Shape derivatives of boundary integral operators in electromagnetic scattering

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Abstract

We develop the shape derivative analysis of solutions to the problem of scattering of time-harmonic electromagnetic waves by a bounded penetrable obstacle. Since boundary integral equations are a classical tool to solve electromagnetic scattering problems, we study the shape differentiability properties of the standard electromagnetic boundary integral operators. To this end, we start with the Gâteaux differentiability analysis with respect to deformations of the obstacle of boundary integral operators with pseudo-homogeneous kernels acting between Sobolev spaces. The boundary integral operators of electromagnetism are typically bounded on the space of tangential vector fields of mixed regularity $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Using Helmholtz decomposition, we can base their analysis on the study of scalar integral operators in standard Sobolev spaces, but we then have to study the Gâteaux differentiability of surface differential operators. We prove that the electromagnetic boundary integral operators are infinitely differentiable without loss of regularity and that the solutions of the scattering problem are infinitely shape differentiable away from the boundary of the obstacle, whereas their derivatives lose regularity on the boundary. We also give a characterization of the first shape derivative as a solution of a new electromagnetic scattering problem.

Keywords : Maxwell's equations, boundary integral operators, surface differential operators, shape derivatives, Helmholtz decomposition.

Contents

Introduction

Consider the scattering of time-harmonic electromagnetic waves by a bounded obstacle Ω in $\mathbb{R}^3$ with a smooth and simply connected boundary Γ filled with an homogeneous dielectric material. This problem is described by the system of Maxwell's equations, valid in the sense of distributions, with two transmission conditions on the boundary or the obstacle guaranteeing the continuity of the tangential components of the electric and magnetic fields across the interface. The transmission problem is completed by the Silver-Müller radiation condition at infinity (see [23] and [24]). Boundary integral equations are an efficient method to solve such problems for low and high frequencies. The dielectric scattering problem is usually reduced to a system of two boundary integral equations for two unknown tangential vector fields on the interface (see [6] and [24]). We refer to [9] and [10] for methods developed by the authors to solve this problem using a single boundary integral equation.

Optimal shape design with the modulus of the far field pattern of the dielectric scattering problem as goal is of practical interest in some important fields of applied mathematics, as for example telecommunication systems and radars. The utilization of shape optimization methods requires the analysis of the dependency of the solution on the shape of the dielectric scatterer. An explicit form of the shape derivatives is required in view of their implementation in a shape optimization algorithms such as gradient methods or Newton's method.

In this paper, we present a complete analysis of the shape differentiability of the solution of the dielectric scattering problem using an integral representation. Even if numerous works exist on the calculus of shape variations [14, 25, 26, 31, 32], in the framework of boundary integral equations the scientific literature is not extensive. However, one can cite the papers [27], [29] and [28], where R. Potthast has considered the question, starting with his PhD thesis [30], for the Helmholtz equation with Dirichlet or Neumann boundary conditions and the perfect conductor problem, in spaces of continuous and Hölder continuous functions. Using the integral representation of the solution, one is lead to study the Gâteaux differentiability of boundary integral operators and potential operators with weakly and strongly singular kernels.

The natural space of distributions (energy space) which occurs in the electromagnetic potential theory is $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, the set of tangential vector fields whose components are in the Sobolev space $H^{-\frac{1}{2}}(\Gamma)$ and whose surface divergence is in $H^{-\frac{1}{2}}(\Gamma)$. We face two main difficulties: On one hand, to be able to construct shape derivatives of the solution – which is given in terms of products of boundary integral operators and their inverses – it is imperative to prove that the derivatives are bounded operators between the same spaces as the boundary integral operators themselves. On the other hand, the very definition of shape differentiability of operators defined on $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ poses non-trivial problems. Our approach consists in using the Helmholtz decomposition of this Hilbert space. In this way, we split the analysis in two main steps: First the Gâteaux differentiability analysis of scalar boundary integral operators and potential operators with pseudo-homogeneous kernels, and second the study of derivatives with respect to smooth deformations of the obstacle of surface differential operators in the classical Sobolev spaces.

This work contains results from the thesis [21] where this analysis has been used to develop a shape optimization algorithm of dielectric lenses in order to obtain a prescribed radiation pattern.

The paper is organized as follows:

In section 1 we define the scattering problem of time-harmonic electromagnetic waves at a dielectric interface and the appropriate spaces. In section 2 we recall some results about trace mappings and boundary integral operators in electromagnetism, following the notation of [6, 24]. We then give an integral representation of the solution following [9]. In section 3, we introduce the notion of shape derivative and its connection to Gâteaux derivatives. We also recall elementary results about differentiability in Fréchet spaces.

The section 4 is dedicated to the Gâteaux differentiability analysis of a class of boundary integral operators with respect to deformations of the boundary. We generalize the results proved in [27, 29] for the standard acoustic boundary integral operators, to the class of integral operators with pseudo-homogenous kernels. We also give higher order Gâteaux derivatives of coefficient functions such as the Jacobian of the change of variables associated with the deformation, or the components of the unit normal vector. These results are new and allow us to obtain explicit forms of the derivatives of the integral operators.

The last section contains the main results of this paper: the shape differentiability properties of the solution of the dielectric scattering problem. We begin by discussing the difficulties of defining the shape dependency of operators defined on the shape-dependent space $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, and we present an altervative using the Helmholtz decomposition (see [11]) on the boundary of smooth domains. We then analyze the differentiability of a family of surface differential operators. Again we prove their infinite Gâteaux differentiability and give an explicit expression of their derivatives. These results are new and important for the numerical implementation of the shape derivatives. Using the chain rule, we deduce the infinite shape differentiability of the solution of the scattering problem away from the boundary and an expression of the shape derivatives. More precisely, we prove that the boundary integral operators are infinitely Gâteaux differentiable without loss of regularity, whereas previous results allowed such a loss [28], and we prove that the shape derivatives of the potentials are smooth far from the boundary but they lose regularity in the neighborhood of the boundary.

These new results generalize existing results: In the acoustic case, using the variational formulation, a characterization of the first Gâteaux derivative was given by A. Kirsch in [20] for the Dirichlet problem and then for a transmission problem by F. Hettlich in [15, 16]. R. Potthast used the integral equation method to obtain a characterisation of the first shape derivative of the solution of the perfect conductor scattering problem.

We end the paper by formulating a characterization of the first shape derivative as the solution of a new electromagnetic scattering problem. We show that both by directly deriving the boundary values and by using the integral representation of the solution, we obtain the same characterization.

1 The dielectric scattering problem

Let Ω denote a bounded domain in $\mathbb{R}^3$ and let Ω^c denote the exterior domain $\mathbb{R}^3\backslash\overline{\Omega}$. In this paper, we will assume that the boundary Γ of Ω is a smooth and simply connected closed surface, so that Ω is diffeomorphic to a ball. Let $\mathbf{n}$ denote the outer unit normal vector on the boundary Γ.

In Ω (resp. Ω^c) the electric permittivity ϵ_i (resp. ϵ_e) and the magnetic permeability μ_i (resp. μ_e) are positive constants. The frequency ω is the same in Ω and in Ω^c. The interior wave number κ_i and the exterior wave number κ_e are complex constants of non negative imaginary part.

Notation: For a domain $G \subset \mathbb{R}^3$ we denote by $H^s(G)$ the usual L^2-based Sobolev space of order $s \in \mathbb{R}$, and by $H^s_{\rm loc}(\overline{G})$ the space of functions whose restrictions to any bounded subdomain B of G belong to $H^s(B)$. Spaces of vector functions will be denoted by boldface letters, thus

$$\mathbf{H}^s(G) = (H^s(G))^3 .$$

If D is a differential operator, we write:

$$\begin{aligned} \mathbf{H}^s(\mathsf{D},\Omega) &= \{u \in \mathbf{H}^s(\Omega) : \mathsf{D}u \in \mathbf{H}^s(\Omega)\} \\ \mathbf{H}^s_{\rm loc}(\mathsf{D},\overline{\Omega^c}) &= \{u \in \mathbf{H}^s_{\rm loc}(\overline{\Omega^c}) : \mathsf{D}u \in \mathbf{H}^s_{\rm loc}(\overline{\Omega^c})\} \end{aligned}$$

The space $\mathbf{H}^s(\mathsf{D},\Omega)$ is endowed with the natural graph norm. When $s = 0$, this defines in particular the Hilbert spaces $\mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{H}(\mathbf{curl}\,\mathbf{curl},\Omega)$. We denote the $\mathbf{L}^2$ scalar product on Γ by $\langle\cdot,\cdot\rangle_\Gamma$.

The time-harmonic Maxwell's sytem can be reduced to second order equations for the electric field only. The time-harmonic dielectric scattering problem is then formulated as follows.

The dielectric scattering problem : Given an incident field $\mathbf{E}^{inc} \in \mathbf{H}_{\rm loc}(\mathbf{curl},\mathbb{R}^3)$ that satisfies $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{inc} - \kappa_e^2\mathbf{E}^{inc} = 0$ in a neighborhood of $\overline{\Omega}$, we seek two fields $\mathbf{E}^i \in \mathbf{H}(\mathbf{curl},\Omega)$ and $\mathbf{E}^s \in \mathbf{H}_{\rm loc}(\mathbf{curl},\overline{\Omega^c})$ satisfying the time-harmonic Maxwell equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^i - \kappa_i^2\mathbf{E}^i = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^s - \kappa_e^2\mathbf{E}^s = 0 \quad \text{in } \Omega^c, \tag{1.2}$$

the two transmission conditions,

$$\mathbf{n}\times\mathbf{E}^i = \mathbf{n}\times(\mathbf{E}^s + \mathbf{E}^{inc}) \quad \text{on } \Gamma \tag{1.3}$$

$$\mu_i^{-1}(\mathbf{n}\times\mathbf{curl}\,\mathbf{E}^i) = \mu_e^{-1}\mathbf{n}\times\mathbf{curl}(\mathbf{E}^s + \mathbf{E}^{inc}) \quad \text{on } \Gamma \tag{1.4}$$

and the Silver-Müller radiation condition:

$$\lim_{|x|\to+\infty} |x| \left| \mathbf{curl}\,\mathbf{E}^s(x)\times\frac{x}{|x|} - i\kappa_e\mathbf{E}^s(x) \right| = 0. \tag{1.5}$$

The interior and exterior magnetic fields are then given by $\mathbf{H}^i = \dfrac{1}{i\omega\mu_i}\mathbf{E}^i$ and $\mathbf{H}^s = \dfrac{1}{i\omega\mu_e}\mathbf{E}^s$. It is well known that this problem admit a unique solution for any positive real values of the exterior wave number [6, 21, 24].

An important quantity, which is of interest in many shape optimization problems, is the far field pattern of the electric solution, defined on the unit sphere of $\mathbb{R}^3$, by

$$\mathbf{E}^\infty(\hat{x}) = \lim_{|x|\to\infty} 4\pi|x|\frac{\mathbf{E}^s(x)}{e^{i\kappa_e|x|}}, \qquad \text{with } \frac{x}{|x|} = \hat{x}.$$

2 Boundary integral operators and main properties

2.1 Traces and tangential differential calculus

We use surface differential operators and traces. More details can be found in [8, 24].

