norm.

2.2 Continuity for non-anticipative functionals

We now define a notion of (left) continuity for non-anticipative functionals.

Definition 2.2 (Continuity at fixed times). A functional F defined on Υ is said to be continuous at fixed times for the d_∞ metric if and only if:

$$\forall t \in [0, T], \quad \forall \epsilon > 0, \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad \exists \eta > 0, (x', v') \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \\ d_\infty((x, v), (x', v')) < \eta \Rightarrow |F_t(x, v) - F_t(x', v')| < \epsilon \quad (10)$$

We now define a notion of joint continuity with respect to time and the underlying path:

Definition 2.3 (Continuous functionals). A non-anticipative functional $F = (F_t)_{t \in [0, T]}$ is said to be continuous at $(x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t$ if

$$\forall \epsilon > 0, \exists \eta > 0, \forall (x', v') \in \Upsilon, \quad d_\infty((x, v), (x', v')) < \eta \Rightarrow |F_t(x, v) - F_t(x', v')| < \epsilon \quad (11)$$

We denote $\mathbb{C}^{0,0}([0, T])$ the set of non-anticipative functionals continuous on Υ .

A non-anticipative functional $F = (F_t, t \in [0, T])$ is said to be left-continuous if for each $t \in [0, T)$, $F_t : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t \rightarrow \mathbb{R}$ in the sup norm and

$$\forall \epsilon > 0, \forall (x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad \exists \eta > 0, \forall h \in [0, t], \quad \forall (x', v') \in D([0, t-h], \mathbb{R}^d) \times \mathcal{S}_{t-h}, \\ d_\infty((x, v), (x', v')) < \eta \Rightarrow |F_t(x, v) - F_{t-h}(x', v')| < \epsilon \quad (12)$$

We denote $\mathbb{C}_l^{0,0}([0, T])$ the set of left-continuous functionals.

We define analogously the class of right continuous functionals $\mathbb{C}_r^{0,0}([0, T])$.

We call a functional ‘‘boundedness preserving’’ if it is bounded on each bounded set of paths:

Definition 2.4 (Boundedness-preserving functionals). Define $\mathbb{B}([0, T])$ as the set of non-anticipative functionals F such that for every compact subset K of \mathbb{R}^d , every $R > 0$ and $t_0 < T$:

$$\exists C_{K,R,t_0} > 0, \quad \forall t \leq t_0, \forall (x, v) \in D([0, t], K) \times \mathcal{S}_t, \quad \sup_{s \in [0, t]} |v(s)| < R \Rightarrow |F_t(x, v)| < C_{K,R,t_0} \quad (13)$$

2.3 Measurability properties

Composing a non-anticipative functional F with the process (X, A) yields an \mathcal{F}_t -adapted process $Y(t) = F_t(X_t, A_t)$. The results below link the measurability and pathwise regularity of Y to the regularity of the functional F .

Lemma 2.5 (Pathwise regularity). *If $F \in \mathbb{C}_l^{0,0}$ then for any $(x, v) \in D([0, T], \mathbb{R}^d) \times \mathcal{S}_T$, the path $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous.*

Proof. Let $F \in \mathbb{C}_l^{0,0}$ and $t \in [0, T)$. For $h > 0$ sufficiently small,

$$d_\infty((x_{t-h}, v_{t-h}), (x_{t-}, v_{t-})) = \sup_{u \in (t-h, t)} |x(u) - x(t-)| + \sup_{u \in (t-h, t)} |v(u) - v(t-)| + h \quad (14)$$

Since x and v are cadlag, this quantity converges to 0 as $h \rightarrow 0+$, so

$$F_{t-h}(x_{t-h}, v_{t-h}) - F_t(x_{t-}, v_{t-}) \xrightarrow{h \rightarrow 0^+} 0$$

so $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous. \square

Theorem 2.6. (i) *If F is continuous at fixed times, then the process Y defined by $Y((x, v), t) = F_t(x_t, v_t)$ is adapted.*

(ii) *If $F \in \mathbb{C}_l^{0,0}([0, T])$, then the process $Z(t) = F_t(X_t, A_t)$ is optional.*

(iii) *If $F \in \mathbb{C}_l^{0,0}([0, T])$, and if either A is continuous or F verifies (6), then Z is a predictable process.*

In particular, any $F \in \mathbb{C}_l^{0,0}$ is a non-anticipative functional in the sense of Definition 2.1. We propose an easy-to-read proof of points (i) and (iii) in the case where A is continuous. The (more technical) proof for the cadlag case is given in the Appendix A.

Continuous case. Assume that F is continuous at fixed times and that the paths of (X, A) are almost-surely continuous. Let us prove that Y is \mathcal{F}_t -adapted: $X(t)$ is \mathcal{F}_t -measurable. Introduce the partition $t_n^i = \frac{iT}{2^n}, i = 0..2^n$ of $[0, T]$, as well as the following piecewise-constant approximations of X and A :

$$\begin{aligned} X^n(t) &= \sum_{k=0}^{2^n} X(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) + X_T 1_{\{T\}}(t) \\ A^n(t) &= \sum_{k=0}^{2^n} A(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) + X_T 1_{\{T\}}(t) \end{aligned} \quad (15)$$

The random variable $Y^n(t) = F_t(X_t^n, A_t^n)$ is a continuous function of the random variables $\{X(t_k^n), A(t_k^n), t_k^n \leq t\}$ hence is \mathcal{F}_t -measurable. The representation above shows in fact that $Y^n(t)$ is \mathcal{F}_t -measurable. X_t^n and A_t^n converge respectively to X_t and A_t almost-surely so $Y^n(t) \xrightarrow{n \rightarrow \infty} Y(t)$ a.s., hence $Y(t)$ is \mathcal{F}_t -measurable.

(i) implies point (iii) since the path of Z are left-continuous by Lemma 2.5. \square

3 Pathwise derivatives of non-anticipative functionals

3.1 Horizontal and vertical derivatives

We now define pathwise derivatives for a functional $F = (F_t)_{t \in [0, T]} \in \mathbb{C}^{0,0}$, following Dupire [9].

Definition 3.1 (Horizontal derivative). The *horizontal derivative* at $(x, v) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t$ of non-anticipative functional $F = (F_t)_{t \in [0, T]}$ is defined as

$$\mathcal{D}_t F(x, v) = \lim_{h \rightarrow 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x_t, v_t)}{h} \quad (16)$$

if the corresponding limit exists. If (16) is defined for all $(x, v) \in \Upsilon$ the map

$$\begin{aligned} \mathcal{D}_t F : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t &\mapsto \mathbb{R}^d \\ (x, v) &\rightarrow \mathcal{D}_t F(x, v) \end{aligned} \quad (17)$$

defines a non-anticipative functional $\mathcal{D}F = (\mathcal{D}_t F)_{t \in [0, T]}$, the *horizontal derivative* of F .

Note that our definition (16) is different from the one in [9] where the case $F(x, v) = G(x)$ is considered.

Dupire [9] also introduced a pathwise spatial derivative for such functionals, which we now introduce. Denote $(e_i, i = 1..d)$ the canonical basis in \mathbb{R}^d .

