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INTERNAL EXPONENTIAL STABILIZATION FOR NAVIER–STOKES EQUATIONS BY MEANS OF FINITE-DIMENSIONAL DISTRIBUTED CONTROLS

VIOREL BARBU, SÉRGIO S. RODRIGUES, ARMEN SHIRIKYAN

ABSTRACT. We consider the Navier–Stokes system in a bounded domain with a smooth boundary. Given a sufficiently regular global solution, we construct a finite-dimensional feedback control that is supported by a given open set and stabilizes the linearized equation. The proof of this fact is based on a truncated observability inequality, the regularizing property for the linearized equation, and some standard techniques of the optimal control theory. We then show that the control constructed for the linear problem stabilizes locally also the full Navier–Stokes system.

KEYWORDS: Navier–Stokes system, exponential stabilization, feedback control

CONTENTS

1. Introduction	1
Notation	3
2. Preliminaries	3
2.1. Functional spaces and reduction to an evolution equation	3
2.2. Setting of the problem	4
3. Main result for linearized system	4
3.1. Existence of a stabilizing control	5
3.2. Feedback control	8
4. Stabilization of the nonlinear problem	11
4.1. Main result	11
4.2. Proof of proposition 4.2	12
5. Appendix	13
5.1. Karush–Kuhn–Tucker theorem	13
5.2. Quadratic functionals with linear constraint	13
5.3. Truncated observability inequality	14
References	16

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$. We consider the controlled Navier–Stokes system in Ω :

$$(1) \quad \partial_t u + \langle u \cdot \nabla \rangle u - \nu \Delta u + \nabla p = h + \zeta, \quad \nabla \cdot u = 0,$$

$$(2) \quad u|_{\Gamma} = 0.$$

Here $u = (u_1, u_2, u_3)$ and p are unknown velocity field and pressure of the fluid, $\nu > 0$ is the viscosity, $\langle u \cdot \nabla \rangle$ stands for the differential operator $u_1 \partial_1 + u_2 \partial_2 + u_3 \partial_3$, h is a fixed function, and ζ is a control taking values in the space \mathcal{E} of square-integrable functions in Ω whose support in x is contained in a given open subset $\omega \subset \Omega$. The problem of exact controllability for (1), (2) was in the focus of attention of many researchers starting from the early nineties, and it is now rather well understood. Namely, it was proved that, given a time $T > 0$ and a smooth solution \hat{u} of (1), (2) with $\zeta \equiv 0$, for any initial function u_0 sufficiently close to $\hat{u}(0)$ one can find a control $\zeta : [0, T] \rightarrow \mathcal{E}$ such that the solution of problem (1), (2) supplemented with the initial condition

$$(3) \quad u(0, x) = u_0(x)$$

is defined on $[0, T]$ and satisfies the relation $u(T) = \hat{u}(T)$. We refer the reader to [6, 10, 11, 12, 13, 7] for the exact statements and the proofs of these results.

Even though the property of exact controllability is quite satisfactory from the mathematical point of view, many problems arising in applications require that the control in question be feedback, because

closed-loop controls are usually more stable under perturbations (e.g., see the introduction to Part 3 in [5]). This question has found a positive answer in the context of stabilization theory. It was intensively studied for the case in which the target solution \hat{u} is stationary (in particular, the external force is independent of time). A typical result in such a situation claims that, given a smooth stationary state \hat{u} of the Navier–Stokes system and a constant $\lambda > 0$, one can construct a continuous linear operator $K_{\hat{u}} : L^2 \rightarrow \mathcal{E}$ with finite-dimensional range such that the solution of problem (1) – (3) with $\zeta = K_{\hat{u}}(u - \hat{u})$ and a function u_0 sufficiently close to \hat{u} is defined for all $t \geq 0$ and converges to \hat{u} at least with the rate $e^{-\lambda t}$. We refer the reader to the papers [8, 9, 16, 17, 18, 1] for boundary stabilization and to [2, 4, 3] for stabilization by a distributed control.

The aim of this paper is to establish a similar result in the case when the target solution \hat{u} depends on time. Namely, we will prove the following theorem, whose exact formulation is given in Section 4.

Main Theorem. *Let (\hat{u}, \hat{p}) be a global smooth solution for problem (1), (2) with $\zeta \equiv 0$ such that*

$$\operatorname{ess\,sup}_{(t,x) \in Q} |\partial_t^j \partial_x^\alpha \hat{u}(t,x)| \leq R \quad \text{for } j = 0, 1, |\alpha| \leq 1,$$

where $Q = \mathbb{R}_+ \times \Omega$ and $R > 0$ is a constant. Then for any $\lambda > 0$ and any open subset $\omega \subset \Omega$ there is an integer $M = M(R, \lambda, \omega) \geq 1$, an M -dimensional space $\mathcal{E} \subset C_0^\infty(\omega, \mathbb{R}^3)$, and a family of continuous linear operators $K_{\hat{u}}(t) : L^2(\Omega, \mathbb{R}^3) \rightarrow \mathcal{E}$, $t \geq 0$, such that the following assertions hold.

- (a) *The function $t \mapsto K_{\hat{u}}(t)$ is continuous in the weak operator topology, and its operator norm is bounded by a constant depending only on R , λ , and ω .*
- (b) *For any divergence free function $u_0 \in H_0^1(\Omega, \mathbb{R}^3)$ that is sufficiently close to $\hat{u}(0)$ in the H^1 -norm problem (1) – (3) with $\zeta = K_{\hat{u}}(t)(u - \hat{u}(t))$ has a unique global strong solution (u, p) , which satisfies the inequality*

$$|u(t) - \hat{u}(t)|_{H^1} \leq C e^{-\lambda t} |u_0 - \hat{u}(0)|_{H^1}, \quad t \geq 0.$$

Note that this theorem remains true for the two-dimensional Navier–Stokes system, and in this case, it suffices to assume that the initial function u_0 is close to $\hat{u}(0)$ in the L^2 -norm. Without going into details, we now describe the main ideas of the proof. We emphasize that the methods developed in the above-mentioned works do not apply to the non-stationary case, because they use essentially the spectral properties of the linearized operator and, hence, the fact that the coefficients of the linearized equation do not depend on time.

Well known methods based on the contraction mapping principle enables one to prove that a control stabilizing the linearized equation locally stabilizes also the nonlinear equation. Therefore, it suffices to consider the following problem obtained by linearizing (1), (2) around \hat{u} :

$$(4) \quad \partial_t v + \langle \hat{u} \cdot \nabla \rangle v + \langle v \cdot \nabla \rangle \hat{u} - \nu \Delta v + \nabla p = \zeta, \quad \nabla \cdot v = 0, \quad v|_\Gamma = 0.$$

Let us assume that, for a sufficiently large integer N , we have constructed a continuous linear operator $\bar{\zeta} : L^2(\Omega, \mathbb{R}^3) \rightarrow L^2((0, 1); \mathcal{E})$ such that, for any initial function v_0 , the solution of (4) with $\zeta = \bar{\zeta}(v_0)$ issued from v_0 satisfies the relation $\Pi_N v(1) = 0$, where Π_N stands for the orthogonal projection in L^2 onto the subspace spanned by the first N eigenfunctions of the Stokes operator in Ω . In this case, using the Poincaré inequality and regularizing property of the resolving operator for (4), we get

$$(5) \quad \begin{aligned} |v(1)|_{L^2} &= |(I - \Pi_N)v(1)|_{L^2} \leq C_1 \alpha_N^{-1/2} |v(1)|_{H^1} \\ &\leq C_2 \alpha_N^{-1/2} (|v_0|_{L^2} + |\bar{\zeta}(v_0)|_{L^2((0,1); \mathcal{E})}) \leq C_3 \alpha_N^{-1/2} |v_0|_{L^2}, \end{aligned}$$

where $\{\alpha_j\}$ denotes the increasing sequence of the eigenvalues for the Stokes operator and C_i , $i = 1, 2, 3$, are some constants not depending on N . The fact that C_3 is independent of N is a crucial property, and its proof is based on a truncated observability inequality (see Section 5.3 in Appendix). It follows from (5) that, if N is sufficiently large, then $|v(1)|_{L^2} \leq e^{-\lambda} |v_0|_{L^2}$. Iterating this procedure, we get an exponentially decaying solution. Once an exponential stabilization of the linearized problem (4) is obtained, the existence of an exponentially stabilizing feedback control can be proved with the help of the dynamic programming principle. We refer the reader to Section 3 for an accurate presentation of the results on the linearized equation.

In conclusion, let us mention that the approach described above works equally well in the case when the control acts via the boundary. This situation will be addressed in a subsequent publication.

The paper is organized as follows. In Section 2, we introduce the functional spaces arising in the theory of the Navier–Stokes equations and recall some well-known facts. Section 3 is devoted to studying the linearized problem. In Section 4, we establish the main result of the paper on local exponential

stabilization of the full Navier–Stokes system. The Appendix gathers some auxiliary results used in the main text.

Notation. We write \mathbb{N} and \mathbb{R} for the sets of non-negative integers and real numbers, respectively, and we define $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$ and $\mathbb{R}_+ = (0, +\infty)$. The partial time derivative $\partial_t u$ of a function $u(t, x)$ will be denoted by u_t .

For Banach spaces X, Y and an interval $I \subseteq \mathbb{R}$, we denote

$$W(I, X, Y) := \{f \in L^2(I, X) \mid f_t \in L^2(I, Y)\},$$

where the derivative $f_t = \frac{df}{dt}$ is taken in the sense of vectorial distributions. This space is endowed with the natural norm

$$|f|_{W(I, X, Y)} := \left(|f|_{L^2(I, X)}^2 + |f_t|_{L^2(I, Y)}^2 \right)^{1/2}.$$

Note that if $X = Y$, then we obtain the Sobolev space $W^{1,2}(I, X)$.

For a given space Z of functions $f = f(t)$ defined on an interval of \mathbb{R} and a constant $\lambda > 0$, we define

$$E^\lambda(Z) := \{f \in Z \mid e^{(\lambda/2)t} f \in Z\}.$$

This space is endowed with the norm

$$|f|_{E^\lambda(Z)} := \left(|f|_Z^2 + |e^{(\lambda/2)t} f|_Z^2 \right)^{1/2}.$$

Given a Banach space X , we denote by X' its dual and by $\langle \cdot, \cdot \rangle_{X', X}$ the duality between X' and X .

