



**HAL**  
open science

# Spectral analysis of the Laplacian on geometrically finite hyperbolic manifolds

Colin Guillarmou, Rafe Mazzeo

► **To cite this version:**

Colin Guillarmou, Rafe Mazzeo. Spectral analysis of the Laplacian on geometrically finite hyperbolic manifolds. 2010. hal-00455589v1

**HAL Id: hal-00455589**

**<https://hal.science/hal-00455589v1>**

Preprint submitted on 10 Feb 2010 (v1), last revised 20 Aug 2012 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SPECTRAL ANALYSIS OF THE LAPLACIAN ON GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

COLIN GUILLARMOU AND RAFFAELLA MAZZEO

ABSTRACT. For geometrically finite hyperbolic manifolds  $\Gamma \backslash \mathbb{H}^{n+1}$ , we prove the meromorphic extension of the resolvent of Laplacian, Poincaré series, Eisenstein series and scattering operator to the whole complex plane. We also deduce the asymptotics of lattice points of  $\Gamma$  in large balls of  $\mathbb{H}^{n+1}$  in terms of the Hausdorff dimension of the limit set of  $\Gamma$ .

## 1. INTRODUCTION

Analysis of the Laplace operator on  $(n + 1)$ -dimensional hyperbolic manifolds which satisfy a geometric finiteness condition commenced in earnest in the early 1980's, inspired by numerous results in the finite volume setting, as well as some extensions of this, by Roelcke and Patterson, to infinite area geometrically finite surfaces. The paper of Lax and Phillips [16], see also [17], shows that the spectrum of the Laplacian on such spaces is equal to  $[n^2/4, \infty) \cup S$  where  $S \subset (0, n^2/4)$  is a finite set of  $L^2$ -eigenvalues, each of finite multiplicity. They also deduce sharp asymptotics of the lattice point counting function for a geometrically finite group of isometries  $\Gamma$  of  $\mathbb{H}^{n+1}$ , under certain assumptions on the dimension of the limit set of  $\Gamma$ . This was followed by an extensive development by many authors concerning the special class of geometrically finite quotients which are convex cocompact, i.e. where  $\Gamma$  has no parabolic elements. In particular, the second author and Melrose [19] proved in the slightly more general setting of conformally compact asymptotically hyperbolic metrics that the resolvent  $R_X(s) = (\Delta - s(n - s))^{-1}$  has a meromorphic continuation<sup>1</sup> to  $s \in \mathbb{C}$ , see [15] for a somewhat simpler proof in the purely constant curvature setting. This was a main step in the development of scattering theory on these spaces, which is an important area in its own right, but also a means to prove various trace formulæ, and a fundamental tool to analyze divisors of Selberg zeta function  $Z_X(s)$ . Analysis of these divisors for convex cocompact hyperbolic manifolds was carried out in great detail by Patterson-Perry [24] and Bunke-Olbrich [5]; a trace formula relating poles of resolvent (called *resonances*) and lengths of closed geodesics is also known [26, 12], the asymptotics for the counting function for lengths of closed geodesics is given in [25, 12], and estimates on the distributions of resonances in  $\mathbb{C}$  are proved in [15, 26, 1].

The two-dimensional case is special because the geometry is much simpler, and all of these results for geometrically finite surfaces are essentially contained in the work of Guillopé-Zworski [14] and Borthwick-Judge-Perry [3]; the book by Borthwick [2] contains a nice unified treatment of this material. In higher dimension, the geometry of

---

<sup>1</sup>see [9] for issues concerning the points  $s \in (n - \mathbb{N})/2$

nonmaximal rank cusps can be more complicated, as we explain below, and this makes the analysis substantially more delicate. The first work in this generality was by the second author and Phillips [20], where the spaces of  $L^2$  harmonic differential forms were studied and interpreted in topological terms. Subsequently, Froese-Hislop-Perry [7] proved the existence of a meromorphic extension to  $s \in \mathbb{C}$  of the scattering operator and resolvent for geometrically finite hyperbolic 3-dimensional manifolds, and recently the first author [10] extended this to higher dimensions when the cusps are ‘rational’ (defined below) and gave a bound on the counting function for resonances. All of these papers are fundamentally analytical. The case left open is where some of the nonmaximal rank cusps have irrational holonomy (which never happens in three dimensions, for example). By contrast, Bunke and Olbrich [6] developed representation theoretic methods to define the scattering operator in the general geometrically finite setting and proved its meromorphic extension to  $\mathbb{C}$ . Their paper is a revision of an older treatment they gave of this subject, but contains a substantially new exposition; nonetheless, this approach is technical and may be difficult to access for those without the representation theory background. In any case, their technique is of a completely different nature to ours and it is not even simple to compare the results.

Here we consider a general geometrically finite hyperbolic manifold  $X := \Gamma \backslash \mathbb{H}^{n+1}$  and give a rather short proof of the meromorphic extension of the resolvent of the Laplacian  $\Delta_X$  to  $s \in \mathbb{C}$ :

**Theorem 1.1.** *Let  $X = \Gamma \backslash \mathbb{H}^{n+1}$  be a smooth geometrically finite hyperbolic manifold and  $\Delta_X$  its Laplacian. Then the resolvent  $R_X(s) := (\Delta_X - s(n-s))^{-1}$ , defined initially as a bounded operator on  $L^2(X)$  when  $\{\operatorname{Re}(s) > n/2, s(n-s) \notin S\}$ , extends to a family of continuous mappings  $\mathcal{C}_0^\infty(X) \rightarrow \mathcal{C}^\infty(X)$  which depends meromorphically on  $s \in \mathbb{C}$ .*

In fact, we shall also give a description of fine mapping properties of  $R_X(s)$  in  $\operatorname{Re}(s) \geq n/2$  (thus in particular on the continuous spectrum), which gives a limiting absorption principle in this setting.

The proof is based as usual on a parametrix construction. As is standard in this subject, it is enough to construct local parametrices with this continuation property near every point of the conformal boundary at infinity (i.e.  $\Gamma \backslash \Omega_\Gamma$  where  $\Omega_\Gamma \subset S^n = \partial \mathbb{H}^{n+1}$  is the domain of discontinuity of  $\Gamma$ ); this is now standard at all but the ‘cusp points’, and hence the main step is to prove this result when  $\Gamma = \Gamma_c$  is a parabolic group of nonmaximal rank, which fixes a single point  $S^n = \partial \mathbb{H}^{n+1}$ . It is clear from our construction that, using [19], the same result holds for more general (non-constant curvature) asymptotically hyperbolic manifolds, which have certain neighbourhoods of infinity which are isometric to a constant curvature cusp model (see Remark 5.2), while the geometry at infinity at all other points is conformally compact, and by more classical methods still, we may also allow the existence of maximal rank cusps. In the interests of simplifying the presentation, we omit any further discussion of these generalizations.

One corollary concerns the Poincaré series and lattice point counting function for the group  $\Gamma$ . Recall that, given  $\Gamma$ , there exists  $\delta = \delta(\Gamma) \in (0, n)$  such that for any  $m, m' \in \mathbb{H}^{n+1}$ , the Poincaré series

$$P_s(m, m') := \sum_{\gamma \in \Gamma} e^{-sd(m, \gamma m')}$$

(where  $d(\cdot, \cdot) = d_{\mathbb{H}^{n+1}}(\cdot, \cdot)$  is distance in  $\mathbb{H}^{n+1}$ ) converges for  $\operatorname{Re}(s) > \delta$ ; by a famous result of Patterson [22] and Sullivan [31], this  $\delta$  is precisely the Hausdorff dimension of the limit set of  $\Gamma$ .

**Corollary 1.2.** *Let  $X = \Gamma \backslash \mathbb{H}^{n+1}$  be a geometrically finite hyperbolic manifold,  $m, m' \in \mathbb{H}^{n+1}$  and  $P_s(m, m')$  the corresponding Poincaré series. Then  $P_s(m, m')$  extends meromorphically from  $\{\operatorname{Re}(s) > \delta\}$  to  $s \in \mathbb{C}$  and*

$$(1.1) \quad \#\{\gamma \in \Gamma; d(m, \gamma m') \leq R\} \sim e^{\delta R} F(m) F(m')$$

as  $R \rightarrow \infty$ , for some function  $F$  on  $\mathbb{H}^{n+1}$  which can be expressed in terms of Patterson-Sullivan measure.

This result was known when  $\delta > n/2$  by the work of Lax-Phillips [16] (with exponential error terms) and in the convex cocompact case by Patterson [23] without any condition on  $\delta$ . Then Roblin [28], using Patterson-Sullivan theory, proves the asymptotic 1.1 of lattice points under weaker assumptions than ours; however he does not prove the continuation of the Poincaré series and his techniques are significantly different. In order to obtain error terms in (1.1) when  $\delta < n/2$ , we would need to prove a gap of resonances, which is for instance known in the convex cocompact case [21, 30].

In the final section, we define the Eisenstein series and scattering operator  $S_X(s)$ , which describe the asymptotic behaviour of generalized eigenfunctions of  $\Delta_X$  near infinity, and prove their meromorphic continuation to  $s \in \mathbb{C}$ . We also establish several typical functional equations for these operators and prove that  $S_X(s)$  is a pseudodifferential operator acting on the manifold  $B := \Gamma \backslash \Omega_\Gamma$ , where  $\Omega_\Gamma \subset S^n$  is the domain of discontinuity of  $\Gamma$ , whose complement on  $S^n$  is the limit set. The manifold  $B$  is the ‘boundary at infinity’ of  $X$ , and is the natural ‘locus’ of scattering. For convex cocompact quotients,  $B$  is compact (and the fact that  $S_X$  is pseudodifferential is well known), but in the geometrically finite case,  $B$  is noncompact with finitely many ends, each of which corresponds to a cusp of  $X$  and is identified with (the end of) a flat vector bundle over a compact flat manifold. We show that  $S_X(s)$  is equal to the sum of the complex power  $\Delta_B^s$  of the Laplacian  $\Delta_B$  and a residual term.

As explained below, if  $F$  is the flat bundle associated to an intermediate rank cusp of  $X$ , then the Laplacian  $\Delta_F$  decomposes as a countable direct sum of operators  $\oplus_I P_I$ , each acting on  $L^2(\mathbb{R}^+)$ , with  $\operatorname{spec}(P_I) = [b_I^2, \infty)$ . Suppose now that the parabolic subgroup  $\Gamma_c$  associated to that end has an irrational parabolic element  $\gamma$ , i.e. no power of  $\gamma$  is a pure translation on an associated horosphere, or equivalently, the flat bundle  $F$  has holonomy representation in  $O(n-k)$  with infinite image, where  $k$  is the rank of the cusp. In this case, in the spectral decomposition above, various subsequences of the  $b_I$  converge to 0 as the index  $I$  varies. The fact that the spectra of the  $P_I$  are not bounded away from zero creates fundamentally new complications, which are the root of the technical difficulties here, and is the reason that the spectral analysis in this general setting has not been treated analytically before.

We do not estimate the growth of the counting function of resonances here, but it is certainly possible to do this using our construction; this will be carried out elsewhere. It is likely that some Diophantine condition on the irrational elements of  $\Gamma$  may be needed to get polynomial growth on this counting function. All this should be a fundamental

step towards proving a Selberg trace formula and the extension of Selberg zeta function for general geometrically finite groups, and would also have applications to the study of the length distribution of geodesics.

## 2. GEOMETRY OF GEOMETRICALLY FINITE HYPERBOLIC MANIFOLDS

We view hyperbolic  $(n+1)$ -space  $\mathbb{H}^{n+1}$  either as the half-space  $\mathbb{R}_x^+ \times \mathbb{R}_y^n$  or as the open unit ball  $B^{n+1} \subset \mathbb{R}^{n+1}$ ; its natural smooth compactification  $\overline{\mathbb{H}}^{n+1} = \overline{B}^{n+1}$  is obtained by gluing on the unit sphere  $S^n \subset \mathbb{R}^{n+1}$ . Let  $\Gamma$  be a discrete torsion-free group of isometries of  $\mathbb{H}^{n+1}$ ,  $X := \Gamma \backslash \mathbb{H}^{n+1}$  is a smooth manifold. Note that  $\Gamma$  also acts on  $\overline{\mathbb{H}}^{n+1}$ , but it necessarily has fixed points on  $S^n$ . If  $m \in \mathbb{H}^{n+1}$ , then the set of accumulation points of the orbit  $\Gamma \cdot m$  in  $\overline{\mathbb{H}}^{n+1}$  is a closed subset  $\Lambda_\Gamma \subset S^n$  which is known as the *limit set* of  $\Gamma$ . Its complement  $\Omega_\Gamma := S^n \setminus \Lambda_\Gamma$  is called the *domain of discontinuity*, and  $\Gamma$  acts properly discontinuously in  $\Omega_\Gamma$ , with quotient  $B = \Gamma \backslash \Omega_\Gamma$ .

We now specialize to the setting where  $\Gamma$  is geometrically finite; good references for this include Bowditch [4] and the monograph by Ratcliffe [27, Chap. 12].

Since  $\Gamma$  has no fixed points in  $\mathbb{H}^{n+1}$ , any  $\gamma \in \Gamma$  has either one or two fixed points on  $S^n$ ; in the former case it is called *parabolic*, and in the latter, *hyperbolic* (or sometimes also *loxodromic*). In either case, the fixed point set of any  $\gamma \in \Gamma$  must lie in  $\Lambda_\Gamma$ . If  $p$  is the fixed point of a parabolic element, then the subgroup  $\Gamma_p \subset \Gamma$  stabilizing  $p$  contains only parabolic elements, and is called an *elementary parabolic group*. Conjugating by a suitable isometry, we may take  $p$  to be  $\infty$  in the upper half-space model. Then  $\Gamma_\infty$  acts on each horosphere  $E_a := \{x = a\} \simeq \mathbb{R}^n$  by Euclidean motions. It was proved by Bieberbach, see [4, Sec. 2.2] and [20], that there is a maximal normal abelian subgroup  $\Gamma'_\infty \subset \Gamma_\infty$  of finite index and an affine subspace  $Z \subset E_1$  of dimension  $k$ , invariant under  $\Gamma_\infty$ , such that  $\Gamma'_\infty$  acts as a group of translations of rank  $k$  on  $Z$ , and hence the quotient  $T' := \Gamma'_\infty \backslash Z$  is a  $k$ -dimensional flat torus. Choose an orthogonal decomposition  $E_1 = Z \times Y$ , with  $Y \simeq \mathbb{R}^{n-k}$  and associated coordinates  $(z, y)$ , so that each element  $\gamma \in \Gamma_\infty$  has the form

$$\gamma(x, y, z) = (x, A_\gamma y, R_\gamma z + b_\gamma), \quad b_\gamma \in \mathbb{R}^k, \quad R_\gamma \in O(n-k), A_\gamma \in O(k),$$

where for each  $\gamma$ ,  $R_\gamma^p = \text{Id}$  for some  $p \in \mathbb{N}$ , with  $m = 1$  if  $\gamma \in \Gamma'_\infty$ . If there exists  $m \in \mathbb{N}$  such that  $\gamma^m(x, y, z) = (x, y, z + c_\gamma)$  for some  $c_\gamma \in \mathbb{R}^k$ , then  $\gamma$  is called *rational*, which is equivalent to saying that  $R_\gamma^m = \text{Id}$  and  $A_\gamma^m = \text{Id}$  for some  $m \in \mathbb{N}$ . Otherwise  $\gamma$  is called *irrational*. The quotients  $\Gamma'_\infty \backslash \mathbb{H}^{n+1}$  and  $\Gamma_\infty \backslash \mathbb{H}^{n+1}$  are both of the form  $\mathbb{R}_x^+ \times F$  and  $\mathbb{R}_x^+ \times F'$  for some flat bundles  $F \rightarrow T$  and  $F' \rightarrow T'$ , where the bases  $T$  and  $T'$  are compact flat manifolds; here  $F = \Gamma'_\infty \backslash E_1$ ,  $F' = \Gamma_\infty \backslash E_1$  and  $T = \Gamma_\infty \backslash Z$ . The hyperbolic metric on  $\mathbb{H}^{n+1}$  descends to a hyperbolic metric  $g_X = (dx^2 + g_F)/x^2$  where  $g_F$  is a flat metric on the bundle  $F$  induced from the restriction of the hyperbolic metric to the horosphere  $\{x = 1\}$ .

Using the identification  $\mathbb{H}^{n+1} = \mathbb{R}^+ \times Z \times Y$ , define the subset of  $\overline{\mathbb{H}}^{n+1}$

$$C_\infty(R) := \{(x, z, y) \in [0, \infty) \times Z \times Y; x^2 + |y|^2 \geq R\};$$

this is invariant under  $\Gamma_\infty$  and is hyperbolically convex. It is called a *standard parabolic region* for  $\Gamma_\infty$ .

