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Nekhoroshev estimates for finitely differentiable quasi-convex Hamiltonians

Bounemoura Abed

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Abstract

A major result concerning perturbations of integrable Hamiltonian systems is given by Nekhoroshev estimates, which ensures exponential stability of all solutions provided the system is analytic and the integrable Hamiltonian not too degenerate. In the particular but important case where the latter is quasi-convex, these exponential estimates have been generalized by Marco and Sauzin if the Hamiltonian is Gevrey regular, using a method introduced by Lochak in the analytic case. In this paper, using the same approach we will investigate the situation where the Hamiltonian is assumed to be only finitely differentiable, it is known that exponential stability does not hold but nevertheless we will prove estimates of polynomial stability.

1 Introduction

In this paper, we are concerned with the stability properties of near-integrable Hamiltonian systems of the form

$$\begin{cases} H(\theta, I) = h(I) + f(\theta, I) \\ |f| < \varepsilon \ll 1 \end{cases}$$

where $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$ are action-angles coordinates for the integrable part h and f is a small perturbation in some suitable topology defined by a norm $|\cdot|$. More precisely, we are interested in the evolution of the action variables $I(t)$, which are trivially constant in the absence of perturbation.

The first known result in this direction is given by an application of the KAM theory on the persistence of quasi-periodic solutions (see [Pös01] for a recent exposition). Assuming h satisfy some non-degeneracy condition and the system is analytic, if ε is sufficiently small there exists a constant c such that

$$|I(t) - I_0| \leq c\sqrt{\varepsilon}, \quad t \in \mathbb{R}$$

for “most” initial actions I_0 , more precisely for a set of large measure but with empty interior. When $n = 2$, this is even true for all solutions but for $n > 3$, the famous example of Arnold ([Arn64]) shows that there exist “unstable” solutions, along which the variation of the actions can be arbitrarily large no matter how small the perturbation is. From its very beginning, KAM theory was known to hold for non-analytic Hamiltonians (see [Mos62] in the context of twist maps). It is now well established in various regularity, including the C^∞ case (essentially by Herman, see [Bos86] and [Féj04]) and the Gevrey case ([Pop04]). Following ideas of Moser, the theorem also holds if H is only of class C^k , with $k > 2n$ (see [Pös82], [Sal04], [SZ89] and [Alb07]), even though the minimal number of derivatives is still an open question, except in a special case for $n = 2$ ([Her86]).

Another fundamental result, which complements KAM theory, is given by Nekhoroshev theorem ([Nek77], [Nek79]). If the integrable part h satisfy some generic condition and the system is analytic, then for ε sufficiently small there exist constants c_1, c_2, c_3, a and b such that

$$|I(t) - I_0| \leq c_1 \varepsilon^b, \quad |t| \leq c_2 \exp(c_3 \varepsilon^{-a})$$

for all initial actions I_0 . Hence all solutions are stable, not for all time, but for an exponentially long time. In the special case where h is strictly quasi-convex, a completely new proof of these estimates was given by Lochak ([Loc92]) using periodic averagings and simultaneous Diophantine approximation. The method of Lochak had many applications, in particular it was used by Marco and Sauzin to extend Nekhoroshev theorem in the Gevrey regular case under the quasi-convexity assumption ([MS02]). However, no such estimates have been studied if the Hamiltonian is merely finitely differentiable, and this is the content of the paper. We will prove below (theorem 2.1) that if H is of class C^k , for $k \geq 2$, and h quasi-convex, then one has the stability estimate

$$|I(t) - I_0| \leq c_1 \varepsilon^{\frac{1}{2n}}, \quad |t| \leq c_2 \varepsilon^{-\frac{k-2}{2n}}.$$

for some constants c_1 and c_2 , and provided that ε is small enough. Of course, under our regularity assumption the exponential estimates have been replaced by polynomial estimates, and earlier examples show that exponential stability cannot possibly hold under such a weak regularity assumption (this was discussed in [MS04]). The proof will use once again the ideas of Lochak which, among other things, reduces the analytic input to its minimum and we will also follow the implementation made by Marco and Sauzin in the Gevrey case.

As we recalled, KAM theory for finitely differentiable Hamiltonians have been widely studied, and so we believe that Nekhoroshev estimates under

weaker regularity assumptions have their own interest. Moreover, for obvious reasons, examples of unstable solutions (the so-called Arnold diffusion) are more easily constructed in the non-analytic case, and it is a natural question to estimate the speed of instability (see [KL08a] and [KL08b] for examples of class C^k with a polynomial speed of diffusion). Finally, one of our motivation is to generalize these estimates using the method of [BN09], where Lochak's ideas are extended to handle analytic but C^k -generic unperturbed Hamiltonians, with $k > 2n + 2$, but this appears more difficult.