For a vector function $\mathbf{v} \in (\mathscr{C}^k(\mathbb{R}^3))^q$ with $k, q \in \mathbb{N}^*$, we note $[\nabla \mathbf{v}]$ the matrix whose the i-th column is the gradient of the i-th component of $\mathbf{v}$ and we set $[\mathsf{D}\mathbf{v}] = {}^T[\nabla \mathbf{v}]$. The tangential gradient of any scalar function $u \in \mathscr{C}^k(\Gamma)$ is defined by

$$\nabla_\Gamma u = \nabla \tilde{u}_{|\Gamma} - \left(\nabla \tilde{u}_{|\Gamma} \cdot \mathbf{n}\right)\mathbf{n}, \tag{2.1}$$

and the tangential vector curl by

$$\mathbf{curl}_\Gamma\, u = \nabla \tilde{u}_{|\Gamma} \times \mathbf{n}, \tag{2.2}$$

where $\tilde{u}$ is an extension of u to the whole space $\mathbb{R}^3$. For a vector function $\mathbf{u} \in (\mathscr{C}^k(\Gamma))^3$, we note $[\nabla_\Gamma \mathbf{u}]$ the matrix whose the i-th column is the tangential gradient of the i-th component of $\mathbf{u}$ and we set $[\mathsf{D}_\Gamma \mathbf{u}] = {}^T[\nabla_\Gamma \mathbf{u}]$.

We define the surface divergence of any vectorial function $\mathbf{u} \in (\mathscr{C}^k(\Gamma))^3$ by

$$\mathrm{div}_\Gamma\, \mathbf{u} = \mathrm{div}\, \tilde{\mathbf{u}}_{|\Gamma} - \left([\nabla \tilde{\mathbf{u}}_{|\Gamma}]\mathbf{n} \cdot \mathbf{n}\right), \tag{2.3}$$

and the surface scalar curl $\mathrm{curl}_{\Gamma_r}\, \mathbf{u}_r$ by

$$\mathrm{curl}_\Gamma\, \mathbf{u} = \mathbf{n} \cdot (\mathbf{curl}\, \tilde{\mathbf{u}}))$$

where $\tilde{\mathbf{u}}$ is an extension of $\mathbf{u}$ to the whole space $\mathbb{R}^3$. These definitions do not depend on the extension.

Definition 2.1 *For a vector function $\mathbf{v} \in (\mathscr{C}^\infty(\overline{\Omega}))^3$ and a scalar function $v \in \mathscr{C}^\infty(\overline{\Omega})$ we define the traces :*

$$\gamma v = v_{|\Gamma},$$
$$\gamma_D \mathbf{v} := (\mathbf{n} \times \mathbf{v})_{|\Gamma} \text{ (Dirichlet) and}$$
$$\gamma_{N_\kappa} \mathbf{v} := \kappa^{-1}(\mathbf{n} \times \mathbf{curl}\, \mathbf{v})_{|\Gamma} \text{ (Neumann).}$$

We introduce the Hilbert spaces $H^s(\Gamma) = \gamma\left(H^{s+\frac{1}{2}}(\Omega)\right)$, and $\mathbf{TH}^s(\Gamma) = \gamma_D\left(\mathbf{H}^{s+\frac{1}{2}}(\Omega)\right)$ For $s > 0$, the traces

$$\gamma : H^{s+\frac{1}{2}}(\Omega) \to H^s(\Gamma),$$

$$\gamma_D : \mathbf{H}^{s+\frac{1}{2}}(\Omega) \to \mathbf{TH}^s(\Gamma)$$

are then continuous. The dual of $H^s(\Gamma)$ and $\mathbf{TH}^s(\Gamma)$ with respect to the L^2 (or $\mathbf{L}^2$) scalar product is denoted by $H^{-s}(\Gamma)$ and $\mathbf{TH}^{-s}(\Gamma)$, respectively.

The surface differential operators defined here above can be extended to the Sobolev spaces: The tangential gradient and the tangential vector curl are linear and continuous from $H^{s+1}(\Gamma)$ to $\mathbf{TH}^s(\Gamma)$, the surface divergence and the surface scalar curl are linear and continuous from $\mathbf{TH}^{s+1}(\Gamma)$ to $H^s(\Gamma)$.

Definition 2.2 *We define the Hilbert space*

$$\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) = \left\{\mathbf{j} \in \mathbf{TH}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \mathbf{j} \in H^{-\frac{1}{2}}(\Gamma)\right\}$$

endowed with the norm

$$||\cdot||_{\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} = ||\cdot||_{\mathbf{TH}^{-\frac{1}{2}}(\Gamma)} + ||\,\mathrm{div}_\Gamma \cdot||_{H^{-\frac{1}{2}}(\Gamma)}.$$

Lemma 2.3 *The operators γ_D and γ_N are linear and continuous from $\mathscr{C}^\infty(\overline{\Omega}, \mathbb{R}^3)$ to $\mathbf{TL}^2(\Gamma)$ and they can be extended to continuous linear operators from $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \Omega)$, respectively, to $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.*

For $\mathbf{u} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \overline{\Omega^c})$ and $\mathbf{v} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}\,\mathbf{curl}, \overline{\Omega^c}))$ we define $\gamma_D^c\mathbf{u}$ and $\gamma_N^c\mathbf{v}$ in the same way and the same mapping properties hold true.

Recall that we assume that the boundary Γ is smooth and topologically trivial. For a proof of the following result, we refer to [3, 8, 24].

Lemma 2.4 *Let $t \in \mathbb{R}$. The Laplace-Beltrami operator*

$$\Delta_\Gamma = \mathrm{div}_\Gamma \nabla_\Gamma = -\,\mathrm{curl}_\Gamma\,\mathbf{curl}_\Gamma \tag{2.4}$$

is linear and continuous from $H^{t+2}(\Gamma)$ to $H^t(\Gamma)$.
It is an isomorphism from $H^{t+2}(\Gamma)/\mathbb{R}$ to the space $H^t_*(\Gamma)$ defined by

$$u \in H^t_*(\Gamma) \quad \Longleftrightarrow \quad u \in H^t(\Gamma) \text{ and } \int_\Gamma u = 0.$$

This result is due to the surjectivity of the operators div_Γ and curl_Γ from $\mathbf{TH}^{t+1}(\Gamma)$ to $H^t_*(\Gamma)$.

We note the following equalities:

$$\mathrm{curl}_\Gamma \nabla_\Gamma = 0 \text{ and } \mathrm{div}_\Gamma\,\mathbf{curl}_\Gamma = 0 \tag{2.5}$$

$$\mathrm{div}_\Gamma(\mathbf{n} \times \mathbf{j}) = -\,\mathrm{curl}_\Gamma\,\mathbf{j} \text{ and } \mathrm{curl}_\Gamma(\mathbf{n} \times \mathbf{j}) = \mathrm{div}_\Gamma\,\mathbf{j} \tag{2.6}$$

2.2 Pseudo-homogeneous kernels

In this paper we are concerned with boundary integral operators of the form :

$$\mathcal{K}_\Gamma u(x) = \text{vp.} \int_\Gamma k(y, x-y)u(y)d\sigma(y),\ x \in \Gamma \tag{2.7}$$

where the integral is assumed to exist in the sense of a Cauchy principal value and the kernel k is weakly singular, regular with respect to the variable $y \in \Gamma$ and quasi-homogeneous with respect to the variable $z = x - y \in \mathbb{R}^3$. We recall the regularity properties of these operators on the Sobolev spaces $H^s(\Gamma)$, $s \in \mathbb{R}$ available also for their adjoints operators:

$$\mathcal{K}_\Gamma^*(u)(x) = \text{vp.} \int_\Gamma k(x, y-x)u(y)d\sigma(y),\ x \in \Gamma. \tag{2.8}$$

We use the class of weakly singular kernel introduced by Nedelec ([24] p. 176). More details can be found in [13, 17, 19, 22, 35, 34].

Definition 2.5 *The homogeneous kernel $k(y,z)$ defined on $\Gamma \times \left(\mathbb{R}^3 \backslash \{0\}\right)$ is said of class $-m$ with $m \geq 0$ if*

$$\begin{cases} \displaystyle\sup_{y\in\mathbb{R}^d} \sup_{|z|=1} \left| \frac{\partial^{|\alpha|}}{\partial y^\alpha} \frac{\partial^{|\beta|}}{\partial z^\beta} k(y,z) \right| \leq C_{\alpha,\beta}, \text{ for all multi-index } \alpha \text{ and } \beta, \\ \dfrac{\partial^{|\beta|}}{\partial z^\beta} k(y,z) \text{ is homogeneous of degree } -2 \text{ with respect to the variable } z \\ \text{for all } |\beta| = m \text{ and } D_z^m k(y,z) \text{ is odd with respect to the variable } z. \end{cases}$$

Definition 2.6 *The kernel $k \in \mathscr{C}^\infty\left(\Gamma \times \left(\mathbb{R}^3 \backslash \{0\}\right)\right)$ is pseudo-homogeneous of class $-m$ for an integer m such that $m \geqslant 0$, if for all integer s the kernel k admit the following asymptotic expansion when z tends to 0:*

$$k(y,z) = k_m(y,z) + \sum_{j=1}^{N-1} k_{m+j}(y,z) + k_{m+N}(y,z), \tag{2.9}$$

where for $j = 0, 1, ..., N-1$ the function k_{m+j} is homogeneous of class $-(m+j)$ and N is chosen such that k_{m+N} is s times differentiables.

For the proof of the following theorem, we refer to [24].

Theorem 2.7 *Let k be a pseudo-homogeneous kernel of class $-m$. The associated operator $\mathcal{K}_\Gamma$ given by (2.7) is linear and continuous from $H^s(\Gamma)$ to $H^{s+m}(\Gamma)$ for all $s \in \mathbb{R}$.*

We have similar results for the adjoint operators $\mathcal{K}_\Gamma^*$.

The following theorem is established in [13].

Theorem 2.8 *Let k be a pseudo-homogeneous kernel of class $-m$. The potential operator $\mathcal{P}$ defined by*

$$\mathcal{P}(u)(x) = \int_\Gamma k(y, x-y)u(y)d\sigma(y),\ x \in \mathbb{R}^3\backslash\Gamma \tag{2.10}$$

is continuous from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^{s+m}(\Omega) \cup H^{s+m}_{loc}(\Omega^c)$ for all positive real number s.

2.3 The electromagnetic boundary integral operators

We use some well known results about electromagnetic potentials. Details can be found in [3, 4, 5, 6, 24].