For any parabolic fixed point  $p$ , there exists an  $R > 0$  so that the parabolic region  $C_p(R)$  satisfies

$$C_p(R) \subset \mathbb{H}^{n+1} \cup \Omega_\Gamma, \quad \text{and} \quad \gamma C_p(R) \cap C_p(R) = \emptyset \text{ for all } \gamma \in \Gamma \setminus \Gamma_p.$$

If these conditions are satisfied, then  $C_p(R)$  descends to a set  $\mathcal{C}_p := \Gamma \setminus (\cup_{\gamma \in \Gamma} \gamma(C_p(R)))$ . This is contained in  $\Gamma \setminus (\mathbb{H}^{n+1} \cup \Omega_\Gamma)$  and has interior isometric to the interior of  $\Gamma_\infty \setminus C_p(R)$ . The set  $\mathcal{C}_p$  is called a *standard cusp region* associated to (the orbit of)  $p$ . The *rank of the cusp* is the rank of  $\Gamma'_\infty$ .

A *geometrically finite hyperbolic quotient* is a quotient  $\Gamma \setminus \mathbb{H}^{n+1}$  by a discrete group  $\Gamma$  such that  $\Gamma \setminus (\mathbb{H}^{n+1} \cup \Omega_\Gamma)$  has a decomposition into the union of a compact set  $K$  and a finite number of standard cusp regions. This is more general than requiring that there exist a convex finite-sided fundamental domain of  $\Gamma$ , although these conditions are equivalent when  $n \leq 3$  or if all  $\gamma \in \Gamma$  are rational (see Prop.5.6 and 5.7 in [4]).

The quotient manifold  $X = \Gamma \setminus \mathbb{H}^{n+1}$  with induced hyperbolic metric  $g$  is a complete noncompact hyperbolic manifold with  $n_c$  cusps, where  $n_c$  is the number of  $\Gamma$ -orbits of fixed points of the parabolic elements of  $\Gamma$ . The conjugacy class of the parabolic subgroup fixing the  $j^{\text{th}}$  cusp point is denoted  $\Gamma_j$ . By geometric finiteness,  $X$  has finitely many ends and there exist a covering of the ends of  $X$  of the form  $\{\mathcal{U}_j^r\}_{j \in J^r} \cup \{\mathcal{U}_j^c\}_{j \in J^c}$ ,  $|J^c| = n_c$ , so that  $X$  minus the union of all these sets is compact, each  $\mathcal{U}_j^r$  is isometric to a half-ball in  $\mathbb{H}^{n+1}$ , and each  $\mathcal{U}_j^c$  is isometric to a cusp region.  $\Gamma_j \setminus C_\infty(R_j)$ . We also assume that the  $\mathcal{U}_j^c$  are disjoint; they are called *cusp neighbourhoods*; the  $\mathcal{U}_j^r$  are called *regular neighbourhoods*.

We shall also consider the smooth manifold with boundary  $\overline{X} := \Gamma \setminus (\mathbb{H}^{n+1} \cup \Omega_\Gamma)$ , which is noncompact, with finitely many ends. There is a compactification of  $\overline{X}$  as a smooth compact manifold with corners, see [20], but we shall not need this here. The boundary  $\partial \overline{X} = \Gamma \setminus \Omega_\Gamma$  is also a noncompact manifold with  $n_c$  ends; if we denote by  $x$  be a boundary defining function of  $\partial \overline{X}$  in  $\overline{X}$  which extends the function  $x$  defined in each  $\mathcal{U}_j^c$  (as transferred to  $X$  via the appropriate isometry), then  $g_{\partial \overline{X}} := (x^2 g_X)|_{T\partial \overline{X}}$  defines a complete metric on  $\partial \overline{X}$  which is flat outside a compact set. Indeed, writing the hyperbolic metric  $g$  in  $\mathcal{U}_j^c$  as  $x^{-2}(dx^2 + g_{F_j})$ , then near the cusp point  $p_j$ ,  $g_{\partial \overline{X}}$  is the metric naturally induced from  $g_{F_j}$  at  $x = 0$ . This also induces a volume form  $dV_{\partial \overline{X}}$ .

We conclude this section with the description of several different function spaces which appear frequently below.

The first is the standard space  $\mathcal{C}_0^\infty(X)$  of compactly supported smooth functions. We also consider  $\mathcal{C}_0^\infty(\overline{X})$ , which by definition consists of compactly supported smooth functions on  $\overline{X}$ , so in particular the intersection of the support of any element  $\phi$  with a cusp neighbourhood  $\mathcal{U}_j^c$  lies in  $\{x^2 + |y|^2 \leq R\}$  for some  $R > R_j > 0$ . The functions in  $\mathcal{C}_0^\infty(\overline{X})$  which vanish to infinite order at the regular boundary  $\partial \overline{X}$  constitute the space  $\dot{\mathcal{C}}_0^\infty(\overline{X})$ .

We next define the  $L^2$ -based Sobolev spaces on  $\partial \overline{X}$  with respect to the metric  $g_{\partial \overline{X}}$ :

$$H^M(\partial \overline{X}) := \{f \in L^2(\partial \overline{X}, dV_{\partial \overline{X}}); \nabla_{\partial \overline{X}}^\ell f \in L^2(\partial \overline{X}, dv_{\partial \overline{X}}), \forall \ell \leq M\},$$

and their intersection  $H^\infty(\partial\bar{X}) = \cap_{M \geq 0} H^M(\partial\bar{X})$ . We also define

$$H^\infty(X) := \{f \in \mathcal{C}^\infty(\bar{X}); f|_{\mathcal{U}_j^c} \in \mathcal{C}_b^\infty([0, \infty)_x, H^\infty(F_j)) \text{ for all } j \in J^c\},$$

where we regard each cusp neighbourhood  $\mathcal{U}_j^c$  as lying in  $[0, \infty)_x \times F_j \simeq \Gamma_j \backslash \mathbb{H}^{n+1}$ , and where for any Frechet space  $E$ ,  $\mathcal{C}_b^\infty([0, \infty), E)$  denotes the set of smooth  $E$ -valued functions  $f$  with all derivatives  $\partial_x^\ell f$  bounded uniformly in  $x$  with respect to each seminorm of  $E$ . In particular, restrictions of functions in this space to  $\partial\bar{X}$  belong to  $H^\infty(\partial\bar{X})$ . Finally we define

$$\dot{H}^\infty(X) := \{f \in H^\infty(X); f = \mathcal{O}(x^\infty) \text{ as } x \rightarrow 0 \text{ and } f = \mathcal{O}(x^{-\infty}) \text{ as } x \rightarrow \infty\}.$$

### 3. SPECTRAL DECOMPOSITION WHEN $\Gamma$ IS AN ELEMENTARY PARABOLIC GROUP

We first tackle the spectral analysis of the Laplacian when  $\Gamma$  is a discrete elementary parabolic subgroup. As before, and using the notation introduced in §2, we assume that the parabolic fixed point is  $\infty$  in the upper half-space model. We use a type of Fourier decomposition for functions on the flat bundle  $F = \Gamma_\infty \backslash \mathbb{R}^n$ . This proceeds in two stages: we first obtain a discrete Fourier decomposition of functions on the compact spherical normal bundle  $SF$ , which is then coupled with a continuous Fourier-Bessel type decomposition for functions in the radial variable on the fibres of  $F$ ; together these reduce  $\Delta_F$  to a multiplication operator.

Recall that we have a maximal abelian normal subgroup  $\Gamma'_\infty \subset \Gamma_\infty$  of finite index and an affine  $k$ -dimensional subspace  $Z \subset E_1$ , where  $k$  is the rank of the cusp, and an orthogonal complement  $Y \simeq \mathbb{R}^{n-k}$  in  $E_1$ . Since  $[\Gamma_\infty : \Gamma'_\infty]$  is finite, it suffices to prove the meromorphic continuation of the resolvent on  $\Gamma'_\infty \backslash \mathbb{H}^{n+1}$ ; the resolvent on the quotient  $\Gamma_\infty \backslash \mathbb{H}^{n+1}$  is simply a finite sum of translates of the resolvent on  $\Gamma'_\infty \backslash \mathbb{H}^{n+1}$ . To simplify exposition we thus assume that  $\Gamma_\infty = \Gamma'_\infty$ . We can also clearly assume that each element of  $\Gamma_\infty$  preserves orientation. The group  $\Gamma_\infty$  is then freely generated by  $k$  elements  $\gamma_1, \dots, \gamma_k$  which can be written in the decomposition  $\mathbb{H}^{n+1} = \mathbb{R}_x^+ \times E_1$ ,  $E_1 = Z \oplus Y$ , as

$$\gamma_\ell(x, y, z) = (x, A_\ell y, z + v_\ell), \quad v_\ell \in \mathbb{R}^k, A_\ell \in \text{SO}(n-k).$$

In other words,  $\gamma_\ell$  is identified with the pair  $(v_\ell, A_\ell)$ . Since the  $A_\ell$  are orthogonal, the Euclidean metric on  $Y$  descends naturally to one on the fibres of  $F := \Gamma_\infty \backslash E_1$ ; in particular, the unit sphere bundle  $SF$  is well-defined.

To complete this picture,  $\{A_1, \dots, A_k\}$  is a commuting set of orthogonal matrices, so there is an orthogonal decomposition  $Y \simeq \mathbb{R}^{n-k} \cong V_0 \oplus \dots \oplus V_s$ , with  $\dim V_0 = r$  and  $\dim V_j = 2$ ,  $j \geq 1$ , such that  $A_\ell$  acts trivially on  $V_0$  and is a rotation by angle  $\theta_{j\ell}$  on  $V_j$  for every  $j, \ell$ . In other words, this decomposition puts each  $A_j$  into block form with each block either the identity or a rotation on each summand  $V_j$ .

Altogether, we have now described  $F$  as the total space of a vector bundle  $\mathcal{V}$  over a compact  $k$ -dimensional torus  $T$ , where  $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \dots \oplus \mathcal{V}_s$ ; here  $\mathcal{V}_0$  is a trivial bundle of rank  $r$  and all the other  $\mathcal{V}_j$  have rank 2. A function  $f$  on  $F$  is identified with a function  $f(z, y)$  on  $Z \oplus Y$  which satisfies

$$f(z + v_\ell, y) = f(z, A_\ell^{-1}y) = f(z, y_0, e^{-i\theta_{1\ell}}y_1, \dots, e^{-i\theta_{s\ell}}y_s), \quad \ell = 1, \dots, k,$$

where  $(y_0, y_1, \dots, y_s)$  are the components of  $y$  with respect to the splitting  $V_0 \oplus V_1 \oplus \dots \oplus V_s$ . We can describe functions  $f$  on  $SF$  in exactly the same way.

**3.1. Fourier decomposition on SF.** The flat Laplacian  $\Delta_Y$  on each fibre  $F_z \simeq Y$  defines an operator on the total space of  $F$  which acts fibrewise; its angular part is the Laplace operator on  $S^{n-k-1}$ . Let

$$(3.1) \quad L^2(S^{n-k-1}) = \bigoplus_{m=0}^{\infty} H_m$$

be the usual irreducible decomposition for the action of  $\mathrm{SO}(n-k)$ , so  $H_m$  is the space of spherical harmonics of degree  $m$ ,

$$H_m = \ker(\Delta_{S^{n-k-1}} - m(m+n-k-2));$$

we also set  $\dim H_m = \mu_m$ .

The vector spaces  $H_m$  on each fibre of  $SF$  (the unit sphere bundle of  $F$ ) fit together to form a flat vector bundle  $\mathcal{H}_m \rightarrow T$ . A section  $\sigma$  of  $\mathcal{H}_m$  is a function  $f(z, y)$  on  $SF$ , identified as above with a function on  $T \times S^{n-k-1}$  satisfying  $f(z + v_j, \omega) = f(z, A_j^{-1}\omega)$  such that for each  $z \in T$ ,  $\omega \mapsto f(z, \omega) \in H_m$ . (This makes sense since the action of  $\mathrm{SO}(n-k)$  preserves each  $H_m$ .) Now identify  $V_j \cong \mathbb{C}$ ,  $j = 1, \dots, s$ , so that  $A_j$  lies in the compact abelian subgroup  $K = \times_{j=1}^s \mathrm{U}(1)$ , which acts by the identity on the first  $r$  components in  $\mathbb{R}^{n-k}$ . The restriction of the irreducible representation of  $\mathrm{SO}(n-k)$  on  $H_m$  to  $K$  is a direct sum of one-dimensional irreducible representations:

$$(3.2) \quad H_m = L_1^{(m)} \oplus \dots \oplus L_{\mu_m}^{(m)},$$

where  $\theta = (\theta_1, \dots, \theta_s) \leftrightarrow (e^{i\theta_1}, \dots, e^{i\theta_s}) \in K$  acts on  $L_p^{(m)}$  by  $\exp(i\alpha_{mp}(\theta))$ ,  $p = 1, \dots, \mu_m$ . It can be shown that

$$\alpha_{mp}(\theta) = \sum_{\ell=1}^s c_{m p \ell} \theta_\ell$$

where the structure constants  $c_{m p \ell}$  are all integers determined by the specific representation  $L_p^{(m)}$ ; the precise formulæ for them are complicated to state and in any case not important here.

Because the  $A_j$  lie in  $K$ , each  $L_p^{(m)}$  determines a flat complex line bundle  $\mathcal{L}_p^{(m)}$  over  $T$ . By construction, a section  $f$  of  $\mathcal{L}_p^{(m)}$  corresponds to a function  $f(z, \omega)$  such that

$$f(z + v_j, \omega) = e^{i\alpha_{mpj}} f(z, \omega),$$

where, by definition,

$$(3.3) \quad \alpha_{mpj} = \alpha_{mp}(\theta_{1j}, \dots, \theta_{sj})$$

in terms of the holonomy angles  $\theta_{\ell j}$  of  $F$ ,

The orthogonal projection  $\Pi_m : L^2(S^{n-k-1}) \rightarrow H_m$  induces a map, which we still call  $\Pi_m$ , on  $L^2(SF)$ . Associated to (3.2) are orthogonal subprojectors  $\Pi_{mp}$ ,  $p = 1, \dots, \mu_m$ , so that  $\Pi_m = \bigoplus_p \Pi_{mp}$ . We often write  $f_{mp} = \Pi_{mp} f$ .



If  $f \in L^2(F)$ , then using polar coordinates on each fibre, we write

$$f(z, y) = f(z, r\omega) = \sum_{m=0}^{\infty} \sum_{p=1}^{\mu_m} f_{mp}(z, r, \omega).$$

Let  $\{v_1^*, \dots, v_k^*\}$  be the basis for the lattice  $\Lambda^*$  dual of  $\Lambda := \{\sum a_j v_j : a_j \in \mathbb{Z}\}$ , i.e.  $\langle v_i, v_j^* \rangle = \delta_{ij}$  for all  $i, j$ , and define

$$f_{mp}(z, r, \omega) = e^{2\pi i \langle z, A_{mp} \rangle} f_{mp}^{\#}(z, r, \omega), \quad \text{where} \quad 2\pi A_{mp} = \sum_{j=1}^k \alpha_{mpj} v_j^*.$$

Then each  $f_{mp}^{\#}$  is simply periodic,

$$f_{mp}^{\#}(z + v_j, r, \omega) = f_{mp}^{\#}(z, r, \omega), \quad j = 1, \dots, k,$$

hence standard Fourier series on  $T$  gives the decomposition

$$(3.4) \quad f_{mp}(z, r, \omega) = \frac{1}{(2\pi)^k} \sum_{v^* \in \Lambda^*} \hat{f}_{mpv^*}(r, \omega) e^{2\pi i \langle z, v^* + A_{mp} \rangle},$$

where the  $\hat{f}_{mpv^*}$  are the Fourier coefficients of  $f_{mp}^{\#}$ . Clearly

$$\int_F |f|^2 dV = \sum_{m=0}^{\infty} \sum_{p=1}^{\mu_m} \sum_{v^* \in \Lambda^*} \int_0^{\infty} \int_{S^{n-k-1}} |\hat{f}_{mpv^*}(r, \omega)|^2 r^{n-k-1} dr d\omega.$$

For simplicity below, we write  $I = (m, p, v^*)$ , so  $I$  ranges over the subset  $\mathcal{J} = \{(m, p, v^*) \in \mathbb{N} \times \mathbb{N} \times \Lambda^* : 1 \leq p \leq \mu_m\}$ , and denote by  $\Pi_I$  the corresponding orthogonal projector on  $L^2(SF)$  and  $\phi_I(z, \omega)$  the associated eigenfunction. We also simply write  $f_I$  instead of  $\hat{f}_I$ .

The Laplacian  $\Delta_F$  is induced from the standard Laplacian on  $T \times Y$  and has the polar coordinate representation

$$\Delta_F = -\partial_r^2 - \frac{n-k-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{n-k-1}} + \Delta_T.$$

For each  $I = (m, p, v^*)$  we have

$$(3.5) \quad (\Delta_F f)_I = \left( -\partial_r^2 - \frac{n-k-1}{r} \partial_r + \frac{m(m+n-k-2)}{r^2} + b_I^2 \right) f_I,$$

where

$$(3.6) \quad b_I = 2\pi |A_{mp} + v^*|.$$

In the following we let  $\Delta_I$  denote the operator on the right in (3.5) acting on the  $I^{\text{th}}$  component.