2 Main result

We consider a Hamiltonian function H defined on the domain

$$\mathcal{D}_R = \mathbb{T}^n \times B_R$$

where B_R is the open ball of \mathbb{R}^n of radius R around the origin. We assume that H is of class C^k , for an integer $k \geq 2$, *i.e.* it is k -times differentiable and all its derivatives up to order k extends continuously to the closure $\overline{\mathcal{D}}_R$. We denote by $C^k(\mathcal{D}_R)$ the space of such functions, it is a Banach space with the norm

$$|H|_{C^k(\mathcal{D}_R)} = \sup_{0 \leq l \leq k} \sup_{|\alpha|=l} \left(\sup_{x \in \overline{\mathcal{D}}_R} |\partial^\alpha H(x)| \right)$$

where $x = (\theta, I)$, $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{N}^{2n}$, $|\alpha| = \alpha_1 + \dots + \alpha_{2n}$ and

$$\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_{2n}^{\alpha_{2n}}.$$

In case the Hamiltonian $H = h$ depends only on the action variables, we will simply write $|h|_{C^k(B_R)}$.

Our Hamiltonian $H \in C^k(\mathcal{D}_R)$ is assumed to be C^k -close to integrable, that is of the form

$$\begin{cases} H(\theta, I) = h(I) + f(\theta, I) \\ |f|_{C^k(\mathcal{D}_R)} < \varepsilon \ll 1 \end{cases} \quad (*)$$

where h is the integrable part and f a small perturbation of size ε in the C^k topology. Moreover, we may assume $|h|_{C^k(B_R)} < M$ for some positive constant M , and we will require that it satisfies the following quasi-convexity assumption :

$$\forall I \in B_R, \forall v \in \mathbb{R}^n, \quad \nabla h(I) \cdot v = 0 \implies \nabla^2 h(I) v \cdot v \geq m|v|^2 \quad (C)$$

for some constant $m > 0$.

Our main theorem is the following.

Theorem 2.1. *Let H as in (*), with $k \geq 2$ and h satisfying (C), and define*

$$a = \frac{k-2}{2n}, \quad b = \frac{1}{2n}.$$

Then there exist ε_0 , c_1 and c_2 such that if $\varepsilon \leq \varepsilon_0$, all solutions $(\theta(t), I(t))$ of H with $I(0) \in B_{R/2}$ satisfy

$$|I(t) - I(0)| \leq c_1 \varepsilon^b, \quad |t| \leq c_2 \varepsilon^{-a}.$$

First note that we have stated our theorem for H of class C^k , $k \geq 2$, but the statement also holds with no changes if H is of class $C^{k-1,1}$, $k \geq 1$, that is H is of class C^{k-1} and its partial derivatives of order $k-1$ are Lipschitz continuous. For $k=2$, this estimate is not useful since trivially all solutions satisfy

$$|I(t) - I(0)| \leq \varepsilon, \quad |t| \leq 1$$

hence the result is only interesting when $k \geq 3$, and this is the case where the regularity allows us to perform at least one averaging step.

Moreover, the time of stability obtained is “optimal” in the sense that one can construct examples of unstable orbits with a polynomial speed of diffusion, but we do not know what should be the optimal value of the exponent of stability a . Note however that using the method of [BM10] one can choose a arbitrarily close to $(k-2)(2(n-1))^{-1}$.

Let us finally point out that if H is C^∞ , then it is an immediate consequence of the above result that the action variables are stable for an interval of time which is longer than any prescribed power of ε^{-1} , but even in this case exponential stability does not hold.

Then, as in the analytic or Gevrey case, we can also state a refined result near resonances. Suppose Λ is a submodule of \mathbb{Z}^n of rank r , $d = n - r$ and let S_Λ be the corresponding resonant manifold, that is

$$S_\Lambda = \{I \in B_R \mid k \cdot \nabla h(I) = 0, k \in \Lambda\}.$$

We can prove the following statement, which actually contains the previous one.

Theorem 2.2. *Under the previous hypotheses, assume $d(I(0), S_\Lambda) \leq \sigma \sqrt{\varepsilon}$ for some constant $\sigma > 0$, and define*

$$a_d = \frac{k-2}{2d}, \quad b_d = \frac{1}{2d}.$$

Then there exist ε'_0 , c'_1 and c'_2 such that if $\varepsilon \leq \varepsilon'_0$, one has

$$|I(t) - I(0)| \leq c'_1 \varepsilon^{b_d}, \quad |t| \leq c'_2 \varepsilon^{-a_d}.$$

For $\Lambda = \{0\}$, $d = n$ and $S_\Lambda = B_R$, we recover theorem 2.1 and therefore it will be enough to prove theorem 2.2.

The constants ε_0 , c_1 and c_2 depend only on h , more precisely they depend on k, n, R, M and m while the constants ε'_0 , c'_1 and c'_2 also depends on σ and Λ . However we will not give explicit value for them to avoid complicated and rather meaningless expressions, hence we shall replace them by the symbol \cdot when it is convenient : for instance, we shall write $u < v$ when there exists a constant c depending on the previous parameters but not on f such that $u < cv$.

Plan of the paper

It is divided into two sections. The next section contains the analytical part of the proof, where we will construct a normal form, that is a system of local coordinates for our Hamiltonian which is more convenient to study the evolution of the action variables. Then, in the last section we will conclude the proof using our convexity assumption and Dirichlet theorem on simultaneous Diophantine approximation.