Let κ be a complex number such that $\mathrm{Im}(\kappa) \geq 0$ and let

$$G(\kappa, |x-y|) = \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}$$

be the fundamental solution of the Helmholtz equation

$$\Delta u + \kappa^2 u = 0.$$

The single layer potential ψ_κ is given by :

$$(\psi_\kappa u)(x) = \int_\Gamma G(\kappa, |x-y|)u(y)d\sigma(y) \qquad x \in \mathbb{R}^3\backslash\Gamma,$$

and its trace by

$$V_\kappa u(x) = \int_\Gamma G(\kappa, |x-y|)u(y)d\sigma(y) \qquad x \in \Gamma.$$

The fundamental solution is pseudo-homogeneous of class -1 (see [18, 24]). As consequence we have the following result :

Lemma 2.9 *Let $s \in \mathbb{R}$. The operators*

$$\begin{aligned} \psi_\kappa &: H^{s-\frac{1}{2}}(\Gamma) \to H^{s+1}_{\mathrm{loc}}(\mathbb{R}^3) \\ V_\kappa &: H^{s-\frac{1}{2}}(\Gamma) \to H^{s+\frac{1}{2}}(\Gamma) \end{aligned}$$

are continuous.

We define the electric potential Ψ_{E_κ} generated by $\mathbf{j} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by

$$\Psi_{E_\kappa}\mathbf{j} := \kappa\psi_\kappa\mathbf{j} + \kappa^{-1}\nabla\psi_\kappa\,\mathrm{div}_\Gamma\,\mathbf{j}$$

This can be written as $\Psi_{E_\kappa}\mathbf{j} := \kappa^{-1}\,\mathbf{curl}\,\mathbf{curl}\,\psi_\kappa\mathbf{j}$ because of the Helmholtz equation and the identity $\mathbf{curl}\,\mathbf{curl} = -\Delta + \nabla\,\mathrm{div}$ (cf [3]).

We define the magnetic potential Ψ_{M_κ} generated by $\mathbf{m} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by

$$\Psi_{M_\kappa}\mathbf{m} := \mathbf{curl}\,\psi_\kappa\mathbf{m}.$$

We denote the identity operator by I.

Lemma 2.10 *The potentials Ψ_{E_κ} et Ψ_{M_κ} are continuous from $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ to $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\mathbb{R}^3)$. For $\mathbf{j}\in\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ we have*

$$(\mathbf{curl}\,\mathbf{curl}-\kappa^2\mathrm{I})\Psi_{E_\kappa}\mathbf{j}=0 \text{ and } (\mathbf{curl}\,\mathbf{curl}-\kappa^2\mathrm{I})\Psi_{M_\kappa}\mathbf{m}=0 \text{ in } \mathbb{R}^3\backslash\Gamma$$

and $\Psi_{E_\kappa}\mathbf{j}$ and $\Psi_{M_\kappa}\mathbf{m}$ satisfy the Silver-Müller condition.

We define the electric and the magnetic far field operators for $\mathbf{j}\in\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ and an element $\hat{x}$ of the unit sphere S^2 of $\mathbb{R}^3$ by

$$\begin{aligned}\Psi^\infty_{E_\kappa}\mathbf{j}(\hat{x})&=\kappa\,\hat{x}\times\left(\int_\Gamma e^{-i\kappa\hat{x}\cdot y}\mathbf{j}(y)d\sigma(y)\right)\times\hat{x},\\ \Psi^\infty_{M_{\kappa_e}}\mathbf{j}(\hat{x})&=i\kappa\,\hat{x}\times\left(\int_\Gamma e^{-i\kappa\hat{x}\cdot y}\mathbf{j}(y)d\sigma(y)\right).\end{aligned}\tag{2.11}$$

These operators are bounded from $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ to $\mathbf{T}(\mathscr{C}^\infty(S^2))^3$.

We can now define the main boundary integral operators:

$$\begin{aligned}C_\kappa&=-\tfrac{1}{2}\{\gamma_D+\gamma_D^c\}\Psi_{E_\kappa}=-\tfrac{1}{2}\{\gamma_N+\gamma_N^c\}\Psi_{M_\kappa},\\ M_\kappa&=-\tfrac{1}{2}\{\gamma_D+\gamma_D^c\}\Psi_{M_\kappa}=-\tfrac{1}{2}\{\gamma_N+\gamma_N^c\}\Psi_{E_\kappa}.\end{aligned}$$

These are bounded operators in $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$. We have

$$\begin{aligned}\mathrm{C}_\kappa\mathbf{j}(x)&=-\kappa\int_\Gamma\mathbf{n}(x)\times(G(\kappa,|x-y|)\mathbf{j}(y))d\sigma(y)+\kappa^{-1}\int_\Gamma\mathbf{curl}^x_\Gamma(G(\kappa,|x-y|)\,\mathrm{div}_\Gamma\mathbf{j}(y))d\sigma(y)\\ &=\left(-\kappa\ \mathbf{n}\times V_\kappa\mathbf{j}+\kappa^{-1}\,\mathrm{curl}_\Gamma V_\kappa\,\mathrm{div}_\Gamma\mathbf{j}\right)(x)\end{aligned}$$

and

$$\begin{aligned}\mathrm{M}_\kappa\mathbf{j}(x)&=-\int_\Gamma\mathbf{n}(x)\times\mathbf{curl}^x(G(\kappa,|x-y|)\mathbf{j}(y))d\sigma(y)\\ &=\ (\mathrm{D}_\kappa\mathbf{j}-\mathrm{B}_\kappa\mathbf{j})(x),\end{aligned}$$

with

$$\begin{aligned}\mathrm{B}_\kappa\mathbf{j}(x)&=\int_\Gamma\nabla^x G(\kappa,|x-y|)\,(\mathbf{j}(y)\cdot\mathbf{n}(x))\,d\sigma(y),\\ \mathrm{D}_\kappa\mathbf{j}(x)&=\int_\Gamma(\nabla^x G(\kappa,|x-y|)\cdot\mathbf{n}(x))\,\mathbf{j}(y)d\sigma(y).\end{aligned}$$

The kernel of D_κ is pseudo-homogeneous of class -1 and the operator M_κ has the same regularity as D_κ on $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$, that is compact.

We describe briefly the boundary integral equation method developped by the autors [9] to solve the dielectric scattering problem.

Boundary integral equation method : This is based on the Stratton-Chu formula, the jump relations of the electromagnetic potentials and the Calderón projector's formula (see [6, 24]).

We need a variant of the operator C_κ defined for $\mathbf{j} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ by :

$$\mathrm{C}_0^* \mathbf{j} = \mathbf{n} \times V_0 \mathbf{j} + \mathbf{curl}_\Gamma V_0 \, \mathrm{div}_\Gamma \mathbf{j}.$$

The operator C_0^* is bounded in $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. We use the following ansatz on the integral representation of the exterior electric field $\mathbf{E}^s$:

$$\mathbf{E}^s = -\Psi_{\mathrm{E}_{\kappa_\mathrm{e}}} \mathbf{j} - i\eta \, \Psi_{\mathrm{M}_{\kappa_\mathrm{e}}} \mathrm{C}_0^* \mathbf{j} \text{ in } \mathbb{R}^3 \backslash \bar{\Omega} \tag{2.12}$$

η is a positive real number and $\mathbf{j} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Thanks to the transmission conditions we have the integral representation of the interior field

$$\mathbf{E}_1 = -\frac{1}{\rho}(\Psi_{E_{\kappa_i}}\{\gamma_{N_e}^c \mathbf{E}^{inc} + \mathrm{N_e}\mathbf{j}\}) - (\Psi_{M_{\kappa_i}}\{\gamma_D^c \mathbf{E}^{inc} + \mathrm{L_e}\mathbf{j}\}) \text{ in } \Omega \tag{2.13}$$

where $\rho = \dfrac{\kappa_i \mu_e}{\kappa_e \mu_i}$ and

$$\mathrm{L_e} = \mathrm{C}_{\kappa_\mathrm{e}} - i\eta \left(\frac{1}{2}\mathrm{I} - \mathrm{M}_{\kappa_\mathrm{e}}\right) \mathrm{C}_0^*,$$

$$\mathrm{N_e} = \left(\frac{1}{2}\mathrm{I} - \mathrm{M}_{\kappa_\mathrm{e}}\right) + i\eta \, \mathrm{C}_{\kappa_\mathrm{e}} \mathrm{C}_0^*.$$

We apply the exterior Dirichlet trace to the righthandside (2.13). The density $\mathbf{j}$ then solves the following boundary integral equation:

$$\mathbf{S}\mathbf{j} = \rho \left(-\frac{1}{2}I + \mathrm{M}_{\kappa_\mathrm{i}}\right) \mathrm{L_e}\mathbf{j} + \mathrm{C}_{\kappa_\mathrm{i}} \mathrm{N_e}\mathbf{j} = -\rho \left(-\frac{1}{2}I + \mathrm{M}_{\kappa_\mathrm{i}}\right) \gamma_D \mathbf{E}^{inc} + \mathrm{C}_{\kappa_\mathrm{i}} \gamma_{N_{\kappa_e}} \mathbf{E}^{inc} \text{ sur } \Gamma.$$

The operator $\mathbf{S}$ is linear, bounded and invertible on $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$.

If we are concerned with the far field pattern $\mathbf{E}^\infty$ of the solution, it suffices to replace the potential operators $\Psi_{E_{\kappa e}}$ and $\Psi_{M_{\kappa e}}$ by the far field operators $\Psi^\infty_{E_{\kappa e}}$ and $\Psi^\infty_{M_{\kappa e}}$ respectively.

The solution $\mathbf{E}(\Omega) = (\mathbf{E}^i(\Omega), \mathbf{E}^s(\Omega))$ and the far field pattern $\mathbf{E}^\infty(\Omega)$ consists of applications defined by integrals on the boundary Γ and if the incident field is a fixed data, these quantities depend on the scatterrer Ω only.

3 Some remarks on shape derivatives

We want to study the dependance of any functionals F with respect to the shape of the dielectric scatterer Ω. The Ω-dependance is highly nonlinear. The standard differential calculus tools need the framework of topological vector spaces which are locally convex at least [33], framework we do not dispose in the case of shape functionals. An interesting approach consists in representing the variations of the domain Ω by elements of a function space. We consider variations generated by transformations of the form

$$x \mapsto x + r(x)$$

of any points x in the space $\mathbb{R}^3$, where r is a vectorial function defined (at least) in the neiborhood of Ω. This transformation deforms the domain Ω in a domain Ω_r of boundary Γ_r. The functions r are assumed to be a small enough elements of a Fréchet space $\mathcal{X}$ in order that $(\mathrm{I}+r)$ is an isomorphism from Γ to

$$\Gamma_r = (\mathrm{I}+r)\Gamma = \{x_r = x + r(x); x \in \Gamma\}.$$

Since we consider smooth surfaces, in the remaining of this paper, the space $\mathcal{X}$ will be the Fréchet space $\mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3) = \bigcap_{k\in\mathbb{N}} \mathscr{C}_b^k(\mathbb{R}^3,\mathbb{R}^3)$ undowed with the set of non decreasing seminorms $(||\cdot||_k)_{k\in\mathbb{N}}$ where $\mathscr{C}_b^k(\mathbb{R}^3,\mathbb{R}^3)$ with $k\in\mathbb{N}$ is the space of k-times continuously differentiable functions whose the derivatives are bounded and

$$||r||_k = \sup_{0\le p\le k} \sup_{x\in\mathbb{R}^3} \left| r^{(p)}(x) \right|.$$

For ϵ small enough we set

$$B_\epsilon^\infty = \left\{ r \in \left(\mathscr{C}^\infty(\overline{\mathbb{R}^3})\right)^3,\ d_\infty(0,r) < \epsilon \right\},$$

where d_∞ is the metric induced by the seminorms.