**3.2. The radial Fourier-Bessel decomposition.** Using the spectral decomposition on  $L^2(SF)$ , we have reduced  $\Delta_F$  to the family of ordinary differential operators  $\{\Delta_I\}_{I \in \mathcal{J}}$ . It follows from standard ODE theory that  $\text{spec}(\Delta_I) = [b_I^2, \infty)$ , and that this spectrum is purely absolutely continuous. Our next goal is to describe the continuous spectral decomposition associated to each  $\Delta_I$ . The fact that the threshold  $b_I^2$  depends on  $I$ , and in particular that the set of all such values has a sequence converging to 0 if the cusp is irrational is the cause of the main difficulties below when summing over  $I$ . However, for the moment, we are still analyzing each operator  $\Delta_I$  in turn.

The spectral decomposition for  $\Delta_I$  is determined by its spectral measure, which in turn, is given via Stone's formula in terms of the resolvent. Thus consider the family of equations

$$(\Delta_I - \lambda)f = 0, \quad \lambda \in \mathbb{C} \setminus [b_I^2, \infty).$$

If we conjugate with  $r^{-(n-k-2)/2}$  and set  $\lambda = t^2 + b_I^2$ , then this can be recognized as a Bessel equation:

$$\Delta_I - \lambda = -r^{-2} r^{-\frac{n-k-2}{2}} \left( (r\partial_r)^2 - \left( \frac{n-k-2}{2} + m \right)^2 + t^2 r^2 \right) r^{\frac{n-k-2}{2}},$$

so the space of homogeneous solutions is spanned by Bessel functions

$$r^{-\frac{n-k-2}{2}} J_{\frac{n-k-2}{2}+m}(rt) \quad \text{and} \quad r^{-\frac{n-k-2}{2}} H_{\frac{n-k-2}{2}+m}^{(1)}(rt).$$

The convention here is that  $\text{Im}(t) > 0$  when  $\lambda \in \mathbb{C} \setminus [b_I^2, \infty)$ , which corresponds to the choice  $\text{Im}(\sqrt{\mu}) > 0$  when  $\mu \in \mathbb{C} \setminus \mathbb{R}^+$ . The Schwartz kernel of the resolvent thus has the explicit expression ( $H$  is the Heaviside function)

$$\begin{aligned} (3.7) \quad R_I(t; r, r') &:= (\Delta_I - t^2 - b_I^2)^{-1} \\ &= (rr')^{-\frac{n-k-2}{2}} J_{\frac{n-k-2}{2}+m}(rt) H_{\frac{n-k-2}{2}+m}^{(1)}(r't) H(r' - r) \\ &\quad + (rr')^{-\frac{n-k-2}{2}} H_{\frac{n-k-2}{2}+m}^{(1)}(rt) J_{\frac{n-k-2}{2}+m}(r't) H(r - r'). \end{aligned}$$

From this, Stone's formula gives the spectral measure of  $\Delta_I - b_I^2$  as

$$\begin{aligned} (3.8) \quad d\Pi_I(t; r, r') &= \frac{1}{i\pi} (R_I(t; r, r') - R_I(-t; r, r')) t dt \\ &= \frac{2}{i\pi} (rr')^{-\frac{n-k-2}{2}} J_{\frac{n-k-2}{2}+m}(rt) J_{\frac{n-k-2}{2}+m}(r't) t dt. \end{aligned}$$

Finally, then, we have the spectral resolution of a function  $f_I \in L^2(\mathbb{R}^+; r^{n-k-1} dr)$ :

$$f_I(r) = \int^{\oplus} \tilde{f}_I(r, t) d\Pi_I(t), \quad \tilde{f}_I(r, t) = \int_{r'=0}^{\infty} f_I(r') d\Pi_I(t, r, r') dr'.$$

In the next subsection we invoke the functional calculus to define functions of  $\Delta_I$  by the formula

$$(3.9) \quad G(\Delta_I) = \int_0^{\infty} G(t^2 + b_I^2) d\Pi_I(t).$$

for a suitable class of functions  $G(t)$ .

4. THE RESOLVENT WHEN  $\Gamma$  IS AN ELEMENTARY PARABOLIC GROUP

We now turn to the construction and analysis of the resolvent of the Laplacian on the quotient  $X_c = \Gamma_c \backslash \mathbb{H}^{n+1} \simeq \mathbb{R}_x^+ \times F$  of hyperbolic space by an elementary parabolic group  $\Gamma_c$ . It is convenient to work with the unitarily equivalent operator

$$P = x^{-n/2} \Delta_{X_c} x^{n/2} = -(x\partial_x)^2 + x^2 \Delta_F + \frac{n^2}{4}$$

acting on  $L^2(\mathbb{R}^+ \times F; \frac{dx}{x} dv_F)$ . This decomposes into components

$$(4.1) \quad P = \bigoplus_I P_I; \quad P_I = -(x\partial_x)^2 + x^2 \Delta_I + \frac{n^2}{4}.$$

where  $\Delta_I$  is the operator of (3.5). These are each symmetric on  $L^2(\mathbb{R}_x^+ \times \mathbb{R}_r^+; r^{n-k-1} \frac{dx}{x} dr)$ . In the following we write

$$d\mu = r^{n-k-1} x^{-1} dx dr.$$

Using the same ODE formalism as above (i.e. Sturm-Liouville theory), along with the formula (3.9), we obtain the Schwartz kernel of the resolvent  $R(s) = (P - s(n-s))^{-1}$  of  $P$

$$(4.2) \quad \begin{aligned} R(s; x, r\omega, z, x', r'\omega', z') = & \\ & \sum_I \int_0^\infty \left( K_{s-n/2} \left( x\sqrt{t^2 + b_I^2} \right) I_{s-n/2} \left( x'\sqrt{t^2 + b_I^2} \right) H(x-x') \right. \\ & \left. + K_{s-n/2} \left( x'\sqrt{t^2 + b_I^2} \right) I_{s-n/2} \left( x\sqrt{t^2 + b_I^2} \right) H(x'-x) \right) \\ & \times \frac{i^{n-k+2m}}{\pi^2} J_{\frac{n-k-2}{2}+m}(rt) J_{\frac{n-k-2}{2}+m}(r't) t dt \phi_I(z, \omega) \phi_I(z', \omega'), \end{aligned}$$

which, we shall show below, is valid when  $\operatorname{Re}(s) > n/2$  as an operator acting on  $L^2$ .

**4.1. Continuation of the resolvent to  $\mathbb{C}$  in weighted  $L^2$  spaces.** We now show that the explicit formula (4.2) is the resolvent  $R(s)$  of  $P$  in  $\operatorname{Re}(s) > n/2$  and that it has a meromorphic continuation to the entire complex  $s$ -plane in weighted spaces.

**Proposition 4.1.** *The resolvent for  $P$  is given in  $\{\operatorname{Re}(s) > n/2\}$  by the expression (4.2) as a continuous operator on  $L^2(X^c, \frac{dx}{x} dv_F)$ . If  $\chi \in C_0^\infty([0, \infty) \times F)$ ,  $N > 0$  and  $\rho := x/(x+1)$ , then the operator  $\chi R(s) \chi$  extends from the half-plane  $\operatorname{Re}(s) > n/2$  to  $\operatorname{Re}(s) > n/2 - N$  as a holomorphic family of bounded operators from  $\rho^N L^2(\frac{dx}{x} dv_F)$  to  $\rho^{-N} L^2(\frac{dx}{x} dv_F)$ .*

*Proof.* Let us first prove the  $L^2$  boundedness in the physical half plane. For  $f \in L^2(X^c, \frac{dx}{x} dv_F)$ , we decompose  $f = \sum_I f_I(x, r) \phi_I(z, \omega)$  where  $f_I(x, r) \in L^2(\mathbb{R}^+ \times \mathbb{R}^+; d\mu)$ . Then

$$R(s)f(x, r\omega, z) = \sum_I (R_I(s)f_I)(x, r) \phi_I(z, \omega)$$

and it suffices to show that for  $\operatorname{Re}(s) > n/2$ ,

$$(4.3) \quad \|R_I(s)f_I\|_{L^2(d\mu)} \leq C \|f_I\|_{L^2(d\mu)}$$

for some  $C$  independent of  $I$ . We thus write

$$R_I(s)f_I(x, \cdot) = \int_0^\infty F_{s,x,x'}(\sqrt{P_I})f_I(x', \cdot) \frac{dx'}{x'},$$

where

$$F_{s,x,x'}(\tau) := K_{s-\frac{n}{2}}(x\tau)I_{s-\frac{n}{2}}(x'\tau)H(x-x') + I_{s-\frac{n}{2}}(x\tau)K_{s-\frac{n}{2}}(x'\tau)H(x'-x).$$

This function is holomorphic in  $\mathbb{C}$  as a function of  $s$ . Let us recall the well-known bounds for Bessel functions

$$(4.4) \quad \begin{aligned} |K_{s-\frac{n}{2}}(x\tau)| &\leq \begin{cases} C(x\tau)^{-|\operatorname{Re}(s)-n/2|} & \text{if } x\tau \leq 1 \\ Ce^{-x\tau}/\sqrt{x\tau} & \text{if } x\tau > 1 \end{cases} \\ |I_{s-\frac{n}{2}}(x\tau)| &\leq \begin{cases} C(x\tau)^{\operatorname{Re}(s)-n/2} & \text{if } x\tau \leq 1 \\ Ce^{x\tau}/\sqrt{x\tau} & \text{if } x\tau > 1, \end{cases} \end{aligned}$$

all for  $s \neq n/2$ . We thus estimate for  $\operatorname{Re}(s) \geq n/2$

$$|F_{s,x,x'}(\tau)| \leq \begin{cases} C(\min(x,x')/(\max(x,x'))^{\frac{1}{2}} & \text{if both } x\tau, x'\tau \geq 1 \\ C(\min(x,x')/\max(x,x'))^{\operatorname{Re}(s)-n/2} & \text{if both } x\tau, x'\tau < 1 \\ C(x'/x)^{\operatorname{Re}(s)-n/2} & \text{if } x\tau < 1 < x'\tau \\ C(x/x')^{\operatorname{Re}(s)-n/2} & \text{if } x'\tau < 1 < x\tau. \end{cases}$$

and in particular  $K_s(x, x') := \sup_{\tau \in \mathbb{R}^+} |F_{s,x,x'}(\tau)|$  is a kernel such that for  $\operatorname{Re}(s) > n/2$

$$\sup_{x \in \mathbb{R}^+} \int_0^\infty K_s(x, x') \frac{dx'}{x'} \leq C, \quad \sup_{x' \in \mathbb{R}^+} \int_0^\infty K_s(x, x') \frac{dx}{x} \leq C$$

for some  $C > 0$  depending on  $s$ . By Schur's lemma, it is the kernel of a bounded operator on  $L^2(dx/x)$  with norm less or equal to  $C$ . In particular this proves (4.3).

Now we study the continuation to  $s \in \mathbb{C}$ . We first decompose the set  $\mathcal{J}$  of indices  $mpv^*$  as  $\mathcal{J} = \mathcal{J}_> \cup \mathcal{J}_0$ , where

$$I \in \mathcal{J}_0 \iff b_I = 0$$

for  $b_I$  as in (3.6). Without loss of generality, we assume that  $\chi(x, y, z) = \varphi(x)\psi(r)$  with  $r = |y|$ , so that this is invariant under  $\Gamma_c$ , hence descends to  $X_c$ . Again for  $f \in L^2(X_c)$ ,  $f = \sum_I f_I(x, r)\phi_I(z, \omega)$  with  $f_I(r, z) \in L^2(\mathbb{R}^+ \times \mathbb{R}^+; d\mu)$ , then for  $N > 0$  fixed, we want to show

$$(4.5) \quad \|\chi\rho^N R_I(s)\rho^N \chi f_I\|_{L^2(d\mu)} \leq C\|f_I\|_{L^2(d\mu)}$$

with holomorphic dependance on  $s$ . Note that  $\rho$  can be replaced by  $x$  since  $x$  is bounded on  $\operatorname{supp}(\chi)$ . We write

$$(\chi R_I(s)\chi f_I)(x, r) = \int_0^\infty \chi(x, r)F_{s,x,x'}(\sqrt{P_I})(f_I\chi) d\mu,$$

and proceed as above. We estimate

$$(4.6) \quad |F_{s,x,x'}(\tau)| \leq \begin{cases} C & \text{if both } x\tau, x'\tau \geq 1 \\ C \max((xx'\tau^2)^{\operatorname{Re}(s)-n/2}, 1) & \text{if both } x\tau, x'\tau < 1 \\ C \max((x\tau)^{\operatorname{Re}(s)-n/2}, 1) & \text{if } x\tau < 1 < x'\tau \\ C \max((x'\tau)^{\operatorname{Re}(s)-n/2}, 1) & \text{if } x'\tau < 1 < x\tau. \end{cases}$$

where  $C$  depends on  $s$  only. If  $x, x' \in \text{supp}(\varphi)$ ,  $\text{Re}(s) > n/2 - N$  and  $\tau \geq b_I$ , then

$$(xx')^N |F_{s,x,x'}(\tau)| \leq C(1 + b_I^{2\text{Re}(s)-n}), \quad \text{if } I \in \mathcal{J}_>.$$

Hence, by the spectral theorem, since  $\sqrt{P_I} \geq b_I$ ,  $(xx')^N F_{s,x,x'}(\sqrt{P_I})$  is bounded on  $L^2(\mathbb{R}^+, r^{n-k-1} dr)$  with norm controlled by  $C(1 + b_I^{2\text{Re}(s)-n})$ .

Now fix  $\epsilon_0 > 0$ . Then there exists some  $m_0 \in \mathbb{N}$  such that if  $m \leq m_0$  and  $I \in \mathcal{J}_>$ , then  $b_I \geq \epsilon_0$ . Let  $\mathcal{J}_{m_0}$  denote the set of all  $I \in \mathcal{J}_>$  with  $m \leq m_0$ . By what we have just established, if  $I \in \mathcal{J}_{m_0}$ , then for  $\epsilon > 0$  small and  $\text{Re}(s) > n/2 - N + \epsilon$

$$\begin{aligned} \|x^{N-\epsilon} \psi R_I(s) \chi x^N f_I\|_{L_r^2}^2 &\leq \int_0^\infty \|x^{N-\epsilon} \psi F_{\lambda,x,x'}(P_I) x^{N-\epsilon} \psi f_I(x', \cdot)\|_{L_r^2} |\varphi(x')| (x')^\epsilon \frac{dx'}{x'} \\ &\leq C \int_0^\infty (x')^\epsilon |\varphi(x')| \|f(x', \cdot)\|_{L_r^2} \frac{dx'}{x'} \leq C \|f_I\|_{L^2(d\mu)} \end{aligned}$$

where  $L_r^2 := L^2(\mathbb{R}^+, r^{n-k-1} dr)$  and the constant  $C$  depends on  $\epsilon_0$ . But  $\varphi(x) x^\epsilon \|f_I\|_{L^2(d\mu)}$  is in  $L^2(\mathbb{R}^+, dx/x)$  with norm bounded by  $C \|f_I\|_{L^2(d\mu)}$  so we conclude that (4.5) is valid uniformly for  $I \in \mathcal{J}_{m_0}$ .

Now, the estimates on  $F_{s,x,x'}$  above also imply that for any  $I$  whatsoever,

$$\|(xx')^N \mathbb{1}_{[1,\infty)}(\sqrt{P_I}) F_{s,x,x'}(\sqrt{P_I})\| \leq C$$

as an operator on  $L^2(r^{n-k-1} dr)$ , so arguing just as above, we see that for all  $I$ ,

$$(4.7) \quad \|x^N \chi R_I(s) \mathbb{1}_{[1,\infty)}(\sqrt{P_I}) \chi x^N f\|_{L^2(d\mu)} \leq C \|f_I\|_{L^2(d\mu)}.$$

The final step, therefore, is to establish the uniform bound for  $\text{Re}(s) > n/2 - N + \epsilon$

$$\|x^{N-\epsilon} \psi \mathbb{1}_{(0,1)}(\sqrt{P_I}) F_{s,x,x'}(\sqrt{P_I}) \chi x^{N-\epsilon} f_I(x', \cdot)\|_{L^2(r^{n-k-1} dr)} \leq C \|f_I(x', \cdot)\|_{L^2(r^{n-k-1} dr)}$$

when  $I \notin \mathcal{J}_{m_0}$  and  $x, x' \in \text{supp}(\varphi)$ . Using (3.8), (3.9) and the bounds (4.6) we can estimate the Schwartz kernel for  $m \geq m_0$

$$\begin{aligned} |\psi(r) \psi(r') (xx')^{N-\epsilon} \mathbb{1}_{[0,1]}(\sqrt{P_I}) F_{s,x,x'}(\sqrt{P_I})(r, r')| \\ \leq C \sup_{m > m_0} \sup_{t \in (0,1)} |J_{\frac{n-k-2}{2}+m}(rt) J_{\frac{n-k-2}{2}+m}(r't)| (1 + t^{2\text{Re}(s)-n}), \end{aligned}$$

again uniformly in  $x, x' \in \text{supp}(\varphi)$ . But now, since  $r, r' \leq C$  in  $\text{supp} \psi$  and  $0 < t < 1$ , we have

$$|J_{\frac{n-k-2}{2}+m}(rt)| \leq \frac{(rt)^{\frac{n-k-2}{2}+m}}{\Gamma(\frac{n-k}{2} + m)},$$

and hence

$$\begin{aligned} |\psi(r) \psi(r') (xx')^{N-\epsilon} \mathbb{1}_{[0,1]}(\sqrt{P_I}) F_{s,x,x'}(\sqrt{P_I})(r, r')| \\ \leq C(1 + t^{n-k-2+2m-2N}) |\psi(r) \psi(r')|, \end{aligned}$$

uniformly in  $x, x' \in \text{supp} \varphi$ . This is bounded provided we have chosen  $m_0 \geq N - \frac{1}{2}(n - k - 2)$ . With this bound on the Schwartz kernel, we obtain directly that

$$(4.8) \quad \|x^N \chi R_I(s) \mathbb{1}_{[0,1]}(\sqrt{P_I}) \chi x^N f\|_{L^2(d\mu)} \leq C \|f_I\|_{L^2(d\mu)}.$$

Notice that the holomorphy in  $s \in \{\operatorname{Re}(s) > n/2 - N\}$  follows immediately from the holomorphy of  $K_{s-\frac{n}{2}}(z)$  and  $I_{s-\frac{n}{2}}(z)$  when  $z \in (0, \infty)$  and the fact that  $|\partial_s K_s(z)|$  and  $|\partial_s I_s(z)|$  satisfy the same type of bounds as  $|K_s(z)|, |I_s(z)|$  by Cauchy's formula.