$\overline{C}_{[a_1, \dots, a_k]}$ denotes a function of non-negative variables a_j that increases in each of its arguments.

$C_i, i = 1, 2, \dots$, stand for unessential positive constants.

2. PRELIMINARIES

2.1. Functional spaces and reduction to an evolution equation. In what follows, we will confine ourselves to the 3D case, although all the results remain valid for the 2D Navier–Stokes equations.

Let $\Omega \subset \mathbb{R}^3$ be a connected bounded domain located locally on one side of its smooth boundary $\Gamma = \partial\Omega$. It is natural to study the incompressible Navier–Stokes system as an evolutionary equation in the subspace H of divergence free vector fields tangent to the boundary:

$$H := \{u \in L^2(T\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega, u \cdot \mathbf{n} = 0 \text{ on } \Gamma\}.$$

Here $L^2(T\Omega) = (L^2(\Omega))^3$ is the space of square integrable vector fields in Ω , $\nabla \cdot u := \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$ is the divergence of the vector field u and \mathbf{n} is the normal vector to the boundary Γ . We will also need the spaces of vector fields on Ω :

$$V := \{u \in H^1(T\Omega) \mid \nabla \cdot u = 0 \text{ in } \Omega, u = 0 \text{ on } \Gamma\}, \quad U := H^2(T\Omega) \cap V.$$

Note that U coincides with the natural domain $D(L)$ of the Stokes operator $L = -\nu\Pi\Delta$, where Π is the orthogonal projection in $L^2(T\Omega)$ onto H . The spaces H, V and U are endowed with the scalar products

$$(u, v)_H := (u, v)_{L^2(T\Omega)}, \quad (u, v)_V := \langle Lu, v \rangle_{V', V}, \quad (u, v)_{D(L)} := (Lu, Lv)_{L^2(T\Omega)},$$

respectively, and we denote by $|\cdot|_H, |\cdot|_V$ and $|\cdot|_{D(L)}$ the corresponding norms. Finally, we introduce the space $\mathcal{W} := W^{1,\infty}(\mathbb{R}_+, W^{1,\infty}(T\Omega))$, where $W^{1,\infty}(T\Omega)$ is the Sobolev space of vector fields in Ω whose components and their first derivatives are essentially bounded.

It is well known that problem (1), (2) is equivalent to the following evolutionary equation in H :

$$(6) \quad u_t + Lu + Bu = \Pi(h + \zeta),$$

where $Bu : V \rightarrow V'$ is defined by $Bu := B(u, u)$ with

$$\langle B(u, v), w \rangle_{V', V} = \sum_{i,j=1}^3 \int_{\Omega} u_i \partial_i u_j w_j dx.$$

In the following, we will deal also with linear equations obtained from (6) after replacing B by one of the operators $\mathbb{B}(\hat{u})$ and $\mathbb{B}^*(\hat{u})$, where $\hat{u} \in \mathcal{W}$ is a fixed function, $\mathbb{B}(\hat{u})v = B(v, \hat{u}) + B(\hat{u}, v)$, and $\mathbb{B}^*(\hat{u})$ stands for the formal adjoint of $\mathbb{B}(\hat{u})$ with respect to the scalar product on H . Namely, let us consider the problem

$$(7) \quad r_t + Lr + \hat{\mathbb{B}}r = f, \quad t \in I_0 = (0, 1),$$

$$(8) \quad r(0) = r_0,$$

where $\hat{\mathbb{B}} = \mathbb{B}(\hat{u})$ or $\mathbb{B}^*(\hat{u})$.

Lemma 2.1. *For any $u_0 \in H$ and $f \in L^2(I_0, V')$, problem (7), (8) has a unique solution $r \in W(I_0, V, V')$, which satisfies the inequality*

$$(9) \quad |r|_{C(\bar{I}_0, H)}^2 + \int_{I_0} |r|_V^2 dt + \int_{I_0} |r_t|_{V'}^2 dt \leq \bar{C}_{[|\hat{u}|_{L^\infty(\bar{I}_0, W^{1,\infty}(T\Omega))}]} |r_0|_H^2 + |f|_{L^2(I_0, V')}^2.$$

Moreover, if $f \in L^2(I_0, H)$, then we have the inclusions $\sqrt{t}v \in C(\bar{I}_0, V)$, $\sqrt{t}v \in L^2(I_0, U)$ and the estimate

$$(10) \quad |\sqrt{t}r|_{C(\bar{I}_0, V)}^2 + \int_{I_0} (\sqrt{t}|r|_U)^2 dt \leq \bar{C}_{[|\hat{u}|_{L^\infty(\bar{I}_0, W^{1,\infty}(T\Omega))}]} (|r_0|_H^2 + |f|_{L^2(I_0, H)}^2).$$

Finally, if $r_0 \in V$ and $f \in L^2(I_0, H)$, then $r \in W(I_0, U, H)$ and

$$(11) \quad |r|_{C(\bar{I}_0, V)}^2 + \int_{I_0} |r|_{D(L)}^2 dt + \int_{I_0} |r_t|_H^2 dt \leq \bar{C}_{[|\hat{u}|_{L^\infty(\bar{I}_0, W^{1,\infty}(T\Omega))}]} (|r_0|_V^2 + |f|_{L^2(I_0, H)}^2).$$

The proof of this lemma is based on a well known argument, and we will not present it here.

2.2. Setting of the problem. Let us fix a function $h \in L^2(\mathbb{R}_+, H)$ and suppose that $\hat{u} \in L^2(\mathbb{R}_+, V) \cap \mathcal{W}$ solves the Navier–Stokes system

$$\hat{u}_t + L\hat{u} + B\hat{u} = h, \quad t > 0.$$

Given a function $u_0 \in H$ and a sub-domain $\omega \subseteq \Omega$, our goal is to find a finite-dimensional subspace $\mathcal{E} \subset L^2(T\omega) = (L^2(\omega))^3$ and a control $\zeta \in L_{\text{loc}}^2(\mathbb{R}_+, \mathcal{E})$ such that the solution of the problem

$$(12) \quad u_t + Lu + Bu = h + \Pi\zeta, \quad u(0) = u_0$$

is defined for all $t > 0$ and converges exponentially to \hat{u} , i.e.,

$$|u(t) - \hat{u}(t)|_H \leq \kappa_1 e^{-\kappa_2 t} \quad \text{for } t \geq 0,$$

where κ_1 and κ_2 are non-negative constants; in this case, we say that u converges κ_2 -exponentially to \hat{u} .

Let us write $L^2(T\Omega)$ as a direct sum $L^2(T\Omega) = H \oplus H^\perp$, where H^\perp denotes the orthogonal complement of H in $L^2(T\Omega)$. For each positive integer N , we now define N -dimensional spaces $E_N \subset L^2(T\Omega)$ and $F_N \subset H$ as follows. Let $\{\phi_i \mid i \in \mathbb{N}_0\}$ be an orthonormal basis in $L^2(T\Omega)$ formed by the eigenfunctions of the Dirichlet Laplacian and let $0 < \beta_1 \leq \beta_2 \leq \dots$ be the corresponding eigenvalues. Furthermore, let $\{e_i \mid i \in \mathbb{N}_0\}$ be the orthonormal basis in H formed by the eigenfunctions of the Stokes operator and let $0 < \alpha_1 \leq \alpha_2 \leq \dots$ be the corresponding eigenvalues. For each $N \in \mathbb{N}_0$, we introduce the N -dimensional subspaces

$$E_N := \text{span}\{\phi_i \mid i \leq N\} \subset L^2(T\Omega), \quad F_N := \text{span}\{e_i \mid i \leq N\} \subset H$$

and denote by $P_N : L^2(T\Omega) \rightarrow E_N$ and $\Pi_N : L^2(T\Omega) \rightarrow F_N$ the corresponding orthogonal projections. We will show that the required control space can be chosen in the form $\mathcal{E}_M = \chi E_M$, where $\chi \in C_0^\infty(\Omega)$ is a given function not identically equal to zero, and the integer M is sufficiently large. In particular, $\chi E_M \subset C_0^\infty(T\omega)$ for any sub-domain $\omega \subseteq \Omega$ containing $\text{supp}(\chi)$.

Let us note that, seeking a solution of (12) in the form $u = \hat{u} + v$, we obtain the following equivalent problem for v :

$$(13) \quad v_t + Lv + Bv + \mathbb{B}(\hat{u})v = \Pi\zeta, \quad v(0) = v_0,$$

where $v_0 = u_0 - \hat{u}(0)$. It is clear that it suffices to consider the problem of exponential stabilization to zero for solutions of (13). Thus, in what follows, we will study problem (13).

3. MAIN RESULT FOR LINEARIZED SYSTEM

In this section, we study the linearized problem. The result established here will be used later for the full nonlinear system.

Let us fix a function $\hat{u} \in L_{\text{loc}}^2(\mathbb{R}_+, V) \cap \mathcal{W}$. In what follows, it will be convenient to write the control ζ entering (13) in the form $\zeta = \chi P_M \eta$, where η takes its values in $L^2(T\Omega)$ and $\chi \in C_0^1(\Omega)$ is a nonzero function not identically equal to zero. Thus, we study the problem

$$(14) \quad v_t + Lv + \mathbb{B}(\hat{u})v = \Pi(\chi P_M \eta),$$

$$(15) \quad v(0) = v_0,$$

where $v_0 \in H$. We refer the reader to [15, 19] for precise definitions of the concept of a solution for (14) (and all other Navier–Stokes type PDE's).

3.1. Existence of a stabilizing control.

Theorem 3.1. *For each $v_0 \in H$ and $\lambda > 0$, there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ and a control $\eta^{\hat{u}, \lambda}(v_0) \in L^2(\mathbb{R}_+, E_M)$ such that the solution v of system (14), (15) satisfies the inequality*

$$(16) \quad |v(t)|_H^2 \leq \kappa |v_0|_H^2 e^{-\lambda t}, \quad t \geq 0,$$

where $\kappa = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} > 0$ is a constant not depending on v_0 . Moreover, the mapping $v_0 \mapsto e^{(\tilde{\lambda}/2)t} \eta^{\hat{u}, \lambda}(v_0)$ is linear and continuous from H to $L^2(\mathbb{R}_+, E_M)$ for all $0 \leq \tilde{\lambda} < \lambda$. Finally, if $v_0 \in V$, then

$$(17) \quad |v(t)|_V^2 \leq \bar{\kappa} |v_0|_V^2 e^{-\lambda t}, \quad t \geq 0,$$

where $\bar{\kappa} = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} > 0$ does not depend on v_0 .