We introduce the application

$$r \in B_\epsilon^\infty \mapsto \mathcal{F}_\Omega(r) = F(\Omega_r).$$

We define the shape derivative of the functional F trough the deformation $\Omega \to \Omega_\xi$ as the Gâteaux derivative of the application $\mathcal{F}_\Omega$ in the direction $\xi \in \mathcal{X}$. We write:

$$DF[\Omega;\xi] = \frac{\partial}{\partial t}_{|t=0} \mathcal{F}_\Omega(t\xi).$$

3.1 Gâteaux differentiability: elementary results

Fréchet spaces are locally convex, metrisable and complete topological vector spaces on which we can extend any elementary results available on Banach spaces. We recall some of them. We refer to the Schwarz's book [33] for more details.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Fréchet spaces and let U be a subset of $\mathcal{X}$.

Definition 3.1 *(**Gâteaux semi-derivatives**) The application $f : U \to \mathcal{Y}$ is said to have Gâteaux semiderivative at $r_0 \in U$ in the direction $\xi \in \mathcal{X}$ if the following limit exists and is finite*

$$\frac{\partial}{\partial r} f[r_0;\xi] = \lim_{t\to 0} \frac{f(r_0+t\xi) - f(r_0)}{t} = \frac{\partial}{\partial t}\Big|_{t=0} f(r_0+t\xi).$$

Definition 3.2 *(**Gâteaux differentiability**) The application $f : U \to \mathcal{Y}$ is said to be Gâteaux differentiable at $r_0 \in U$ if it has Gâteaux semiderivatives in all direction $\xi \in \mathcal{X}$ and if the map*

$$\xi \in \mathcal{X} \mapsto \frac{\partial}{\partial r} f[r_0; \xi] \in \mathcal{Y}$$

is linear and continuous.

We say that f is continuously (or $\mathscr{C}^1$-) Gâteaux differentiable if it is Gâteaux differentiable at all $r_0 \in U$ and the application

$$\frac{\partial}{\partial r} f : (r_0; \xi) \in U \times \mathcal{X} \mapsto \frac{\partial}{\partial r} f[r_0; \xi] \in \mathcal{Y}$$

is continuous.

Remark 3.3 *Let us come to shape functionals. In calculus of shape variation, we usually consider the Gâteaux derivative in $r = 0$ only. This is due to the result : If $\mathcal{F}_\Omega$ is Gâteaux differentiable on B_ϵ^∞ then for all $\xi \in \mathcal{X}$ we have*

$$\frac{\partial}{\partial r} \mathcal{F}_\Omega[r_0; \xi] = \mathsf{D}F(\Omega_{r_0}; \xi \circ (\mathrm{I} + r_0)^{-1}) = \frac{\partial}{\partial r} \mathcal{F}_{\Omega_{r_0}}[0; \xi \circ (\mathrm{I} + r_0)^{-1}].$$

Definition 3.4 *(**higher order derivatives**) Let $m \in \mathbb{N}$. We say that f is $(m+1)$-times continuously (or $\mathscr{C}^{m+1}$-) Gâteaux differentiable if it is $\mathscr{C}^m$-Gâteaux differentiable and*

$$r \in U \mapsto \frac{\partial^m}{\partial r^m} f[r; \xi_1, \ldots, \xi_m]$$

is continuously Gâteaux differentiable for all m-uple $(\xi_1, \ldots, \xi_m) \in \mathcal{X}^m$. Then for all $r_0 \in U$ the application

$$(\xi_1, \ldots, \xi_{m+1}) \in \mathcal{X}^{m+1} \mapsto \frac{\partial^{m+1}}{\partial r^{m+1}} f[r_0; \xi_1, \ldots, \xi_{m+1}] \in \mathcal{Y}$$

is $(m+1)$-linear, symetric and continuous. We say that f is $\mathscr{C}^\infty$-Gâteaux differentiable if it is $\mathscr{C}^m$-Gâteaux differentiable for all $m \in \mathbb{N}$.

We use the notation

$$\frac{\partial^m}{\partial r^m} f[r_0, \xi] = \frac{\partial^m}{\partial t^m}\Big|_{t=0} f(r_0 + t\xi). \tag{3.1}$$

If it is $\mathscr{C}^m$-Gâteaux differentiable we have

$$\frac{\partial^m}{\partial r^m} f[r_0, \xi_1, \ldots, \xi_m] = \frac{1}{m!} \sum_{p=1}^{m} (-1)^{m-p} \sum_{1 \le i_1 < \cdots < i_p \le m} \frac{\partial^m}{\partial r^m} f[r_0; \xi_{i_1} + \ldots + \xi_{i_p}]. \tag{3.2}$$

To determine higher order Gâteaux derivatives it is more easy to use this equality.

The chain and product rules and the Taylor expansion with integral remainder are still available for $\mathscr{C}^m$-Gâteaux differentiable maps ([33] p. 30). We use the following lemma to study the Gâteaux differentiability of any applications mapping r on the inverse of an element in a unitary topological algebra.

Lemma 3.5 *Let $\mathcal{X}$ be a Fréchet space and $\mathcal{Y}$ be a unitary Fréchet algebra. Let U be an open set of $\mathcal{X}$. Assume that the application $f : U \to \mathcal{Y}$ is Gâteaux differentiable at $r_0 \in U$ and that $f(r)$ is invertible in $\mathcal{Y}$ for all $r \in U$ and that the application $g : r \mapsto f(r)^{-1}$ is continuous at r_0. Then g is Gâteaux differentiable at r_0 and its first derivative in the direction $\xi \in \mathcal{X}$ is*

$$\frac{\partial}{\partial r} f[r_0, \xi] = -f(r_0)^{-1} \circ \frac{\partial}{\partial r} f[r_0, \xi] \circ f(r_0)^{-1}. \tag{3.3}$$

Moreover if f is $\mathscr{C}^m$-Gâteaux differentiable then g is too.

PROOF. Let $\xi \in \mathcal{X}$ and $t > 0$ small enough such that $(r_0 + t\xi) \in U$, on a:

$$\begin{aligned} g(r_0 + t\xi) - g(r_0) = & \quad f(r_0)^{-1} \circ f(r_0) \circ f(r_0 + t\xi)^{-1} - f(r_0)^{-1} \circ f(r_0 + t\xi) \circ f(r_0 + t\xi)^{-1} \\ = & \quad f(r_0)^{-1} \circ (f(r_0) - f(r_0 + t\xi)) \circ f(r_0 + t\xi)^{-1} \\ = & \quad f(r_0)^{-1} \circ (f(r_0) - f(r_0 + t\xi)) \circ f(r_0)^{-1} \\ & + f(r_0)^{-1} \circ (f(r_0) - f(r_0 + t\xi)) \circ \left(f(r_0 + t\xi)^{-1} - f(r_0)^{-1} \right). \end{aligned}$$

Since g is continuous in r_0, we have $\lim_{t \to 0} \left(f(r_0 + t\xi)^{-1} - f(r_0)^{-1} \right) = 0$ and since f is Gâteaux differentiable in r_0 we have

$$\lim_{t \to 0} \frac{f(r_0)^{-1} \circ (f(r_0) - f(r_0 + t\xi)) \circ f(r_0)^{-1}}{t} = - \left(f(r_0) \right)^{-1} \circ \frac{\partial}{\partial r} f[r_0, \xi] \circ \left(f(r_0) \right)^{-1}.$$

As a consequence

$$\lim_{t \to 0} \frac{g(r_0 + t\xi) - g(r_0)}{t} = - \left(f(r_0) \right)^{-1} \circ \frac{\partial}{\partial r} f[r_0, \xi] \circ \left(f(r_0) \right)^{-1}.$$

■

4 Gâteaux differentiability of pseudo-homogeneous kernels

Let x_r denote an element of Γ_r and let $\mathbf{n}_r$ be the outer unit normal vector to Γ_r. When $r = 0$ we write $\mathbf{n}_0 = \mathbf{n}$. We note again $d\sigma$ the area element on Γ_r.

In this section we want to study the Gâteaux differentiability of the application mapping $r \in B_\epsilon^\infty$ to the integral operator $\mathcal{K}_{\Gamma_r}$ defined for a function $u_r \in H^s(\Gamma_r)$ by:

$$\mathcal{K}_{\Gamma_r} u_r(x_r) = \text{vp.} \int_{\Gamma_r} k_r(y_r, x_r - y_r) u_r(y_r) d\sigma(y_r), \ x_r \in \Gamma_r \tag{4.1}$$

and of the application mapping $r \in B_\epsilon^\infty$ to the potential operator $\mathcal{P}_r$ defined for a function $u_r \in H^s(\Gamma_r)$ by:

$$\mathcal{P}_r u_r(x) = \int_{\Gamma_r} k_r(y_r, x - y_r) u_r(y_r) d\sigma(y_r), \ x \in K, \tag{4.2}$$

where $k_r \in \mathscr{C}^\infty\left(\Gamma_r \times \left(\mathbb{R}^3 \backslash \{0\}\right)\right)$ is a pseudo-homogeneous kernel of class $-m$ with $m \in \mathbb{N}$.

We want to differentiate applications of the form $r \mapsto \mathcal{F}_\Omega(r)$ where the domain of definition of $\mathcal{F}_\Omega(r)$ varies with r. How do we do? A first idea, quite classical (see [25, 27, 29]), is that instead of studying the application

$$r \in B_\epsilon^\infty \mapsto \mathcal{F}_\Omega(r) \in \mathscr{C}^k(\Gamma_r)$$

we consider the application

$$r \in B_\epsilon^\infty \mapsto \mathcal{F}_\Omega(r) \circ (\mathrm{I} + r) \in \mathscr{C}^k(\Gamma).$$

An example is $r \mapsto \mathbf{n}_r$. This point of view can be extended to Sobolev spaces $H^s(\Gamma)$, $s \in \mathbb{R}$. From now we use the transformation τ_r which maps a function u_r defined on Γ_r to the function $u_r \circ (\mathrm{I} + r)$ defined on Γ. For all $r \in B_\epsilon^\infty$, this transformation τ_r admit an inverse. We have

$$(\tau_r u_r)(x) = u_r(x + r(x)) \text{ and } (\tau_r^{-1} u)(x_r) = u(x).$$

Then, instead of studying the application

$$r \in B_\epsilon^\infty \mapsto \mathcal{K}_{\Gamma_r} \in \mathscr{L}_c\left(H^s(\Gamma_r), H^{s+m}(\Gamma_r)\right)$$

we consider the conjugate application

$$r \in B_\epsilon^\infty \mapsto \tau_r \mathcal{K}_{\Gamma_r} \tau_r^{-1} \in \mathscr{L}_c\left(H^s(\Gamma), H^{s+m}(\Gamma)\right).$$

In the framework of boundary integral equations, this approach is sufficient to obtain the shape differentability of any solution to scalar boundary value problems [27, 29].