It finally remains to deal with the terms with indices in  $\{I = mpv^* \in \mathcal{J}_0; m \leq m_0\}$ . If  $I = mpv^* \in \mathcal{J}_0$ , then  $\Delta_I$  acting on  $L^2(\mathbb{R}^+, r^{n-k-1} dr)$  is unitarily equivalent to the Laplacian acting on  $\mathbb{R}^{n-k}$  but restricted on the subspace  $L^2(\mathbb{R}^+, r^{n-k-1} dr; H_m)$  under the decomposition  $L^2(\mathbb{R}^{n-k}) = \bigoplus_{m=0}^{\infty} L^2(\mathbb{R}^+, r^{n-k-1} dr; H_m)$ . In addition, we can rewrite

$$P_I - s(n-s) = -(x\partial_x)^2 + x^2\Delta_I + \frac{(n-k)^2}{4} - t(n-k-t), \quad \text{with } t := s - k/2,$$

and hence deduce that  $x^{\frac{n-k}{2}} R_I(s) x^{-\frac{n-k}{2}}$  is unitarily equivalent, in  $\operatorname{Re}(s) > n/2$ , to  $R_{\mathbb{H}^{n-k+1}}(t) := (\Delta_{\mathbb{H}^{n-k+1}} - t(n-k-t))^{-1}$  acting on  $L^2(\mathbb{R}^+, \frac{dx}{x^{n+1}}; L^2(\mathbb{R}^+, r^{n-k-1} dr; H_m))$ , with  $t = s - k/2$ , under the decomposition

$$L^2(\mathbb{H}^{n-k+1}) = L^2(\mathbb{R}^+, \frac{dx}{x^{n+1}}; L^2(\mathbb{R}^{n-k})) \simeq \bigoplus_{m=0}^{\infty} L^2\left(\mathbb{R}^+, \frac{dx}{x^{n+1}}; L^2(\mathbb{R}^+, r^{n-k-1} dr; H_m)\right).$$

But it is known [15] that the resolvent  $x^N \chi R_{\mathbb{H}^{n-k+1}}(t) \chi x^N$  has a meromorphic (resp. holomorphic) extension if  $n-k+1$  is even (resp. odd), with simple poles at  $t \in -\mathbb{N}_0$  and finite rank residues. In particular, since  $\chi(x, r)$  commutes with the decomposition into spherical harmonics, this implies that  $x^N \chi R_I(s) \chi x^N$  has a meromorphic continuation with the same property (the poles then lie in  $k/2 - \mathbb{N}_0$ ). The proof is now complete.  $\square$

We also prove a technical lemma which is useful later, the proof of which follows the same lines as the argument above.

**Lemma 4.2.** *Let  $\varphi \in \mathcal{C}_0^\infty(X_c)$  and  $\chi \in \mathcal{C}_0^\infty([0, \infty) \times F)$ ; then for any  $N \in \mathbb{N}$ , there exist functions  $M_\ell(s) \in L^2(F)$  such that*

$$(4.9) \quad (\chi R_{X_c}(s)\varphi)(x, y, z) - \chi \sum_{\ell=0}^N x^{s+2k} M_\ell(s; y, z) \in x^{\operatorname{Re}(s)+2N} L^2(X_c)$$

and  $\Gamma(s - n/2 + \ell + 1) M_\ell(s)$  is meromorphic in  $s \in \mathbb{C}$ , with at most simple poles at  $s_0 \in k/2 - \mathbb{N}_0$ , and finite rank residues.

*Proof.* When  $\operatorname{Re}(s) > n/2$ ,  $x^{-s} R_{X_c}(s)\varphi$  can be written, using (4.2), as

$$(4.10) \quad (x^{-s} R_{X_c}(s)\varphi)(x, \cdot) = x^{\frac{n}{2}-s} \int_\epsilon^\infty I_{s-\frac{n}{2}}(x\sqrt{\Delta_F}) K_{s-\frac{n}{2}}(x'\sqrt{\Delta_F}) x'^{-\frac{n}{2}} \varphi(x', \cdot) \frac{dx'}{x'}$$

if  $\epsilon > 0$  is such that  $\operatorname{supp}(\varphi) \subset \{x > \epsilon\}$ . Now, for any  $N \in \mathbb{N}$  and  $\tau \in (0, \infty)$ , the modified Bessel function  $I_{s-n/2}(\tau)$  satisfies

$$\left| I_{s-n/2}(\tau) - (\tau/2)^{s-\frac{n}{2}} \sum_{\ell=0}^N \frac{2^{-2\ell} \tau^{2\ell}}{\ell! \Gamma(s - n/2 + \ell + 1)} \right| \leq C \min(\tau, 1)^{\operatorname{Re}(s)+2N+2} e^\tau$$

for some  $C$  depending on  $s$ . Then, by mimicking the proof of Proposition 4.1, we obtain directly that if we set

$$M_\ell(s) = \frac{2^{-2\ell}}{\ell! \Gamma(s - n/2 + \ell + 1)} \int_\epsilon^\infty \psi \Delta_F^\ell K_{s-n/2}(x' \sqrt{\Delta_F}) x'^{-\frac{n}{2}} \varphi(x', \cdot) \frac{dx'}{x'},$$

then (4.9) holds, where  $\psi \in \mathcal{C}_0^\infty(F)$  and  $\chi\psi = \chi$ ; the integral has meromorphic extension in  $s \in \mathbb{C}$  as a function in  $L^2(F)$  by the same arguments as in the proof of Proposition 4.1. Poles can arise only when  $I \in \mathcal{J}_0$ , hence lie in  $k/2 - \mathbb{N}_0$ , and their residues have finite rank.  $\square$

**4.2. Regularity of the Green kernel up to  $\partial\overline{X}$ .** We have now established that the family of operators  $R_{X_c}(s)$  has an analytic continuation to  $\mathbb{C}$ , albeit with a rather minimal description of the regularity of its integral kernel. Obviously, the structure of this kernel is standard in any compact set of  $X_c$  by usual elliptic regularity, but near infinity it needs a more involved approach. For the analysis in any relatively compact open set of  $\overline{X}_c$ , one can use the pseudodifferential 0-calculus from [19], but in fact, the simpler ‘constant curvature’ adaptation used by Guillopé-Zworski [15] is sufficient here.

To be more precise, let  $W := \{x^2 + r^2 \leq R^2\} \subset X_c$  and its partial closure  $\overline{W}_0 = W \cup (\overline{W} \cap \{x = 0\}) \subset \overline{X}_c$ .

**Proposition 4.3.** *Fix  $\psi_1, \psi_2 \in \mathcal{C}_0^\infty(\overline{W}_0)$  with disjoint supports; then  $\psi_2 R_{X_c}(s) \psi_1$  has an integral kernel which lies in  $x^s (x')^s \mathcal{C}_0^\infty(\overline{W}_0 \times \overline{W}_0)$ .*

*Proof.* We actually show something slightly more refined since we give a sharp characterization of the structure of the resolvent kernel in  $\overline{W}_0 \times \overline{W}_0$ . Recall the parametrix construction from [15]. Cover  $W$  by finitely many open charts  $\mathcal{U}_1, \dots, \mathcal{U}_\ell$  such that  $\overline{W} \subset \mathcal{U} := \cup_{j=1}^\ell \mathcal{U}_j$ , where each  $\mathcal{U}_j$  is identified by an isometry  $\iota_j$  to the half-ball  $B := \{(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}^n; x_0^2 + |y_0|^2 < 1\}$  in  $\mathbb{H}^{n+1}$  with hyperbolic metric  $(x_0)^{-2}(dx_0^2 + |dy_0|^2)$ .

Using this isometry, we can systematically identify operators on  $\mathcal{U}_j$  with their counterparts on  $B$ . In particular, we denote by  $R_j(s)$  the operator obtained in this way from the restriction to  $B$  of the meromorphically extended resolvent  $R_{\mathbb{H}^{n+1}}(s; m, m')$  on all of  $\mathbb{H}^{n+1}$ . This Schwartz kernel is a hypergeometric function of  $\cosh d(m, m')$ , depending meromorphically on  $s \in \mathbb{C}$ , with simple poles at  $-\mathbb{N}_0$  and finite rank residues if  $n$  is odd, but no poles if  $n$  is even. Now fix  $\chi_j, \hat{\chi}_j \in \mathcal{C}_0^\infty(\overline{W}_0)$  with  $\chi := \sum_{j=1}^\ell \chi_j$  equal to 1 on  $\overline{W}_0$  and  $\hat{\chi}_j = 1$  on the support of  $\chi_j$ .

As the initial parametrix we define  $Q_0(s) = \sum_{j=1}^\ell \hat{\chi}_j R_j(s) \chi_j$ . This satisfies

$$(\Delta_{X_c} - s(n-s))Q_0(s) = \chi + K_0(s), \quad K_0(s) := \sum_{j=1}^\ell [\Delta_{X_c}, \hat{\chi}_j] R_j(s) \chi_j.$$

The function  $\hat{\chi}_j$  can be taken as a smooth function of  $(x_0^2, y_0)$  in  $B$ . Furthermore, as a Schwartz kernel,

$$[\Delta_X, \hat{\chi}_j] R_j(s) \chi_j^r \psi(x_0, y_0; x'_0, y'_0) = x_0^{s+2} (x'_0)^s G(s; x_0^2, y_0, (x'_0)^2, y'_0)$$

for some function  $G(s; m)$  which is smooth in  $m \in ([0, 1] \times \mathbb{R}^n)^2$  and meromorphic in  $s$  with poles of finite multiplicities at  $-\mathbb{N}_0$  if  $n$  is even.

Next, for any smooth function  $F$  on  $[0, 1] \times \mathbb{R}^n$ , we have

$$(4.11) \quad (\Delta_{\mathbb{H}^{n+1}} - s(n-s)) \frac{x^{s+2\ell} F(x_0^2, y_0)}{2\ell(n-2s-2\ell)} = x_0^{s+2\ell} (F(x_0^2, y_0) + x_0^2 H_{s,\ell}(x_0^2, y_0)),$$

where  $H_{s,\ell} \in \mathcal{C}^\infty([0, 1] \times \mathbb{R}^n)$  depends meromorphically on  $s$  with first order poles lying in  $n/2 - \mathbb{N}$ . This ‘indicial equation’ allows us to inductively solve away terms in the Taylor expansion in  $x$  of the Schwartz kernel of  $K_0(s)$ . In other words, for any  $N > 0$  we can construct a kernel  $Q_{N,j}(s)$  such that  $(x_0 x_0')^{-s} Q_{N,j}(s)$  is a smooth function of  $(x_0^2, y_0, (x_0')^2, y_0')$  down to  $x_0 = 0$  and  $x_0' = 0$ , and

$$(\Delta_{X_c} - s(n-s)) Q_{N,j}(s) = \chi_j + K_{N,j}(s)$$

for some  $K_{N,j}(s) \in x^{s+2N} (x')^s C^\infty(\overline{\mathcal{U}}_j \times \overline{\mathcal{U}}_j)$ , which are therefore compact on  $x^N L^2$ . Moreover, the poles of  $Q_{N,j}(s), K_{N,j}(s)$  are contained in  $-\mathbb{N}_0$  and have finite rank residues. Finally, set  $Q_N(s) := \sum_{j=1}^\ell Q_{N,j}(s)$ , so that

$$(\Delta_{X_c} - s(n-s)) Q_N(s) = \chi + K_N(s), \quad K_N(s) := \sum_{j=1}^\ell K_{N,j}(s),$$

where  $K_N(s) \in x^{s+2N} x'^s C_0^\infty(\overline{\mathcal{U}}_0 \times \overline{\mathcal{U}}_0)$  (here  $\mathcal{U}_0 := \mathcal{U} \cup (\overline{\mathcal{U}} \cap \{x = 0\})$  is the partial closure of  $\mathcal{U}$ ). We can go further and use Borel’s lemma to construct an asymptotic limit  $Q_\infty(s)$  of the  $Q_N$ , which satisfies

$$(\Delta_{X_c} - s(n-s)) Q_\infty(s) = \chi + K_\infty(s), \quad K_\infty(s) \in x^\infty (x')^s \mathcal{C}_0^\infty(\overline{\mathcal{U}}_0 \times \overline{\mathcal{U}}_0)$$

Exchanging the functions  $\hat{\chi}_j$  and  $\chi_j$  and applying the same method yields a right parametrix, i.e. operators  $Q'_\infty(s)$  and  $K'_\infty(s)$  such that

$$Q'_\infty(s) (\Delta_{X_c} - s(n-s)) = \chi + K'_\infty(s), \quad K'_\infty(s) \in x^s (x')^\infty \mathcal{C}_0^\infty(\overline{\mathcal{U}}_0),$$

where  $Q'_\infty(s)$  and  $K'_\infty(s)$  have the same meromorphic properties as  $Q_\infty(s)$  and  $K_\infty(s)$  with respect to  $s$ . The operators  $Q_\infty(s), Q'_\infty(s)$  are elements of the space  $\Psi_0^{-2,s,s}(\overline{\mathcal{U}}_0)$  in the calculus of 0-pseudodifferential operators.

The error terms  $K_\infty(s)$  and  $K'_\infty(s)$  are residual in the sense that for any  $N > |\operatorname{Re}(s)|$  and any  $N' \geq 0$ ,

$$(4.12) \quad K_\infty(s) : x^N L^2(\mathcal{U}) \longrightarrow x^{N'} \mathcal{C}_0^\infty(\overline{\mathcal{U}}_0), \quad K'_\infty(s) : x^{-N'} L^2(\mathcal{U}) \longrightarrow x^s \mathcal{C}_0^\infty(\overline{\mathcal{U}}_0)$$

are bounded. By a standard argument,  $R(s)$  agrees with  $Q_\infty(s)$  up to some more regular term: indeed, when  $\operatorname{Re}(s) > n/2$ , we have that

$$\begin{aligned} R_{X_c}(s) (\Delta_{X_c} - s(n-s)) Q_\infty(s) &= Q_\infty(s) = R_{X_c}(s) (\chi - K_\infty(s)) \\ Q'_\infty(s) (\Delta_{X_c} - s(n-s)) R_{X_c}(s) &= Q'_\infty(s) = (\chi - K'_\infty(s)) R_{X_c}(s) \end{aligned}$$

which shows that

$$\chi R_{X_c}(s) \chi = \chi Q_\infty(s) + Q'_\infty(s) K_\infty(s) + K'_\infty(s) R_{X_c}(s) K_\infty(s)$$

Using Proposition 4.1 and (4.12), this last identity extends meromorphically to  $s \in \mathbb{C}$ . Moreover, [18, Th. 3.18] shows that  $Q'_\infty(s) K_\infty(s)$  and  $K'_\infty(s) R_{X_c}(s) K_\infty(s)$  are residual, hence

$$(4.13) \quad \chi R_{X_c}(s) \chi - \chi Q_\infty(s) \in x^s (x')^s \mathcal{C}_0^\infty(\overline{\mathcal{U}}_0 \times \overline{\mathcal{U}}_0).$$



for all  $s \in \mathbb{C} \setminus -\mathbb{N}_0$ . To finish the proof it suffices to note that  $Q_\infty(s)$  is a sum of explicit terms, each of which are in  $x^s(x')^s \mathcal{C}_0^\infty(\overline{\mathcal{U}}_0 \times \overline{\mathcal{U}}_0 \setminus \text{diag})$ .

We refer to [15] for all further details.  $\square$

## 5. CONTINUATION OF THE RESOLVENT ON $X$

In this section we pass from the ‘local’ result, i.e. the continuation of the resolvent on the model cusp  $X_c$  to its continuation on an arbitrarily geometrically finite quotient  $X$ . This argument is essentially a recapitulation of the proof of Proposition 4.3.