3 Analytical part

Given an action $I \in B_R$ and denoting by $\omega = \nabla h(I)$ its frequency, we know from classical averaging theory that the relevant part of the perturbation

$$f(\theta, I) = \sum_{k \in \mathbb{Z}^n} \hat{f}_k(I) e^{i2\pi k \cdot \theta}$$

is given by those harmonics associated to integers $k \in \mathbb{Z}^n$ that are in resonance with ω , that is such that $k \cdot \omega = 0$. Actually one can construct a symplectic, close to identity transformation Φ , defined around I such that

$$H \circ \Phi = h + g + \tilde{f}$$

where g contains only harmonics that resonates with ω and \tilde{f} is a small remainder, which can be made exponentially small if the system is analytic as was first shown by Nekhoroshev. These are usually called resonant normal forms, and to obtain them one has to deal with small divisors $k \cdot \omega$ and therefore technical estimates.

It is a remarkable fact discovered by Lochak ([Loc92]) that to prove exponential estimates in the quasi-convex case, it is enough to average along periodic frequencies, which are frequencies ω such that $T\omega \in \mathbb{Z}^n \setminus \{0\}$ for some $T > 0$ (see also [BN09] for an extension of this method for generic integrable Hamiltonians). These periodic frequencies correspond to periodic orbits of the unperturbed Hamiltonian. In this approach no small divisors arise and therefore this special resonant normal is much more easy to prove. Moreover, this method appears useful for non-analytic Hamiltonians, as was

shown in the Gevrey case in [MS02] and as we will show here for the C^k case.

In this section we will construct such a normal form, up to a polynomial remainder, in 3.3 but before we will recall some useful estimates concerning the C^k norm in 3.1 and then prove an intermediate statement in 3.2.

3.1 Elementary estimates

Let us begin by recalling some easy estimates. Given two functions $f, g \in C^k(\mathcal{D}_R)$, their product fg belongs to $C^k(\mathcal{D}_R)$ and by Leibniz rule

$$|fg|_{C^k(\mathcal{D}_R)} \leq |f|_{C^k(\mathcal{D}_R)} |g|_{C^k(\mathcal{D}_R)}$$

while their Poisson Bracket $\{f, g\}$ belongs to $C^{k-1}(\mathcal{D}_R)$, and by its definition and Leibniz rule one gets

$$|\{f, g\}|_{C^{k-1}(\mathcal{D}_R)} \leq |f|_{C^k(\mathcal{D}_R)} |g|_{C^k(\mathcal{D}_R)}$$

where the above implicit constants depend only on k (in fact in the first estimate one can trivially modify the definition of the C^k norm so as to have a constant equal to one, but this will not be important for us). These are very elementary facts, but we shall also need estimates concerning vector fields, canonical transformations and compositions.

First, given a vector-valued function $F \in C^k(\mathcal{D}_R, \mathbb{R}^l)$, $F = (F_1, \dots, F_l)$ and $l \in \mathbb{N}$, we extend the norm componentwise, that is

$$|F|_{C^k(\mathcal{D}_R)} = \sup_{1 \leq i \leq l} |F_i|_{C^k(\mathcal{D}_R)}.$$

Now for a function $f \in C^k(\mathcal{D}_R)$, we define its Hamiltonian vector field X_f by

$$X_f = (\partial_I f, -\partial_\theta f)$$

where

$$\partial_I f = (\partial_{I_1} f, \dots, \partial_{I_n} f), \quad \partial_\theta f = (\partial_{\theta_1} f, \dots, \partial_{\theta_n} f).$$

Obviously $X_f \in C^{k-1}(\mathcal{D}_R, \mathbb{R}^{2n})$, and if we denote

$$|X_f|_{C^{k-1}(\mathcal{D}_R)} = \sup \left\{ |\partial_I f|_{C^{k-1}(\mathcal{D}_R)}, |\partial_\theta f|_{C^{k-1}(\mathcal{D}_R)} \right\}$$

then one has trivially

$$|X_f|_{C^{k-1}(\mathcal{D}_R)} \leq |f|_{C^k(\mathcal{D}_R)}.$$

Moreover, by classical theorems on ordinary differential equations, if X_f is of class C^{k-1} then so is Φ_t^f , the time- t map of the vector field X_f when it exists.

Assuming $|\partial_\theta f|_{C^0(\mathcal{D}_R)} < r$ for some $r < R$ (for example $|X_f|_{C^0(\mathcal{D}_R)} < r$), then by the mean value theorem

$$\Phi^f = \Phi_1^f : \mathcal{D}_{R-r} \longrightarrow \mathcal{D}_R$$

is a well-defined C^{k-1} -embedding. In the case f is integrable, one can choose $r = 0$.

In the sequel, we will need to estimate the C^k norm of Φ^f in terms of the C^k norm of the vector field X_f . More precisely we need the rather natural fact that Φ^f is C^k close to the identity when X_f is C^k close to zero. This is trivial for $k = 0$, but in the general case, this follows by induction on k using on one hand the relation

$$\Phi_t^f = Id + \int_0^t X_f \circ \Phi_s^f ds$$

and on the other the formula of Faà di Bruno (see [AR67] for example), which gives bounds of the form

$$|F \circ G|_{C^k} \leq |F|_{C^k} |G|_{C^k}^k$$

and also

$$|F \circ G|_{C^k} \leq |F|_{C^1} |G|_{C^k}^k + |F|_{C^k} |G|_{C^{k-1}}^k$$

for C^k vector-valued functions on appropriate domains (once again, the above implicit constants depend only on k). Let us state this a lemma, for which we refer to [DH09], Lemma 3.15 and appendix C, for a detailed proof.