Definition 3.2. A non-anticipative functional $F = (F_t)_{t \in [0, T]}$ is said to be *vertically differentiable* at $(x, v) \in D([0, t], \mathbb{R}^d) \times D([0, t], S_d^+)$ if

$$\begin{aligned} \mathbb{R}^d &\mapsto \mathbb{R} \\ e &\rightarrow F_t(x_t^e, v_t) \end{aligned}$$

is differentiable at 0. Its gradient at 0

$$\nabla_x F_t(x, v) = (\partial_i F_t(x, v), i = 1..d) \quad \text{where} \quad \partial_i F_t(x, v) = \lim_{h \rightarrow 0} \frac{F_t(x_t^{he_i}, v) - F_t(x, v)}{h} \quad (18)$$

is called the *vertical derivative* of F_t at (x, v) . If (18) is defined for all $(x, v) \in \Upsilon$, the maps

$$\begin{aligned} \nabla_x F : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t &\mapsto \mathbb{R}^d \\ (x, v) &\rightarrow \nabla_x F_t(x, v) \end{aligned} \quad (19)$$

define a non-anticipative functional $\nabla_x F = (\nabla_x F_t)_{t \in [0, T]}$, the *vertical derivative* of F . F is then said to be *vertically differentiable* on Υ .

Remark 3.3. $\partial_i F_t(x, v)$ is simply the directional derivative of F_t in direction $(1_{\{t\}}e_i, 0)$. Note that this involves examining cadlag perturbations of the path x , even if x is continuous.

Remark 3.4. If $F_t(x, v) = f(t, x(t))$ with $f \in C^{1,1}([0, T] \times \mathbb{R}^d)$ then we retrieve the usual partial derivatives:

$$\mathcal{D}_t F(x, v) = \partial_t f(t, X(t)) \quad \nabla_x F_t(X_t, A_t) = \nabla_x f(t, X(t)).$$

Remark 3.5. Bismut [3] considered directional derivatives of functionals on $D([0, T], \mathbb{R}^d)$ in the direction of purely discontinuous (e.g. piecewise constant) functions with finite variation, which is similar to Def. 3.2. This notion, used in [3] to derive an integration by parts formula for pure-jump processes, is natural in the context of discontinuous semimartingales. We will show that the directional derivative (18) also intervenes naturally when the underlying process X is *continuous*, which is less obvious.

Definition 3.6 (Regular functionals). Define $\mathbb{C}^{1,k}([0, T])$ as the set of functionals $F \in \mathbb{C}_l^{0,0}$ which are

- horizontally differentiable with $\mathcal{D}_t F$ continuous at fixed times,
- k times vertically differentiable with $\nabla_x^j F \in \mathbb{C}_l^{0,0}([0, T])$ for $j = 1..k$.

Define $\mathbb{C}_b^{1,k}([0, T])$ as the set of functionals $F \in \mathbb{C}^{1,2}$ such that $\mathcal{D}F, \nabla_x F, \dots, \nabla_x^k F \in \mathbb{B}([0, T])$.

We denote $\mathbb{C}^{1,\infty}([0, T]) = \bigcap_{k \geq 1} \mathbb{C}^{1,k}([0, T])$.

Note that this notion of regularity only involves directional derivatives with respect to *local* perturbations of paths, so $\nabla_x F$ and $\mathcal{D}_t F$ seems to contain *less* information on the behavior of F than, say, the Fréchet derivative which consider perturbations in all directions in $C_0([0, T], \mathbb{R}^d)$ or the Malliavin derivative [21, 22] which examines perturbations in the direction of all absolutely continuous functions. Nevertheless we will show in Section 4 that knowledge of $\mathcal{D}F, \nabla_x F, \nabla_x^2 F$ along the paths of X derivatives are sufficient to reconstitute the path of $Y(t) = F_t(X_t, A_t)$.

Example 1 (Smooth functions). In the case where F reduces to a smooth *function* of $X(t)$,

$$F_t(x_t, v_t) = f(t, x(t)) \quad (20)$$

where $f \in C^{1,k}([0, T] \times \mathbb{R}^d)$, the pathwise derivatives reduces to the usual ones: $F \in \mathbb{C}_b^{1,k}$ with:

$$\mathcal{D}_t F(x_t, v_t) = \partial_t f(t, x(t)) \quad \nabla_x^j F_t(x_t, v_t) = \partial_x^j f(t, x(t)) \quad (21)$$

In fact to have $F \in \mathbb{C}^{1,k}$ we just need f to be right-differentiable in the time variable, with right-derivative $\partial_t f(t, \cdot)$ which is continuous in the space variable and $f, \nabla f$ and $\nabla^2 f$ to be jointly left-continuous in t and continuous in the space variable.

Example 2 (Cylindrical functionals). Let $g \in C^0(\mathbb{R}^d, \mathbb{R}), h \in C^k(\mathbb{R}^d, \mathbb{R})$ with $h(0) = 0$. Then

$$F_t(\omega) = h(\omega(t) - \omega(t_n-)) \quad \mathbf{1}_{t \geq t_n} \quad g(\omega(t_1-), \omega(t_2-), \dots, \omega(t_n-))$$

is in $\mathbb{C}_b^{1,k}$ with $\mathcal{D}_t F(\omega) = 0$ and

$$\forall j = 1..k, \quad \nabla_{\omega}^j F_t(\omega) = h^{(j)}(\omega(t) - \omega(t_n-)) \quad \mathbf{1}_{t \geq t_n} g(\omega(t_1-), \omega(t_2-), \dots, \omega(t_n-))$$

Example 3 (Integrals with respect to quadratic variation). A process $Y(t) = \int_0^t g(X(u))d[X](u)$ where $g \in C_0(\mathbb{R}^d)$ may be represented by the functional

$$F_t(x_t, v_t) = \int_0^t g(x(u))v(u)du \quad (22)$$

It is readily observed that $F \in \mathbb{C}_b^{1,\infty}$, with:

$$\mathcal{D}_t F(x_t, v_t) = g(x(t))v(t) \quad \nabla_x^j F_t(x_t, v_t) = 0 \quad (23)$$

Example 4. The martingale $Y(t) = X(t)^2 - [X](t)$ is represented by the functional

$$F_t(x_t, v_t) = x(t)^2 - \int_0^t v(u)du \quad (24)$$

Then $F \in \mathbb{C}_b^{1,\infty}$ with:

$$\begin{aligned} \mathcal{D}_t F(x, v) &= -v(t) & \nabla_x F_t(x_t, v_t) &= 2x(t) \\ \nabla_x^2 F_t(x_t, v_t) &= 2 & \nabla_x^j F_t(x_t, v_t) &= 0, j \geq 3 \end{aligned} \quad (25)$$

Example 5. $Y = \exp(X - [X]/2)$ may be represented as $Y(t) = F(X_t)$

$$F_t(x_t, v_t) = e^{x(t) - \frac{1}{2} \int_0^t v(u)du} \quad (26)$$

Elementary computations show that $F \in \mathbb{C}_b^{1,\infty}$ with:

$$\mathcal{D}_t F(x, v) = -\frac{1}{2}v(t)F_t(x, v) \quad \nabla_x^j F_t(x_t, v_t) = F_t(x_t, v_t) \quad (27)$$

Note that, although A_t may be expressed as a functional of X_t , this functional is not continuous and without introducing the second variable $v \in \mathcal{S}_t$, it is not possible to represent Examples 3, 4 and 5 as a left-continuous functional of x alone.