As before, let  $x$  be a smooth function on  $X$  which equals the upper half-space coordinate  $x$  in each cusp neighbourhood  $\mathcal{U}_j^c$  and which is a global boundary defining function on  $\overline{X}$ . We also use  $\rho = x/(1+x)$ , which is still a boundary defining function, but is bounded in the cusp neighbourhoods.

**Theorem 5.1.** *Let  $X = \Gamma \backslash \mathbb{H}^{n+1}$  be geometrically finite and  $R_X(s) := (\Delta_X - s(n-s))^{-1}$  the resolvent of  $\Delta_X$ , defined as a bounded operator on  $L^2(X)$  for  $\text{Re}(s) > n/2$  and  $s(n-s) \notin \sigma_{\text{pp}}(\Delta_X)$ . Fix  $\psi \in \mathcal{C}_0^\infty(\overline{X})$ ; then for each  $N > 0$ ,  $\psi R_X(s)\psi : \rho^N L^2(X) \rightarrow \rho^{-N} L^2(X)$  extends as a bounded operator meromorphically to  $\{\text{Re}(s) > n/2 - N\}$  with all poles of finite rank.*

*Proof.* We use the regular and cusp neighbourhoods  $\mathcal{U}_j^r$  and  $\mathcal{U}_j^c$ , and a corresponding subordinate partition of unity  $\{\chi_0, \chi_j^{r/c}\}$  for this cover, with  $\chi_0 \in \mathcal{C}_0^\infty(X)$  and  $\text{supp } \chi_j^{r/c} \subset \mathcal{U}_j^{r/c}$ . We also choose functions  $\hat{\chi}_0 \in \mathcal{C}_0^\infty(X)$ ,  $\hat{\chi}_j^{r/c} \in \mathcal{C}^\infty(X)$  with similar supports which are equal to 1 on the supports of the corresponding ‘unhatted’ functions.

Let  $R_j^r(s)$  denote the kernel of the resolvent on  $\mathbb{H}^{n+1}$ , restricted to a standard half-ball  $B$  and transferred back to  $\mathcal{U}_j^r$ , and  $R_j^c(s)$  the kernel of the resolvent on the model cusp  $\Gamma_j \backslash \mathbb{H}^{n+1}$  as constructed in the previous section, again transferred back to  $\mathcal{U}_j^c$ . (Here  $\Gamma_j$  is a representative of the conjugacy class of the parabolic subgroup fixing the  $j$ -th cusp.)

Fix a large value  $s_0 \gg n$  and some  $\psi \in \mathcal{C}^\infty(\overline{X})$  which equals 1 on the supports of  $\nabla \hat{\chi}_0$  and every  $\nabla \hat{\chi}_j^{r/c}$  and with compact support in  $\overline{X}$ . As in §4.2, the initial parametrix is

$$Q_0(s) := \hat{\chi}_0 R_X(s_0) \chi_0 + \sum_{j \in J^r} \hat{\chi}_j^r R_j^r(s) \chi_j^r + \sum_{j \in J^c} \hat{\chi}_j^c R_j^c(s) \chi_j^c.$$

The point of the first term, of course, is simply to capture the interior singularity of the resolvent, whereas the other terms also capture the dependence of the parametrix in  $s$  near all boundaries. This satisfies

$$(\Delta_X - s(n-s))Q_0(s)\psi = \psi(1 + \psi L_0(s)\psi + \psi K_0(s)\psi)$$

where

$$K_0(s) := \sum_{j \in J^r} K_{0,j}^r(s) + \sum_{j \in J^c} K_{0,j}^c(s) \quad \text{with } K_{0,j}^{r/c} := \sum_{j \in J^{r/c}} [\Delta_X, \hat{\chi}_j^{r/c}] R_j^{r/c}(s) \chi_j^{r/c},$$

$$L_0(s) := [\Delta_X, \hat{\chi}_0] R_X(s_0) \chi_0 + (s_0(n-s_0) - s(n-s)) \hat{\chi}_0 R_X(s_0) \chi_0.$$

Since  $\chi_0, \hat{\chi}_0$  are compactly supported,  $L_0(s)$  is compact on any weighted space  $\rho^N L^2(X)$ .

The next step is to improve the part coming from the sum of the  $K_{0,j}^r(s)$  terms as in the proof of Proposition 4.3. We construct operators  $Q_{N,j}^r(s)$  supported in  $\mathcal{U}_j^r \times \mathcal{U}_j^r$  which satisfy

$$(\Delta_X - s(n-s))Q_{N,j}^r(s) = \chi_j^r + K_{N,j}^r(s)$$

with  $K_{N,j}^r(s) \in x_0^{s+2N}(x'_0)^s \mathcal{C}^\infty(\overline{\mathcal{U}}_j^r \times \overline{\mathcal{U}}_j^r)$ . Poles of  $Q_{N,j}^r(s), K_{N,j}^r(s)$  lie in  $-\mathbb{N}_0$  with finite rank residues, and furthermore the error terms  $K_{N,j}^r(s)$  are compact on  $\rho^N L^2$ .

To improve the parametrix in the cusp neighbourhoods, we use that  $[\Delta_X, \chi_j^c]$  is a first order differential operator with coefficients supported in a neighbourhood  $\mathcal{V}_j^c \subset \mathcal{U}_j^c$  which is relatively compact in  $\overline{X}$ . Using the coordinates  $(x, y, z)$  in the model cusp,

$$\Delta_X = -(x\partial_x)^2 + nx\partial_x + x^2(\Delta_y + \Delta_z),$$

we can certainly take  $\hat{\chi}_j^p$  to be a smooth function of  $(x^2, |y|^2)$  (in particular, independent of  $y/|y|$  and  $z$ ), and hence

$$(5.1) \quad [\Delta_X, \hat{\chi}_j^c] = x^2(a_j(x^2, y) + b_j(x^2, y)\partial_x + [\Delta_y, \chi_j^c](x^2, y))$$

where  $a_j$  and  $b_j$  are smooth as functions of  $x^2$  and  $y$ . Using Proposition 4.3, we deduce directly that

$$K_{0,j}^c(s) = [\Delta_X, \hat{\chi}_j^c]R_j^c(s)\chi_j^c \in x^{s+2}(x')^s \mathcal{C}^\infty(\overline{W}_j \times \overline{W}_j)$$

where  $W_j \subset \mathcal{U}_j^c$  is some open set containing  $\text{supp}(\hat{\chi}_j^c)$  and  $\overline{W}_j$  is its partial closure which includes its boundary at  $x = 0$ .

We now claim that, with  $m = (x, y, z)$ , there is an expansion as  $x \rightarrow 0$  of the form

$$(5.2) \quad K_{0,j}^c(s; m, m') = \psi(m) \sum_{\ell=1}^N x^{s+2\ell} M_{j,\ell}(s; y, z, m') + \mathcal{O}(x^{\text{Re}(s)+2N})$$

for some  $\psi \in \mathcal{C}_0^\infty(\overline{W}_j)$  which equals 1 on  $\text{supp}(\hat{\chi}_j^c)$ , and with coefficient functions  $M_{j,\ell}(s) \in (x')^s \mathcal{C}_0^\infty((\overline{W}_j \cap \{x=0\}) \times \overline{W}_j)$  such that  $\Gamma(s - n/2 + \ell)M_{j,\ell}(s)$  is meromorphic in  $s \in \mathbb{C}$  with at most simple poles at  $s \in \frac{n}{2} - \frac{1}{2}\mathbb{N}$ , and with finite rank residues. Indeed, we already know that  $x^{-s-2}K_{0,j}^c(s; m, m')$  has an expansion as  $x \rightarrow 0$  in powers of  $x$  and with meromorphic coefficients. We must then check that only even powers of  $x$  occur and that we can factor out the Gamma function from each term. However, taking any  $\varphi \in \mathcal{C}_0^\infty(\mathcal{U}_j^c)$  and  $\psi \in \mathcal{C}_0^\infty(\overline{W}_j)$ , then we can deduce these properties from Lemma 4.2 using the expansions at  $x = 0$  of  $x^{-s}\psi(R_j(s)\varphi)$  and of (5.1).

As before, we solve away the expansion of  $K_j^c(s; m, m')$  at  $x = 0$ : for any  $k \in \mathbb{N}_0$  and  $F \in \mathcal{C}^\infty(\overline{W}_j)$  which depends smoothly on  $x^2$ , we have

$$(5.3) \quad (\Delta_X - s(n-s)) \frac{x^{s+2k} F(x^2, y, z)}{2k(n-2s-2k)} = x^{s+2k} F(x^2, y, z) + \frac{x^{s+2k+2} H_k(x^2, y, z)}{2k(n-2s-2k)}$$

with  $H_k \in \mathcal{C}^\infty(\overline{W}_j)$  smooth in  $x^2$  and independent of  $s$ . Combining (5.3) with (5.2) and the fact that each  $\Gamma(s - n/2 - \ell)M_{j,\ell}(s)$  has at most first order poles with finite rank residues, we can construct for all  $N \in \mathbb{N}$  an operator  $Q_{N,j}^c(s)$  which is holomorphic in  $\{\text{Re}(s) > n/2 - N\}$  with  $Q_{N,j}^c(s) - \hat{\chi}_j^c R_j^c(s)\chi_j^c \in x^{s+2N}(x')^s \mathcal{C}_0^\infty(\overline{W}_j \times \overline{W}_j)$  and

$$(\Delta_X - s(n-s))Q_{N,j}^c(s) = \chi_j^c + K_{N,j}^c(s)$$

for some  $K_{N,j}^c(s) \in x^{s+2N}(x')^s \mathcal{C}_0^\infty(\overline{W}_j \times \overline{W}_j)$  holomorphic in  $\{\operatorname{Re}(s) > n/2 - N\}$ .

We finally obtain a good parametrix

$$Q_N(s) := \sum_{j \in J^r} Q_{N,j}^r(s) + \sum_{j \in J^c} Q_{N,j}^c(s) + \hat{\chi}_0 R(s_0) \chi_0$$

since

$$(\Delta_X - s(n-s))Q_N(s)\psi = \psi(1 + \psi(L_0(s) + K_N(s))\psi) =: \psi(1 + \tilde{K}_N(s))$$

$$\text{with } K_N(s) = \sum_{j \in J^c} K_{N,j}^c(s) + \sum_{j \in J^r} K_{N,j}^r(s) \in x^{s+2N}(x')^s \mathcal{C}_0^\infty(\overline{X} \times \overline{X}).$$

Notice that the support of  $K_N(s)$  is compactly supported and does not intersect the cusps in either set of variables. Hence  $\tilde{K}_N(s)$  is compact on  $\rho^N L^2(X) \subset L^2(X)$  and meromorphic in  $\{\operatorname{Re}(s) > n/2 - N\}$  with poles of finite multiplicity. Using standard arguments, we can modify  $Q_N(s)$  by a finite rank operator if  $1 + \tilde{K}_N(s_0)$  is not invertible, to make the new remainder invertible, and this can be done without changing the regularity properties of  $Q_N(s)$ ,  $\tilde{K}_N(s)$ .

We then invoke the analytic Fredholm theorem to show that  $(1 + \tilde{K}_N(s))^{-1}$  has a meromorphic extension to  $\{\operatorname{Re}(s) > n/2 - N\}$  with poles of finite multiplicity, as an operator bounded on  $\rho^N L^2(X)$ . Thus

$$\psi R_X(s)\psi = \psi Q_N(s)\psi(1 + \tilde{K}_N(s))^{-1}$$

gives the meromorphic extension of the resolvent in  $s \in \{\operatorname{Re}(s) > n/2 - N; s \notin n/2 - \mathbb{N}\}$ , as an operator from  $\rho^N L^2(X)$  to  $\rho^{-N} L^2(X)$ .

As in §4.2 again, we can obtain the extension to all of  $\mathbb{C}$  directly, rather than only to any half-plane, by using Borel summation solve away the entire expansion as  $x \rightarrow 0$ , and a standard pseudodifferential parametrix construction to correct the compactly supported error part  $L_0(s)$ . This yields  $Q_\infty(s)$ , with the all same properties as  $Q_N(s)$ , which satisfies

$$(\Delta_X - s(n-s))Q_\infty(s)\psi = \psi(1 + K_\infty(s))$$

where  $K_\infty(s) \in \rho^\infty(\rho')^s \mathcal{C}_0^\infty(\overline{X} \times \overline{X})$  is a residual term with support contained in  $\operatorname{supp}(\psi) \times \operatorname{supp}(\psi)$  and  $(1 + \psi K_\infty(s_0)\psi)$  invertible. By the analytic Fredholm theorem again,

$$R_X(s)\psi = Q_\infty(s)\psi(1 + K_\infty(s))^{-1}$$

and this is meromorphic in  $s \in \mathbb{C}$  with poles of finite multiplicity. The term  $(1 + K_\infty(s))^{-1}$  can be written as  $(1 + S_\infty(s))$  where  $S_\infty(s) = -K_\infty(s) + K_\infty(s)(1 + K_\infty(s))^{-1}K_\infty(s)$  lies in  $\rho^\infty(\rho')^s \mathcal{C}_0^\infty(\overline{X} \times \overline{X})$  and has support contained in  $\operatorname{supp}(\psi) \times \operatorname{supp}(\psi)$ . This gives that

$$(5.4) \quad R_X(s)\psi = Q_\infty(s)\psi + Q_\infty(s)\psi S_\infty(s)$$

and the operator  $R_X(s)\psi$  has the same mapping properties as  $Q_\infty(s)\psi$ .  $\square$

**Remark 5.2.** *Let  $(X, g)$  be a manifold which admits a decomposition  $X = K \cup_{i \in J^c} E_j^c \cup_{j \in J^r} E_j^r$  where  $(K, g)$  is a smooth compact manifold with boundary,  $(E_j^c, g)$  are isometric to standard cusp neighbourhoods (and thus have constant curvature) and  $(E_j^r, g)$*

are isometric to

$$\{(x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^n; x_0^2 + |y_0|^2 \leq 1\} \text{ with metric } g = (dx_0^2 + h(x_0^2))/x_0^2$$

where  $u \in [0, 1] \rightarrow h(u)$  is a one parameter smooth family of tensors on  $\{|y| \leq 1\}$ . Combining this with the parametrization construction from [19], the same proof as above yields the meromorphic continuation of the resolvent  $R(s)$  of  $\Delta_g$  to  $s \in \mathbb{C}$ , with poles of finite multiplicities.

## 6. FINER DESCRIPTION OF THE RESOLVENT

We have shown that the resolvent  $R_X(s)$  continues meromorphically in  $s$  as an operator acting on weighted  $L^2$  spaces. As part of this, we obtained detailed information about the Schwartz kernel  $R_X(s; m; m')$  on any compact region  $K \subset \overline{X} \times \overline{X}$ . We now show how to obtain alternate descriptions of  $R_X(s)$  valid in certain regions of  $\mathbb{C}$  and which are more precise in certain asymptotic regimes: first, we obtain a representation of  $R_X(s)$  as a sum of translates by group elements of the free space resolvent, which converges when  $\operatorname{Re}(s) > (n-1)/2$  and provides good asymptotics in the cusp region; after that, we examine its Fourier analytic description more closely to obtain better information about asymptotics on  $\partial\overline{X}$  when  $s$  lies in the closed half-plane  $\{\operatorname{Re}(s) \geq n/2\}$ .

**6.1. The resolvent  $R_X(s)$  as a sum over  $\Gamma$ .** Let  $X_c = \Gamma_c \backslash \mathbb{H}^{n+1}$  with  $\Gamma_c$  an elementary parabolic group of rank  $k$ , fixing  $\infty$ , with generators  $(\gamma_1, \dots, \gamma_k)$ . As in §2, we assume (by passing to a finite index subgroup) that each  $\gamma_j$  acts on  $(x, y, z) \in \mathbb{H}^{n+1}$  by  $\gamma_j(x, y, z) = (x, A_j y, z + v_j)$  where  $z_j \in \mathbb{R}^k$  and  $A_j \in SO(n-k)$ . For any  $\gamma \in \Gamma_c$ , we also write  $\gamma(x, y, z) = (x, A_\gamma y, z + v_\gamma)$  where  $v_\gamma$  is in the lattice generated by the  $v_j$ . A convenient fundamental domain for this action is  $\mathcal{F} = \mathbb{R}^+ \times \mathbb{R}^{n-k} \times \mathcal{F}_T$ , where  $\mathcal{F}_T$  is a (compact) fundamental Dirichlet domain for the induced lattice on  $\mathbb{R}^k$ . We sometimes abuse notation by identifying a point  $w \in \mathcal{F}$  with its image in  $X_c$ . Both  $x$  and  $r = |y|$  descend to  $X_c$ .