Using the change of variable $x \mapsto x_r = x + r(x)$, we have for $u \in H^s(\Gamma)$:

$$\tau_r \mathcal{K}_r \tau_r^{-1}(u)(x) = \int_\Gamma k_r(y + r(y), x + r(x) - y - r(y)) u(y) J_r(y) d\sigma(y), \ x \in \Gamma$$

where J_r is the jacobian (the determinant of the Jacobian matrix) of the change of variable mapping $x \in \Gamma$ to $x + r(x) \in \Gamma_r$. The differentiablility analysis of these operators begins with the jacobian one. We have

$$J_r = \mathrm{Jac}_\Gamma(\mathrm{I} + r) = ||\omega_r|| \text{ with } \omega_r = \mathrm{com}(\mathrm{I} + \mathsf{D} r_{|\Gamma}) \mathbf{n}_0 = \det(\mathrm{I} + \mathsf{D} r_{|\Gamma})^T (\mathrm{I} + \mathsf{D} r_{|\Gamma})^{-1} \mathbf{n},$$

and the normal vector $\mathbf{n}_r$ is given by

$$\mathbf{n}_r = \tau_r^{-1}\left(\frac{\omega_r}{\|\omega_r\|}\right).$$

The first derivative at $r = 0$ of these applications are well known [12, 25]. Here we present one method to obtain higher order derivative.

Lemma 4.1 *The application J mapping $r \in B_\epsilon^\infty$ to the jacobian $J_r \in \mathscr{C}^\infty(\Gamma, \mathbb{R})$ is $\mathscr{C}^\infty$ Gâteaux differentiable and its first derivative at r_0 is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$ by:*

$$\frac{\partial J}{\partial r}[r_0, \xi] = J_{r_0}(\tau_{r_0} \operatorname{div}_{\Gamma_{r_0}} \tau_{r_0}^{-1})\xi.$$

PROOF. We just have to prove the $\mathscr{C}^\infty$-Gâteaux differentiability of $W : r \mapsto w_r$. We do the proof for hypersurfaces Γ of $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$. We use local coordinate system. Assume that Γ is parametrised by an atlas $(\mathcal{O}_i, \phi_i)_{1 \leq i \leq p}$ then Γ_r can be parametrised by the atlas $(\mathcal{O}_i, (\mathrm{I} + r) \circ \phi_i)_{1 \leq i \leq p}$. For any $x \in \Gamma$, let us note $e_1(x), e_2(x), \ldots, e_{n-1}(x)$ the vector basis of the tangent plane to Γ at x. The vector basis of the tangent plane to Γ_r at $x + r(x)$ are given by

$$e_i(r, x) = [(\mathrm{I} + \mathsf{D}r)(x)]e_i(x) \quad \text{for } i = 1, \ldots, n-1.$$

Thus, we have $\omega_r(x) = \dfrac{\bigwedge_{i=1}^{n-1} e_i(r, x)}{\left| \bigwedge_{i=1}^{n-1} e_i(x) \right|}$. Since the applications $r \mapsto e_i(r, x)$, for $i = 1, \ldots, n-1$ are $\mathscr{C}^\infty$-Gâteaux differentiable, the application W is too. Now want to compute the derivatives using the formula (3.2). Let $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and t small enough. We have at $r_0 \in B_\epsilon^\infty$

$$\frac{\partial^m W}{\partial r^m}[r_0, \xi] = \frac{\partial^m}{\partial t^m}\Big|_{t=0} \frac{\bigwedge_{i=1}^{n-1} (\mathrm{I} + Dr_0 + tD\xi)e_i(x)}{\left| \bigwedge_{i=1}^{n-1} e_i(x) \right|}.$$

To simplify this expression one have to note that

$$\begin{aligned}
[\mathsf{D}\xi(x)]e_i(x) &= [\mathsf{D}\xi(x)][(\mathrm{I} + \mathsf{D}r_0)(x)]^{-1}[(\mathrm{I} + \mathsf{D}r_0)(x)]e_i(x) \\
&= [\mathsf{D}\xi(x)][\mathsf{D}(\mathrm{I} + r_0)^{-1}(x + r_0(x))][(\mathrm{I} + \mathsf{D}r_0)(x)]e_i(x) \\
&= [(\tau_{r_0}\mathsf{D}\tau_{r_0}^{-1})\xi(x)]e_i(r_0, x) = [(\tau_{r_0}\mathsf{D}_{\Gamma_{r_0}}\tau_{r_0}^{-1})\xi(x)]e_i(r_0, x).
\end{aligned}$$

NB: given a $(n \times n)$ matrix A we have

$$\sum_{i=1}^{n-1} \cdots \times e_{i-1} \times Ae_i \times e_{i+1} \times \cdots = (\operatorname{Trace}(A)\mathrm{I} - {}^T A) \bigwedge_{i=1}^{n-1} e_i.$$

Thus we have with $A = [\tau_{r_0} \mathsf{D}_{\Gamma_{r_0}} \tau_{r_0}^{-1} \xi]$ and $B_0 = \mathrm{I}$, $B_1(A) = \mathrm{Trace}(A)\mathrm{I} - {}^T A$

$$(\#) \begin{cases} W(r_0) & = & J_{r_0}(\tau_{r_0}\mathbf{n}_{r_0}), \\ \dfrac{\partial W}{\partial r}[r_0,\xi] & = & J_{r_0}\left((\tau_{r_0}\,\mathrm{div}_{\Gamma_{r_0}}\,\tau_{r_0}^{-1})\xi\cdot\tau_{r_0}\mathbf{n}_{r_0} - [(\tau_{r_0}\nabla_{\Gamma_{r_0}}\tau_{r_0}^{-1})\xi]\tau_{r_0}\mathbf{n}_{r_0}\right) \\ & = & [B_1(A)\xi]W(r_0), \\ \dfrac{\partial^m W}{\partial r^m}[r_0,\xi] & = & [B_m(A)\xi]W(r_0) \\ & = & \displaystyle\sum_{i=1}^{m}(-1)^{i+1}\frac{(m-1)!}{(m-i)!}[B_1(A^i)B_{m-i}(A)\xi]W(r_0) \ \text{ for } m=1,\dots,n-1, \\ \dfrac{\partial^m W}{\partial r^m}[r_0,\xi] & \equiv & 0 \text{ for all } m \geq n. \end{cases}$$

It follows that

$$\frac{\partial J}{\partial r}[r_0,\xi] = \frac{1}{\|W(r_0)\|}\frac{\partial W}{\partial r}[r_0,\xi]\cdot W(r_0) = \frac{\partial W}{\partial r}[r_0,\xi]\cdot\tau_{r_0}\mathbf{n}_{r_0} = J_{r_0}(\tau_{r_0}\,\mathrm{div}_{\Gamma_{r_0}}\,\tau_{r_0}^{-1})\xi.$$

■

Thanks to (#) we deduce easily the Gâteaux differentiability of $r \mapsto \tau_r\mathbf{n}_r$.

Lemma 4.2 *The application N mapping $r \in B_\epsilon^\infty$ to $\tau_r\mathbf{n}_r = \mathbf{n}_r \circ (\mathrm{I}+r) \in \mathscr{C}^\infty(\Gamma,\mathbb{R}^3)$ is $\mathscr{C}^\infty$ Gâteaux-differentiable and its first derivative at r_0 is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$ by:*

$$\frac{\partial N}{\partial r}[r_0,\xi] = -\left[\tau_{r_0}\nabla_{\Gamma_{r_0}}\tau_{r_0}^{-1}\xi\right] N(r_0).$$

Proof. This results from the precedent proof and we have

$$\begin{aligned} \frac{\partial N}{\partial r}[r_0,\xi] &= \frac{1}{\|W(r_0)\|}\frac{\partial W}{\partial r}[r_0,\xi] - \frac{1}{\|W(r_0)\|^3}\left(\frac{\partial W}{\partial r}[r_0,\xi]\cdot W(r_0)\right)W(r_0) \\ &= J_{r_0}^{-1}\left(\frac{\partial W}{\partial r}[r_0,\xi] - \left(\frac{\partial W}{\partial r}[r_0,\xi]\cdot(\tau_{r_0}\mathbf{n}_{r_0})\right)\right)\tau_{r_0}\mathbf{n}_{r_0} \\ &= -\left[\tau_{r_0}\nabla_{\Gamma_{r_0}}\tau_{r_0}^{-1}\xi\right]\tau_{r_0}\mathbf{n}_{r_0}. \end{aligned}$$

■

To obtain higher order shape derivatives of these applications one can use the equalities (#) and

$$(*) \begin{cases} \|\tau_r\mathbf{n}_r\| & \equiv & 1, \\ \dfrac{\partial^m N\cdot N}{\partial r^m}[r_0,\xi] & \equiv & 0 \text{ for all } m \geq 1. \end{cases}$$

As for exemple we have at $r=0$ in the direction $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$:

$$\frac{\partial J}{\partial r}[0,\xi] = \mathrm{div}_\Gamma\,\xi \text{ and } \frac{\partial N}{\partial r}[0,\xi] = -[\nabla_\Gamma\xi]\mathbf{n},$$

$$\frac{\partial^2 J}{\partial r^2}[0,\xi_1,\xi_2] = -\operatorname{Trace}([\nabla_\Gamma \xi_2][\nabla_\Gamma \xi_1]) + \operatorname{div}_\Gamma \xi_1 \cdot \operatorname{div}_\Gamma \xi_2 + ([\nabla_\Gamma \xi_1]\mathbf{n} \cdot [\nabla_\Gamma \xi_2]\mathbf{n})\,.$$

Notice that $\operatorname{Trace}([\nabla_\Gamma \xi_2][\nabla_\Gamma \xi_1]) = \operatorname{Trace}([\nabla_\Gamma \xi_1][\nabla_\Gamma \xi_2])$,

$$\frac{\partial^2 N}{\partial r^2}[0,\xi_1,\xi_2] = [\nabla_\Gamma \xi_2][\nabla_\Gamma \xi_1]\mathbf{n} + [\nabla_\Gamma \xi_1][\nabla_\Gamma \xi_2]\mathbf{n} - ([\nabla_\Gamma \xi_1]\mathbf{n} \cdot [\nabla_\Gamma \xi_2]\mathbf{n})\,\mathbf{n}.$$

For $n \geq 3$ it needs too long calculations to simplify the expression of the derivatives and we only obtain the quadratic expression. In the last section we give a second method to obtain higher order derivatives using the Gâteaux derivatives of the surface differential operators.

Remark 4.3 *We do not need more than the first derivative of the deformations ξ. As a consequence for hypersurfaces of class $\mathscr{C}^{k+1}$, it suffice to consider deformations of class $\mathscr{C}^{k+1}$ to conserve the regularity of the jacobian and of the normal vector by differentiation.*

The following theorem establish sufficient conditions for the Gâteaux differentiability of the boundary integral operators described here above and that we obtain their derivatives by deriving their kernels.