To prove this theorem, we will need two auxiliary lemmas. For each $\tau \geq 0$, consider equation (14) on the time interval $I_\tau = (\tau, \tau + 1)$ and supplement it with the initial condition

$$(18) \quad v(\tau) = w_0.$$

Let us denote by $S_{\hat{u}, \tau}(w_0, \eta)$ the operator that takes the pair (w_0, η) to the solution of (14), (18). It is well known that the operator $S_{\hat{u}, \tau}$ is continuous from H to $C(\bar{I}_\tau, H) \cap L^2(I_\tau, V)$ and from V to $C(\bar{I}_\tau, V) \cap L^2(I_\tau, U)$. We will write $S_{\hat{u}, \tau}(w_0, \eta)(t)$ for the value of the solution at time t .

Lemma 3.2. *For each $N \in \mathbb{N}$ there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ such that, for every $w_0 \in H$ and an appropriate control $\eta \in L^2(I_\tau, E_M)$ we have*

$$\Pi_N S_{\hat{u}, \tau}(w_0, \eta)(\tau + 1) = 0.$$

Moreover, there is a constant C_χ depending only on $|\hat{u}|_{\mathcal{W}}$ (but not on N and τ) such that

$$(19) \quad |\eta|_{L^2(I_\tau, E_M)}^2 \leq C_\chi |w_0|_H^2.$$

Proof. Let us fix $\epsilon > 0$ and consider the following minimization problem.

Problem 3.3. *Given $M, N \in \mathbb{N}$ and $w_0 \in H$, find the minimum of the quadratic functional*

$$J_\epsilon(v, \eta) := |\eta|_{L^2(I_\tau, L^2(T\Omega))}^2 + \frac{1}{\epsilon} |\Pi_N S_{\hat{u}, M, \tau}(w_0, \eta)(\tau + 1)|_H^2$$

on the set of functions $(v, \eta) \in W(I_\tau, V, V') \times L^2(I_\tau, L^2(T\Omega))$ that satisfy (14) and (18).

Theorem 5.2 implies immediately that problem 3.3 has a unique minimizer $(\bar{v}_\epsilon, \bar{\eta}_\epsilon)$, which linearly depends on $w_0 \in H$. We now derive some estimates for the norm of the optimal control $\bar{\eta}^\epsilon$.

To this end, the general theory of linear-quadratic optimal control problems is applicable. We use here a version of the Karush–Kuhn–Tucker theorem (see theorem 5.1). Let us define the affine mapping

$$F : W(\bar{I}_\tau, V, V') \times L^2(I_\tau, L^2(T\Omega)) \rightarrow H \times L^2(I_\tau, V'),$$

$$(v, \eta) \mapsto (v(0) - w_0, v_t + Lv + \mathbb{B}(\hat{u})v - \Pi(\chi P_M \eta))$$

and note that its derivative is surjective. Hence, by the Karush–Kuhn–Tucker theorem, there is a Lagrange multiplier $(\mu^\epsilon, q^\epsilon) \in H \times L^2(I_\tau, V)$ such that¹

$$J'_\epsilon(\bar{v}^\epsilon, \bar{\eta}^\epsilon) - (\mu^\epsilon, q^\epsilon) \circ F'(\bar{v}^\epsilon, \bar{\eta}^\epsilon) = 0.$$

It follows that, for all $(z, \xi) \in W(\bar{I}_\tau, V, V') \times L^2(I_\tau, L^2(T\Omega))$, we have

$$(20) \quad \frac{2}{\epsilon} (\Pi_N \bar{v}^\epsilon(\tau + 1), z(\tau + 1))_H + (z(\tau), \mu^\epsilon)_H + \int_{I_\tau} \langle z_t + Lz + \mathbb{B}(\hat{u})z, q^\epsilon \rangle_{V', V} dt = 0,$$

$$(21) \quad 2 \int_{I_\tau} (\bar{\eta}^\epsilon, \xi)_{L^2(T\Omega)} dt + \int_{I_\tau} \langle -\Pi(\chi P_M \xi), q^\epsilon \rangle_{V', V} dt = 0.$$

Relation (20) implies that q^ϵ is the solution of the problem

$$(22) \quad q_t^\epsilon - Lq^\epsilon - \mathbb{B}^*(\hat{u})q^\epsilon = 0, \quad t \in I_\tau,$$

$$(23) \quad q^\epsilon(\tau + 1) = -2\epsilon^{-1} \Pi_N \bar{v}^\epsilon(\tau + 1).$$

Furthermore, it follows from (21) that

$$(24) \quad 2\bar{\eta}_\epsilon = P_M(\chi q^\epsilon).$$

¹The space $H \times L^2(I_\tau, V)$ is regarded as the dual of $H \times L^2(I_\tau, V')$, and the sign \circ stands for the composition of two linear operators.

Combining (14), (22), and (24), we derive

$$\begin{aligned} \frac{d}{dt}(q^\epsilon, \bar{v}^\epsilon)_H &= (q_t^\epsilon, \bar{v}^\epsilon)_H + (q^\epsilon, \bar{v}_t^\epsilon)_H \\ &= (Lq^\epsilon + \mathbb{B}^*(\hat{u})q^\epsilon, \bar{v}^\epsilon)_H + (q^\epsilon, -L\bar{v}^\epsilon - \mathbb{B}(\hat{u})\bar{v}^\epsilon + \Pi(\chi P_M \bar{\eta}^\epsilon))_H \\ &= (q^\epsilon, \Pi(\chi P_M \bar{\eta}^\epsilon))_H = \frac{1}{2} |P_M(\chi q^\epsilon)|_{L^2(T\Omega)}^2. \end{aligned}$$

Integrating in time over the interval I_τ , we obtain

$$\int_{I_\tau} |P_M(\chi q^\epsilon)|_{L^2(T\Omega)}^2 dt = 2((q^\epsilon(\tau+1), \bar{v}^\epsilon(\tau+1))_H - (q^\epsilon(\tau), \bar{v}^\epsilon(\tau))_H).$$

Recalling now (23), we see that $2(q^\epsilon(\tau+1), \bar{v}^\epsilon(\tau+1))_H = -\epsilon |q^\epsilon(\tau+1)|_H^2$ and therefore

$$(25) \quad \int_{I_\tau} |P_M(\chi q^\epsilon)|_{L^2(T\Omega)}^2 dt + \epsilon |q^\epsilon(\tau+1)|_H^2 = -2(q^\epsilon(\tau), \bar{v}^\epsilon(\tau))_H.$$

We now use the truncated observability inequality (76) to estimate the right-hand side of (25). For every $\alpha > 0$, we have

$$\begin{aligned} \int_{I_\tau} |P_M(\chi q^\epsilon)|_{L^2(T\Omega)}^2 dt + \epsilon |q^\epsilon(\tau+1)|_H^2 &\leq \alpha |q^\epsilon(\tau)|_H^2 + \alpha^{-1} |\bar{v}^\epsilon(\tau)|_H^2 \\ &\leq \alpha D_\chi \int_{I_\tau} |P_M \chi q^\epsilon|_{L^2(T\Omega)}^2 dt + \alpha^{-1} |\bar{v}^\epsilon(\tau)|_H^2. \end{aligned}$$

Setting $\alpha = (2D_\chi)^{-1}$, we obtain

$$(26) \quad \int_{I_\tau} |P_M(\chi q^\epsilon)|_{L^2(T\Omega)}^2 dt + 2\epsilon |q^\epsilon(\tau+1)|_H^2 \leq 4D_\chi |w_0|_H^2.$$

In particular, the family of functions $\{P_M(\chi q^\epsilon) \mid \epsilon > 0\}$ is bounded in $L^2(I_\tau, L^2(T\Omega))$, and the family of solutions $\{\bar{v}^\epsilon \mid \epsilon > 0\}$ for problem (14), (18) is bounded in $L^2(I_\tau, V)$. It follows that the family $\{\bar{v}_t^\epsilon \mid \epsilon > 0\}$ is bounded in $L^2(I_\tau, V')$. Thus, we can find a sequence $\epsilon_n \rightarrow 0^+$ such that

$$\begin{aligned} \eta^{\epsilon_n} = \frac{1}{2} P_M(\chi q^{\epsilon_n}) &\rightharpoonup \eta^0 \quad \text{in } L^2(I_\tau, E_M), \\ \bar{v}^{\epsilon_n} &\rightharpoonup v^0 \quad \text{in } L^2(I_\tau, V), \\ \bar{v}_t^{\epsilon_n} &\rightharpoonup v_t^0 \quad \text{in } L^2(I_\tau, V'), \end{aligned}$$

where $\eta^0 \in L^2(I_\tau, E_M)$ and $v^0 \in W(I_\tau, V, V')$ are some functions. A standard limiting argument shows that v^0 is a solution of problem (14), (18) with $\eta = \eta^0$. Furthermore, it follows from (26) and (23) that

$$|\Pi_N \bar{v}^\epsilon(\tau+1)|_H^2 = \frac{\epsilon^2}{4} |q^\epsilon(\tau+1)|_H^2 \leq \frac{\epsilon D_\chi}{2} |w_0|_H^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

This convergence implies that $\Pi_N v^0(\tau+1) = 0$. It remains to establish inequality (19).

To this end, note that, in view of (26), we have

$$|\eta^0|_{L^2(I_\tau, E_M)}^2 \leq 4D_\chi |w_0|_H^2.$$

Combining this with the property of continuity of the resolving operator for (14), (18) (see lemma 2.1), we arrive at the required estimate. The proof of the lemma is complete. \square

In view of lemma 3.2, it makes sense to consider the following minimization problem.

Problem 3.4. *Given integers $M, N \geq 1$ and a function $w_0 \in H$, find the minimum of the quadratic functional*

$$J(\eta) := |\eta|_{L^2(I_\tau, L^2(T\Omega))}^2$$

on the set of functions $(v, \eta) \in W(I_\tau, V, V') \times L^2(I_\tau, E_M)$ satisfying equations (14), (18) and the condition $\Pi_N v(\tau+1) = 0$.