**Proposition 6.1.** *If  $w, w' \in \mathcal{F}$ , then the resolvent kernel  $R_{X_c}(s; w, w')$  can be written as*

$$(6.1) \quad R_{X_c}(s; w, w') = \sum_{\gamma \in \Gamma_c} R_{\mathbb{H}^{n+1}}(s; w, \gamma w');$$

this converges in  $\mathcal{C}^\infty$  (apart from the diagonal singularity) locally on compact sets of  $\mathcal{F} \times \mathcal{F}$  for  $\operatorname{Re}(s) > k/2$  and agrees with the continuation of  $R_{X_c}(s; w, w')$  there. If  $\chi \in \mathcal{C}_0^\infty(\overline{X_c})$  and  $\psi \in \mathcal{C}^\infty(\overline{X_c})$  have disjoint supports, then for  $\operatorname{Re}(s) > k/2$  and any  $j \geq 0$  and multi-indices  $\alpha, \beta$ , there exists  $C > 0$  such that

$$|\partial_x^j \partial_y^\alpha \partial_z^\beta [(xx')^{-s} \psi(w) R_{X_c}(s; w, w') \chi(w')]| \leq C(1 + x^2 + |y|^2)^{(k-|\alpha|-|\beta|-j)/2 - \operatorname{Re}(s)}.$$

*Proof.* If  $w = (x, y, z), w' = (x', y', z') \in \mathcal{F}$  and  $d(w, w')$  is hyperbolic distance, then

$$\cosh d(w, w') = \frac{x^2 + (x')^2 + |y - y'|^2 + |z - z'|^2}{2xx'} := \frac{1}{\theta(w, w')}.$$

Furthermore, it is well known (see [19]) that the free space resolvent is a function of this elementary point-pair invariant:

$$(6.2) \quad R_{\mathbb{H}^{n+1}}(s; w, w') = \theta(w, w')^s F_s(\theta(w, w')),$$

where  $F_s \in \mathcal{C}^\infty([0, 1])$  depends holomorphically on  $s$  in  $\operatorname{Re}(s) > 0$ , has a polyhomogeneous singularity at  $\theta = 1$  with leading term  $(\theta - 1)^{1-n}$  and  $F_s(0) \neq 0$ . Any  $\gamma \in \Gamma_c$  acts by  $(x, y, z) \mapsto (x, A_\gamma y, z + z_\gamma)$ , where  $z_\gamma$  is in the lattice generated by the  $z_j$ . Note that there exists a subset  $\Gamma'_c \subset \Gamma_c$  with  $\Gamma_c \setminus \Gamma'_c$  finite and an  $\epsilon > 0$  such that  $|z - z' - z_\gamma|^2 \geq \epsilon |z_\gamma|^2$  for all  $\gamma \in \Gamma'_c$ . This gives that for  $\gamma \in \Gamma'_c$

$$\begin{aligned} \frac{1}{\theta(w, \gamma w')} - 1 &= \frac{(x - x')^2 + |y - A_\gamma y'|^2 + |z - z' - z_\gamma|^2}{2xx'} \\ &\geq \frac{(x - x')^2 + |y - A_\gamma y'|^2 + \epsilon^2 \langle z_\gamma \rangle^2}{2xx'}. \end{aligned}$$

This is clearly bounded below uniformly by a positive constant depending on  $L$  if  $x' \leq L$  and so is  $\theta(w, \gamma w') - 1$ . Consequently, for  $x' \leq L$ , we get

$$(6.3) \quad \begin{aligned} \left| (xx')^{-s} \sum_{\gamma \in \Gamma_c} R_{\mathbb{H}^{n+1}}(s; w, \gamma w') \right| &\leq \\ C \sum_{\gamma \in \Gamma'_c} (x^2 + (x')^2 + |y - A_\gamma y'|^2 + |z - z' - z_\gamma|^2)^{-\operatorname{Re}(s)} & \end{aligned}$$

for some  $C$  depending on  $s$  and  $L$  only. If in addition we assume that  $(w, w') \in \mathcal{F} \times \mathcal{F}$  are such that  $(x')^2 + |y'|^2 \leq L^2$  and  $x^2 + |y|^2 \leq 4L^2$ , then the series on the right in (6.3) converges uniformly in  $w, w'$  as long as  $\operatorname{Re}(s) > k/2$ , and for  $w$  and  $w'$  subject to these constraints, is bounded by a constant. If  $(w, w') \in \mathcal{F} \times \mathcal{F}$  are such that  $(x')^2 + |y'|^2 \leq L^2$  and  $x^2 + |y|^2 > 4L^2$ , then we obtain the estimate for  $\operatorname{Re}(s) > k/2$

$$\begin{aligned} \left| (xx')^{-s} \sum_{\gamma \in \Gamma_c} R_{\mathbb{H}^{n+1}}(s; w, \gamma w') \right| &\leq C(1 + x^2 + |y|^2)^{-\operatorname{Re}(s)} \sum_{\gamma \in \Gamma'_c} \left(1 + \frac{\langle z_\gamma \rangle^2}{x^2 + |y|^2}\right)^{-\operatorname{Re}(s)} \\ &\leq C(1 + x^2 + |y|^2)^{k/2 - \operatorname{Re}(s)} \end{aligned}$$

for some  $C$  depending only on  $s, L, L'$ . Here we have used the fact that  $z_\gamma$  run over a lattice in  $\mathbb{R}^k$  to obtain the bound in the second line. The same type of bounds are easily obtained for derivatives of any order with respect to  $w, w'$  in compact sets of  $\mathcal{F} \times \{w' \in \mathcal{F}; (x')^2 + |y'|^2 \leq L^2\}$ , we omit the details.

Since for any  $\chi \in L^\infty(\mathcal{F})$  supported in  $(x')^2 + |y'|^2 \leq L^2$ , the operator with kernel  $(xx')^{\operatorname{Re}(s)}(1 + x^2 + |y|^2)^{-\operatorname{Re}(s)}\chi(w')$  is bounded as an operator on  $L^2(\mathcal{F})$  for  $\operatorname{Re}(s) > n/2$ , and so is the operator with kernel  $K_s(w, w') := \sum_{\gamma \in \Gamma'_c} R_{\mathbb{H}^{n+1}}(s; w, \gamma w')\chi(w')$ , moreover  $K_s(w, w')$  clearly solves  $(\Delta_{\mathbb{H}^{n+1}} - s(n - s))K_s(w, w') = 0$  for  $w, w'$  in compact sets of  $\mathcal{F} \times \{w' \in \mathcal{F}; (x')^2 + |y'|^2 \leq L^2\}$ .

There remains to analyze  $K'_s(w, w') := \sum_{\gamma \in \Gamma_c \setminus \Gamma'_c} R_{\mathbb{H}^{n+1}}(s; w, \gamma w')\chi(w')$  which contains the diagonal singularity (at least in the region where  $(x')^2 + |y'|^2 \leq L^2$ ). By the fact that  $R_{\mathbb{H}^{n+1}}(s)$  is bounded on  $L^2(\mathbb{H}^{n+1})$  for  $\operatorname{Re}(s) > n/2$  and that  $\Gamma_c \setminus \Gamma'_c$  is finite, it

is clear that  $K'_s(m, m')$  is the kernel of a bounded operator on  $L^2(\mathcal{F})$ , moreover it solves, in the distribution sense,  $(\Delta_{\mathbb{H}^{n+1}} - s(n-s))K'_s(w, w')\chi(w') = \delta(w - w')\chi(w')$ .

Combining the discussions for  $K_s$  and  $K'_s$ , we have thus proved that  $R_{X_c}(s)\chi$  has Schwartz kernel given by  $K_s + K'_s$  in  $\text{Re}(s) > n/2$ , and moreover that  $K_s(w, w') + K'_s(w, w')$  is well defined as a locally uniformly converging series on compact sets in  $(m, m')$  away from the diagonal, as long as  $\text{Re}(s) > k/2$ . The last statement about the estimate of the kernel of  $\psi R_{X_c}(s)\chi$  is a straightforward consequence of what we have just discussed.  $\square$

Now use the inverted coordinate system  $(u, v, z)$  and the polar variable  $R$ , where

$$(6.4) \quad u := \frac{x}{x^2 + |y|^2}, \quad v := \frac{-y}{x^2 + |y|^2}, \quad R := \sqrt{u^2 + |v|^2} = \frac{1}{\sqrt{x^2 + |y|^2}}.$$

These functions too descend to  $X_c$ , and the cusp itself is at  $R = 0$ , while the regular part of  $\partial\bar{X}_c$  is  $\{u = 0, v \neq 0\}$ . A simple calculation gives that

$$\frac{1}{\theta(w, w')} = \frac{u^2 + (u')^2 + |y - y'|^2 + (RR')^2|z - z'|^2}{2uu'}.$$

Therefore, repeating the same arguments of the proof of Proposition 6.1, we easily obtain that for any  $\epsilon > 0$ , there exists  $C$  with

$$(6.5) \quad |(uu')^{-s} R_{X_c}(s; w, w')| \leq C(RR')^{-k},$$

in the region  $\{(u - u')^2 + (|v| - |v'|)^2 > \epsilon\}$ , with the analogous estimate holding if we apply any number of derivatives  $\partial_u, \partial_v$  and  $\partial_z$ .

We can now use this to estimate the kernel of the resolvent on the entire geometrically finite quotient  $X$ .

**Corollary 6.2.** *Let  $X = \Gamma \backslash \mathbb{H}^{n+1}$  be a geometrically finite quotient, and let  $\bar{k}$  be the maximum rank of all the nonmaximal rank cusps. Then  $R_X(s)$  continues meromorphically to the half-plane  $\text{Re}(s) > \bar{k}/2$ . If  $\chi \in \mathcal{C}_0^\infty(\bar{X})$  and  $\psi \in \mathcal{C}^\infty(\bar{X})$  have disjoint support, then in a cusp neighbourhood  $\mathcal{U}_j^c$  of a cusp of rank  $k$ , and using the inverted coordinates of (6.4),*

$$(6.6) \quad \begin{aligned} & \left| (uu')^{-s} \psi(w) R_X(s; w, w') \chi(w') \right| \leq C(u^2 + |v|^2)^{-k/2}, \\ & \left| (uu')^{-s} \psi(w) (u\partial_u - s) R_X(s; w, w') \chi(w') \right| \leq C(u^2 + |v|^2)^{-k/2}. \end{aligned}$$

*Proof.* From (5.4) we can write  $R_X(s)\chi = Q_\infty(s)\chi + Q_\infty(s)\chi S_\infty(s)$  for a suitably chosen parametrix  $Q_\infty(s)$  and with  $S_\infty(s) \in x^\infty(x')^s \mathcal{C}_0^\infty(\bar{X} \times \bar{X})$ . By the construction in the proof of Theorem 5.1, if  $\hat{\chi}_j^c R_j^c(s) \chi_j^c$  are the model resolvents in  $\mathcal{U}_j^c$ , then  $L(s) := Q_\infty(s) - \sum_{j \in J^c} \hat{\chi}_j^c R_j^c(s) \chi_j^c$  has compact support in  $\bar{X} \times \bar{X}$ . Let  $\epsilon > 0$  be small so that  $\chi(w')$  has support in  $(u')^2 + |v'|^2 > 2\epsilon$  in each cusp neighbourhood, we can then combine the result in Proposition 4.3 for the region  $u^2 + |v|^2 > \epsilon$  far from the cusps and the estimate (6.5) for the region  $u^2 + |v|^2 \leq \epsilon$  near the cusp to deduce that each  $\psi R_j^c \chi$  satisfies (6.6), and  $\psi L(s) \in x^{s+2}(x')^s \mathcal{C}_0^\infty(\bar{X} \times \bar{X})$  satisfies this same estimate in  $\mathcal{U}_j^c$  as well. Finally, using the residual structure of  $S_\infty(s)$ , we obtain the same estimate also for the composition  $\psi Q_\infty(s)\chi S_\infty(s)$ .  $\square$

**6.2. Poincaré series.** We now review a standard argument which shows that the meromorphic continuation of the resolvent  $R_X(s)$  implies a corresponding extension for the Poincaré series of  $\Gamma$ :

$$P_s(m, m') := \sum_{\gamma \in \Gamma \setminus \text{Id}} e^{-sd(m, \gamma m')}, \quad m, m' \in \mathbb{H}^{n+1}$$

Here, notice that we have removed the term in the series corresponding to  $\gamma = \text{Id}$ , this obviously does not change any result about meromorphic continuation of  $P_s$  and is only done for notational simplicity below. Recall that this sum converges to a holomorphic function in  $\text{Re}(s) > \delta$ , where  $\delta = \delta(\Gamma) \in (0, n)$  equals the Hausdorff dimension of the limit set of  $\Gamma$  ([22, 31]).

**Theorem 6.3.** *The series  $P_s(m, m')$  admits a meromorphic continuation to the entire complex plane.*

*Proof.* To simplify exposition, we shall suppose that  $m, m' \in \text{int}(\mathcal{F})$  where  $\mathcal{F}$  is a fundamental domain of  $\Gamma$ . Define, for  $\text{Re}(s) > n$ ,

$$\tilde{R}_s(m, m') := R_X(s; m, m') - R_{\mathbb{H}^{n+1}}(s; m, m') = \sum_{\gamma \in \Gamma \setminus \text{Id}} R_{\mathbb{H}^{n+1}}(s; m, \gamma m').$$

By Theorem 5.1, this extends meromorphically to  $\mathbb{C}$ , and (by elliptic regularity) is smooth in  $\text{int}(\mathcal{F}) \times \text{int}(\mathcal{F})$ . Now, for any  $N \in \mathbb{N}$ , we can write

$$R_{\mathbb{H}^{n+1}}(s; m, m') = \sum_{j=0}^N c_{s,j} Q^{s+j}(m, m') + Q^{s+N+1}(m, m') L_s(Q(m, m'))$$

where  $Q(m, m') = e^{-d(m, m')}$ , and the scalar functions  $c_{s,j}$  and  $L_s \in C^\infty([0, 1])$  are meromorphic in  $\mathbb{C}$  with  $c_{s,0} \neq 0$ . Now sum over translates by  $\gamma \in \Gamma \setminus \text{Id}$ :

$$P_s(m, m') = c_{s,0}^{-1} \left( \tilde{R}_s(m, m') - \sum_{j=1}^N c_{s,j} P_{s+j}(m, m') - \sum_{\gamma \in \Gamma \setminus \text{Id}} Q^{s+N+1}(m, \gamma m') L_s(Q(m, \gamma m')) \right),$$

initially at least for  $\text{Re}(s) > n$ . The infinite series on the right converges to a meromorphic function in  $\text{Re}(s) > n - N - 1$ . Assuming that  $P_s(m, m')$  is meromorphic in  $\text{Re}(s) > n - M$  for some  $M \geq 0$ , then all terms on the right are meromorphic in  $\text{Re}(s) > n - M - 1$ , provided  $N > M + 1$ . By induction, this provides the continuation of  $P_s(m, m')$  to all of  $\mathbb{C}$ .  $\square$

Combining this result with Theorem 2 of [23] and the Wiener-Ikehara Tauberian theorem, we obtain the proof of Corollary 1.2, which was stated in the Introduction.

**6.3. Mapping properties of the resolvent.** We now come back to the description of resolvent on a cusp neighbourhood using Fourier decomposition, as in §3, to obtain finer mapping properties of  $R_X(s)$  all the way up to the critical line, i.e. in the closed half-plane  $\{\text{Re}(s) \geq n/2\}$ . This is necessary in order to analyze the scattering operator and prove that it satisfies a functional equation.

Let  $X_c = \Gamma_c \backslash \mathbb{H}^{n+1} = \mathbb{R}^+ \times F$  be a cusp of rank  $k$ . We use standard coordinates  $(x, y, z)$  as before, and the Sobolev spaces  $H^\ell(F)$ ,  $H^\infty(F)$ , and  $H^\infty(X_c) = \mathcal{C}_b^\infty([0, \infty); H^\infty(F))$ , as defined at the end of §2. We also use the variable  $\rho = x/(1+x)$ , and set

$$\dot{H}^\infty(X_c) := \{u \in H^\infty(X_c) : u = \mathcal{O}((x/(1+x^2))^\infty)\}.$$

**Lemma 6.4.** *Let  $\operatorname{Re}(s - \frac{n}{2}) \geq 0$  and  $f \in \dot{H}^\infty(X_c)$ ; then  $R_{X_c}(s)f \in \rho^{s - \frac{n}{2}} H^\infty(X_c)$ .*

*Proof.* Using the Fourier decomposition of Section 3.1, any  $f \in H^\infty(X_c)$  decomposes as  $f = \sum_I f_I(x, r) \phi_I(z)$ ,  $\phi_I(z) := \exp(2\pi i \langle z, v^* + A_{mp} \rangle)$ , where each  $I = (m, p, v^*)$ , and  $f_I(x, r) \in \Pi_{mp}(L^2(SF))$  for every  $x$  and  $r$ .