Theorem 4.4 *Let $k \in \mathbb{N}$. We set $(\Gamma \times \Gamma)^* = \{(x,y) \in \Gamma \times \Gamma;\ x \neq y\}$. Assume that*

1) For all fixed $(x,y) \in (\Gamma \times \Gamma)^*$ the function

$$\begin{array}{rccl} f: & B_\epsilon^\infty & \to & \mathbb{C} \\ & r & \mapsto & k_r(y+r(y), x+r(x)-y-r(y))J_r(y) \end{array}$$

is $\mathscr{C}^{k+1}$-Gâteaux differentiable.

2) The functions $(y, x-y) \mapsto f(r_0)(y,x-y)$ and

$$(y,x-y) \mapsto \frac{\partial^\alpha}{\partial r^\alpha} f[r_0,\xi_1,\ldots,\xi_\alpha](y,x-y)$$

are pseudo-homogeneous of class $-m$ for all $r_0 \in B_\epsilon^\infty$, for all $\alpha = 1,\ldots,k+1$ and for all $\xi_1,\ldots,\xi_{k+1} \in \mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$.

Then the application

$$\begin{array}{rcl} B_\epsilon^\infty & \to & \mathscr{L}_c(H^s(\Gamma), H^{s+m}(\Gamma)) \\ r & \mapsto & \tau_r \mathcal{K}_{\Gamma_r} \tau_r^{-1} \end{array}$$

is $\mathscr{C}^k$-Gâteaux differentiable and

$$\frac{\partial^k}{\partial r^k}\left\{\tau_r \mathcal{K}_{\Gamma_r} \tau_r^{-1}\right\}[r_0,\xi_1,\ldots,\xi_k]u(x) = \int_\Gamma \frac{\partial^k}{\partial r^k} f[r_0,\xi_1,\ldots,\xi_k](y,x-y)u(y)d\sigma(y).$$

PROOF. We use the linearity of the integral and Taylor expansion with integral remainder. We do the proof $k = 1$ only. Let $r_0 \in B_\epsilon^\infty$, $\xi \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and t small enough such that $r_0 + t\xi \in B_\epsilon^\infty$. We have

$$f(r_0+t\xi, x, y)-f(r_0, y, x-y) = t\frac{\partial f}{\partial r}[r_0, \xi](y, x-y)+t^2\int_0^1 (1-\lambda)\frac{\partial^2 f}{\partial r^2}[r_0+\lambda t\xi, \xi](y, x-y)d\lambda.$$

We have to verify that each terms in the equality here above is a kernel of an operator mapping $H^s(\Gamma)$ onto $H^{s+m}(\Gamma)$. The two first terms in the left hand side are kernels of class $-m$ and by hypothesis $\dfrac{\partial^2 f}{\partial r^2}[0, \xi]$ is also a kernel of class $-m$. It remains to prove that the operator with kernel

$$(x, y) \mapsto \int_0^1 (1-\lambda)\frac{\partial^2 f}{\partial r^2}[r_0 + \lambda t\xi, \xi](x, y)$$

acts from $H^s(\Gamma)$ to $H^{s+m}(\Gamma)$. Since $\dfrac{\partial^2 f}{\partial r^2}[r_0 + \lambda t\xi, \xi]$ is pseudo-homogeneous of class $-m$ for all $\lambda \in [0, 1]$, it suffice to use Lebesgue's theorem in order to invert the integration with respect to the variable λ and the integration with respect to y on Γ.

$$\begin{aligned}
& \left\| \int_\Gamma \left(\int_0^1 (1-\lambda)\frac{\partial^2 f}{\partial r^2}[r_0 + \lambda t\xi, \xi](x, y)d\lambda \right) u(y)d\sigma(y) \right\|_{H^{s+m}(\Gamma)} \\
= & \left\| \int_0^1 (1-\lambda) \left(\int_\Gamma \frac{\partial^2 f}{\partial r^2}[r_0 + \lambda t\xi, \xi](x, y)u(y)d\sigma(y) \right) d\lambda \right\|_{H^{s+m}(\Gamma)} \\
\leq & \sup_{\lambda\in[0,1]} \left\| \left(\int_\Gamma \frac{\partial^2 f}{\partial r^2}[r_0 + \lambda t\xi, \xi](x, y)u(y)d\sigma(y) \right) \right\|_{H^{s+m}(\Gamma)} \\
\leq & C||u||_{H^s(\Gamma)}.
\end{aligned}$$

We then have

$$\begin{aligned}
& \frac{1}{t}\left(\int_\Gamma f(r_0 + t\xi, x, y)u(y)d\sigma(y) - \int_\Gamma f(r_0, x, y)u(y)d\sigma(y) \right) \\
= & \int_\Gamma \frac{\partial f}{\partial r}[r_0, \xi](x, y)u(y)d\sigma(y) + t\int_\Gamma \left(\int_0^1 (1-\lambda)\frac{\partial^2 f}{\partial r^2}[r_0 + \lambda t\xi, \xi](x, y)d\lambda \right) u(y)d\sigma(y).
\end{aligned}$$

We pass to the limit in $t = 0$ and we obtain the first Gâteaux derivative. For higher order derivative it suffice to write the proof with $\dfrac{\partial^k}{\partial r^k} f[r_0, \xi_1, \ldots, \xi_k]$ instead of f. The linearity, the symetry and the continuity of the first derivative is deduced from the kernel one. ■

Now we will consider some particular classes of pseudo-homogeneous kernels.

Corollary 4.5 *Assume that the kernels k_r are of the form*

$$k_r(y_r, x_r - y_r) = G(x_r - y_r)$$

where G is pseudo-homogeneous kernel which do not depend on r. Then the application

$$\begin{array}{ccc} B_\epsilon^\infty & \to & \mathscr{L}_c(H^s(\Gamma), H^{s+m}(\Gamma)) \\ r & \mapsto & \tau_r \mathcal{K}_{\Gamma_r} \tau_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable and the kernel of the first derivative at $r = 0$ is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$ by

$$\frac{\partial \{G(x + r(x) - y - r(y))\}}{\partial r}[0, \xi] = (\xi(x) - \xi(y)) \cdot \nabla^z G(x - y) + G(x - y) \operatorname{div}_\Gamma \xi(y).$$

Proof. For all fixed $(x, y) \in (\Gamma \times \Gamma)^*$, consider the application

$$f : U \mapsto f(r, x, y) = G(x + r(x) - y - r(y)) J_r(y) \in \mathbb{C}.$$

We have to prove that $r \mapsto f(r)$ is $\mathscr{C}^\infty$-Gâteaux differentiable and that each derivative define a pseudo-homogeneous kernel of class $-m$.

▷Step 1:

First of all we prove that for $(x, y) \in (\Gamma \times \Gamma)^*$ fixed the application $r \mapsto f(r, x, y)$ is infinitely Gâteaux differentiable on B_ϵ^∞. By lemma 4.1 the application $r \mapsto J_r(y)$ is infinitely Gâteaux differentiable on B_ϵ^∞, the application $r \mapsto x + r(x)$ is also infinitely Gâteaux differentiable on B_ϵ^∞ and the kernel G is of class $\mathscr{C}^\infty$ on $\mathbb{R}^3 \backslash \{0\}$. Being composed of applications infinitely Gâteaux differentiable, the application $r \mapsto f(r, x, y)$ is too and using Leibniz formula we have :

$$\frac{\partial^k}{\partial r^k} f[r_0, \xi_1, \ldots, \xi_k](x, y) =$$
$$\sum_{\alpha=0}^{k} \sum_{\sigma \in \mathcal{S}_k^+} \frac{\partial^\alpha}{\partial r^\alpha} \{G(x + r(x) - y - r(y))\} [r_0, \xi_{\sigma(1)}, \ldots, \xi_{\sigma(\alpha)}] \frac{\partial^{k-\alpha} J_r(y)}{\partial r^{k-\alpha}} [r_0, \xi_{\sigma(\alpha+1)}, \ldots, \xi_{\sigma(k)}]$$

where $\mathcal{S}_k^+$ denote the non decreasing permutations of $\{1, \ldots, k\}$ and

$$\frac{\partial^\alpha}{\partial r^\alpha} \{G(x_r - y_r)\} [r_0; \xi_1, \ldots, \xi_\alpha] = D_z^\alpha G[x + r_0(x) - y - r_0(y); \xi_1(x) - \xi_1(y), \ldots, \xi_\alpha(x) - \xi_\alpha(y)].$$

▷Step 2:

We then prove that each derivative define a new pseudo-homogeneous kernel of class $-m$ that is to say that for all $k \in \mathbb{N}$ and for all k-uple $(\xi_1, \ldots, \xi_k)$ the application

$$(x, y) \mapsto \frac{\partial^k}{\partial r^k} f[r_0, \xi_1, \ldots, \xi_k](x, y)$$

is pseudo-homogeneous of class $-m$. Since $\dfrac{\partial^{k-\alpha} J_r}{\partial r^{k-\alpha}}[r_0, \xi_1, \ldots, \xi_{k-\alpha}] \in \mathscr{C}^\infty(\Gamma, \mathbb{R})$ we have to prove that

$$(x, y) \mapsto \frac{\partial^\alpha}{\partial r^\alpha} \{G(x + r(x) - y - r(y))\} [r_0, \xi_1, \ldots, \xi_\alpha]$$

defines a pseudo-homogeneous kernel of class $-m$. By definition, $G(z)$ admit the following asymptotic expansion when z tends to zero:

$$G(z) = G_m(z) + \sum_{j=1}^{N-1} G_{m+j}(z) + G_{m+N}(h, z)$$

where G_{m+j} is homogeneous of class $-(m+j)$ for $j = 0, \ldots, N-1$ and G_{m+N} is of arbitrary regularity. Using Taylor formula we obtain the following result :

Proposition 4.6 *Let $G_m(z)$ be an homogeneous kernel of class $-m$ and any deformations $\xi = (\xi_1, \ldots, \xi_\alpha) \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$. The function*

$$(x, y) \mapsto D^\alpha G_m[x + r_0(x) - y - r_0(y); \xi_1(x) - \xi_1(y), \ldots, \xi_\alpha(x) - \xi_\alpha(y)]$$

is pseudo-homogeneous of class $-m$.

The application mapping $(\xi_1, \ldots, \xi_\alpha) \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$ to the integral operator of kernel

$$\frac{\partial^k \{G(x + r(x) - y - r(y))\}}{\partial r^m}[r_0, \xi_1, \ldots, \xi_k]$$

is clearly linear and continuous for all $r_0 \in B_\epsilon^\infty$.