Lemma 3.1. *Let $X_f \in C^k(\mathcal{D}_R, \mathbb{R}^{2n})$, and assume that $|X_f|_{C^0(\mathcal{D}_R)} < r$ and*

$$|X_f|_{C^k(\mathcal{D}_R)} < 1. \tag{1}$$

Then we have

$$|\Phi^f - Id|_{C^k(\mathcal{D}_{R-r})} \leq |X_f|_{C^k(\mathcal{D}_R)}.$$

The above implicit constant depends only on k and R . Now if $H \in C^k(\mathcal{D}_R)$, under the above hypotheses we have $H \circ \Phi^f \in C^k(\mathcal{D}_{R-r})$ and the estimate

$$|H \circ \Phi^f|_{C^k(\mathcal{D}_{R-r})} \leq |H|_{C^k(\mathcal{D}_R)} |\Phi^f|_{C^k(\mathcal{D}_{R-r})}^k \tag{2}$$

follows trivially from Faà di Bruno formula.

3.2 The linear case

Following [MS02], we change for a moment our setting and we consider a perturbation of a linear Hamiltonian, more precisely the Hamiltonian

$$\begin{cases} H(\theta, I) = l(I) + f(\theta, I) \\ |f|_{C^k(\mathcal{D}_\rho)} < \mu \ll 1 \end{cases} \tag{**}$$

where ρ is a fixed number and $l(I) = \omega \cdot I$ is a linear Hamiltonian with a T -periodic frequency ω . Recall that the latter means that $T\omega \in \mathbb{Z}^n \setminus \{0\}$ for some $T > 0$, and in this context our small parameter is μ .

In the sequel, ω will be chosen of the form $\nabla h(I_*)$ for some action $I_* \in B_R$, hence we may assume $|\omega| < 1$ and therefore we may write

$$|\Phi_t^l|_{C^k(\mathcal{D}_\rho)} < 1 \quad (3)$$

for $t \in [0, T]$. In the proposition below, we will construct a “global” normal form for the Hamiltonian (**), that we will use in the next section to produce a “local” normal form around periodic orbits for our original Hamiltonian (*).

Proposition 3.2. *Consider H as in (**) with $k \geq 2$, and assume*

$$T\mu < 1. \quad (4)$$

Then there exists a C^2 symplectic transformation

$$\Phi : \mathcal{D}_{\rho/2} \rightarrow \mathcal{D}_\rho$$

with $|\Phi - Id|_{C^2(\mathcal{D}_{\rho/2})} < T\mu$ such that

$$H \circ \Phi = l + g + f$$

with $\{g, l\} = 0$ and the estimate

$$|g|_{C^2(\mathcal{D}_{\rho/2})} < \mu, \quad |f|_{C^2(\mathcal{D}_{\rho/2})} < (T\mu)^{k-2}\mu.$$

First note that $\{g, l\} = 0$ means exactly that $\partial_\theta g \cdot \omega = 0$ and expanding g in Fourier series, one easily sees that it contains only harmonics associated to integers k satisfying $k \cdot \omega = 0$. Therefore the proposition below is indeed a resonant normal form, up to a polynomial remainder.

Note also that we need our transformed Hamiltonian $H \circ \Phi$, and hence our transformation Φ , to be at least of class C^2 simply because we need our transformed vector field to be of class C^1 to have existence and uniqueness of solutions (or course, a Lipschitz regularity would have been enough). This explains the factor $(k - 2)$ in our stability exponent a .

Proof. Our transformation Φ will be obtained by a finite composition of averaging transformations. Let us define

$$r = \frac{\rho}{2(k-2)}$$

and for $j \in \{0, \dots, k-2\}$, let

$$\rho_j = \rho - jr.$$

Then we claim that for any $j \in \{0, \dots, k-2\}$, there exists C^{k-j} symplectic transformation $\Phi_j : \mathcal{D}_{\rho_j} \rightarrow \mathcal{D}_\rho$ with $|\Phi_j - Id|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll T\mu$ such that

$$H \circ \Phi_j = l + g_j + f_j$$

with $\{g_j, l\} = 0$ and the estimate

$$|g_j|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll \mu, \quad |f_j|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll (T\mu)^j \mu.$$

The proposition follows easily by taking $\Phi = \Phi_{k-2}$, $g = g_{k-2}$ and $f = f_{k-2}$.