Lemma 3.5. *For any $N \in \mathbb{N}$ there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} \geq 1$ such that for any $w_0 \in H$ problem 3.4 has a unique minimizer $(\bar{v}^{\hat{u}, \tau}, \bar{\eta}^{\hat{u}, \tau}) \in W(I_\tau, V, V') \times L^2(I_\tau, E_M)$. Moreover, the mapping $w_0 \mapsto (\bar{v}^{\hat{u}, \tau}, \bar{\eta}^{\hat{u}, \tau})$ is linear and continuous in the corresponding spaces, and there is a constant C_χ depending only on $|\hat{u}|_{\mathcal{W}}$ (but not on N and τ) such that*

$$(27) \quad |\bar{\eta}^{\hat{u}, \tau}|_{\mathcal{L}(H, L^2(I_\tau, E_M))}^2 \leq C_\chi.$$

Proof. Let us fix $N \in \mathbb{N}$ and choose an integer M as in lemma 3.2. We set

$$W_N(I_\tau, V, V') := \{v \in W(I_\tau, V, V') \mid \Pi_N v(\tau + 1) = 0\}$$

and define \mathcal{X} as the space of functions $(v, \eta) \in W_N(I_\tau, V, V') \times L^2(I_\tau, E_M)$ that satisfy equation (14). In view of lemma 3.2 and the linearity of (14), \mathcal{X} is a nontrivial Banach space, and the operator $A : \mathcal{X} \rightarrow H$ taking (v, η) to $v(0)$ is surjective. Thus, by theorem 5.2, problem 3.4 has a unique minimizer $(\bar{v}^{\hat{u}, \tau}, \bar{\eta}^{\hat{u}, \tau})$, which linearly depends on w_0 . Inequality (27) follows immediately from (19), because the norm of $\bar{\eta}^{\hat{u}, \tau}(w_0)$ in the space $L^2(I_\tau, E_M)$ is necessarily smaller than the norm of the control function η constructed in lemma 3.2. \square

Proof of theorem 3.1. The main idea of the proof was outlined in the introduction: we use the operator $\bar{\eta}^{\hat{u}, \tau}$ constructed in lemma 3.5 with $N \gg 1$ to define an exponentially stabilizing control $\eta^{\hat{u}, \lambda}$ consecutively on the intervals $I_n = (n, n + 1)$, $n \geq 0$. Namely, let us fix an initial function $v_0 \in H$ and an integer $N = N(\lambda) \geq 1$, and set²

$$\eta^{\hat{u}, \lambda}(t) = \bar{\eta}^{\hat{u}, 0}(v_0)(t) \quad \text{for } t \in I_0.$$

Assuming that $\eta^{\hat{u}, \lambda}$ is constructed on the interval $(0, n)$ and denoting by $v(t)$ the corresponding solution on $[0, n]$, we define

$$\eta^{\hat{u}, \lambda}(t) = \bar{\eta}^{\hat{u}, n}(v(n))(t) \quad \text{for } t \in I_n.$$

By construction, $\eta^{\hat{u}, \lambda}$ is an E_M -valued function square integrable on every bounded interval. Moreover, the linearity of $\bar{\eta}^{\hat{u}, \tau}$ implies that $\eta^{\hat{u}, \lambda}$ linearly depends on v_0 . We claim that, if $N \in \mathbb{N}$ is sufficiently large, then the solution v of system (14), (15) with $\eta = \eta^{\hat{u}, \lambda}$ goes λ -exponentially to 0 as $t \rightarrow +\infty$.

Indeed, it follows from (10) that

$$|v(1)|_V^2 = |S_{\hat{u}, 0}(v_0, \bar{\eta}^{\hat{u}, 0}(v_0))(1)|_V^2 \leq \bar{C}_{[\hat{u}|w]} \left(|v_0|_H^2 + 3|\chi|_{L^\infty(\Omega)}^2 |\bar{\eta}^{\hat{u}, 0}(v_0)|_{L^2(I_0, E_M)}^2 \right).$$

Since $\Pi_N v(1) = 0$, we obtain

$$\alpha_N |v(1)|_H^2 \leq |v(1)|_V^2 \leq \bar{C}_{[\hat{u}|w]} \left(|v_0|_H^2 + 3|\chi|_{L^\infty(\Omega)}^2 |\bar{\eta}^{\hat{u}, 0}(v_0)|_{L^2(I_0, E_M)}^2 \right).$$

Using the continuity of the operator $\bar{\eta}^{\hat{u}, 0}$ (see lemma 3.5) and setting $C'_\chi := C_\chi |\chi|_{L^\infty(\Omega)}^2$, we derive

$$|v(1)|_H^2 \leq \alpha_N^{-1} |v(1)|_V^2 \leq \alpha_N^{-1} (\bar{C}_{[\hat{u}|w]} + 3C'_\chi) |v_0|_H^2.$$

Taking N so large that $\alpha_N \geq e^\lambda (\bar{C}_{[\hat{u}|w]} + 3C'_\chi)$, we obtain

$$|v(1)|_H^2 \leq e^{-\lambda} |v_0|_H^2.$$

We may repeat the above argument on every the interval I_n and conclude that, applying the control $\bar{\eta}^{\hat{u}, 1}(v(n))$ with $\alpha_N \geq e^\lambda (\bar{C}_{[\hat{u}|w]} + 3C'_\chi)$, we have

$$|v(n+1)|_H^2 \leq e^{-\lambda} |v(n)|_H^2.$$

By induction, we see that the solution v of problem (14), (15) with $\eta = \eta^{\hat{u}, \lambda}$ satisfies the inequality

$$(28) \quad |v(n)|_H^2 \leq e^{-\lambda n} |v_0|_H^2.$$

On the other hand, in view of (10), we have

$$|v|_{C(\bar{I}_n, H)}^2 \leq \bar{C}_{[\hat{u}|w]} |v(n)|_H^2 + |\chi|_{L^\infty}^2 |\bar{\eta}^{\hat{u}, n}(v(n))|_{L^2(I_n, E_M)}^2 \leq (\bar{C}_{[\hat{u}|w]} + C'_\chi) |v(n)|_H^2.$$

Combining this with (9), we see that

$$|v(t)|_H^2 \leq (\bar{C}_{[\hat{u}|w]} + C'_\chi) e^{-\lambda[t]} |v_0|_H^2 \leq e^\lambda (\bar{C}_{[\hat{u}|w]} + C'_\chi) e^{-\lambda t} |v_0|_H^2.$$

where $[t]$ is the largest integer not exceeding t . Thus, v satisfies inequality (16).

We now prove the continuity of the map $v_0 \mapsto e^{(\tilde{\lambda}/2)t} \eta^{\hat{u}, \lambda}$ in the corresponding spaces. It follows from (27) and (28) that, for any $\tilde{\lambda} < \lambda$, we have

$$(29) \quad \begin{aligned} |e^{(\tilde{\lambda}/2)t} \eta^{\hat{u}, \lambda}|_{L^2(\mathbb{R}_+, E_M)}^2 &= \sum_{n \in \mathbb{N}} |e^{(\tilde{\lambda}/2)t} \bar{\eta}^{\hat{u}, n}(v(n))|_{L^2(I_n, E_M)}^2 \leq C'_\chi \sum_{n \in \mathbb{N}} e^{\tilde{\lambda}(n+1)} |v(n)|_H^2 \\ &\leq C'_\chi e^{\tilde{\lambda}} \sum_{n \in \mathbb{N}} e^{(\tilde{\lambda}-\lambda)n} |v_0|_H^2 \leq C_{\chi, \lambda} |v_0|_H^2. \end{aligned}$$

It remains to prove inequality (17). In view of (10), we have

$$|\sqrt{t-n} v|_{C(\bar{I}_n, V)}^2 \leq \bar{C}_{[\hat{u}|w]} \left(|v(n)|_H^2 + 3|\chi|_{L^\infty(\Omega)}^2 |\bar{\eta}^{\hat{u}, n}(v(n))|_{L^2(I_n, E_M)}^2 \right) \leq (\bar{C}_{[\hat{u}|w]} + 3C'_\chi) |v(n)|_H^2.$$

²Recall that the operator $\bar{\eta}^{\hat{u}, \tau}$ depends on N .

Combining this with (28), we see that

$$|v(n+1)|_V^2 \leq C_1 |v(n)|_H^2 \leq C_1 e^{-\lambda n} |v_0|_H^2.$$

In view of (11), for $n \geq 1$ we derive

$$|v|_{C(\bar{I}_n, V)}^2 \leq \overline{C}_{[\hat{u}|\mathcal{W}]} |v(n)|_V^2 + 3|\chi|_{L^\infty(\Omega)}^2 |\bar{\eta}^{\hat{u}, n}(v(n))|_{L^2(I_n, E_M)}^2 \leq C_2 e^{-\lambda(n-1)} |v_0|_H^2,$$

whence it follows that

$$|v(t)|_V^2 \leq C_3 e^{-\lambda t} |v_0|_H^2 \quad \text{for } t \geq 1.$$

Using again inequality (11), we conclude that (17) holds. The proof of the theorem is complete. \square

3.2. Feedback control. In this section, we show that exponentially stabilizing control constructed in theorem 3.1 can be chosen in a feedback form. Namely, let us fix a nonzero function $\chi \in C_0^1(\Omega)$ and denote by \mathcal{E}_M the vector space spanned by the functions $\chi\phi_j$, $j = 1, \dots, M$. We will prove the following theorem.

Theorem 3.6. *For any $\hat{u} \in \mathcal{W}$ and $\lambda > 0$ there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|\mathcal{W}]} \in \mathbb{N}$, a family of continuous operators $K_{\hat{u}}^\lambda(t) : H \rightarrow \mathcal{E}_M$, and a constant $\kappa = \overline{C}_{[\lambda, |\hat{u}|\mathcal{W}]}$ such that the following properties hold.*

(i) *The function $t \mapsto K_{\hat{u}}^\lambda(t)$ is continuous in the weak operator topology, and its operator norm is bounded by κ .*

(ii) *For any $s \geq 0$ and $v_0 \in H$, the solution of the problem*

$$(30) \quad v_t + Lv + \mathbb{B}(\hat{u})v = \Pi K_{\hat{u}}^\lambda(t)v,$$

$$(31) \quad v(s) = v_0$$

exists on the time interval $(s, +\infty)$ and satisfies the inequality

$$(32) \quad e^{\lambda(t-s)} |v(t)|_H^2 + \int_s^t e^{\lambda(\tau-s)} (|v(\tau)|_V^2 + |v_t(\tau)|_{V'}^2) d\tau \leq \kappa |v_0|_H^2, \quad t \geq s,$$

Moreover, if $v_0 \in V$, then

$$(33) \quad e^{\lambda(t-s)} |v(t)|_V^2 + \int_s^t e^{\lambda(\tau-s)} (|v(\tau)|_{D(L)}^2 + |v_t(\tau)|_H^2) d\tau \leq \kappa |v_0|_V^2, \quad t \geq s.$$

To prove this theorem, we will need two auxiliary lemmas. Recall that, given a constant $\lambda > 0$ and a space Z of functions $f(t)$ on an interval $I \subseteq \mathbb{R}$, the space $E^\lambda(Z)$ was defined in the Introduction. Let us consider the following problem.