■

Example 4.7 ***(Single layer kernel)*** *We note V_κ^r the integral operator defined for $u_r \in H^s(\Gamma_r)$ by*

$$V_\kappa^r u_r(x) = \int_{\Gamma_r} G(\kappa, |x_r - y_r|) u_r(y_r) d\sigma(y_r).$$

The application

$$\begin{array}{rcl} B_\delta & \to & \mathscr{L}_c(H^s(\Gamma), H^{s+1}(\Gamma)) \\ r & \mapsto & \tau_r V_\kappa^r \tau_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable and its first derivative at $r = 0$ in the direction $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$ is

$$\frac{\partial \tau_r V_\kappa^r \tau_r^{-1}}{\partial r}[0, \xi] u(x) = \int_\Gamma k'(y, x - y) u(y) d\sigma(y) \tag{4.3}$$

where in $\mathbb{R}^3$ we have

$$k'(x, y) = G(\kappa, |x - y|) \left(\frac{(\xi(x) - \xi(y)) \cdot (x - y)}{|x - y|} \left(i\kappa - \frac{1}{|x - y|} \right) + \operatorname{div}_\Gamma \xi(y) \right).$$

Example 4.8 ***(Double layer kernel)*** *We note D_κ^r the integral operator defined for $u_r \in H^s(\Gamma_r)$ by*

$$D_\kappa^r u_r(x) = \int_{\Gamma_r} \mathbf{n}_r(x_r) \cdot \nabla^z G(\kappa, |x_r - y_r|) u_r(y_r) d\sigma(y_r).$$

The application

$$\begin{array}{rcl} B_\delta & \to & \mathscr{L}_c(H^s(\Gamma), H^{s+1}(\Gamma)) \\ r & \mapsto & \tau_r D_\kappa^r \tau_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$ Gâteaux-differentiable .

PROOF. We have

$$\mathbf{n}_r(x_r)\cdot\nabla^z G(\kappa,|x_r-y_r|)u_r(y_r)=\mathbf{n}_r(x_r)\cdot(x_r-y_r)\frac{G(\kappa,|x_r-y_r|)}{|x_r-y_r|}\left(i\kappa-\frac{1}{|x_r-y_r|}\right).$$

We have to prove that

$$r\in B_\epsilon^\infty\mapsto\frac{(\tau_r\mathbf{n}_r)(x)\cdot(x+r(x)-y-r(y))}{|x+r(x)-y-r(y)|^3}$$

is $\mathscr{C}^\infty$ Gâteaux differentiable and that the derivatives are pseudo-homogeneous of class -1. To do so we use local coordinates as Potthast did in [29] and prove that

$$\frac{\partial^k\ (\tau_r\mathbf{n}_r)(x)\cdot(x+r(x)-y-r(y))}{\partial r^k}[r_0,\xi_1,\ldots,\xi_k]$$

behaves as $|x-y|^2$ when $x-y$ tends to zero. ■

Each domain Ω is a countable union of compact subset of Ω: $\Omega=\bigcup_{p\geq 1}K_p$. Instead of studying the application

$$r\in B_\epsilon^\infty\mapsto\mathcal{F}_\Omega(r)\in\mathscr{L}_c\left(H^s(\Gamma_r),H^{s+m}(\Omega_r)\right)$$

we consider the application

$$r\in B_\epsilon^\infty\mapsto\mathcal{F}_\Omega(r)\tau_r^{-1}\in\mathscr{L}_c\left(H^s(\Gamma),H^{s+m}(K_p)\right).$$

We use this approach for potential operators. We have for $u\in H^{s-\frac{1}{2}}(\Gamma)$

$$\mathcal{P}_r\tau_r^{-1}(u)(x)=\int_\Gamma k_r(y+r(y),x-y-r(y))u(y)J_r(y)d\sigma(y),\ x\in K_p.$$

Theorem 4.9 *Let $s\in\mathbb{R}$. Let $G(z)$ be a pseudo-homogeneous kernel of class $-(m+1)$ with $m\in\mathbb{N}$. Assume that for all $r\in B_\epsilon^\infty$, we have $k_r(y_r,x-y_r)=G(x-y_r)$. Then the application*

$$\begin{array}{ccc}B_\epsilon^\infty&\to&\mathscr{L}_c\left(H^{s-\frac{1}{2}}(\Gamma),\mathscr{C}^\infty(K_p)\right)\\ r&\mapsto&\mathcal{P}_r\tau_r^{-1}\end{array}$$

is infinitely Gâteaux differentiable and

$$\frac{\partial^k\mathcal{P}_r\tau_r^{-1}}{\partial r^k}[r_0,\xi_1,\ldots,\xi_k]u(x)=\int_\Gamma\frac{\partial^k}{\partial r^k}\left\{G(x-y-r(y))J_r(y)\right\}[r_0,\xi_1,\ldots,\xi_k]u(y)d\sigma(y).$$

Its first derivative at $r=0$ in the direction $\xi\in\mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$ is the integral operator denoted by $\mathcal{P}^1$ with the kernel

$$\frac{\partial}{\partial r}\left\{G(x-y-r(y))\right\}[r_0,\xi]=-\xi(y)\cdot\nabla^z G(x-y)+G(x-y)\,\mathrm{div}_\Gamma\,\xi(y).\tag{4.4}$$

The operator $\mathcal{P}^{(1)}$ can be extended in a linear and continuous integral operator from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^{s+m}(\Omega)$ and $H_{loc}^{s+m}(\Omega)$.

PROOF. The kernel and its higher order derivatives are of class $\mathscr{C}^\infty$ on K_p.
Since Ω is an increasing union of compact manifolds we can define a shape derivative on the whole domain Ω. Let us look at the first derivative : the term $G(x-y)\,\mathrm{div}_\Gamma\,\xi(y)$ has the same regularity than $G(x-y)$ when $x-y$ tends to zero wheareas $\xi(y)\cdot\nabla G(x-y)$ loose one order of regularity. As a consequence the kernel must be of class $-m+1$ in order that its first derivative acts from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^{s+m}(\Omega)$ and $H^{s+m}_{loc}(\Omega^c)$. ■

Remark 4.10 *We conclude that the boundary integral operators are smooth with respect to the domain whereas the potential operators loose one order of regularity at each derivation. We point out that we do not need more than the first derivative of the deformations ξ to compute the Gâteaux derivatives of these integral operators.*

Example 4.11 ***(Single layer potential)*** *We denote by ψ^r_κ the single layer potential defined for $u_r \in H^s(\Gamma_r)$ by*

$$\psi^r_\kappa u_r(x) = \int_{\Gamma_r} G(\kappa, |x-y_r|)u_r(y_r)d\sigma(y_r),\ x\in\mathbb{R}^3\backslash\Gamma_r.$$

The application

$$\begin{array}{ccc} B^\infty_\epsilon & \to & \mathscr{L}_c\left(H^s(\Gamma),\mathscr{C}^\infty(K_p)\right)\\ r & \mapsto & \tau_r\psi^r_\kappa\tau_r^{-1}\end{array}$$

is infinitely Gâteaux differentiable. Its first derivative at $r=0$ can be extended in a linear and continuous operator from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^s(\Omega)\cup H^s_{loc}(\Omega^c)$.

Since the potential operators are infinitely Gâteaux differentiable far from the boundary we have the following result by inverting the derivation with respect to r and the passage to the limit $|x|\to\infty$.

Example 4.12 *Let $s\in\mathbb{R}$. We denote by $\psi^{\infty,r}_\kappa$ the far field operator associated to the single layer potential defined for $u_r\in H^s(\Gamma_r)$ by*

$$\psi^{\infty,r}_\kappa u_r(\hat{x}) = \int_{\Gamma_r} e^{-i\kappa\hat{x}\cdot y_r}u_r(y_r)d\sigma(y_r),\ \hat{x}\in S^2.$$

The application

$$\begin{array}{ccc} B^\infty_\epsilon & \to & \mathscr{L}_c(H^s(\Gamma),\mathscr{C}^\infty(S^2))\\ r & \mapsto & \Psi^{\infty,r}_\kappa\tau_r^{-1}\end{array}$$

is infinitely Gâteaux differentiable and its first deriavtive at $r=0$ is defined for $u\in H^s(\Gamma)$ by:

$$\frac{\partial\Psi^{\infty,r}_\kappa\tau_r^{-1}}{\partial r}[0,\xi]u(\hat{x}) = \left(\int_\Gamma e^{-i\kappa\hat{x}\cdot y}\left(\mathrm{div}_\Gamma\,\xi(y) - i\kappa\hat{x}\cdot\xi(y)\right)u(y)d\sigma(y)\right).$$

5 Shape differentiability of the solution

Let $\mathbf{E}^{inc}$ be an incident electric field which is a fixed data. The aim of this section is to study the shape differentiation properties of the application $\mathbf{E}$ mapping the bounded scatterer Ω to the solution $\mathbf{E}(\Omega) = \left(\mathbf{E}^i(\Omega), \mathbf{E}^s(\Omega)\right) \in \mathbf{H}_{loc}(\mathbf{curl}, \mathbb{R}^3)$ to the dielectric scattering problem by the obstacle Ω lit by the incident field $\mathbf{E}^{inc}$ established in section 1. To do so we use the integral representation of the solution.

We set $\mathscr{E}^i(r) = \mathbf{E}^i(\Omega_r)$ and $\mathscr{E}^s(r) = \mathbf{E}^s(\Omega_r)$ and we denote $\Psi^r_{E_\kappa}$, $\Psi^r_{M_\kappa}$, C_0^{*r}, C^r_κ and M^r_κ the potential operators and the boundary integral operators on Γ_r and γ^r_D, $\gamma^r_{N_\kappa}$, $\gamma^{c,r}_D$ the $\gamma^{c,r}_{N_\kappa}$ trace mappings on Γ_r. We have :

$$\mathscr{E}^{tot}(r) = \mathbf{E}^{inc} + \mathscr{E}^s(r) \tag{5.1}$$

with

$$\mathscr{E}^s(r) = \left(-\Psi^r_{E_{\kappa e}} - i\eta\Psi^r_{M_{\kappa e}} C_0^{*r}\right)\mathbf{j}_r \qquad \text{dans } \Omega^c_r = \mathbb{R}^3\backslash\overline{\Omega_r} \tag{5.2}$$

where $\mathbf{j}_r$ solves the integral equation

$$\mathbf{S}^r\mathbf{j}_r = -\rho\left(-\frac{1}{2}\mathrm{I} + M^r_{\kappa_i}\right)\gamma^r_D\mathbf{E}^{inc} - C^r_{\kappa_i}\gamma^r_{N_{\kappa e}}\mathbf{E}^{inc},$$

and

$$\mathscr{E}^i(r) = -\frac{1}{\rho}\Psi^r_{E_{\kappa_i}}\gamma^{c,r}_{N_{\kappa e}}\mathscr{E}^{tot}(r) - \Psi^r_{M_{\kappa_i}}\gamma^{c,r}_D\mathscr{E}^{tot}(r) \qquad \text{dans } \Omega_r \tag{5.3}$$

Recall that the operator $\mathbf{S}^r$ is composed of the operators $C^r_{\kappa e}$, $M^r_{\kappa e}$, $C^r_{\kappa i}$ et $M^r_{\kappa i}$ and that these last ones are defined on the space $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$.