We will prove the claim by induction on $j \in \{0, \dots, k-2\}$. For $j = 0$, there is nothing to prove since we can simply write $H = l + g_0 + f_0$ with $g_0 = 0$, $f_0 = f$ and therefore Φ_0 is the identity. Now assume the claim is true for some $j \in \{0, \dots, k-3\}$, and consider

$$H_j = H \circ \Phi_j = l + g_j + f_j.$$

Let us define

$$[f_j] = \frac{1}{T} \int_0^T f_j \circ \Phi_t^l dt$$

and

$$\chi_j = \frac{1}{T} \int_0^T (f_j - [f_j]) \circ \Phi_t^l dt.$$

We have

$$|[f_j]|_{C^{k-j}(\mathcal{D}_{\rho_j})} \leq |f_j \circ \Phi_t^l|_{C^{k-j}(\mathcal{D}_{\rho_j})}$$

hence by (2) and (3) we obtain

$$|[f_j]|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll |f_j|_{C^{k-j}(\mathcal{D}_{\rho_j})}$$

and with our hypotheses of induction, this gives

$$|[f_j]|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll (T\mu)^j \mu.$$

Similarly we have

$$|\chi_j|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll T(T\mu)^j \mu = (T\mu)^{j+1}$$

and the latter can also be estimated by

$$|\chi_j|_{C^{k-j}(\mathcal{D}_{\rho_j})} \ll T\mu$$

by our condition (4).

If we let Φ^{χ_j} be the time-one map of the Hamiltonian vector field generated by χ_j , then we will show that the map

$$\Phi_{j+1} = \Phi_j \circ \Phi^{\chi_j}$$

satisfies the assumptions.

Indeed, thanks to our condition (4), we can ensure that Φ^{X_j} , which is of class C^{k-j-1} , is a well-defined embedding

$$\Phi^{X_j} : \mathcal{D}_{\rho_{j+1}} \longrightarrow \mathcal{D}_{\rho_j}.$$

Moreover, as $|X_\chi|_{C^{k-j}(\mathcal{D}_{\rho_j})} \leq T\mu$ and using once again (4), we can arrange condition (1) and apply lemma 3.1 to obtain

$$|\Phi^{X_j} - Id|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \leq T\mu.$$

Now we have

$$\begin{aligned} |\Phi_{j+1} - Id|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} &= |\Phi_j \circ \Phi^{X_j} - \Phi^{X_j} + \Phi^{X_j} - Id|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \\ &\leq |(\Phi_j - Id) \circ \Phi^{X_j}|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \\ &\quad + |\Phi^{X_j} - Id|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \\ &\leq |\Phi_j - Id|_{C^{k-j}(\mathcal{D}_{\rho_j})} + |\Phi^{X_j} - Id|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \end{aligned}$$

where we have used (2) in the last line. By our hypotheses of induction, this eventually gives

$$|\Phi_{j+1} - Id|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \leq T\mu.$$

Now by Taylor formula with integral remainder, we can expand

$$H_{j+1} = H \circ \Phi_{j+1} = h + g_{j+1} + f_{j+1}$$

with

$$g_{j+1} = g_j + [f_j], \quad f_{j+1} = \int_0^1 \{g_j + f_j^t, \chi\} \circ \Phi_t^X dt$$

where $f_j^t = tf_j + (1-t)[f_j]$, as one can check by a standard calculation. Since $\{g_j, l\} = 0$ by our hypothesis of induction and obviously $\{[f_j], l\} = 0$, we have $\{g_{j+1}, l\} = 0$ together with the estimate

$$\begin{aligned} |g_{j+1}|_{C^{k-j}(\mathcal{D}_{\rho_j})} &\leq |g_j|_{C^{k-j}(\mathcal{D}_{\rho_j})} + |[f_j]|_{C^{k-j}(\mathcal{D}_{\rho_j})} \\ &\leq \mu + (T\mu)^j \mu \\ &\leq \mu(1 + (T\mu)^j) \\ &\leq \mu \end{aligned}$$

using (4). For the remainder, using (2) and the trivial estimate concerning

the Poisson bracket we compute

$$\begin{aligned}
|f_{j+1}|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} &\leq |\{g_j + f_j^t, \chi\} \circ \Phi_t^X|_{C^{k-j-1}(\mathcal{D}_{\rho_{j+1}})} \\
&\leq |\{g_j + f_j^t, \chi\}|_{C^{k-j-1}(\mathcal{D}_{\rho_j})} \\
&\leq |\{g_j, \chi\}|_{C^{k-j-1}(\mathcal{D}_{\rho_j})} + |\{f_j^t, \chi\}|_{C^{k-j-1}(\mathcal{D}_{\rho_j})} \\
&\leq |g_j|_{C^{k-j}(\mathcal{D}_{\rho_j})} |\chi|_{C^{k-j}(\mathcal{D}_{\rho_j})} + |f_j^t|_{C^{k-j}(\mathcal{D}_{\rho_j})} |\chi|_{C^{k-j}(\mathcal{D}_{\rho_j})} \\
&\leq \mu(T\mu)^{j+1} + \mu(T\mu)^j (T\mu)^{j+1} \\
&\leq \mu(T\mu)^{j+1} (1 + (T\mu)^j) \\
&\leq \mu(T\mu)^{j+1}
\end{aligned}$$

using once again (4). This concludes the proof. \square

3.3 Normal form

Now let us come back to our original setting which is the Hamiltonian

$$\begin{cases} H(\theta, I) = h(I) + f(\theta, I) \\ |f|_{C^k(\mathcal{D}_R)} < \varepsilon \ll 1. \end{cases}$$

We will say that an action $I_* \in B_R$ is T -periodic if its frequency vector $\nabla h(I_*)$ is T -periodic. In the proposition below, we will fix a T -periodic action I_* , l will be the linear integrable Hamiltonian $l(I) = \omega \cdot I$ associated to its periodic frequency $\omega = \nabla h(I_*)$ and we denote by

$$\Pi_I : \mathbb{T}^n \times B_R \rightarrow B_R$$

the projection onto action space.