Problem 3.7. *Given $s \geq 0$, $\lambda > 0$, $M \in \mathbb{N}$ and $w_0 \in H$, find the minimum of the functional*

$$M_s^\lambda(v, \eta) := \int_{(s, +\infty)} e^{\lambda t} (|v|_V^2 + |\eta|_{L^2(T\Omega)}^2) dt$$

on the set of functions $(v, \eta) \in E^\lambda(W([s, +\infty), V, V']) \times E^\lambda(L^2([s, +\infty), L^2(T\Omega)))$ that satisfy equation (14) and the initial condition

$$(34) \quad v(s) = w_0.$$

The following lemma establishes the existence of an optimal solution and gives a formula for the optimal cost.

Lemma 3.8. *For any $\hat{u} \in \mathcal{W}$ and $\lambda > 0$ there is an integer $M = \overline{C}_{[\lambda, |\hat{u}|\mathcal{W}]} \geq 1$ such that problem 3.7 has a unique minimizer (v_s^*, η_s^*) . Moreover, there is a continuous operator $Q_{\hat{u}}^{s, \lambda} : H \rightarrow H$ such that*

$$(35) \quad M_s^\lambda(v_s^*, \eta_s^*) = (Q_{\hat{u}}^{s, \lambda} w_0, w_0),$$

$$(36) \quad |Q_{\hat{u}}^{s, \lambda}|_{\mathcal{L}(H)} \leq C e^{\lambda s},$$

where $C = \overline{C}_{[\lambda, |\hat{u}|\mathcal{W}]} > 0$ is a constant. Finally, $Q_{\hat{u}}^{s, \lambda}$ continuously depends on s in the weak operator topology.

Proof. Let \mathcal{X} be the space of functions $(v, \eta) \in E^\lambda(W([s, +\infty), V, V']) \times E^\lambda(L^2([s, +\infty), L^2(T\Omega)))$ that satisfy (14), (34) and endow it with the norm $M_s^\lambda(v, \eta)^{1/2}$. It is straightforward to see that \mathcal{X} is a Hilbert space. Moreover, using theorem 3.1 with a constant $\hat{\lambda} > \lambda$ and the initial point moved to s , we can construct an integer $M = \overline{C}_{[\hat{\lambda}, |\hat{u}|\mathcal{W}]} \geq 1$ such that, for any $w_0 \in H$ and an appropriate control $\eta \in E^{\hat{\lambda}}(L^2([s, \infty), E_M))$, we have

$$|v(t)|_H^2 \leq \kappa e^{-\hat{\lambda}(t-s)} |w_0|_H^2, \quad |\eta(t)|_{E_M}^2 \leq C_1 e^{-\hat{\lambda}(t-s)} |w_0|_H^2,$$

where v stands for the solution of (14), (34). Combining this with lemma 2.1, we conclude that

$$(37) \quad M_s^\lambda(v, \eta) = \int_{(s, +\infty)} e^{\lambda t} (|v|_V^2 + |\eta|_{L^2(T\Omega)}^2) dt \leq \overline{C}_{[\hat{\lambda}, (\hat{\lambda}-\lambda)^{-1}, |\hat{u}|_W]} e^{\lambda s} |w_0|_H^2.$$

It follows that \mathcal{X} is nonempty, and the mapping $A : \mathcal{X} \rightarrow H$ taking (v, η) to $v(0)$ is surjective. Thus, by theorem 5.2, Problem 3.7 has a unique minimizer $(v_s^*, \eta_s^*) = (v_s^*(w_0), \eta_s^*(w_0))$, which linearly depends on w_0 .

We now prove (35) and (36). It follows from (37) that the mapping

$$(a, b) \mapsto \int_{(s, +\infty)} e^{\lambda t} ((v_s^*(a), v_s^*(b))_V + (\eta_s^*(a), \eta_s^*(b))_{L^2(T\Omega)}) dt$$

is a continuous bilinear form on H which is bounded by $C_2 e^{\lambda s}$ on the unit ball. Therefore, the optimal cost can be written as (35), where $Q_{\hat{u}}^{s, \lambda}$ is a bounded self-adjoint operator in H whose norm satisfies (36).

It remains to establish the continuity of $Q_{\hat{u}}^{s, \lambda}$ in the weak operator topology. It is easy to see that the optimal cost $M_s^\lambda(v_s^*, \eta_s^*)$ continuously depends on s . It follows that

$$(Q_{\hat{u}}^{s, \lambda} w, w) \rightarrow (Q_{\hat{u}}^{s_0, \lambda} w, w) \quad \text{as } s \rightarrow s_0,$$

where $w \in H$ is an arbitrary function. This convergence immediately implies the required assertion. \square

We now consider another minimization problem closely related to problem 3.7 with $s = 0$.

Problem 3.9. *Given $\lambda > 0$ and $v_0 \in H$, find the minimum of the functional*

$$N_s^\lambda(v, \eta) := \int_{(0, s)} e^{\lambda t} (|v|_V^2 + |\eta|_{L^2(T\Omega)}^2) dt + (Q_{\hat{u}}^{s, \lambda} v(s), v(s))$$

on the set of functions $(v, \eta) \in W([0, s], V, V') \times L^2((0, s), L^2(T\Omega))$ that satisfy (14), (15), where M is the integer constructed in lemma 3.8.

Theorem 5.2 implies that problem 3.9 has a unique minimizer $(v_s^\bullet, \eta_s^\bullet)$, which is a linear function of $v_0 \in H$. The following lemma is the dynamic programming principle for Problem 3.7 with $s = 0$.

Lemma 3.10. *Under the hypotheses of lemma 3.8, the restriction of (v_0^*, η_0^*) to the interval $(0, s)$ coincides with $(v_s^\bullet, \eta_s^\bullet)$ and the restriction of (v_0^*, η_0^*) to the interval $(s, +\infty)$ coincides with $(v_s^*, \eta_s^*)(v_0(s))$.*

Proof. We will confine ourselves to the proof of the first assertion, because the second one is obvious. Let us define the function

$$(z_0^*, \theta_0^*)(t) := \begin{cases} (v_s^\bullet, \eta_s^\bullet)(v_0)(t) & \text{for } t \in (0, s), \\ (v_s^*, \eta_s^*)(v_0(s))(t) & \text{for } t \in (s, +\infty). \end{cases}$$

Then we have

$$M_0^\lambda(z_0^*, \theta_0^*) = N_s^\lambda(v_s^\bullet, \eta_s^\bullet).$$

On the other hand, the definition of $(v_s^\bullet, \eta_s^\bullet)$ implies that

$$N_s^\lambda(v_s^\bullet, \eta_s^\bullet) \leq N_s^\lambda((v_0^*, \eta_0^*)|_{(0, s)}) \leq M_0^\lambda(v_0^*, \eta_0^*),$$

whence it follows that

$$M_0^\lambda(z_0^*, \theta_0^*) \leq M_0^\lambda(v_0^*, \eta_0^*).$$

The uniqueness of minimizer for problem 3.7 with $s = 0$ implies that $(z_0^*, \theta_0^*) = (v_0^*, \eta_0^*)$, and the required assertion follows. \square

Proof of theorem 3.6. Step 1. It is straightforward to see that 3.9 satisfies the hypotheses of the Karush–Kuhn–Tucker theorem 5.1, in which

$$\mathcal{X} = W([0, s], V, V') \times L^2((0, s), L^2(T\Omega)), \quad \mathcal{Y} = H \times L^2((0, s), V),$$

$J = N_s^\lambda$, and $F : \mathcal{X} \rightarrow \mathcal{Y}$ is the affine operator taking (v, η) to $(v(0) - w_0, v_t + Lv + \mathbb{B}(\hat{u})v - \Pi(\chi P_M \eta))$. Hence, there is a Lagrange multiplier $(\mu_s, q_s) \in H \times L^2((0, s), V)$ such that

$$(N_s^\lambda)'(v_s^\bullet, \eta_s^\bullet) - (\mu_s, q_s) \circ F'(v_s^\bullet, \eta_s^\bullet) = 0.$$

It follows that, for all $z \in W([0, s], V, V')$ and $\xi \in L^2((0, s), L^2(T\Omega))$, we have

$$(38) \quad 2 \int_0^s e^{\lambda t} (v_s^\bullet, z)_V dt + 2(Q_{\hat{u}}^{s,\lambda} v_s^\bullet(s), z(s))_H + (z(0), \mu_s)_H + \int_0^s \langle z_t + Lz + \mathbb{B}(\hat{u})z, q_s \rangle_{V',V} dt = 0,$$

$$(39) \quad 2 \int_0^s e^{\lambda t} (\eta_s^\bullet, \xi)_{L^2(T\Omega)} dt + \int_0^s \langle -\Pi(\chi P_M \xi), q_s \rangle_{V',V} dt = 0.$$

In particular, from (38) we may conclude that

$$(40) \quad (q_s)_t - Lq_s - \mathbb{B}^*(\hat{u})q_s = e^{\lambda t} L v_s^\bullet(t).$$

Since $q_s, v_s^\bullet \in L^2((0, s), V)$, we see that $\partial_t q_s \in L^2((0, s), V')$, and therefore $q_s \in W([0, s], V, V')$, whence it follows that $q_s \in C([0, s], H)$. Using again (38), we derive

$$(41) \quad q_s(s) = -2Q_{\hat{u}}^{s,\lambda} v_s^\bullet(s).$$

On the other hand, relation (39) implies that

$$(42) \quad \eta_s^\bullet = \frac{1}{2} e^{-\lambda t} P_M (\chi q_s).$$

In particular, $\eta_s^\bullet(t)$ is a continuous function of t with range in E_M . Combining (42) and (41), we derive

$$\eta_s^\bullet(s) = -e^{-\lambda s} P_M \chi Q_{\hat{u}}^{s,\lambda} v_s^\bullet(s).$$