3.2 Obstructions to regularity

It is instructive to observe what prevents a functional from being regular in the sense of Definition 3.6. The examples below illustrate the fundamental obstructions to regularity:

Example 6 (Delayed functionals). Let $\epsilon > 0$. $F_t(x_t, v_t) = x(t - \epsilon)$ defines a $\mathbb{C}_b^{0,\infty}$ functional. All vertical derivatives are 0. However, F fails to be horizontally differentiable.

Example 7 (Jump of x at the current time). $F_t(x_t, v_t) = x(t) - x(t-)$ defines a functional which is infinitely differentiable and has regular pathwise derivatives:

$$\mathcal{D}_t F(x_t, v_t) = 0 \quad \nabla_x F_t(x_t, v_t) = 1 \quad (28)$$

However, the functional itself fails to be $\mathbb{C}_l^{0,0}$.

Example 8 (Jump of x at a fixed time). $F_t(x_t, v_t) = 1_{t \geq t_0}(x(t_0) - x(t_0-))$ defines a functional in $\mathbb{C}_l^{0,0}$ which admits horizontal and vertical derivatives at any order at each point (x, v) . However, $\nabla_x F_t(x_t, v_t) = 1_{t=t_0}$ fails to be either right- or left-continuous so F is not $\mathbb{C}^{0,1}$ in the sense of Definition 3.2.

Example 9 (Maximum). $F_t(x_t, v_t) = \sup_{s \leq t} x(s)$ is $\mathbb{C}_l^{0,0}$ but fails to be vertically differentiable on the set

$$\{(x_t, v_t) \in D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \quad x(t) = \sup_{s \leq t} x(s)\}.$$

4 Functional Ito calculus

4.1 Functional Ito formula

We are now ready to prove our first main result, which is a change of variable formula for non-anticipative functionals of a semimartingale [6, 9]:

Theorem 4.1. *For any non-anticipative functional $F \in \mathbb{C}_b^{1,2}$ verifying (6) and any $t \in [0, T)$,*

$$\begin{aligned} F_t(X_t, A_t) - F_0(X_0, A_0) &= \int_0^t \mathcal{D}_u F(X_u, A_u) du + \int_0^t \nabla_x F_u(X_u, A_u) \cdot dX(u) \\ &+ \int_0^t \frac{1}{2} \text{tr}({}^t \nabla_x^2 F_u(X_u, A_u) d[X](u)) \quad a.s. \end{aligned} \quad (29)$$

In particular, for any $F \in \mathbb{C}_b^{1,2}$, $Y(t) = F_t(X_t, A_t)$ is a semimartingale.

We note that:

- the dependence of F on the second variable A does not enter the formula (29). Indeed, under the assumption (6) variations in A lead to “higher order” terms which do not contribute to (29).
- (29) shows that, for a regular functional $F \in \mathbb{C}^{1,2}([0, T))$, the process $Y = F(X, A)$ may be reconstructed from the second-order jet $(\mathcal{D}F, \nabla_x F, \nabla_x^2 F)$ of F along the paths of X .

Proof. Let us first assume that X does not exit a compact set K and that $\|A\|_\infty \leq R$ for some $R > 0$. Let us introduce a sequence of random partitions $(\tau_k^n, k = 0..k(n))$ of $[0, t]$, by adding the jump times of A to the dyadic partition $(t_i^n = \frac{it}{2^n}, i = 0..2^n)$:

$$\tau_0^n = 0 \quad \tau_k^n = \inf\{s > \tau_{k-1}^n | 2^n s \in \mathbb{N} \text{ or } |A(s) - A(s-)| > \frac{1}{n}\} \wedge t \quad (30)$$

The following arguments apply pathwise. Lemma A.3 ensures that

$$\eta_n = \sup\{|A(u) - A(\tau_i^n)| + |X(u) - X(\tau_i^n)| + \frac{t}{2^n}, i \leq 2^n, u \in [\tau_i^n, \tau_{i+1}^n]\} \xrightarrow{n \rightarrow \infty} 0.$$

Denote ${}_n X = \sum_{i=0}^{\infty} X(\tau_{i+1}^n) 1_{[\tau_i^n, \tau_{i+1}^n)} + X(t) 1_{\{t\}}$ which is a cadlag piecewise constant approximation of X_t , and ${}_n A = \sum_{i=0}^{\infty} A(\tau_i^n) 1_{[\tau_i^n, \tau_{i+1}^n)} + A(t) 1_{\{t\}}$ which is an adapted cadlag piecewise constant approximation of A_t . Denote $h_i^n = \tau_{i+1}^n - \tau_i^n$. Start with the decomposition:

$$\begin{aligned} F_{\tau_{i+1}^n}({}_n X_{\tau_{i+1}^n}, {}_n A_{\tau_{i+1}^n}) - F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n}) &= F_{\tau_{i+1}^n}({}_n X_{\tau_{i+1}^n}, {}_n A_{\tau_i^n, h_i^n}) - F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n}) \\ &+ F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n}) - F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n}) \end{aligned} \quad (31)$$

where we have used the fact that F has predictable dependence in the second variable to have $F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n}) = F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n})$. The first term in can be written $\psi(h_i^n) - \psi(0)$ where:

$$\psi(u) = F_{\tau_i^n + u}({}_n X_{\tau_i^n, u}, {}_n A_{\tau_i^n, u}) \quad (32)$$

Since $F \in \mathbb{C}^{1,2}([0, T])$, ψ is right-differentiable and left-continuous by Lemma 2.5, so:

$$F_{\tau_{i+1}^n}({}_n X_{\tau_i^n, h_i^n}, {}_n A_{\tau_i^n, h_i^n}) - F_{\tau_i^n}({}_n X_{\tau_i^n}, {}_n A_{\tau_i^n}) = \int_0^{\tau_{i+1}^n - \tau_i^n} \mathcal{D}_{\tau_i^n + u} F({}_n X_{\tau_i^n, u}, {}_n A_{\tau_i^n, u}) du \quad (33)$$