Proposition 3.3 (Normal form). *Suppose H as in (*), and under the previous hypotheses, let $\mu > 0$ such that*

$$\varepsilon < \mu^2, \quad \mu < 1, \quad T\mu < 1. \quad (5)$$

Then there exists a C^2 symplectic transformation

$$\Phi : \mathbb{T}^n \times B(I_*, \mu) \rightarrow \mathbb{T}^n \times B(I_*, 2\mu)$$

with $|\Pi_I \Phi - Id_I|_{C^0(B(I_, \mu))} < T\mu$ such that*

$$H \circ \Phi = h + g + \tilde{f}$$

with $\{g, l\} = 0$ and the estimate

$$|g + \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} < \mu^2, \quad |\partial_{\tilde{g}} \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} < (T\mu)^{k-2} \mu^2.$$

Let us immediately explain how such coordinates will be used in the sequel. If we denote them by $(\tilde{\theta}, \tilde{I})$, then writing down the equation of motions for $\tilde{H} = H \circ \Phi$, using the fact that $\partial_{\tilde{\theta}} g \cdot \omega = 0$ and the mean value theorem, one shows that $\tilde{I}(t)$ remains close to the line generated by ω passing through $\tilde{I}(0)$, for an interval of time governed by the size of $\partial_{\tilde{\theta}} \tilde{f}$. Hence any only potential drift has to occur in the direction transversal to ω . When we come back to the original coordinates (θ, I) , this picture is only slightly distorted since the projection of Φ onto action space is close to identity.

Proof. Without loss of generality, we may assume $h(I_*) = 0$. To analyze our Hamiltonian H in a neighbourhood of size μ around I^* , we translate and rescale the action variables using the map

$$\sigma_\mu : (\theta^\mu, I^\mu) \mapsto (\theta, I) = (\theta^\mu, I_* + \mu I^\mu)$$

which sends the domain $\mathcal{D}_2 = \mathbb{T}^n \times B_2$ onto $\mathbb{T}^n \times B(I_*, 2\mu)$, and note that by the second part of condition (5), we can assume that the latter domain is included in \mathcal{D}_R . Let

$$H_\mu = \mu^{-1}(H \circ \sigma_\mu)$$

be the rescaled Hamiltonian, so that H_μ is defined on \mathcal{D}_2 and it can be written as

$$H_\mu(\theta, I^\mu) = \mu^{-1}H(\theta, I_* + \mu I^\mu) = \mu^{-1}h(I_* + \mu I^\mu) + \mu^{-1}f(\theta, I_* + \mu I^\mu)$$

for $(\theta, I^\mu) \in \mathcal{D}_2$. Now using Taylor formula we can expand h around I_* to obtain

$$\begin{aligned} h(I_* + \mu I^\mu) &= \mu \omega \cdot I^\mu + \mu^2 \int_0^1 (1-t) \nabla^2 h(I_* + t\mu I^\mu) I^\mu \cdot I^\mu dt \\ &= \mu \omega \cdot I^\mu + \mu^2 h_\mu(I^\mu) \end{aligned}$$

where we have set

$$h_\mu(I^\mu) = \int_0^1 (1-t) \nabla^2 h(I_* + t\mu I^\mu) I^\mu \cdot I^\mu dt$$

and therefore we can write

$$H_\mu = l + f_\mu$$

with

$$f_\mu = \mu h_\mu + \mu^{-1}(f \circ \sigma_\mu).$$

Now we know that $|f|_{C^k(\mathcal{D}_R)} < \varepsilon < \mu^2$ from the first part of condition (5) and $|h|_{C^k(\mathcal{D}_R)} < 1$ so we obviously have

$$|f_\mu|_{C^k(\mathcal{D}_2)} < \mu.$$

Therefore using the last part of our condition (5) we can apply the proposition 3.2, with $\rho = 2$, to the Hamiltonian $H_\mu = l + f_\mu$: there exists a C^2 symplectic transformation $\Phi_\mu : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ with $|\Phi_\mu - Id|_{C^2(\mathcal{D}_1)} \leq T\mu$ such that

$$H_\mu \circ \Phi_\mu = l + g_\mu + \tilde{f}_\mu$$

with $\{g_\mu, l\} = 0$ and the estimate

$$|g_\mu|_{C^2(\mathcal{D}_1)} \leq \mu, \quad |\tilde{f}_\mu|_{C^2(\mathcal{D}_1)} \leq (T\mu)^{k-2}\mu.$$