Recalling lemma 3.10 and using the fact that s is arbitrary, we conclude that η_0 is a continuous function of time with range in E_M and that

$$\eta_0^*(t) = -e^{-\lambda t} P_M (\chi Q_{\hat{u}}^{t,\lambda} v_0^*(t)) \quad \text{for all } t \geq 0.$$

Thus, the optimal trajectory v_0^* for problem 3.7 with $s = 0$ solves the system

$$(43) \quad v_t + Lv + B(\hat{u}, v) + B(v, \hat{u}) = \Pi(K_{\hat{u}}^\lambda v), \quad t \in \mathbb{R}_+,$$

$$(44) \quad v(0) = v_0,$$

where we set

$$K_{\hat{u}}^\lambda(t) := -e^{-\lambda t} \chi P_M \chi Q_{\hat{u}}^{t,\lambda}.$$

It is clear that $K_{\hat{u}}^\lambda(t)$ is a linear continuous operator from H to \mathcal{E}_M . Moreover, it continuously depends on t in the weak operator topology, because so does the family $Q_{\hat{u}}^{t,\lambda}$. Finally, it follows from (36) that the norm of $K_{\hat{u}}^\lambda(t)$ is bounded by a constant depending only on λ and $|\hat{u}|_{\mathcal{W}}$. We have thus constructed a feedback control $K_{\hat{u}}^\lambda(t)$ such that $M_0^\lambda(v, \eta) \leq C|v_0|^2$, where v denotes the solution of (30), (31) with $s = 0$. Moreover, the explicit formula for $K_{\hat{u}}^\lambda$ and the definition of $Q_{\hat{u}}^{t,\lambda}$ imply that if v is the solution of problem (30), (31), then $M_s^\lambda(v, \eta) \leq C|v_0|^2$, where $\eta(t) = K_{\hat{u}}^\lambda(t)v(t)$.

Step 2. We now prove inequalities (32) and (33) for solutions of problem (30), (31). Let us fix $v_0 \in H$ and denote by v the solution of (30), (31). It is straightforward to see that the function $z(t) = e^{(\lambda/2)t} v(t)$ satisfies the equation

$$(45) \quad z_t + Lz + \mathbb{B}(\hat{u})z = \frac{\lambda}{2}z + K_{\hat{u}}^\lambda(t)z.$$

Taking the scalar product of (45) with $2z$ and using the uniform boundedness of the family $K_{\hat{u}}^\lambda(t)$, we derive

$$(46) \quad \frac{d}{dt} |z(t)|_H^2 + 2|z(t)|_V^2 = \lambda|z(t)|_H^2 + 2(K_{\hat{u}}^\lambda(t)z, z) - 2(\mathbb{B}(\hat{u})z(t), z(t))_H \leq \overline{C}_{[\lambda, |\hat{u}|_{\mathcal{W}}]} |z(t)|_H^2$$

Integrating this inequality over the interval (s, t) with $t > s$, recalling the definition of z and using lemmas 3.8 and 3.10, we obtain

$$(47) \quad \begin{aligned} e^{\lambda t} |v(t)|_H^2 + 2 \int_{(s,t)} e^{\lambda \tau} |v(\tau)|_V^2 d\tau &\leq e^{\lambda s} |v_0|_H^2 + C_1 \int_{(s,t)} e^{\lambda \tau} |v(\tau)|_H^2 d\tau \\ &\leq e^{\lambda s} |v_0|_H^2 + C_1 \int_{(s,+\infty)} e^{\lambda \tau} |v(\tau)|_H^2 d\tau \\ &\leq e^{\lambda s} |v_0|_H^2 + C_2 (Q_{\hat{u}}^{s,\lambda} v_0, v_0) \leq C_3 e^{\lambda s} |v_0|_H^2. \end{aligned}$$

Furthermore, it follows from (43) that

$$e^{\lambda t} |v_t(t)|_V^2 \leq C_4 e^{\lambda t} |v(t)|_V^2.$$

Combining this with (47), we arrive at

$$e^{\lambda t}|v(t)|_H^2 + 2 \int_{(s,t)} e^{\lambda \tau} (|v(\tau)|_V^2 + |v_t(\tau)|_{V'}^2) d\tau \leq C_5 e^{\lambda s} |v_0|_H^2.$$

This is equivalent to (32).

We now assume that $v_0 \in V$. Taking the scalar product of (45) with $2Lz$, and using the Schwarz inequality and the uniform boundedness of the family $K_{\hat{u}}^\lambda(t)$, we derive

$$\begin{aligned} \frac{d}{dt}|z(t)|_V^2 + 2|z(t)|_{D(L)}^2 &= \lambda(z, Lz)_H + 2(K_{\hat{u}}^\lambda(t)z, Lz) - 2(\mathbb{B}(\hat{u})z(t), Lz(t))_H \\ &\leq |Lz|_H^2 + C(\lambda, |\hat{u}|_{\mathcal{W}})|z(t)|_V^2. \end{aligned}$$

Integrating this inequality over the interval (s, t) and using (32), we obtain

$$(48) \quad e^{\lambda t}|v(t)|_V^2 + \int_{(s,t)} e^{\lambda \tau}|v(\tau)|_{D(L)}^2 d\tau \leq C_6 \int_{(s,t)} e^{\lambda \tau}|v(\tau)|_V^2 d\tau \leq C_7 e^{\lambda s}|v_0|_H^2.$$

Furthermore, relation (43) implies that

$$e^{\lambda t}|v_t(t)|_H^2 \leq C_7 e^{\lambda t}|v(t)|_{D(L)}^2.$$

Combining this with (48), we arrive at (33). \square

4. STABILIZATION OF THE NONLINEAR PROBLEM

4.1. Main result. Let us consider the nonlinear problem

$$(49) \quad v_t + Lv + Bv + \mathbb{B}(\hat{u})v = K_{\hat{u}}^\lambda(t)v, \quad t \in \mathbb{R}_+;$$

$$(50) \quad v(0) = v_0,$$

where the operator $K_{\hat{u}}^\lambda(t)$ is constructed in theorem 3.6. Given a constant $\lambda > 0$, we denote by \mathcal{Z}^λ the space of functions $z \in C(\mathbb{R}_+, V) \cap L_{\text{loc}}^2(\mathbb{R}_+, U)$ such that

$$|z|_{\mathcal{Z}^\lambda} := \sup_{t \geq 0} \left(e^{\lambda t}|z(t)|_V^2 + \int_{(t,t+1)} e^{\lambda \tau}|z(\tau)|_{D(L)}^2 d\tau \right)^{1/2} < \infty.$$

The following theorem is the main result of this paper.

Theorem 4.1. *Let $\hat{u} \in \mathcal{W}$ be an arbitrary function, let $\lambda > 0$, and let $M = \overline{C}_{[|\hat{u}|_{\mathcal{W}}, \lambda]}$ be the integer constructed in theorem 3.6. Then are positive constants ϑ and ϵ depending only on $|\hat{u}|_{\mathcal{W}}$ and λ such that for $|v_0|_V \leq \epsilon$ the solution v of system (49), (50) is well defined for all $t \geq 0$ and satisfies the inequality*

$$(51) \quad |v(t)|_V^2 \leq \vartheta e^{-\lambda t} |v_0|_V^2 \quad \text{for } t \geq 0.$$

Proof. We will use the contraction mapping principle. We fix a constant $\vartheta > 0$ and a function $v_0 \in V$ and introduce the following subset of \mathcal{Z}^λ :

$$\mathcal{Z}_\vartheta^\lambda := \{z \in \mathcal{Z}^\lambda \mid z(0) = v_0, |z|_{\mathcal{Z}^\lambda}^2 \leq \vartheta |v_0|_V^2\}.$$

We define a mapping $\Xi : \mathcal{Z}_\vartheta^\lambda \rightarrow C(\mathbb{R}_+, V) \cap L_{\text{loc}}^2(\mathbb{R}_+, U)$ that takes a function $a \in \mathcal{Z}^\lambda$ to the solution of the problem

$$(52) \quad b_t + Lb + \mathbb{B}(\hat{u})b = K_{\hat{u}}^\lambda b - Ba, \quad t \in \mathbb{R}_+,$$

$$(53) \quad b(0) = v_0.$$

Suppose we have shown the following proposition.

Proposition 4.2. *Under the hypotheses of theorem 4.1, there exists $\vartheta > 0$ such that for any $\gamma \in (0, 1)$ and an appropriate constant $\epsilon = \epsilon_\gamma > 0$ the mapping Ξ takes the set $\mathcal{Z}_\vartheta^\lambda$ into itself and satisfies the inequality*

$$(54) \quad |\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda} \leq \gamma |a_1 - a_2|_{\mathcal{Z}^\lambda} \quad \text{for all } a_1, a_2 \in \mathcal{Z}_\vartheta^\lambda,$$

provided that $|v_0|_V \leq \epsilon$.