The second term in (31) can be written $\phi(X(\tau_{i+1}^n) - X(\tau_i^n)) - \phi(0)$ where $\phi(u) = F_{\tau_i^n}({}_n X_{\tau_i^n, u}, {}_n A_{\tau_i^n})$. Since $F \in \mathbb{C}_b^{1,2}$, ϕ is a C^2 function and $\phi'(u) = \nabla_x F_{\tau_i^n}({}_n X_{\tau_i^n, u}, {}_n A_{\tau_i^n, h_i^n})$, $\phi''(u) = \nabla_x^2 F_{\tau_i^n}({}_n X_{\tau_i^n, u}, {}_n A_{\tau_i^n, h_i^n})$. Applying the Ito formula to ϕ between 0 and $\tau_{i+1}^n - \tau_i^n$ and the $(\mathcal{F}_{\tau_i^n + s})_{s \geq 0}$ continuous semimartingale $(X(\tau_i^n + s))_{s \geq 0}$, yields:

$$\begin{aligned} \phi(X(\tau_{i+1}^n) - X(\tau_i^n)) - \phi(0) &= \int_{\tau_i^n}^{\tau_{i+1}^n} \nabla_x F_{\tau_i^n}({}_n X_{\tau_i^n, X(s) - X(\tau_i^n)}, {}_n A_{\tau_i^n}) dX(s) \\ &+ \frac{1}{2} \int_{\tau_i^n}^{\tau_{i+1}^n} \text{tr} \left[{}^t \nabla_x^2 F_{\tau_i^n}({}_n X_{\tau_i^n, X(s) - X(\tau_i^n)}, {}_n A_{\tau_i^n}) d[X](s) \right] \end{aligned} \quad (34)$$

Summing over $i \geq 0$ and denoting $i(s)$ the index such that $s \in [\tau_{i(s)}^n, \tau_{i(s)+1}^n)$, we have shown:

$$\begin{aligned} F_t({}_n X_t, {}_n A_t) - F_0(X_0, A_0) &= \int_0^t \mathcal{D}_s F({}_n X_{\tau_{i(s)}^n, s - \tau_{i(s)}^n}, {}_n A_{\tau_{i(s)}^n, s - \tau_{i(s)}^n}) ds \\ &+ \int_0^t \nabla_x F_{\tau_{i(s)+1}^n}({}_n X_{\tau_{i(s)}^n, X(s) - X(\tau_{i(s)}^n)}, {}_n A_{\tau_{i(s)}^n, h_{i(s)}^n}) dX(s) \\ &+ \frac{1}{2} \int_0^t \text{tr} \left[\nabla_x^2 F_{\tau_{i(s)}^n}({}_n X_{\tau_{i(s)}^n, X(s) - X(\tau_{i(s)}^n)}, {}_n A_{\tau_{i(s)}^n}) \cdot d[X](s) \right] \end{aligned} \quad (35)$$

$F_t({}_n X_{t,n}, A_t)$ converges to $F_t(X_t, A_t)$ almost surely. Since all approximations of (X, A) appearing in the various integrals have a d_∞ -distance from (X_s, A_s) less than $\eta_n \rightarrow 0$, the continuity at fixed times of $\mathcal{D}F$ and left-continuity $\nabla_x F$, and $\nabla_x^2 F$ imply that the integrands appearing in the above integrals converge respectively to $\mathcal{D}_s F(X_s, A_s)$, $\nabla_x F_s(X_s, A_s)$, $\nabla_x^2 F_s(X_s, A_s)$ as $n \rightarrow \infty$. Since the derivatives are in \mathbb{B} the integrands in the various above integrals are bounded by a constant dependant only on F, K and R and t does not depend on s nor on ω . The dominated convergence and the dominated convergence theorem for the stochastic integrals [28, Ch.IV Theorem 32] then ensure that the Lebesgue-Stieltjes integrals converge almost surely, and the stochastic integral in probability, to the terms appearing in (29) as $n \rightarrow \infty$.

Consider now the general case where X and A may be unbounded. Let K_n be an increasing sequence of compact sets with $\bigcup_{n \geq 0} K_n = \mathbb{R}^d$ and denote the optional stopping times

$$\tau_n = \inf\{s < t \mid X_s \notin K^n \text{ or } |A_s| > n\} \wedge t.$$

Applying the previous result to the stopped process $(X_{t \wedge \tau_n}, A_{t \wedge \tau_n})$ and noting that, by (6), $F_t(X_t, A_t) = F_t(X_t, A_{t-})$ leads to:

$$\begin{aligned} F_t(X_{t \wedge \tau_n}, A_{t \wedge \tau_n}) - F_0(Z_0, A_0) &= \int_0^{t \wedge \tau_n} \mathcal{D}_u F_u(X_u, A_u) du + \frac{1}{2} \int_0^{t \wedge \tau_n} \text{tr}({}^t \nabla_x^2 F_u(X_u, A_u) d[X](u)) \\ &\quad + \int_0^{t \wedge \tau_n} \nabla_x F_u(X_u, A_u) \cdot dX + \int_{t \wedge \tau_n}^t D_u F(X_{u \wedge \tau_n}, A_{u \wedge \tau_n}) du \end{aligned}$$

The terms in the first line converges almost surely to the integral up to time t since $t \wedge \tau_n = t$ almost surely for n sufficiently large. For the same reason the last term converges almost surely to 0. \square

Remark 4.2. The above proof is probabilistic and makes use of the (classical) Ito formula [15]. In the companion paper [5] we give a non-probabilistic proof of Theorem 4.1, using the analytical approach of Föllmer [12], which allows X to have discontinuous (cadlag) trajectories.

Example 10. If $F_t(x_t, v_t) = f(t, x(t))$ where $f \in C^{1,2}([0, T] \times \mathbb{R}^d)$, (29) reduces to the standard Itô formula.

Example 11. For the functional in Example 5) $F_t(x_t, v_t) = e^{x(t) - \frac{1}{2} \int_0^t v(u) du}$, the formula (29) yields the well-known integral representation

$$\exp(X(t) - \frac{1}{2}[X](t)) = \int_0^t e^{X(u) - \frac{1}{2}[X](u)} dX(u) \quad (36)$$

An immediate corollary of Theorem 4.1 is that, if X is a local martingale, any $\mathbb{C}_b^{1,2}$ functional of X which has finite variation is equal to the integral of its horizontal derivative:

Corollary 4.3. *If X is a local martingale and $F \in \mathbb{C}_b^{1,2}$, the process $Y(t) = F_t(X_t, A_t)$ has finite variation if only if $\nabla_x F_t(X_t, A_t) = 0$ $d[X] \times d\mathbb{P}$ -almost everywhere.*

Proof. $Y(t)$ is a continuous semimartingale by Theorem 4.1, with semimartingale decomposition given by (29). If Y has finite variation, then by formula (29), its continuous martingale component should be zero i.e. $\int_0^t \nabla_x F_t(X_t, A_t) \cdot dX(t) = 0$ a.s. Computing its quadratic variation, we obtain

$$\int_0^T \text{tr}({}^t \nabla_x F_t(X_t, A_t) \cdot \nabla_x F_t(X_t, A_t) \cdot d[X]) = 0$$

which implies in particular that $\|\partial_i F_t(X_t, A_t)\|^2 = 0$ $d[X^i] \times d\mathbb{P}$ -almost everywhere for $i = 1..d$. Thus, $\nabla_x F_t(X_t, A_t) = 0$ for $(t, \omega) \notin A \subset [0, T] \times \Omega$ where $\int_A d[X^i] \times d\mathbb{P} = 0$ for $i = 1..d$. \square

4.2 Vertical derivative of an adapted process

For a (\mathcal{F}_t) -adapted process Y , the the functional representation (41) is not unique, and the vertical $\nabla_x F$ depends on the choice of representation F . However, Theorem 4.1 implies that the *process* $\nabla_x F_t(X_t, A_t)$ has an intrinsic character i.e. independent of the chosen representation:

Corollary 4.4. *Let $F^1, F^2 \in \mathbb{C}_b^{1,2}([0, T])$, such that:*

$$\forall t \in [0, T], \quad F_t^1(X_t, A_t) = F_t^2(X_t, A_t) \quad \mathbb{P} - a.s. \quad (37)$$

Then, outside an evanescent set:

$${}^t[\nabla_x F_t^1(X_t, A_t) - \nabla_x F_t^2(X_t, A_t)]A(t-)[\nabla_x F_t^1(X_t, A_t) - \nabla_x F_t^2(X_t, A_t)] = 0 \quad (38)$$

Proof. Let $X(t) = B(t) + M(t)$ where B is a continuous process with finite variation and M is a continuous local martingale. There exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and for $\omega \in \Omega$ the path of $t \mapsto X(t, \omega)$ is continuous and $t \mapsto A(t, \omega)$ is cadlag. Theorem 4.1 implies that the local martingale part of $0 = F^1(X_t, A_t) - F^2(X_t, A_t)$ can be written:

$$0 = \int_0^t [\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)] dM(u) \quad (39)$$

Considering its quadratic variation, we have, on Ω_1

$$0 = \int_0^t \frac{1}{2} {}^t[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]A(u-)[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]du \quad (40)$$

By Lemma 2.5 ($\nabla_x F^1(X_t, A_t) = \nabla_x F^1(X_{t-}, A_{t-})$) since X is continuous and F verifies (6). So on Ω_1 the integrand in (40) is left-continuous; therefore (40) implies that for $t < T$ and $\omega \in \Omega_1$,

$${}^t[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]A(u-)[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)] = 0$$

. \square

In the case where for all $t < T$, $A(t-)$ is almost surely positive definite, Corollary 4.4 allows to define intrinsically the pathwise derivative of a process Y which admits a functional representation $Y(t) = F_t(X_t, A_t)$:

Definition 4.5 (Vertical derivative of a process). Define $\mathcal{C}_b^{1,2}(X)$ the set of \mathcal{F}_t -adapted processes Y which admit a functional representation in $\mathbb{C}_b^{1,2}$:

$$\mathcal{C}_b^{1,2}(X) = \{Y, \quad \exists F \in \mathbb{C}_b^{1,2} \quad Y(t) = F_t(X_t, A_t) \quad \mathbb{P} - a.s.\} \quad (41)$$

If $A(t)$ is non-singular i.e. $\det(A(t)) \neq 0$ $dt \times d\mathbb{P}$ almost-everywhere then for any $Y \in \mathcal{C}_b^{1,2}(X)$, the predictable process:

$$\nabla_X Y(t) = \nabla_x F_t(X_t, A_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$ in the representation (41). We will call $\nabla_X Y$ the *vertical derivative* of Y with respect to X .

In particular this construction applies to the case where X is a standard Brownian motion, where $A = I_d$, so we obtain the existence of a vertical derivative process for $\mathbb{C}_b^{1,2}$ Brownian functionals:

Definition 4.6 (Vertical derivative of non-anticipative Brownian functionals). Let W be a standard d -dimensional Brownian motion. For any $Y \in \mathbb{C}_b^{1,2}(W)$ with representation $Y(t) = F_t(W_t, t)$, the predictable process

$$\nabla_W Y(t) = \nabla_x F_t(W_t, t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$.

5 Martingale representation formulas

Consider now the case where X is a Brownian martingale:

Assumption 5.1. $X(t) = X(0) + \int_0^t \sigma(u).dW(u)$ where σ is a process adapted to \mathcal{F}_t^W verifying

$$\det(\sigma(t)) \neq 0 \quad dt \times d\mathbb{P} - a.e. \quad (42)$$

The functional Ito formula (Theorem 4.1) then leads to an explicit martingale representation formula for \mathcal{F}_t -martingales in $\mathbb{C}_b^{1,2}(X)$. This result may be seen as a non-anticipative counterpart of the Clark-Haussmann-Ocone formula [4, 25, 14] and generalizes other constructive martingale representation formulas previously obtained using Markovian functionals [7, 10, 11, 17, 26], Malliavin calculus [2, 18, 14, 25, 24] or other techniques [1, 27].

Consider an \mathcal{F}_T measurable random variable H with $E|H| < \infty$ and consider the martingale $Y(t) = E[H|\mathcal{F}_t]$.

5.1 A martingale representation formula

If Y admits a representation $Y(t) = F_t(X_t, A_t)$ where $F \in \mathbb{C}_b^{1,2}$, we obtain the following stochastic integral representation for Y in terms of its derivative $\nabla_X Y$ with respect to X :

Theorem 5.2. *If $Y(t) = F_t(X_t, A_t)$ for some functional $F \in \mathbb{C}_b^{1,2}$, then:*

$$Y(T) = Y(0) + \int_0^T \nabla_x F_t(X_t, A_t) dX(t) = Y(0) + \int_0^T \nabla_X Y.dX \quad (43)$$

Note that regularity assumptions are not on $H = Y(T)$ but on the functionals $Y(t) = E[H|\mathcal{F}_t]$, $t < T$, which is typically more regular than H itself.

Proof. Theorem 4.1 implies that for $t \in [0, T)$:

$$\begin{aligned} Y(t) = & \left[\int_0^t \mathcal{D}_u F(X_u, A_u) du + \frac{1}{2} \int_0^t \text{tr} [{}^t \nabla_x^2 F_u(X_u, A_u) d[X](u)] \right. \\ & \left. + \int_0^t \nabla_x F_u(X_u, A_u) dX(u) \right] \end{aligned} \quad (44)$$

Given the regularity assumptions on F , the first term in this sum is a continuous process with finite variation while the second is a continuous local martingale. However, Y is a martingale and its

decomposition as sum of a finite variation process and a local martingale is unique [29]. Hence the first term is 0 and: $Y(t) = \int_0^t F_u(X_u, A_u) dX_u$. Since $F \in \mathbb{C}_l^{0,0}([0, T])$ $Y(t)$ has limit $F_T(X_T, A_T)$ as $t \rightarrow T$, so the stochastic integral also converges. \square

Example 12.

If $e^{X(t) - \frac{1}{2}[X](t)}$ is a martingale, applying Theorem 5.2 to the functional $F_t(x_t, v_t) = e^{x(t) - \int_0^t v(u) du}$ yields the familiar formula:

$$e^{X(t) - \frac{1}{2}[X](t)} = 1 + \int_0^t e^{X(s) - \frac{1}{2}[X](s)} dX(s) \quad (45)$$

5.2 Extension to square-integrable functionals

Let $\mathcal{L}^2(X)$ be the Hilbert space of progressively-measurable processes ϕ such that:

$$\|\phi\|_{\mathcal{L}^2(X)}^2 = E \left[\int_0^t \phi_s^2 d[X](s) \right] < \infty \quad (46)$$

and $\mathcal{I}^2(X)$ be the space of square-integrable stochastic integrals with respect to X :

$$\mathcal{I}^2(X) = \left\{ \int_0^\cdot \phi(t) dX(t), \phi \in \mathcal{L}^2(X) \right\} \quad (47)$$

endowed with the norm $\|Y\|_2^2 = E[Y(T)^2]$. The Ito integral $I_X : \phi \mapsto \int_0^\cdot \phi_s dX(s)$ is then a bijective isometry from $\mathcal{L}^2(X)$ to $\mathcal{I}^2(X)$.