If  $f \in L^2(X_c)$ , then  $R(s)f = \sum_I (R_I(s)f_I)(x, r) \phi_I(z) = \sum_I u_I(x, r) \phi_I(z)$ , where

$$(6.7) \quad \begin{aligned} u_I(x, r) &= \int_x^\infty I_\lambda(x\sqrt{\Delta_I}) K_\lambda(x'\sqrt{\Delta_I}) f_I(x', \cdot, \omega) \frac{dx'}{x'} \\ &\quad + \int_0^x K_\lambda(x\sqrt{\Delta_I}) I_\lambda(x'\sqrt{\Delta_I}) f_I(x', \cdot, \omega) \frac{dx'}{x'}, \end{aligned}$$

and we have set  $\lambda := s - n/2$ . This can be rewritten using the Fourier transform  $\mathcal{F}$  in  $y$ , with dual variable  $\xi$ , as

$$(6.8) \quad \begin{aligned} u_I(x, y) &= \Pi_{mp} \int_x^\infty \int_{\mathbb{R}^{n-k}} e^{iy \cdot \xi} I_\lambda(x\sqrt{|\xi|^2 + b_I^2}) K_\lambda(x'\sqrt{|\xi|^2 + b_I^2}) \mathcal{F}(f_I)(x', \xi) d\xi \frac{dx'}{x'} \\ &\quad + \Pi_{mp} \int_0^x \int_{\mathbb{R}^{n-k}} e^{iy \cdot \xi} K_\lambda(x\sqrt{|\xi|^2 + b_I^2}) I_\lambda(x'\sqrt{|\xi|^2 + b_I^2}) \mathcal{F}(f_I)(x', \xi) d\xi \frac{dx'}{x'} \\ &= u_I^< + u_I^>. \end{aligned}$$

Integrating by parts yields

$$(6.9) \quad \begin{aligned} &\partial_y^\alpha (u_I^<)(x, y) = \\ &\int_x^\infty \int_{\mathbb{R}^{n-k}} e^{iy \cdot \xi} \left( I_\lambda(x\sqrt{|\xi|^2 + b_I^2}) K_\lambda(x'\sqrt{|\xi|^2 + b_I^2}) \mathcal{F}((-\partial_y)^\alpha f_I)(x', \xi) \right) d\xi \frac{dx'}{x'}, \end{aligned}$$

with a corresponding identity for  $u_I^>$ .

To obtain  $L^2$  bounds in  $y$  of  $\partial_y^\alpha \partial_x^{\beta_1} (\rho^{-\lambda} u_I^<)(x, y)$ , we must bound the  $L^\infty$  norm in  $\xi$  of

$$\partial_x^{\beta_1} \left( \rho^{-\lambda} I_\lambda(x\sqrt{|\xi|^2 + b_I^2}) \right) \partial_{x'}^{\beta_2} K_\lambda(x'\sqrt{|\xi|^2 + b_I^2}), \quad \beta_1 + \beta_2 \leq \beta$$

when  $x \leq x'$ . Recall the standard estimates on Bessel functions:

$$(6.10) \quad |\partial_t^\alpha K_\lambda(t)| \leq \begin{cases} C_\alpha e^{-t} & \text{if } t > 1 \\ C_\alpha t^{-|\operatorname{Re}(\lambda)| - \alpha} & \text{if } t \leq 1 \end{cases}, \quad |\partial_t^\alpha (t^{-\lambda} I_\lambda(t))| \leq C_\alpha e^t,$$

valid for  $t \in \mathbb{R}^+$  and  $\operatorname{Re}(\lambda) = A \geq 0$ : thus when  $x \leq x'$ , we have

$$(6.11) \quad \left| \partial_x^{\beta_1} \left( \rho^{-\lambda} I_\lambda(x\sqrt{|\xi|^2 + b_I^2}) \right) \partial_{x'}^{\beta_2} K_\lambda(x'\sqrt{|\xi|^2 + b_I^2}) \right| \leq C \langle |\xi| + b_I \rangle^{A+\beta} \rho'^{-A-\beta},$$



where  $C$  depends on  $A$  and  $\beta$ . Using (6.9) and (6.11). Using Cauchy-Schwarz in the  $x'$  integral, we immediately obtain the bound

$$(6.12) \quad \|\partial_y^\alpha \partial_x^\beta (\rho^{-\lambda} u_I^<)(x, y)\|_{L^2(dy)} \leq C \max_{\beta' \leq \beta} \left\| \frac{\langle x \rangle^{A+\epsilon}}{\rho^{A+\beta+\epsilon}} (-\Delta_F + 1)^{\frac{A+\beta}{2}} \partial_x^{\beta'} \partial_y^\alpha f_I \right\|_{L_x^\infty L_F^2}$$

when  $\epsilon > 0$ , and  $\operatorname{Re}(\lambda) = A \geq 0$ ; here  $C$  now depends on  $K, \beta$  and  $A$ , and we have set  $L_x^\infty L_F^2 := L^\infty(\mathbb{R}^+; L^2(F, dv_F))$ .

The estimates are similar for  $u_I^>$ , so we omit the details.  $\square$

The corresponding fact the resolvent of  $\Delta_X$  is a direct consequence:

**Corollary 6.5.** *Let  $X = \Gamma \backslash \mathbb{H}^{n+1}$  be a geometrically finite quotient, and let  $\operatorname{Re}(s - \frac{n}{2}) \geq 0$  and  $f \in \dot{H}^\infty(X)$ ; then  $R_X(s)f \in \rho^{s - \frac{n}{2}} H^\infty(X)$ .*

*Proof.* The proof follows from Lemma 6.11 and the parametrix construction in (5.4), just as in the proof of Corollary 6.2. The crucial fact is that  $R_X(s)$  equals  $R_j^c(s)$  in the cusp neighbourhood  $\mathcal{U}_c^j$ , up to very residual terms.  $\square$

## 7. SCATTERING THEORY

Using the estimates and various properties of the resolvent we have obtained above, we now construct the Eisenstein (or Poisson) and scattering operators. The scheme is the same as for convex co-compact hyperbolic quotients [24, 8] and for quotients with rational cusps [11].

**7.1. The Poisson operator for a pure parabolic group.** The Poisson operator for a boundary problem, including the asymptotic one considered here, is the mapping which carries the (asymptotic) boundary value to the solution of the equation in question in the interior which has this boundary value. For hyperbolic manifolds, the Schwartz kernel of this operator can be identified with the Eisenstein series for the group, hence in this setting the Poisson operator is sometimes also called the Eisenstein operator.

Let  $X_c = \Gamma_c \backslash \mathbb{H}^{n+1}$  with  $\Gamma_c$  an elementary discrete parabolic group of rank  $k < n$ . The Poisson operator for this space,  $P^c(s)$ , is defined by

$$(7.1) \quad P^c(s)f(x, y, z) = \frac{2^{-\lambda+1}}{\Gamma(\lambda)} x^{\frac{n}{2}} \Delta_F^{\frac{\lambda}{2}} K_\lambda(x\sqrt{\Delta_F}) f(y, z), \quad \lambda = s - n/2.$$

This evidently satisfies  $(\Delta_{X_c} - s(n-s))P^c(s)f = 0$ .

**Proposition 7.1.** *Let  $\lambda = s - n/2$ . The operator*

$$(7.2) \quad P^c(s) : H^\infty(F) \rightarrow \rho^s H^\infty(X_c) + \rho^{n-s} H^\infty(X_c)$$

*is a holomorphic family of bounded operators for  $\{\operatorname{Re}(\lambda) \geq 0; \lambda \notin \mathbb{N}\}$  which satisfies the estimate  $\|\partial_x^\beta P^c(s)f(x, \cdot)\|_{L^2(F)} \leq C x^{\frac{n}{2}-\lambda} \|f\|_{H^{2\operatorname{Re}(\lambda)}}$  for  $x > 1$  and  $\beta \in \mathbb{N}_0$ .*

*If  $f \in H^\infty(F)$ , then there are functions  $F_\pm \in H^\infty(X_c)$  such that  $P^c(s)f = \rho^{n-s} F^- + \rho^s F^+$ , with*

$$F^-|_{x=0} = f, \quad F^+|_{x=0} = 2^{-2\lambda} \frac{\Gamma(-\lambda)}{\Gamma(\lambda)} \Delta_F^\lambda f.$$

*Proof.* The holomorphy in  $\operatorname{Re}(s) \geq n/2$ ,  $s \notin n/2 + \mathbb{N}$ , is obvious.

To check the estimate for  $x \geq 1$ , fix  $\chi \in \mathcal{C}^\infty(\mathbb{R}^+)$  which equals 1 in  $(2, \infty)$  and vanishes in  $[0, 1]$ . Then

$$(1 + \Delta_F)^N \partial_x^\beta (\chi(x\sqrt{\Delta_F}) P^c(s) f) = \partial^\beta (\chi K_\lambda)(x\sqrt{\Delta_F}) \Delta_F^{(\lambda+\beta)/2} (1 + \Delta_F)^N f.$$

Since  $\sup_{t \in \mathbb{R}_+} |t^{\pm L} \partial^\beta (\chi K_\lambda)(t)| < \infty$  for any  $L \in \mathbb{R}$ , we deduce that for all  $L > 0$

$$\|\partial_x^\beta (\chi(x\sqrt{\Delta_F}) P^c(s) f)\|_{H^{2N}(F)} \leq C x^{n/2-L} \|f\|_{H^{\operatorname{Re}(\lambda)+\beta+2N-L}}.$$

On the other hand, observe that  $2^{-\lambda+1} K_\lambda(t)/\Gamma(\lambda) = t^{-\lambda} G_\lambda^-(t) + t^\lambda G_\lambda^+(t)$  for some smooth functions  $G_\lambda^\pm$  on  $[0, \infty)$  with  $G_\lambda^-(0) = 1$ , and hence

$$(1 - \chi(x\sqrt{\Delta_F})) P^c(s) = (1 - \chi(x\sqrt{\Delta_F})) (x^{-\lambda} G_\lambda^-(x\sqrt{\Delta_F}) - x^\lambda \Delta_F^\lambda G_\lambda^+(x\sqrt{\Delta_F})).$$

It is straightforward that  $(1 - \chi(x\sqrt{\Delta_F})) G_\lambda^\pm(x\sqrt{\Delta_F}) f \in H^\infty(X_c)$  and for  $x > 1$

$$\|\partial_x^\beta ((x^2 \Delta_F)^\lambda (1 - \chi(x\sqrt{\Delta_F})) G_\lambda^+(x\sqrt{\Delta_F}) f)\|_{H^{2N}(F)} \leq C \|f\|_{H^{2N+\beta+2\operatorname{Re}(\lambda)}},$$

which proves (7.2).

The asymptotic limits when  $x \rightarrow 0$  come from the asymptotic expansion of  $K_\lambda(t)$  at  $t = 0$ , which gives that  $G_\lambda^+(0) = 1$  and  $G_\lambda^-(0) = 2^{-\lambda} \Gamma(-\lambda)/\Gamma(\lambda)$ .  $\square$

**Remark 7.2.** *The functions  $F_\pm$  in the Proposition above have a Taylor expansion at  $x = 0$  with only even powers of  $x$ , this is a consequence of the fact that the functions  $G^\pm(z)$  defined in the proof of this Proposition are smooth functions of  $z^2 \in [0, \infty)$ .*

**7.2. Scattering theory on  $X$ .** We now proceed to define the scattering operator in the usual way.

**Proposition 7.3.** *Let  $X = \Gamma \backslash \mathbb{H}^{n+1}$  be a geometrically finite hyperbolic manifold. Suppose that  $s \in \{\operatorname{Re}(s) > n/2, s \notin (n/2 + \mathbb{N}_0), s(n-s) \notin \sigma_{\text{pp}}(\Delta_X)\}$ , and fix any  $f \in H^\infty(\partial \bar{X})$ . Then there is a unique solution  $u_s \in \mathcal{C}^\infty(X)$  to the equation  $(\Delta_X - s(n-s))u_s = 0$  for which  $u_s|_{x \geq 1} \in L^2(X)$ , and such that there exist functions  $G_\pm \in H^\infty(X)$  with*

$$u_s = \rho^s G_+ + \rho^{n-s} G_-, \quad \text{where } G_-|_{x=0} = f.$$

*Proof.* The problem is solved in each cusp neighbourhood  $\mathcal{U}_j^c$  using the Poisson operator  $P_{c,j}^c(s) : H^\infty(F_j) \rightarrow \mathcal{C}^\infty(\Gamma_j \backslash \mathbb{H}^{n+1})$  in each model space  $\Gamma_j \backslash \mathbb{H}^{n+1}$  given by (7.1). Fix cutoff functions  $\hat{\chi}_j^c$ ,  $\chi_j^c$  and  $\chi_j^r$  as above and write  $\phi_j^c = \chi_j^c|_{x=0}$ ; then set

$$u^c = \sum_{j \in J^c} \hat{\chi}_j^c P_{c,j}^c(s) \phi_j^c f, \quad u^r = \sum_{j \in J^r} \chi_j^r(x^s f).$$

These satisfy

$$(\Delta_{X_c} - s(n-s))u^c = \sum_{j \in J^c} [\Delta_X, \hat{\chi}_j^c] P_j^c(s) \phi_j^c f := q^c,$$

$$(\Delta_{X_c} - s(n-s))u^r = \sum_{j \in J^r} [\Delta_X, \chi_j^r] f := q^r$$

(where  $(x_0, y_0) \in \mathcal{U}_j^r \simeq B$  are coordinates in each regular neighbourhood).

By Proposition 7.1 and equation (5.1),  $q^c \in x^{s+2}\mathcal{C}_0^\infty(\overline{X}) + x^{n-s+2}\mathcal{C}_0^\infty(\overline{X})$ , while  $q^r \in x^2\mathcal{C}^\infty(\overline{X})$ . Remark 7.2 shows that the  $\mathcal{C}_0^\infty(\overline{X})$  functions in the expansion of  $q^c$  are smooth functions of  $x^2$  in  $\mathcal{U}_j^c$ , while in  $\mathcal{U}_r^j$ , the function  $x^{-s-2}q^r$  has an even Taylor expansion in powers of  $x$ . Then, using the indicial equation (5.3) in each of these neighbourhoods, we remove all terms in the expansion of the remainder terms  $q^c, q^r$  at  $x = 0$ , just as we did in the resolvent parametrix construction. This gives a function  $v_s = x^s v^+ + x^{n-s} v^-$  with  $v^\pm \in \mathcal{C}_0^\infty(\overline{X})$ , such that

$$(\Delta_X - s(n-s))v_s \in \dot{\mathcal{C}}_0^\infty(\overline{X}), \quad v^-|_{\partial\overline{X}} = f \quad \text{and } v^\pm \in H^\infty(X).$$

Finally, set

$$u_s := v_s - R(s)(\Delta_X - s(n-s))v_s;$$

the mapping properties of  $R(s)$  from Corollary 6.5 and the expansion of the terms involving  $P_c(s)$  from Proposition 7.1 show that this is indeed a solution to the problem.

We conclude by proving uniqueness. First note that, using the indicial equations (4.11) and (5.3), the expansion of a solution of the problem at  $x = 0$  is determined entirely by  $G_+|_{x=0}$  and  $G_-|_{x=0}$ . Therefore, if  $w$  is the difference of two such solutions, then  $(\Delta_X - s(n-s))w = 0$  and  $w \in L^2(X)$ . Since  $\operatorname{Re}(s) > n/2$  and  $s \notin \sigma_{\text{pp}}(\Delta_X)$ , we have  $w = 0$ , which concludes the proof.  $\square$

We can now define the Poisson operator for  $\{s \in \mathbb{C}; \operatorname{Re}(s) > n/2, s \notin \frac{n}{2} + \mathbb{N}\}$  by  $P_X(s)f = u_s$ , where  $u_s$  is the solution obtained in Proposition 7.3. With  $\rho = x/(x+1)$  as usual, and for this range of  $s$ , we see that

$$(7.3) \quad P_X(s) : H^\infty(\partial\overline{X}) \longrightarrow \rho^s H^\infty(X) + \rho^{n-s} H^\infty(X).$$

The next result shows that this extends to the closed half-plane:

**Proposition 7.4.** *The operator  $P_X(s)$ , which is holomorphic in  $\{s \in \mathbb{C}; \operatorname{Re}(s) > n/2, s \notin \frac{n}{2} + \mathbb{N}\}$ , admits a meromorphic continuation to the entire complex plane as an operator  $\mathcal{C}_0^\infty(\partial\overline{X}) \rightarrow \mathcal{C}^\infty(X)$ . Moreover,*

$$P_X(s) : H^\infty(\partial\overline{X}) \rightarrow \rho^s H^\infty(X) + \rho^{n-s} H^\infty(X), \quad \text{if } \operatorname{Re}(s) \geq n/2,$$

*Proof.* The existence of the meromorphic continuation will follow from a slight variant of the construction of  $P_X(s)$ .