Thus, if $|v_0|_V$ is sufficiently small, then the contraction mapping principle implies that there is a unique fixed point $v \in \mathcal{Z}_\vartheta^\lambda$ for Ξ . It follows from the definition of Ξ and $\mathcal{Z}_\vartheta^\lambda$ that v is a solution of problem (49), (50) and satisfies the required inequality (51). Furthermore, standard methods show that problem (49), (50) has at most one solution in the space $C(\mathbb{R}_+, V) \cap L_{\text{loc}}^2(\mathbb{R}_+, U)$. Hence, to complete the proof of the theorem, it suffices to establish the above proposition. \square

4.2. Proof of proposition 4.2. *Step 1.* We first derive an estimate for solutions of the equation

$$(55) \quad z_t + Lz + \mathbb{B}(\hat{u})z = K_{\hat{u}}^\lambda z + f(t),$$

where $f \in L_{\text{loc}}^2(\mathbb{R}_+, H)$. Namely, we will show that

$$(56) \quad \sup_{t \geq 0} \left(e^{\lambda t} |z(t)|_V^2 + \int_{(t, t+1)} e^{\lambda s} |z(s)|_{D(L)}^2 ds \right) \leq C_1 \left(|z(0)|_V^2 + \sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda s} |f(s)|_H^2 ds \right),$$

where $C_1 = \overline{C}_{[|\hat{u}|_{\mathcal{W}}, \lambda]}$ is a constant. Indeed, let $U(s, t)$ be the operator that takes $v_0 \in H$ to $v(t)$, where v stands for the solution of (30), (31). By the Duhamel formula, we can write z as

$$(57) \quad z(t) = U(0, t)z(0) + \int_{(0, t)} U(s, t)f(s) ds.$$

Combining this with (32), we derive

$$(58) \quad \begin{aligned} |z(t)|_H^2 &= 2|U(0, t)z(0)|_H^2 + 2 \left(\int_{(0, t)} |U(s, t)f(s)|_H ds \right)^2 \\ &= 2\kappa e^{-\lambda t} |z(0)|_H^2 + 2\kappa e^{-\lambda t} \left(\int_{(0, t)} e^{(\lambda/2)s} |f(s)|_H ds \right)^2. \end{aligned}$$

Now note that, for any nonnegative function $c(t)$ and any $\lambda > 0$, we have

$$\begin{aligned} \sup_{t \geq 0} \int_{(0, t)} e^{(\lambda/2)s} c(s) ds &\leq \int_{(0, +\infty)} e^{(\lambda/2)s} c(s) ds = \sum_{k=1}^{\infty} \int_{(k-1, k)} e^{(\lambda/2)s} c(s) ds \\ &\leq \sum_{k=1}^{\infty} e^{(\lambda/2)k} \left(\int_{(k-1, k)} |c(s)|^2 ds \right)^{1/2} \\ &\leq \sum_{k=1}^{\infty} e^{-(\lambda/2)(k-2)} \left(\int_{(k-1, k)} e^{2\lambda s} |c(s)|^2 ds \right)^{1/2} \\ &\leq C_2 \left(\sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda s} |c(s)|^2 ds \right)^{1/2}. \end{aligned}$$

Substituting this inequality with $c(t) = |f(t)|_H$ into (58), we derive

$$(59) \quad \sup_{t \geq 0} \left(e^{\lambda t} |z(t)|_H^2 \right) \leq 2\kappa \left(|z(0)|_H^2 + C_2^2 \sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda s} |f(s)|_H^2 ds \right)$$

On the other hand, it is easy to see that the analogue of lemma 2.1 is true for equation (55). In particular, for any $s \geq 0$ we have the estimates

$$(60) \quad (t-s)|U(s, t)z_0|_V^2 \leq C_3 \left(|z_0|_H^2 + \int_{(s, t)} |f(\tau)|_H^2 d\tau \right),$$

$$(61) \quad |U(s, t)z_0|_V^2 + \int_{(s, s+1)} |U(s, \tau)z_0|_{D(L)}^2 d\tau \leq C_3 \left(|z_0|_V^2 + \int_{(s, t)} |f(\tau)|_H^2 d\tau \right),$$

where $s \leq t \leq s+1$, and $C_3 > 0$ does not depend on s . Combining (59) with inequality (60) in which $z_0 = U(0, s)z(0)$ and $t = s+1$, we obtain

$$\begin{aligned} |z(s+1)|_V^2 &\leq C_3 \left(|U(0, s)z(0)|_H^2 + \int_{(s, s+1)} |f(\tau)|_H^2 d\tau \right) \\ &\leq C_4 e^{-\lambda s} \left(|z(0)|_H^2 + \sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda \tau} |f(\tau)|_H^2 d\tau \right). \end{aligned}$$

Using now (61), for $s \geq 1$ we derive

$$(62) \quad |z(s)|_V^2 + \int_{(s, s+1)} |z(\tau)|_{D(L)}^2 d\tau \leq C_5 e^{-\lambda s} \left(|z(0)|_H^2 + \sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda \tau} |f(\tau)|_H^2 d\tau \right).$$

On the other hand, it follows from (61) that

$$(63) \quad \sup_{0 \leq s \leq 1} |z(s)|_V^2 + \int_{(0, 1)} |z(\tau)|_{D(L)}^2 d\tau \leq C_3 \left(|z_0|_V^2 + \int_{(0, 1)} |f(\tau)|_H^2 d\tau \right).$$

The required inequality (56) follows immediately from (62) and (63).

Step 2. We now prove that Ξ maps the set $\mathcal{Z}_\vartheta^\lambda$ into itself. Inequality (56) with $f(t) = -Ba(t)$ implies that

$$(64) \quad |\Xi(a)|_{\mathcal{Z}^\lambda}^2 \leq C_1 \left(|v_0|_V^2 + \sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda s} |Ba(s)|_H^2 ds \right).$$

Now note that $|Ba|_H \leq C_6 |a|_V |a|_{D(L)}$, whence it follows that

$$\sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda s} |Ba(s)|_H^2 ds \leq C_6^2 \sup_{t \geq 0} \int_{(t, t+1)} (e^{\lambda s} |a|_V^2) (e^{\lambda s} |a|_{D(L)}^2) ds \leq C_6^2 |a|_{\mathcal{Z}^\lambda}^4.$$

Substituting this into (64), we see that if $a \in \mathcal{Z}_\vartheta^\lambda$, then

$$(65) \quad |\Xi(a)|_{\mathcal{Z}^\lambda} \leq C_7 (|v_0|_V + |a|_{\mathcal{Z}^\lambda}^2) \leq C_7 (1 + \vartheta |v_0|_V) |v_0|_V.$$

Setting $\vartheta = 2C_7$ and choosing $\epsilon > 0$ so small that $C_7(1 + \vartheta\epsilon) \leq \vartheta$, we see that if $|v_0|_V \leq \epsilon$, then Ξ maps the set $\mathcal{Z}_\vartheta^\lambda$ into itself.

Step 3. It remains to prove that Ξ satisfies inequality (54). Let us take two functions $a_1, a_2 \in \mathcal{Z}_\vartheta^\lambda$ and set $a = a_1 - a_2$ and $z = \Xi(a_1) - \Xi(a_2)$. Then the function z satisfies the initial condition $z(0) = 0$ and equation (55) with $f = Ba_2 - Ba_1$. Therefore, by inequality (56), we have

$$(66) \quad |\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda}^2 \leq \sup_{t \geq 0} \int_{(t, t+1)} e^{2\lambda s} |Ba_1 - Ba_2|_H^2 ds.$$

Using a standard estimate for $B(u, v)$ and the interpolation inequality $|u|_{L^\infty}^2 \leq C |u|_V |u|_{D(L)}$, we derive

$$\begin{aligned} |Ba_1 - Ba_2|_H^2 &= |B(a_1, a) - B(a, a_2)|_H^2 \\ &\leq C_8 (|a_1|_{L^\infty} |a|_V + |a|_{L^\infty} |a_2|_V)^2 \\ &\leq C_9 (|a_1|_V |a_1|_{D(L)} |a|_V^2 + |a|_V |a|_{D(L)} |a_2|_V). \end{aligned}$$

It follows that

$$(67) \quad \int_{(t, t+1)} e^{2\lambda s} |Ba_1 - Ba_2|_H^2 ds \leq C_{10} (|a_1|_{\mathcal{Z}^\lambda}^2 + |a_2|_{\mathcal{Z}^\lambda}^2) |a|_{\mathcal{Z}^\lambda}^2.$$

Substituting (67) into (66) and recalling the definition of $\mathcal{Z}_\vartheta^\lambda$, we obtain

$$|\Xi(a_1) - \Xi(a_2)|_{\mathcal{Z}^\lambda}^2 \leq 2\vartheta C_{10} |v_0|_V^2 |a_1 - a_2|_{\mathcal{Z}^\lambda}^2.$$

Choosing $\epsilon > 0$ so small that $2\vartheta C_{10} \epsilon^2 \leq \gamma^2$, we see that if $|v_0|_V \leq \epsilon$, then (54) holds. This completes the proof of the proposition.

5. APPENDIX

5.1. Karush–Kuhn–Tucker theorem. Let \mathcal{X} and \mathcal{Y} be Banach spaces and let $J : \mathcal{X} \rightarrow \mathbb{R}$ and $F : \mathcal{X} \rightarrow \mathcal{Y}$ be two continuously differentiable functions. Consider the following minimization problem with constraints:

$$(68) \quad J(x) \rightarrow \min, \quad F(x) = 0.$$

We will say that $\bar{x} \in \mathcal{X}$ is a *local minimum* for (68) if $F(\bar{x}) = 0$ and there is a neighborhood $U \ni \bar{x}$ such that $J(\bar{x}) \leq J(x)$ for any $x \in U$ such that $F(x) = 0$. A proof of the following theorem can be found in [14].

Theorem 5.1. *Let $\bar{x} \in \mathcal{X}$ be a local minimum for (68) and let the derivative $F'(\bar{x}) : \mathcal{X} \rightarrow \mathcal{Y}$ be a surjective operator. Then there is $y^* \in \mathcal{Y}^*$ such that*

$$(69) \quad J'(\bar{x}) + y^* \circ F'(\bar{x}) = 0.$$

5.2. Quadratic functionals with linear constraint. Let \mathcal{X} and \mathcal{Y} be normed vector spaces, let $J(x, y)$ be a bounded symmetric bilinear form on \mathcal{X} that is weakly continuous with respect to each of its arguments, and let $A : \mathcal{X} \rightarrow \mathcal{Y}$ be a continuous surjective linear operator. Given a vector $y \in \mathcal{Y}$, consider the minimization problem

$$(70) \quad J(x) \rightarrow \min, \quad Ax = y,$$

where $J(x) = J(x, x)$. We will say that $\bar{x} \in \mathcal{X}$ is a *global minimum* for (70) if $A\bar{x} = y$ and $J(\bar{x}) \leq J(x)$ for $x \in \mathcal{X}$ such that $Ax = y$. The following result is rather standard in the optimal control theory.

Theorem 5.2. *Suppose that $J(x)$ is nonnegative and vanishes only for $x = 0$, and the set $\{x \in \mathcal{X} : J(x) \leq c\}$ is weakly compact for any $c > 0$. Then problem (70) has a unique global minimum $\bar{x} \in \mathcal{X}$, and the function $L : \mathcal{Y} \rightarrow \mathcal{X}$ taking y to \bar{x} is linear.*

Proof. Existence. Let $m \geq 0$ be the infimum of J on $A^{-1}(y)$ and let $\{x_n\} \subset A^{-1}(y)$ be a sequence such that $J(x_n) \rightarrow m$. Since the set $\{x \in \mathcal{X} : J(x) \leq m + 1\}$ is weakly compact, we can assume that $\{x_n\}$ converges weakly to a vector $\bar{x} \in \mathcal{X}$. Now note that

$$0 \leq J(x_n - \bar{x}) = J(x_n) - 2J(x_n, \bar{x}) + J(\bar{x}).$$

Combining this with the weak continuity of J , we see that

$$J(\bar{x}) \leq \liminf_{n \rightarrow \infty} J(x_n) = m.$$

Thus, \bar{x} is a global minimum for (70).