We will now show that the operator $\nabla_X : \mathcal{I}^2(X) \rightarrow \mathcal{L}^2(X)$ admits a suitable extension $\mathcal{I}^2(X)$ which verifies

$$\forall \phi : \mathcal{I}^2(X) \in \mathcal{L}^2(X), \quad \nabla_X \left(\int \phi \cdot dX \right) = \phi, \quad dt \times d\mathbb{P} - a.s. \quad (48)$$

i.e. ∇_X is the inverse of the Ito stochastic integral with respect to X .

Definition 5.3 (Space of test processes). The space of *test processes* $D(X)$ is defined as

$$D(X) = \mathcal{C}_b^{1,2}(X) \cap \mathcal{I}^2(X) \quad (49)$$

Theorem 5.2 allows to define intrinsically the vertical derivative of a process in $D(X)$ as an element of $\mathcal{L}^2(X)$.

Definition 5.4. Let $Y \in D(X)$, define the process $\nabla_X Y \in \mathcal{L}^2(X)$ as the equivalence class of $\nabla_x F_t(X_t, A_t)$, which does not depend on the choice of the representation functional $Y(t) = F_t(X_t, A_t)$

Proposition 5.5 (Integration by parts on $D(X)$). *Let $Y, Z \in D(X)$. Then:*

$$E[Y(T)Z(T)] = E \left[\int_0^T \nabla_X Y(t) \nabla_X Z(t) d[X](t) \right] \quad (50)$$

Proof. Let $Y, Z \in D(X) \subset \mathcal{C}_b^{1,2}(X)$. Then Y, Z are martingales with $Y(0) = Z(0) = 0$ and $E[|Y(T)|^2] < \infty, E[|Z(T)|^2] < \infty$. Applying Theorem 5.2 to Y and Z , we obtain

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y dX \int_0^T \nabla_X Z dX\right]$$

Applying the Ito isometry formula yields the result. \square

Using this result, we can extend the operator ∇_X in a weak sense to a suitable space of the space of (square-integrable) stochastic integrals, where $\nabla_X Y$ is characterized by (50) being satisfied against all test processes.

The following definition introduces the Hilbert space $\mathcal{W}^{1,2}(X)$ of martingales on which ∇_X acts as a weak derivative, characterized by integration-by-part formula (50). This definition may be also viewed as a non-anticipative counterpart of Wiener-Sobolev spaces in the Malliavin calculus [22, 30].

Definition 5.6 (Martingale Sobolev space). The Martingale Sobolev space $\mathcal{W}^{1,2}(X)$ is defined as the closure in $\mathcal{I}^2(X)$ of $D(X)$.

The Martingale Sobolev space $\mathcal{W}^{1,2}(X)$ is in fact none other than $\mathcal{I}^2(X)$, the set of square-integrable stochastic integrals:

Lemma 5.7. $\{\nabla_X Y, Y \in D(X)\}$ is dense in $\mathcal{L}^2(X)$ and

$$\mathcal{W}^{1,2}(X) = \mathcal{I}^2(X).$$

Proof. We first observe that the set U of ‘‘cylindrical’’ processes of the form

$$\phi_{n,f,(t_1,\dots,t_n)}(t) = f(X(t_1), \dots, X(t_n))1_{t>t_n}$$

where $n \geq 1, 0 \leq t_1 < \dots < t_n \leq T$ and $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ is a total set in $\mathcal{L}^2(X)$ i.e. the linear span of U is dense in $\mathcal{L}^2(X)$. For such an integrand $\phi_{n,f,(t_1,\dots,t_n)}$, the stochastic integral with respect to X is given by the martingale

$$Y(t) = I_X(\phi_{n,f,(t_1,\dots,t_n)})(t) = F_t(X_t, A_t)$$

where the functional F is defined on Υ as:

$$F_t(x_t, v_t) = f(x(t_1-), \dots, x(t_n-))(x(t) - x(t_n))1_{t>t_n}$$

so that:

$$\nabla_x F_t(x_t, v_t) = f(x_{t_1-}, \dots, x_{t_n-})1_{t>t_n}, \nabla_x^2 F_t(x_t, v_t) = 0, \mathcal{D}_t F(x_t, v_t) = 0$$

which shows that $F \in \mathcal{C}_b^{1,2}$ (see Example 2). Hence, $Y \in \mathcal{C}_b^{1,2}(X)$. Since f is bounded, Y is obviously square integrable so $Y \in D(X)$. Hence $I_X(U) \subset D(X)$.

Since I_X is a bijective isometry from $\mathcal{L}^2(X)$ to $\mathcal{I}^2(X)$, the density of U in $\mathcal{L}^2(X)$ entails the density of $I_X(U)$ in $\mathcal{I}^2(X)$, so $\mathcal{W}^{1,2}(X) = \mathcal{I}^2(X)$. \square

Theorem 5.8 (Extension of ∇_X to $\mathcal{W}^{1,2}(X)$). *The vertical derivative $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$. Its closure defines a bijective isometry*

$$\begin{aligned} \nabla_X : \mathcal{W}^{1,2}(X) &\mapsto \mathcal{L}^2(X) \\ \int_0^\cdot \phi \cdot dX &\mapsto \phi \end{aligned} \quad (51)$$

characterized by the following integration by parts formula: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that

$$\forall Z \in D(X), \quad E[Y(T)Z(T)] = E \left[\int_0^T \nabla_X Y(t) \nabla_X Z(t) d[X](t) \right]. \quad (52)$$

In particular, ∇_X is the adjoint of the Ito stochastic integral

$$\begin{aligned} I_X : \mathcal{L}^2(X) &\mapsto \mathcal{W}^{1,2}(X) \\ \phi &\mapsto \int_0^\cdot \phi \cdot dX \end{aligned} \quad (53)$$

in the following sense:

$$\forall \phi \in \mathcal{L}^2(X), \quad \forall Y \in \mathcal{W}^{1,2}(X), \quad E[Y(T) \int_0^T \phi \cdot dX] = E \left[\int_0^T \nabla_X Y \phi d[X] \right] \quad (54)$$

Proof. Any $Y \in \mathcal{W}^{1,2}(X)$ may be written as $Y(t) = \int_0^t \phi(s) dX(s)$ with $\phi \in \mathcal{L}^2(X)$, which is uniquely defined $d[X] \times d\mathbb{P}$ a.e. The Ito isometry formula then guarantees that (52) holds for ϕ . To show that (52) uniquely characterizes ϕ , consider $\psi \in \mathcal{L}^2(X)$ which also satisfies (52), then, denoting $I_X(\psi) = \int_0^\cdot \psi dX$ its stochastic integral with respect to X , (52) then implies that

$$\forall Z \in D(X), \quad \langle I_X(\psi) - Y, Z \rangle_{\mathcal{W}^{1,2}(X)} = E \left[(Y(T) - \int_0^T \psi dX) Z(T) \right] = 0$$

which implies $I_X(\psi) = Y \cdot d[X] \times d\mathbb{P}$ a.e. since by construction $D(X)$ is dense in $\mathcal{W}^{1,2}(X)$. Hence, $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$.