Moreover, as we had $f_\mu = \mu h_\mu + \mu^{-1}(f \circ \sigma_\mu)$ with h_μ integrable, we can write

$$g_\mu = \mu h_\mu + \hat{f}_\mu$$

with $\{\hat{f}_\mu, l\} = 0$ and $|\hat{f}_\mu|_{C^2(\mathcal{D}_1)} \leq \mu$, and so the transformed Hamiltonian can also be written

$$H_\mu \circ \Phi_\mu = l + \mu h_\mu + \hat{f}_\mu + \tilde{f}_\mu.$$

Now scaling back to our original Hamiltonian, we define $\Phi = \sigma_\mu \circ \Phi_\mu \circ \sigma_\mu^{-1}$, therefore

$$\Phi : B(I_*, \mu) \longrightarrow B(I_*, 2\mu)$$

and one has

$$\begin{aligned} H \circ \Phi &= \mu H_\mu \circ \Phi_\mu \circ \sigma_\mu^{-1} \\ &= \mu(l + \mu h'_\mu + \hat{f}_\mu + \tilde{f}_\mu) \circ \sigma_\mu^{-1} \\ &= (\mu l + \mu^2 h_\mu) \circ \sigma_\mu^{-1} + \mu \hat{f}_\mu \circ \sigma_\mu^{-1} + \mu \tilde{f}_\mu \circ \sigma_\mu^{-1}. \end{aligned}$$

Observe that $(\mu l + \mu^2 h_\mu) \circ \sigma_\mu^{-1} = h$, so we may define

$$g = \mu \hat{f}_\mu \circ \sigma_\mu^{-1}, \quad \tilde{f} = \mu \tilde{f}_\mu \circ \sigma_\mu^{-1}$$

and write

$$H \circ \Phi = h + g + \tilde{f}.$$

It is obvious that $\{g, l\} = 0$ with

$$|g|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} \leq \mu |\hat{f}_\mu|_{C^0(\mathcal{D}_1)} \leq \mu^2$$

and similarly

$$|\tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} \leq \mu |\tilde{f}_\mu|_{C^0(\mathcal{D}_1)} \leq (T\mu)^{k-2}\mu^2 \leq \mu^2$$

so that

$$|g + \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} \leq \mu^2.$$

Moreover, as $\partial_{\tilde{\theta}} \tilde{f} = \mu \partial_{\tilde{\theta}} \tilde{f}_\mu$ then

$$|\partial_{\tilde{\theta}} \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} \leq (T\mu)^{k-2}\mu^2$$

and finally

$$|\Pi_I \Phi - Id_I|_{C^0(B(I_*, \mu))} \leq T\mu.$$

is trivial. This ends the proof. \square

4 Proof of theorem 2.2

Now we can complete the proof of our theorem 2.2 in the spirit of Lochak, following three elementary steps that use successively some arithmetic (simultaneous Diophantine approximation), some analysis (normal form around periodic orbits) and some geometry (quasi-convexity). The analysis has been done in the previous section, and the arithmetic and geometry are exactly the same as in the analytic case or the Gevrey case. Therefore instead of rewriting proofs which are well-known, we will merely explain the ideas and state the relevant results which can be found in [Loc92] and [MS02].

Let us begin the arithmetic part, since in order to use our proposition 3.3 (the normal form), we will need to show that any action $I_0 \in B_{R/2}$, which is close to some resonant surface S_Λ , can be approximated by a periodic action. Let $\omega_0 = \nabla h(I_0)$, then using the isoenergetic non-degeneracy of h , that is the non-degeneracy of the map h restricted to an energy level (which is easily implied by quasi-convexity), it is enough to approximate ω_0 by a periodic vector ω . If Λ has rank $n - 1$, this is totally obvious, since necessarily we can write $\Lambda = \omega^\perp \cap \mathbb{Z}^n$ for some periodic vector ω . Now in the case where Λ has rank $r = n - d$ with $d > 1$, this approximation is given by a theorem of Dirichlet, which moreover gives an explicit bound on its period T .

Proposition 4.1. *Let $I_0 \in B_{R/2}$, Λ a submodule of \mathbb{Z}^n of rank $r = n - d$, with $d > 1$ and Q be a real number such that*

$$Q \cdot > 1. \tag{6}$$

Then there exists a T -periodic action $I_ \in B_R$ such that*

$$|I_0 - I_*| \leq \max \left(d(I_0, S_\Lambda), T^{-1} Q^{-\frac{1}{d-1}} \right)$$

and the period satisfy

$$1 < T < Q.$$

The proof of the above proposition can be found in [MS02], Corollary 3.2.

Now it remains to explain how the quasi-convexity hypothesis (C) on h together with the normal form obtained in proposition 3.3 will enable us to bound the variation $|I(t) - I_0|$ for an initial action I_0 close to some periodic action I_* . The idea goes as follows.

Let $\omega = \nabla h(I_*)$. We have already explained after proposition 3.3 how in those new coordinates the evolution of the actions is bounded in the directions generated by ω , that is if F is the hyperplane orthogonal to ω , then $\tilde{I}(t)$ remains close to the affine subspace $\tilde{I}_0 + F$ for an interval of

time $|t| \leq \tau$ which is governed by the size of the remainder in the normal form. But by preservation of energy, for all time $\tilde{I}(t)$ remains close to the unperturbed energy hypersurface $E_0 = \{\tilde{I} \in B_R \mid h(\tilde{I}) = h(\tilde{I}_0)\}$, and as the latter is strictly convex by quasi-convexity of h , the connected component of \tilde{I}_0 in $E_0 \cap (\tilde{I}_0 + F)$ is in fact bounded and so is the variation $|\tilde{I}(t) - \tilde{I}_0|$ for $|t| \leq \tau$.