Uniqueness. Since the only point of \mathcal{X} at which J vanishes is $x = 0$, a standard argument proves that J is strictly convex, that is,

$$J\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}(J(x_1) + J(x_2)) \quad \text{for all } x_1, x_2 \in \mathcal{X},$$

and the equality holds if and only if $x_1 = x_2$. This immediately implies that the global minimum is unique.

Linearity. Let $y \in \mathcal{Y}$ and $z \in A^{-1}(0)$. For all $\lambda > 0$, we have $A(Ly \pm \lambda z) = y$, and the definition of L implies that $J(Ly) \leq J(Ly \pm \lambda z)$. It follows that $0 \leq \lambda J(z) \pm 2J(Ly, z)$ for all $\lambda > 0$. Letting λ go to 0, we see that

$$(71) \quad J(Ly, z) = 0 \quad \text{for all } y \in \mathcal{Y}, z \in A^{-1}(0).$$

For $a, b \in \mathcal{Y}$ and $\alpha, \beta \in \mathbb{R}$, let us set

$$k := \alpha La + \beta Lb - L(\alpha a + \beta b).$$

Then $Ak = 0$, and by (71), we have $J(k) = J(k, k) = \alpha J(La, k) + \beta J(Lb, k) - J(L(\alpha a + \beta b), k) = 0$. It follows that $k = 0$, and therefore L is linear. \square

5.3. Truncated observability inequality. We first recall a well-known observability inequality for the linearized Navier–Stokes system. Let us fix a function $\hat{u} \in \mathcal{W}_\tau \cap L^2(I_\tau, V)$ with $I_\tau = (\tau, \tau + 1)$ and $\mathcal{W}_\tau := W^{1,\infty}(I_\tau, W^{1,\infty}(T\Omega))$, and consider the problem

$$(72) \quad q_t - Lq - \mathbb{B}^*(\hat{u})q = 0, \quad t \in I_\tau,$$

$$(73) \quad q(\tau + 1) = q_1,$$

where $q_1 \in H$. By Theorem 2.2 in [13] (see also [7]), for any open subset $\omega \subset \Omega$ there is a constant C_ω such that

$$(74) \quad |q(\tau)|_H^2 \leq C_\omega \int_{I_\tau} |q|_{L^2(T\omega)}^2 dt,$$

Since $\text{supp } \chi \cap \Omega \neq \emptyset$, the domain $\omega_\chi := \{x \in \Omega \mid |\chi(x)| > \rho\}$ is nonempty for a sufficiently small $\rho > 0$. It follows from (74) that

$$|q(\tau)|_H^2 \leq C_{\omega_\chi} \int_{I_\tau} |q|_{L^2(T\omega_\chi)}^2 dt \leq C_{\omega_\chi} \rho^{-2} \int_{I_\tau} |\chi q|_{L^2(T\Omega)}^2 dt.$$

Thus, setting $D'_\chi := C_{\omega_\chi} \rho^{-2}$, for any solution of system (72), (73), we have the observability inequality

$$(75) \quad |q(\tau)|_H^2 \leq D'_\chi \int_{I_\tau} |\chi q|_{L^2(T\Omega)}^2 dt.$$

The following proposition shows that if q_1 belongs to a finite-dimensional subspace of H , then the function χq on the right-hand side of (75) can be replaced by $P_M(\chi q)$ with a sufficiently large M .

Proposition 5.3. *For any integer $N \geq 1$ there is $M = \overline{C}_{[N, |\hat{u}|_{\mathcal{W}_\tau}]} \in \mathcal{N}$ such that any solution q for system (72), (73) with $q_1 \in F_N = \Pi_N H$ satisfies the inequality*

$$(76) \quad |q(\tau)|_H^2 \leq D_\chi \int_{I_\tau} |P_M(\chi q)|_{L^2(T\Omega)}^2 dt$$

for a suitable constant D_χ depending only on χ .

To prove the proposition, we need the following lemma.

Lemma 5.4. *For any solution q of system (72), (73) with $q_1 \in F_N$, we have*

$$(77) \quad \int_{I_\tau} |\chi q|_{H^1(T\Omega)}^2 dt \leq \bar{C} \int_{I_\tau} |\chi q|_{L^2(T\Omega)}^2 dt,$$

where the constant $C = \bar{C}_{[N, |\hat{u}|_{\mathcal{W}_\tau}]}$ does not depend on the shape of the trajectory \hat{u} .

Proof. We argue by contradiction. Suppose there is a sequence $(q_1^n, \hat{u}^n) \in F_N \times (L^2(I_\tau, V) \cap \mathcal{W}_\tau)$, with $(|\hat{u}^n|_{\mathcal{W}_\tau})$ bounded, such that the solution q^n of the problem

$$(78) \quad q_t^n - Lq^n - \mathbb{B}^*(\hat{u}^n)q^n = 0, \quad t \in I_\tau,$$

$$(79) \quad q^n(\tau + 1) = q_1^n$$

satisfies the inequality

$$(80) \quad \int_{I_\tau} |\chi q^n|_{H^1(T\Omega)}^2 dt > n \int_{I_\tau} |\chi q^n|_{L^2(T\Omega)}^2 dt.$$

Since the equations are linear, there is no loss of generality in assuming that $|q_1^n| = 1$. The boundedness of $(|\hat{u}^n|_{\mathcal{W}_\tau})$ implies that both (\hat{u}^n) and (\hat{u}_t^n) are bounded in $L^\infty(I_\tau, W^{1,\infty}(T\Omega))$. After a change of time variable $s \in I_0$, $t = \tau + 1 - s$ we may apply lemma 2.1 and see that $(|q^n|_{L^2(I_\tau, D(L))})$ and $(|q_t^n|_{L^2(I_\tau, H)})$ are bounded.

What has been said implies that there is a subsequence of (q_1^n, q^n, \hat{u}^n) (for which we preserve the same notation), a unit vector $q_1^\infty \in F_N$, and functions $q^\infty \in W(I_\tau, D(L), H)$ and $\hat{u}^\infty \in \mathcal{W}_\tau$ such that

$$\begin{aligned} q_1^n &\rightarrow q_1^\infty \quad \text{in } F_N, \\ q^n &\rightarrow q^\infty \quad \text{in } L^2(I_\tau, V), \\ \partial_t q^n &\rightarrow \partial_t q^\infty \quad \text{in } L^2(I_\tau, H), \\ \hat{u}^n &\rightarrow \hat{u}^\infty \quad \text{in } L^2(I_\tau, H), \\ \partial_t^j \partial_x^\alpha \hat{u}^n &\rightharpoonup_* \partial_t^j \partial_x^\alpha \hat{u}^\infty \quad \text{in } L^\infty(I_\tau, L^\infty(\Omega)), \end{aligned}$$

where $j = 0, 1$ and $|\alpha| \leq 1$. Combining this with the boundedness of the sequences (\hat{u}^n) and (q^n) in the corresponding spaces, we can easily pass to the limit in (78), (79) and derive the equations

$$(81) \quad q_t^\infty - Lq^\infty - \mathbb{B}^*(\hat{u}^\infty)q^\infty = 0, \quad t \in I_\tau,$$

$$(82) \quad q^\infty(\tau + 1) = q_1^\infty.$$

Furthermore, since the operator of multiplication by χ is continuous in $L^2(I_\tau, H^1(T\Omega))$, we also have

$$(83) \quad \chi q^n \rightarrow \chi q^\infty \quad \text{in } L^2(I_\tau, H^1(T\Omega)).$$

Therefore, passing to the limit in inequality (80) as $n \rightarrow \infty$, we conclude that

$$(84) \quad \int_{I_\tau} |\chi q^\infty|_{L^2(T\Omega)}^2 dt = 0.$$

Applying now the observability inequality (75) to equation (81) considered on the interval $(\tau + r, \tau + 1)$ with $0 \leq r < 1$, we conclude that $q^\infty(t) = 0$ for $\tau \leq t < \tau + 1$. Since $q^\infty \in C(\bar{I}_\tau, V)$, we obtain $q_1^\infty = q^\infty(\tau + 1) = 0$. This contradicts the fact that $q_1^\infty \in F_N$ is a unit vector. The contradiction obtained proves that (77) holds. \square

Proof of proposition 5.3. We use lemma 5.4 to derive

$$\begin{aligned} \int_{I_\tau} |\chi q|_{L^2(T\Omega)}^2 dt &\leq \int_{I_\tau} |P_M(\chi q)|_{L^2(T\Omega)}^2 dt + \int_{I_\tau} |(1 - P_M)\chi q|_{L^2(T\Omega)}^2 dt \\ &\leq \int_{I_\tau} |P_M(\chi q)|_{L^2(T\Omega)}^2 dt + \beta_M^{-1} \int_{I_\tau} |(1 - P_M)(\chi q)|_{H^1(T\Omega)}^2 dt \\ &\leq \int_{I_\tau} |P_M(\chi q)|_{L^2(T\Omega)}^2 dt + \beta_M^{-1} \int_{I_\tau} |\chi q|_{H^1(T\Omega)}^2 dt \\ &\leq \int_{I_\tau} |P_M(\chi q)|_{L^2(T\Omega)}^2 dt + \beta_M^{-1} \bar{C}_{[N, |\hat{u}|_{\mathcal{W}_\tau}]} \int_{I_\tau} |\chi q|_{L^2(T\Omega)}^2 dt. \end{aligned}$$

Choosing the integer M so large that $\beta_M^{-1} \bar{C}_{[N, |\hat{u}|_{\mathcal{W}_\tau}]} \leq \frac{1}{2}$, we obtain

$$\int_{I_\tau} |\chi q|_{L^2(T\Omega)}^2 dt \leq 2 \int_{I_\tau} |P_M(\chi q)|_{L^2(T\Omega)}^2 dt.$$

Combining this with (75), we arrive the required inequality (76). \square

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