This construction shows that $\nabla_X : \mathcal{W}^{1,2}(X) \mapsto \mathcal{L}^2(X)$ is a bijective isometry which coincides with the adjoint of the Ito integral on $\mathcal{W}^{1,2}(X)$. \square

Thus, the Ito integral I_X with respect to X

$$I_X : \mathcal{L}^2(X) \mapsto \mathcal{W}^{1,2}(X)$$

admits an inverse on $\mathcal{W}^{1,2}(X)$ which is an extension of the (pathwise) vertical derivative ∇_X operator introduced in Definition 3.2, and

$$\forall \phi \in \mathcal{L}^2(X), \quad \nabla_X \left(\int_0^\cdot \phi dX \right) = \phi \quad (55)$$

holds in the sense of equality in $\mathcal{L}^2(X)$.

The above results now allow us to state a general version of the martingale representation formula, valid for all square-integrable martingales:

Theorem 5.9 (Martingale representation formula: general case). *For any square-integrable \mathcal{F}_t^X -martingale Y ,*

$$Y(T) = Y(0) + \int_0^T \nabla_X Y dX \quad \mathbb{P} - a.s.$$

6 Relation with the Malliavin derivative

The above results hold in particular in the case where $X = W$ is a Brownian motion. In this case, the vertical derivative ∇_W may be related to the *Malliavin derivative* [22, 2, 3, 31] as follows.

Consider the canonical Wiener space $(\Omega_0 = C_0([0, T], \mathbb{R}^d), \|\cdot\|_\infty, \mathbb{P})$ endowed with its Borelian σ -algebra, the filtration of the canonical process. Consider an \mathcal{F}_T -measurable functional $H = H(X(t), t \in [0, T]) = H(X_T)$ with $E[|H|^2] < \infty$. If H is differentiable in the Malliavin sense [2, 22, 24, 31] e.g. $H \in \mathbf{D}^{1,2}$ with Malliavin derivative $\mathbb{D}_t H$, then the Clark-Haussmann-Ocone formula [25, 24] gives a stochastic integral representation of H in terms of the Malliavin derivative of H :

$$H = E[H] + \int_0^T {}^p E[\mathbb{D}_t H | \mathcal{F}_t] dW_t \quad (56)$$

where ${}^p E[\mathbb{D}_t H | \mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. This yields a stochastic integral representation of the martingale $Y(t) = E[H | \mathcal{F}_t]$:

$$Y(t) = E[H | \mathcal{F}_t] = E[H] + \int_0^t {}^p E[\mathbb{D}_u H | \mathcal{F}_u] dW_u$$

Related martingale representations have been obtained under a variety of conditions [2, 7, 11, 18, 26, 24].

Denote by

- $L^2([0, T] \times \Omega)$ the set of (anticipative) processes ϕ on $[0, T]$ with $E \int_0^T \|\phi(t)\|^2 dt < \infty$.
- \mathbb{D} the Malliavin derivative operator, which associates to a random variable $H \in \mathbf{D}^{1,2}(0, T)$ the (anticipative) process $(\mathbb{D}_t H)_{t \in [0, T]} \in L^2([0, T] \times \Omega)$.

Theorem 6.1 (Lifting theorem). *The following diagram is commutative in the sense of $dt \times d\mathbb{P}$ equality:*

$$\begin{array}{ccc} \mathcal{I}^2(W) & \xrightarrow{\nabla_W} & \mathcal{L}^2(W) \\ \uparrow (E[\cdot | \mathcal{F}_t])_{t \in [0, T]} & & \uparrow (E[\cdot | \mathcal{F}_t])_{t \in [0, T]} \\ \mathbf{D}^{1,2} & \xrightarrow{\mathbb{D}} & L^2([0, T] \times \Omega) \end{array}$$

In other words, the conditional expectation operator intertwines ∇_W with the Malliavin derivative:

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \quad \nabla_W (E[H | \mathcal{F}_t]) = E[\mathbb{D}_t H | \mathcal{F}_t] \quad (57)$$

Proof. The Clark-Haussmann-Ocone formula [25] gives

$$\forall H \in \mathbf{D}^{1,2}, \quad H = E[H] + \int_0^T {}^p E[\mathbb{D}_t H | \mathcal{F}_t] dW_t \quad (58)$$

where ${}^pE[\mathbb{D}_t H|\mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. On other hand theorem 5.2 gives:

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \quad H = E[H] + \int_0^T \nabla_W Y(t) dW(t) \quad (59)$$

where $Y(t) = E[H|\mathcal{F}_t]$. Hence ${}^pE[\mathbb{D}_t H|\mathcal{F}_t] = \nabla_W E[H|\mathcal{F}_t]$, $dt \times d\mathbb{P}$ almost everywhere. \square

Thus, the conditional expectation operator (more precisely: the *predictable* projection on \mathcal{F}_t) can be viewed as a morphism which “lifts” relations obtained in the framework of Malliavin calculus into relations between non-anticipative quantities, where the Malliavin derivative and the Skorokhod integral are replaced, respectively, by the vertical derivative ∇_W and the Ito stochastic integral.

From a computational viewpoint, unlike the Clark-Haussmann-Ocone representation which requires to simulate the *anticipative* process $\mathbb{D}_t H$ and compute conditional expectations, $\nabla_X Y$ only involves non-anticipative quantities which can be computed in a pathwise manner. It is thus more amenable to numerical computations. This topic is further explored in a forthcoming work.

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A Proof of Theorem 2.6

In order to prove theorem 2.6 in the general case where A is only required to be cadlag, we need the following three lemmas. The first lemma states a property analogous to 'uniform continuity' for cadlag functions:

Lemma A.1. *Let f be a cadlag function on $[0, T]$ and define $\Delta f(t) = f(t) - f(t-)$. Then*

$$\forall \epsilon > 0, \quad \exists \eta(\epsilon) > 0, \quad |x - y| \leq \eta \Rightarrow |f(x) - f(y)| \leq \epsilon + \sup_{t \in (x, y]} \{|\Delta f(t)|\} \quad (60)$$

Proof. If (60) does not hold, then there exists a sequence $(x_n, y_n)_{n \geq 1}$ such that $x_n \leq y_n$, $y_n - x_n \rightarrow 0$ but $|f(x_n) - f(y_n)| > \epsilon + \sup_{t \in [x_n, y_n]} \{|\Delta f(t)|\}$. We can extract a convergent subsequence $(x_{\psi(n)})$ such that $x_{\psi(n)} \rightarrow x$. Noting that either an infinity of terms of the sequence are less than x or an infinity are more than x , we can extract *monotone* subsequences $(u_n, v_n)_{n \geq 1}$ which converge to x . If $(u_n), (v_n)$ both converge to x from above or from below, $|f(u_n) - f(v_n)| \rightarrow 0$ which yields a contradiction. If one converges from above and the other from below, $\sup_{t \in [u_n, v_n]} \{|\Delta f(t)|\} > |\Delta f(x)|$ but $|f(u_n) - f(v_n)| \rightarrow |\Delta f(x)|$, which results in a contradiction as well. Therefore (60) must hold. \square

Lemma A.2. *If $\alpha \in \mathbb{R}$ and V is an adapted cadlag process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and σ is a optional time, then:*

$$\tau = \inf\{t > \sigma, \quad |V(t) - V(t-)| > \alpha\} \quad (61)$$

is a stopping time.