This idea is formalized in the proposition below. Once again, I_* is a T -periodic action and l is the linear Hamiltonian with periodic frequency $\omega = \nabla h(I_*)$.

Proposition 4.2. *Under the previous hypotheses, let $r > 0$, $\tau > 0$ and*

$$\tilde{H} = h + g + \tilde{f} \in C^2(\mathbb{T}^n \times B(I_*, r))$$

with h satisfying (C), $\{g, l\} = 0$ and the estimates

$$|g + \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, r))} < r^2, \quad |\partial_{\tilde{f}} \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, r))} < r^2 \tau^{-1}.$$

If

$$r \ll 1 \tag{7}$$

then for any $\tilde{I}_0 \in B(I_*, r)$, the solution satisfies

$$|\tilde{I}(t) - \tilde{I}_0| \ll r, \quad |t| \leq \tau.$$

Once again, we refer to [MS02], Corollary 3.1, for a complete proof.

Let us not conclude the proof of theorem 2.2. In a first step we will use proposition 4.1 to find a periodic action close to our initial action, then in a second step we will apply proposition 3.3 to find adapted coordinates and the third step will consist in applying proposition 4.2 to bound the evolution of the action variables in those coordinates, and hence in the original coordinates.

Proof of theorem 2.2. Let $I_0 \in B_{R/2}$, Λ a submodule of \mathbb{Z}^n of rank $m = n - d$, and we assume that $d(I_0, S_\Lambda) \ll \sqrt{\varepsilon}$ (note that this last assumption is void if Λ is trivial).

First step. In the case $d = 1$, any action $I \in S_\Lambda$ has a frequency $\nabla h(I)$ which is a multiple of some non-zero vector $k_* \in \mathbb{Z}^n$, therefore we can choose a periodic action $I_* \in S_\Lambda$ so that

$$|I_0 - I_*| = d(I_0, S_\Lambda) \ll \sqrt{\varepsilon}$$

and the period T trivially satisfy $T \ll 1$. In the case $d > 1$, we apply proposition 4.1 with

$$Q = \cdot \varepsilon^{-\frac{d-1}{2d}}.$$

and the condition (6) gives a first smallness condition on ε . Observe that $Q^{-\frac{1}{d-1}} = \cdot \varepsilon^{\frac{1}{2d}}$, hence the periodic action I_* given by the proposition satisfy

$$|I_0 - I_*| \leq \max\left(d(I_0, S_\Lambda), T^{-1}Q^{-\frac{1}{d-1}}\right) \leq T^{-1}\varepsilon^{\frac{1}{2d}}.$$

and the period T is estimated by

$$1 \leq T \leq \varepsilon^{-\frac{d-1}{2d}}. \quad (8)$$

Second step. Having found a periodic action, we will now apply proposition 3.3 with

$$\mu = \cdot T^{-1}\varepsilon^{\frac{1}{2d}}.$$

With this choice, the first part of condition (5) is satisfied thanks to the estimate (8) on the period T and the others give only further smallness conditions on ε . Applying the proposition, we have a C^2 symplectic transformation

$$\Phi : \mathbb{T}^n \times B(I_*, \mu) \rightarrow \mathbb{T}^n \times B(I_*, 2\mu)$$

with $|\Pi_I \Phi - Id|_{C^0(B(I_*, \mu))} \leq T\mu$ such that

$$H \circ \Phi = h + g + \tilde{f}$$

with $\{g, l\} = 0$ and the estimate

$$|g + \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} \leq \mu^2, \quad |\partial_{\tilde{\theta}} \tilde{f}|_{C^0(\mathbb{T}^n \times B(I_*, \mu))} \leq (T\mu)^{k-2} \mu^2.$$

Let us write $\tilde{H} = H \circ \Phi$ and $(\tilde{\theta}, \tilde{I})$ are the new coordinates in $\mathbb{T}^n \times B(I_*, \mu)$.

Third step. Now we set

$$r = \cdot \mu, \quad \tau = \cdot (T\mu)^{k-2}.$$

and we apply proposition 4.2 to the Hamiltonian \tilde{H} . To do so, we need to impose condition (7) and this gives our last smallness condition on ε . Therefore we find

$$|\tilde{I}(t) - \tilde{I}_0| \leq \mu, \quad |t| \leq (T\mu)^{k-2}$$

and recalling that

$$\mu = \cdot T^{-1}\varepsilon^{\frac{1}{2d}} \leq \varepsilon^{\frac{1}{2d}}$$

this gives

$$|\tilde{I}(t) - \tilde{I}_0| \leq \varepsilon^{\frac{1}{2d}}, \quad |t| \leq \varepsilon^{\frac{k-2}{2d}}.$$

Now since

$$|\Pi_I \Phi - Id|_{C^0(B(I_*, \mu))} \leq T\mu \leq \varepsilon^{\frac{1}{2d}}$$

standard arguments gives the conclusion

$$|I(t) - I_0| \leq \varepsilon^{\frac{1}{2d}}, \quad |t| \leq \varepsilon^{\frac{k-2}{2d}}.$$

This ends the proof. \square

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