Fix any  $f \in \mathcal{C}_0^\infty(\partial\overline{X})$  and construct (using the indicial equation in each chart and Borel summation) a function  $\Phi(s) \in x^{n-s}\mathcal{C}_0^\infty(\overline{X})$  which satisfies

$$(\Delta_X - s(n-s))\Phi(s) \in \dot{\mathcal{C}}_0^\infty(\overline{X}), \quad \text{with } x^{s-n}\Phi(s)|_{x=0} = f.$$

Now use the resolvent to solve away this error term. This leads to the formula

$$P_X(s)f := \Phi(s) - R_X(s)(\Delta_X - s(n-s))\Phi(s).$$

The right hand side obviously continues meromorphically to  $\mathbb{C}$  with finite rank poles. The fact that this lies in  $x^{n-s}\mathcal{C}_0^\infty(\overline{X}) + \rho^s H^\infty(X)$  when  $\operatorname{Re}(s) \geq n/2$  follows from Corollary 6.5.  $\square$

**Lemma 7.5.** *The integral kernel of  $P_X(s)$  is related to the integral kernel of  $R_X(s)$  by*

$$P_X(s; m, b') = (2s - n)[x(m')^{-s} R_X(s; m, m')]|_{m'=b'}, \quad m \in X, \quad b' \in \partial\overline{X}.$$

*Proof.* This relationship is derived almost exactly as in the convex cocompact case; we sketch it for the convenience of the reader. Green's formula and the equation  $(\Delta_X - s(n-s))R_X(s; m, m') = \delta(m - m')$  imply

$$\begin{aligned} P_X(s)f(m) &= \Phi(s; m) - \lim_{\epsilon \rightarrow 0} \int_{x(m') \geq \epsilon} R_X(s; m, m')(\Delta_X - s(n-s))\Phi(s; m') dv_g(m') \\ &= \lim_{\epsilon \rightarrow 0} \int_{x(m') = \epsilon} \left( \partial_{n'} R_X(s; m, m')\Phi(s; m') - R_X(s; m, m')\partial_{n'}\Phi(s; m') \right) dv_g(m'); \end{aligned}$$

here  $\partial_{n'}$  is the inner unit normal to  $\{x(m') = \epsilon\}$  acting on the  $m'$  variable. Note too that the integration is over a compact set  $K$  in  $\overline{X}$  because  $\Phi(s) \in \mathcal{C}_0^\infty(\overline{X})$ . It is not hard to check that in terms of local coordinates  $(x, y)$ ,  $\partial_{n'} = x\partial_x + \alpha x^2\partial_x + \sum_i \beta_i x\partial_{y_i}$  with  $\alpha, \beta_i \in \mathcal{C}_0^\infty(\overline{X})$ . Hence considering the asymptotic expansions of  $R_X(s; m, m')$  and  $\Phi(s; m')$  and their derivatives with respect to  $\partial_{n'}$  as  $x(m') \rightarrow 0$ , we obtain

$$P_X(s)f(m) = (2s - n) \int_{\partial\overline{X}} [x(m')^{-s} R_X(s; m, m')]_{|m'=b'} f(b') dv_{\partial\overline{X}}(b'),$$

as desired.  $\square$

Combining this Lemma with Proposition 4.3 we obtain that

$$(7.4) \quad P_X(s) \in \rho^s C^\infty((\overline{X} \times \partial\overline{X}) \setminus \text{diag}_{\partial\overline{X}}),$$

where  $\text{diag}_{\partial\overline{X}} := \{(b, b) \in \partial\overline{X} \times \partial\overline{X}\}$ . Moreover, Corollary 6.2 gives that for all  $m \in X$ ,

$$(7.5) \quad P(s; m, \cdot) \in L^2(\partial\overline{X}, dv_{\partial\overline{X}}), \quad \text{if } \text{Re}(s) > \bar{k}/2,$$

where  $\bar{k}$  is the maximum of the ranks of all nonmaximal rank cusps of  $X$ . Indeed, using the coordinates  $(v, z)$  from (6.4) on the boundary of a cusp neighbourhood  $\mathcal{U}_j^c \cap \{x = 0\}$  of a rank  $k$  cusp, the measure on  $\partial\overline{X}$  equals  $|v|^{2k} dv dz$ , and  $P_X(s; m, v) = \mathcal{O}(|v|^{-k})$  as  $|v| \rightarrow 0$ .

The resolvent and Poisson kernels are also related by a functional equation.

**Lemma 7.6.** *There is an identity*

$$(7.6) \quad \begin{aligned} &R_X(s; m, m') - R_X(n-s; m, m') = \\ &\frac{1}{(2s-n)} \int_{\partial\overline{X}} P_X(s; m, b) P_X(n-s; m', b) dv_{\partial\overline{X}}(b). \end{aligned}$$

which holds for any  $m, m' \in X$  when  $|\text{Re}(s) - n/2| < 1/2$ .

*Proof.* The proof is much the same as the one of Proposition 2.1 in [13] or Theorem 1.3 in [7]. Use the coordinates  $(u, v, z)$  from (6.4) in each cusp neighbourhood  $\mathcal{U}_j^c$ ; thus  $u$  is a boundary defining function of  $\partial\overline{X}$  in  $\mathcal{U}_j^c \cap \overline{X}$ . We extend it to a global boundary defining function, still denoted  $u$ , for  $\partial\overline{X}$  on all of  $\overline{X}$ . For  $\epsilon > 0$  small, we use Green's formula as in [13, Prop 2.1] to get

$$(7.7) \quad \begin{aligned} &R_X(s; m, m') - R_X(n-s; m, m') = \\ &-\int_{u(b)=\epsilon} \left( R_X(s; m, b) \partial_n R_X(n-s; b, m') - \partial_n R_X(s; b, m') R_X(n-s; b, m') \right) dv_g(b), \end{aligned}$$

where  $\partial_n$  is the inner unit normal to  $\{u = \epsilon\}$ . The metric in each regular neighbourhood  $\mathcal{U}_j^r$  has the form  $g = (du^2 + h_0 + O(u))/u^2$ , where  $h_0$  is a metric on  $\partial\bar{X}$ , while in each cusp neighbourhood  $\mathcal{U}_j^c$  it appears as

$$g = \frac{du^2 + |dv|^2 + (u^2 + |v|^2)^2 |dz|^2}{u^2}.$$

Thus  $\partial_n$  equals  $u\partial_u$  in cusp neighbourhoods and  $u\partial_u + \alpha u^2\partial_u + \sum_i \beta_i u\partial_{y_i}$ , with  $\alpha, \beta_i \in \mathcal{C}_0^\infty(\partial\bar{X})$ , in regular neighbourhoods.

Introduce a partition of unity to localize to these different neighbourhoods. From the Lemma 7.5, the structure of  $R_X(s)$  in  $U_j^r$  and its symmetry  $R_X(s; m, b) = R_X(s; b, m)$ , we obtain the contribution to the integrand from  $\{u = \epsilon\} \cap \mathcal{U}_j^r$  in the limit as  $\epsilon \rightarrow 0$  is given by  $(2s - n)^{-1} P_X(s; m, b) P_X(n - s; m', b) dv_{\partial\bar{X}}(b)$ . Applying analogous arguments, using Lemma 7.5, Corollary 6.2 and dominated convergence (the measure restricted on  $\{u = \epsilon\}$  is  $dv_g = (\epsilon^2 + |v|^2)^k dv dz$  and so  $P_X(s; m, \cdot) \in L^2$  on this hypersurface), we find that the contribution from the cusp neighbourhoods equals exactly the same thing.  $\square$

We can now define the scattering operator  $S_X(s) : H^\infty(\partial\bar{X}) \rightarrow H^\infty(\partial\bar{X})$  for  $\operatorname{Re}(s) \geq n/2$  and  $s \notin n/2 + \mathbb{N}$  by

$$(7.8) \quad S_X(s)f := G_+|_{\partial\bar{X}}$$

where  $G_+ \in H^\infty(X)$  is the function appearing defined in Propositions 7.3 and 7.4 for the expansion of  $P_X(s)f$  at  $\partial\bar{X}$ . From Theorem 5.1 and the construction of  $P_X(s)f$ , the operator  $S_X(s)$  has a meromorphic continuation as a continuous operator  $\mathcal{C}_0^\infty(\partial\bar{X}) \rightarrow \mathcal{C}^\infty(\partial\bar{X})$  to  $\mathbb{C} \setminus (n/2 + \mathbb{N})$  with finite rank poles.

**Lemma 7.7.** *The Schwartz kernel of  $S_X(s)$  is given by*

$$S_X(s; b, b') = [x(m)^{-s} P_X(s; m, b')]|_{m=b}, \quad b, b' \in \partial\bar{X}.$$

Furthermore, for any  $\varphi \in \mathcal{C}_0^\infty(\partial\bar{X})$ ,  $\varphi S_X(s)\varphi$  is a classical pseudodifferential operator of order  $2s - n$ .

*Proof.* The first statement follows from the relationship  $S_X(s)f = \lim_{x \rightarrow 0} (x^{-s} P_X(s)f)$  when  $\operatorname{Re}(s) < n/2$  and the meromorphic extension. That  $S_X(s)$  is pseudodifferential follows from (4.13) and the explicit formula for  $R_j(s)$  in the proof of Proposition 4.3; indeed, by [24, Prop 4.12],  $[(x(m)x(m'))^{-s} R_j(s; m, m')]_{(m, m')=(y_0, y'_0)} = f_s(y_0) f_s(y'_0) |y_0 - y'_0|^{-2s}$  for some smooth functions  $f_s$  in the chart  $\mathcal{U}_j$  defined in the proof of Proposition 4.3.  $\square$

**Lemma 7.8.** *For  $\operatorname{Re}(s) = n/2$  and  $s \neq n/2$ , there are identities*

$$P_X(s) = P_X(n - s)S_X(s), \quad S_X(n - s)S_X(s) = S_X(s)S_X(n - s) = \operatorname{Id}.$$

*Proof.* By Lemma 7.7, the Schwartz kernel of  $S_X(s)$  in  $\{\operatorname{Re}(s) < n/2\}$  lies in  $L_{\text{loc}}^1(\partial\bar{X} \times \partial\bar{X})$ , and from Corollary 6.2, we also have  $S_X(s; b, \cdot) \in L^2(\partial\bar{X} \setminus B_\epsilon(b), dv_{\partial\bar{X}})$  for all  $b \in \partial\bar{X}$ , where  $B_\epsilon(b)$  is a ball of small radius  $\epsilon > 0$  in  $\partial\bar{X}$ . Now fix  $m \in X, b \in \partial\bar{X}$  and

<sup>2</sup>Notice that  $(u/x)|_{\partial\bar{X}} = |v|^2$  and the terms involving extra powers of  $|v|$  from writing  $P_X(s), P_X(n - s)$  as weighted restrictions to  $\partial\bar{X}$  cancel out with the extra powers of  $|v|$  coming from writing the volume measure  $dv_g(b)$  in terms of  $dv_{\partial\bar{X}} = \operatorname{dvol}_{h_0}$ .

$\operatorname{Re}(s) \in ((n-1)/2, n/2)$ , and multiply (7.6) by  $(2s-n)x(m')^{-s}$  and let  $m' \rightarrow b$ . By (7.4), (7.5) and the decay and regularity properties of  $S(s; b, b')$  stated above, we deduce that

$$P_X(s; m, b) = \int_{\partial\bar{X}} P_X(n-s; m, b') S_X(s; b, b') dv_{\partial\bar{X}}(b').$$

In particular, the integral converges. From the symmetry of the resolvent, we also have  $S_X(s; b, b') = S_X(s; b', b)$ , so  $P_X(s) = P_X(n-s)S_X(s)$  for  $\operatorname{Re}(s) \in ((n-1)/2, n/2)$  as operators  $\mathcal{C}_0^\infty(\partial\bar{X}) \rightarrow \mathcal{C}^\infty(X)$ . However, this extends to  $|\operatorname{Re}(s) - n/2| \leq 1/2$  meromorphically, in view of the mapping properties of  $S(s)$  and  $P(s)$ . The functional equation for  $S(s)$  is an easy consequence: one has for  $\operatorname{Re}(s) = n/2$  (and  $s \neq n/2$ )

$$P_X(n-s) = P_X(s)S_X(n-s) = P_X(n-s)S_X(s)S_X(n-s)$$

as operators acting on  $H^\infty(\partial\bar{X})$ , but  $P_X(n-s)$  is injective on  $H^\infty(\partial\bar{X})$  by construction.  $\square$

**Acknowledgement.** The first author is grateful to Gilles Carron for explaining him the Fourier decomposition of a flat 3-dimensional bundle, which was somehow fundamental in our approach to the general case. Part of this work was done while C.G. was visiting Stanford Math. Dept.. C.G. is partially supported by grant ANR-09-JCJC-0099-01. R.M. is partially supported by NSF grant DMS-0805529.

#### REFERENCES

- [1] D. Borthwick, *Upper and lower bounds on resonances for manifolds hyperbolic near infinity*, Comm. PDE **33** (2008), 1507–1539.
- [2] D. Borthwick, *Spectral Theory of Infinite-Area Hyperbolic Surfaces*, Progress in Mathematics, Vol. 256 Birkhauser Boston, 2007.
- [3] D. Borthwick, C. Judge, P. Perry, *Selberg's zeta function and the spectral geometry of geometrically finite hyperbolic surfaces*, Comment. Math. Helv. **80** (2005), 483–515.
- [4] B.H. Bowditch, *Geometrical finiteness for hyperbolic groups*, Jour. Funct. Anal. **113** (1993), 245–317.
- [5] U. Bunke, M. Olbrich, *Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group*, Ann. Math **149** (1999), 627–689
- [6] U. Bunke, M. Olbrich, *Scattering theory for geometrically finite groups*, Geometry, analysis and topology of discrete groups, 40–136, Adv. Lect. Math. (ALM), **6**, Int. Press, Somerville, MA, 2008..
- [7] R. Froese, P. Hislop, P. Perry, *The Laplace operator on hyperbolic three-manifolds with cusps of non-maximal rank*, Invent. Math. **106** (1991), 295–333.
- [8] C.R. Graham, M. Zworski, *Scattering matrix in conformal geometry*, Invent. Math. **152** (2003), 89–118.
- [9] C. Guillarmou, *Meromorphic properties of the resolvent for asymptotically hyperbolic manifolds*, Duke Math. J. **129** no 1 (2005), 1–37.
- [10] C. Guillarmou, *Resonances on some geometrically finite hyperbolic manifolds*, Comm. Part. Diff. Eq. **31** (2006), 445–467.
- [11] C. Guillarmou, *Scattering on geometrically finite hyperbolic quotients with rational cusps*, Cubo Journal. **11** (2009), no. 05, 129–172.
- [12] C. Guillarmou, F. Naud, *Wave 0-Trace and length spectrum on convex co-compact hyperbolic manifolds*, Comm. Anal. Geom. **14** (2006), no. 5, 945–967.
- [13] L. Guillopé, *Fonctions Zêta de Selberg et surfaces de géométrie finie*, Adv. Stud. Pure Math. **21** (1992), 33–70.
- [14] L. Guillopé, M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. Math. **145** (1997), 597–660.

- [15] L. Guillopé, M. Zworski, *Polynomial bounds on the number of resonances for some complete spaces of constant negative curvature near infinity*, *Asymp. Anal.* **11** (1995), 1-22.
- [16] P. Lax, R. Phillips, *The asymptotic distribution of lattice points and noneuclidean spaces*, *J. Funct. Anal.* **46** (1982), no. 3, 280-350.
- [17] P. Lax, R. Phillips *Translation representation for automorphic solutions of the wave equation in non-Euclidean spaces, IV* *Comm. Pure Appl. Math.* **45** (1992), No. 2, 179-201.
- [18] R. Mazzeo, *Elliptic theory of differential edge operators. I*, *Comm. P.D.E.* **16** (1991), 1615-1664.
- [19] R. Mazzeo, R. Melrose, *Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature*, *J. Funct. Anal.* **75** (1987), 260-310.
- [20] R. Mazzeo, R. Phillips, *Hodge theory for hyperbolic manifolds*, *Duke Math. J.* **60** (1990), no. 2, 509-559.
- [21] F. Naud, *Expanding maps on Cantor sets and analytic continuation of zeta functions*. *Ann. Sci. cole Norm. Sup. (4)* **38** (2005), no. 1, 116-153.
- [22] S.J. Patterson, *The limit set of a Fuchsian group*. *Acta Math* **136** (1976), 241-273.
- [23] S.J. Patterson, *On a lattice-point problem for hyperbolic space and related questions in spectral theory*, *Arxiv für Math.* **26** (1988), 167-172.
- [24] S. Patterson, P. Perry, *The divisor of Selberg's zeta function for Kleinian groups. Appendix A by Charles Epstein.*, *Duke Math. J.* **106** (2001) 321-391.
- [25] P. Perry, *Asymptotics of the length spectrum for hyperbolic manifolds of infinite volume*, *GAFAGeom. funct. anal.*, **11** (2001), 132-141.
- [26] P. Perry, *A Poisson formula and lower bounds for resonances on hyperbolic manifolds*, *Int. Math. Res. Notices* **34** (2003), 1837-1851.
- [27] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Springer GTM 149.
- [28] T. Roblin, *Ergodicité et équidistribution en courbure négative*. *Mém. Soc. Math. Fr. No.* **95** (2003), 96 pp
- [29] A. Selberg, *Collected papers*, t. 1, Springer, Heidelberg, 1989.
- [30] L. Stoyanov *Ruelle zeta functions and spectra of transfer operators for some Axiom A flows.*, preprint 2005.
- [31] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, *Publ. Math. de l'IHES.*, **50** (1979), 171-202.

DMA, U.M.R. 8553 CNRS, ECOLE NORMALE SUPÉRIEURE, 45 RUE D'ULM, F 75230 PARIS CEDEX 05, FRANCE

*E-mail address:* `cguillar@dma.ens.fr`

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305, USA.

*E-mail address:* `mazzeo@math.stanford.edu`.