Proof. We can write that:

$$\{\tau \leq t\} = \bigcup_{q \in \mathbb{Q} \cap [0, t)} (\{\sigma \leq t - q\} \cap \{\sup_{t \in (t-q, t]} |V(u) - V(u-)| > \alpha\}) \quad (62)$$

and, using Lemma A.1,

$$\{\sup_{u \in (t-q, t]} |V(u) - V(u-)| > \alpha\} = \bigcup_{n_0 > 1} \bigcap_{n > n_0} \left\{ \sup_{1 \leq i \leq 2^n} |V(t - q \frac{i-1}{2^n}) - V(t - q \frac{i}{2^n})| > \alpha \right\}. \quad (63)$$

\square

Lemma A.3 (Uniform approximation of cadlag functions by step functions).

Let $f \in D([0, T], \mathbb{R}^d)$ and $\pi^n = (t_i^n)_{n \geq 1, i=0..k_n}$ a sequence of partitions ($0 = t_0^n < t_1 < \dots < t_{k_n}^n = T$) of $[0, T]$ such that:

$$\sup_{0 \leq i \leq k_n - 1} |t_{i+1}^n - t_i^n| \xrightarrow{n \rightarrow \infty} 0 \quad \sup_{u \in [0, T] \setminus \pi^n} |\Delta f(u)| \xrightarrow{n \rightarrow \infty} 0$$

then

$$\sup_{u \in [0, T]} |f(u) - \sum_{i=0}^{k_n-1} f(t_i^n) \mathbf{1}_{[t_i^n, t_{i+1}^n)}(u) + f(t_{k_n}^n) \mathbf{1}_{\{t_{k_n}^n\}}(u)| \xrightarrow{n \rightarrow \infty} 0 \quad (64)$$

Proof. Denote $h^n = f - \sum_{i=0}^{k_n-1} f(t_i^n) \mathbf{1}_{[t_i^n, t_{i+1}^n)} + f(t_{k_n}^n) \mathbf{1}_{\{t_{k_n}^n\}}$. Since $f - h^n$ is piecewise constant on π^n and $h^n(t_i^n) = 0$ by definition,

$$\sup_{t \in [0, T]} |h^n(t)| = \sup_{i=0..k_n-1} \sup_{[t_i^n, t_{i+1}^n)} |h^n(t)| = \sup_{t_i^n < t < t_{i+1}^n} |f(t) - f(t_i^n)|$$

Let $\epsilon > 0$. For $n \geq N$ sufficiently large, $\sup_{u \in [0, T] \setminus \pi^n} |\Delta f(u)| \leq \epsilon/2$ and $\sup_i |t_{i+1}^n - t_i^n| \leq \eta(\epsilon/2)$ using the notation of Lemma A.1. Then, applying Lemma A.1 to f we obtain, for $n \geq N$,

$$\sup_{t \in [t_i^n, t_{i+1}^n)} |f(t) - f(t_i^n)| \leq \frac{\epsilon}{2} + \sup_{t_i^n < t < t_{i+1}^n} |\Delta f(u)| \leq \epsilon.$$

□

We can now prove Theorem 2.6 in the case where A is a cadlag adapted process.

Proof of Theorem 2.6: Let us first show that $F_t(X_t, A_t)$ is adapted. Define:

$$\tau_0^N = 0 \quad \tau_k^N = \inf \{t > \tau_{k-1}^N | 2^N t \in \mathbb{N} \text{ or } |A(t) - A(t-)| > \frac{1}{N}\} \wedge t \quad (65)$$

From lemma A.2, τ_k^N are stopping times. Define the following piecewise constant approximations of X_t and A_t along the partition $(\tau_k^N, k \geq 0)$:

$$\begin{aligned} X^N(s) &= \sum_{k \geq 0} X_{\tau_k^N} \mathbf{1}_{[\tau_k^N, \tau_{k+1}^N)}(s) + X(t) \mathbf{1}_{\{t\}}(s) \\ A^N(s) &= \sum_{k=0} A_{\tau_k^N} \mathbf{1}_{[\tau_k^N, \tau_{k+1}^N)}(s) + A(t) \mathbf{1}_{\{t\}}(s) \end{aligned} \quad (66)$$

as well as their truncations of rank K :

$${}_K X^N(s) = \sum_{k=0}^K X_{\tau_k^N} \mathbf{1}_{[\tau_k^N, \tau_{k+1}^N)}(s) \quad {}_K A^N(t) = \sum_{k=0}^K A_{\tau_k^N} \mathbf{1}_{[\tau_k^N, \tau_{k+1}^N)}(t) \quad (67)$$

Since $({}_K X_t^N, {}_K A_t^N)$ coincides with (X_t^N, A_t^N) for K sufficiently large,

$$F_t(X_t^N, A_t^N) = \lim_{K \rightarrow \infty} F_t({}_K X_t^N, {}_K A_t^N). \quad (68)$$

The truncations $F_t^n(X_t^N, A_t^N)$ are \mathcal{F}_t -measurable as they are continuous functions of the random variables:

$$\{(X(\tau_k^N)1_{\tau_k^N \leq t}, A(\tau_k^N)1_{\tau_k^N \leq t}), k \leq K\}$$

so their limit $F_t(X_t^N, A_t^N)$ is also \mathcal{F}_t -measurable. Thanks to lemma A.3, X_t^N and A_t^N converge uniformly to X_t and A_t , hence $F_t(X_t^N, A_t^N)$ converges to $F_t(X_t, A_t)$ since $F_t : (D([0, t], \mathbb{R}^d) \times \mathcal{S}_t, \|\cdot\|_\infty) \rightarrow \mathbb{R}$ is continuous.

To show optionality of $Z(t)$ in point (ii), we will exhibit it as limit of right-continuous adapted processes. For $t \in [0, T]$, define $i^n(t)$ to be the integer such that $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$. Define the process: $Z_t^n = F_{\frac{(i^n(t))T}{n}}(X_{\frac{(i^n(t))T}{n}}, A_{\frac{(i^n(t))T}{n}})$, which is piecewise-constant and has right-continuous trajectories, and is also adapted by the first part of the theorem. Since $F \in \mathbb{C}_l^{0,0}$, $Z^n(t) \rightarrow Z(t)$ almost surely, which proves that Z is optional. Point (iii) follows from (i) and lemma 2.5, since in both cases $F_t(X_t, A_t) = F_t(X_{t-}, A_{t-})$ hence Z has left-continuous trajectories.