5.1 Variations of Helmholtz decomposition

We have to turn out many difficulties. On one hand, to be able to construct shape derivatives of the solution it is necessary to prove that the derivatives are defined on the same spaces than the boundary integral operators themselves, that is $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ (if we derive at $r = 0$). On the other hand, the very definition of the differentiability of operators defined on $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ raises non-trivial questions. The first one is : **How to derive applications defined on the variable space $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$?**
A first idea is to insert the identity $\tau_r^{-1}\tau_r = \mathrm{I}_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_r)}$ between each operator in the integral representation of the solution in order to consider integral operators on the fixed boundary Γ only and to study the differentiability of the applications

$$\begin{array}{rcl} r & \mapsto & \tau_r C^r_\kappa \tau_r^{-1}, \\ r & \mapsto & \tau_r M^r_\kappa \tau_r^{-1}, \\ r & \mapsto & \Psi^r_{E_\kappa}\tau_r^{-1}, \\ r & \mapsto & \Psi^r_{M_\kappa}\tau_r^{-1} \end{array}$$

but many difficulties persist as Potthast pointed out [28]. The electromagnetic boundary integral operators are defined and bounded on tangential functions to Γ_r.The restriction

of the operator $\tau_r M_\kappa^r \tau_r^{-1}$ to tangential densities to Γ_r, has the same regularity of the double layer potential operator. If we differentiate $\tau_r M_\kappa^r \tau_r^{-1}$, we will not obtain an operator with the same regularity than M_κ and acting on $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ since:

$$\tau_r(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)) \neq \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma).$$

The incident field $\mathbf{E}^{inc}$ is analytic in the neighborhood of Γ thus $\gamma_D^r \mathbf{E}^{inc} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ for all $r \in B_\epsilon^\infty$. Set $f(r) = \tau_r \left(\gamma_D^r \mathbf{E}^{inc}\right)$. For $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$, the Gâteaux semiderivative $\dfrac{\partial f(t\xi)}{\partial t}_{|t=0}$ is not tangent to Γ anymore it follows that $M_\kappa \dfrac{\partial f(t\xi)}{\partial t}_{|t=0}$ is not defined. We have the same difficulties we the Neuman trace $\gamma_{N_\kappa}^r$ and the other operators.

As an alternative, the idea of R. Potthast was to introduce projectors on the tangent planes of the surfaces Γ and Γ_r. Let us note $\pi(r)$ the orthogonal projection of any functions defined on Γ_r onto the tangent plane to Γ. This is a linear and continuous operator from the continous vector function space on Γ_r to the the space of continuous tangential function to Γ and for $\mathbf{u}_r \in (\mathscr{C}(\Gamma_r))^3$ we have

$$(\pi(r)\mathbf{u}_r)(x) = \mathbf{u}_r(x + r(x)) - (\mathbf{n}(x) \cdot \mathbf{u}_r(x + r(x)))\, \mathbf{n}(x).$$

Proposition 5.1 *The restriction of $\pi(r)$ to the continuous and tangential functions to Γ_r admit an inverse, denoted by $\pi^{-1}(r)$. The application $\pi^{-1}(r)$ is defined for a tangential function $\mathbf{u}$ to Γ by*

$$(\pi^{-1}(r)\mathbf{u})(x + r(x)) = \mathbf{u}(x) - \mathbf{n}(x)\frac{\mathbf{n}_r(x + r(x)) \cdot \mathbf{u}(x)}{\mathbf{n}_r(x + r(x)) \cdot \mathbf{n}(x)}.$$

And we have $\pi^{-1}(r)\mathbf{u} \in \mathbf{TH}^s(\Gamma_r)$ if and only if $\mathbf{u} \in \mathbf{TH}^s(\Gamma)$.

In the framework of the space of tangential continuous functions it suffices to insert the product $\pi^{-1}(r)\pi(r) = \mathrm{I}_{\mathbf{T}\mathscr{C}^0(\Gamma_r)}$ in the integral representation of the solution to lead us to study boundary integral operators defined on $\mathbf{T}\mathscr{C}^0(\Gamma)$ which do not depend on r anymore but here we would obtain operators defined on

$$\pi(r)\left(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)\right) = \left\{u \in \mathbf{TH}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_{\Gamma_r}(\pi^{-1}(r)\mathbf{u}) \in H^{-\frac{1}{2}}(\Gamma_r)\right\}.$$

This space depends again on the variable r and do not correspond to $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Our approach consist in using the Helmholtz decomposition of the spaces $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ for $r \in B_\epsilon^\infty$ and to introduce a new invertible operator $\mathbf{P}_r$ defined on $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ and which is not a projection operator.

We have the following decomposition. We refer to [11] for the proof.

Theorem 5.2 *The Hilbert space $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ admit the following Helmholtz decomposition:*

$$\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) = \nabla_\Gamma\left(H^{\frac{3}{2}}(\Gamma)/\mathbb{R}\right) \bigoplus \mathbf{curl}_\Gamma\left(H^{\frac{1}{2}}(\Gamma)/\mathbb{R}\right). \tag{5.4}$$

Since the real ϵ is chosen such that for all $r \in B_\epsilon^\infty$ the surfaces Γ_r are still regular and simply connected, then the spaces $\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ admit the same decomposition.
Let $\mathbf{j}_r \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)$ and let $\nabla_{\Gamma_r}\, p_r + \mathbf{curl}_{\Gamma_r}\, q_r$ its Helmholtz decomposition. Since $p_r \in H^{\frac{3}{2}}(\Gamma_r)$ and $q_r \in H^{\frac{1}{2}}(\Gamma_r)$, their change of variables from Γ_r to Γ, $\tau_r(p_r)$ and $\tau_r(q_r)$, are in $H^{\frac{3}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ respectively. The following operator :

$$\begin{array}{rccc} \mathbf{P}_r: & \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r) & \longrightarrow & \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}, \Gamma) \\ & \mathbf{j}_r = \nabla_{\Gamma_r}\, p_r + \mathbf{rot}_{\Gamma_r}\, q_r & \mapsto & \mathbf{j} = \nabla_\Gamma\, \tau_r p_r + \mathbf{rot}_\Gamma\, \tau_r q_r \end{array} \tag{5.5}$$

is well-defined.

The operator $\mathbf{P}_r$ transforms a tangential vector field $\mathbf{j}_r$ to Γ_r in a tangential vector field $\mathbf{j}$ to Γ. This operator is linear, continuous and admit an inverse $\mathbf{P}_r^{-1}$ given by :

$$\begin{array}{rccc} \mathbf{P}_r^{-1}: & \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma}, \Gamma) & \longrightarrow & \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r) \\ & \mathbf{j} = \nabla_\Gamma\, p + \mathbf{rot}_\Gamma\, q & \mapsto & \mathbf{j}_r = \nabla_{\Gamma_r}\, \tau_r^{-1}(p) + \mathbf{rot}_{\Gamma_r}\, \tau_r^{-1}(q). \end{array} \tag{5.6}$$

Obviously we have when $r = 0$ that $\mathbf{P}_r = \mathbf{P}_r^{-1} = \mathrm{I}_{\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)}$. We insert the identity $\mathrm{I}_{\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r}, \Gamma_r)} = \mathbf{P}_r^{-1}\mathbf{P}_r$ between each operator in the integral representation of the solution $(\mathscr{E}^i(r), \mathscr{E}^s(r))$. Finaly we have to study the Gâteaux differentiality properties of the following applications :

$$\begin{array}{rcll} B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \mathbf{H}(\mathbf{curl}, K_p)) & : \quad r \mapsto \Psi_{E_\kappa}^r \mathbf{P}_r^{-1} \\ B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \mathbf{H}(\mathbf{curl}, K_p)) & : \quad r \mapsto \Psi_{M_\kappa}^r \mathbf{P}_r^{-1} \\ B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{TH}^{s}(\mathrm{div}_\Gamma, \Gamma), \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)) & : \quad r \mapsto \mathbf{P}_r M_\kappa^r \mathbf{P}_r^{-1} \\ B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)) & : \quad r \mapsto \mathbf{P}_r C_\kappa^r \mathbf{P}_r^{-1} \end{array} \tag{5.7}$$

where K_p is a compact subset of $\mathbb{R}^3 \backslash \Gamma$.

Now let us look at the integral representation of these operators .

▷ **Integral representation of $\Psi_{E_\kappa}^r \mathbf{P}_r^{-1}$.**

The operator $\Psi_{E_\kappa}^r {\mathbf{P}_\mathbf{r}}^{-1}$ is defined for $\mathbf{j} = \nabla_\Gamma\, p + \mathbf{curl}_\Gamma\, q \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and $x \in K_p$ by:

$$\begin{aligned} \Psi_{E_\kappa}^r \mathbf{P}_r^{-1}\mathbf{j}(x) = \kappa \int_{\Gamma_r} & G(\kappa, |x - y_r|) \left(\nabla_{\Gamma_r} \tau_r^{-1} p\right)(y_r) d\sigma(y_r) \\ & + \kappa \int_{\Gamma_r} G(\kappa, |x - y_r|) \left(\mathbf{rot}_{\Gamma_r} \tau_r^{-1} q\right)(y_r) d\sigma(y_r) \\ & - \kappa^{-1} \nabla \int_{\Gamma_r} G(\kappa, |x - y_r|) \left(\Delta_{\Gamma_r} \tau_r^{-1} p\right)(y_r) d\sigma(y_r). \end{aligned}$$

▷ **Integral representation of $\Psi_{M_\kappa}^r \mathbf{P}_r^{-1}$.**

The operator $\Psi_{M_\kappa}^r {\mathbf{P}_\mathbf{r}}^{-1}$ is defined for $\mathbf{j} = \nabla_\Gamma\, p + \mathbf{curl}_\Gamma\, q \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and $x \in K_p$

by:

$$\Psi^r_{M_\kappa}\mathbf{P}_r^{-1}\mathbf{j}(x) = \mathbf{curl}\int_{\Gamma_r} G(\kappa,|x-y_r|)\left(\nabla_{\Gamma_r}\tau_r^{-1}p\right)(y_r)d\sigma(y_r) + \mathbf{curl}\int_{\Gamma_r} G(\kappa,|x-y_r|)\left(\mathbf{curl}_{\Gamma_r}\tau_r^{-1}q\right)(y_r)(y_r)d\sigma(y_r).$$

▷ **Integral representation of $\mathbf{P}_r C^r_\kappa \mathbf{P}_r^{-1}$.**
Recall that for $\mathbf{j}_r \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_{\Gamma_r},\Gamma_r)$, the operator C^r_κ is defined by

$$C^r_\kappa\mathbf{j}_r(x_r) = -\kappa\,\mathbf{n}_r(x_r)\times\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\mathbf{j}_r(y_r)d\sigma(y_r) - \kappa^{-1}\mathbf{n}_r(x_r)\times\nabla^{x_r}_{\Gamma_r}\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\,\mathrm{div}_{\Gamma_r}\mathbf{j}_r(y_r)d\sigma(y_r).$$

We want to write $\mathrm{C}^{\mathrm{r}}_\kappa\mathbf{j}_r$ of the form $\nabla_{\Gamma_r}P_r + \mathbf{curl}_{\Gamma_r}Q_r$. Using the formula (2.5)-(2.6) we deduce that :

$$\mathrm{div}_{\Gamma_r}\mathrm{C}^{\mathrm{r}}_\kappa\mathbf{j}_r = \Delta_{\Gamma_r}P_r \ \text{ et } \ \mathrm{curl}_{\Gamma_r}\mathrm{C}^{\mathrm{r}}_\kappa\mathbf{j}_r = -\Delta_{\Gamma_r}Q_r.$$

As a consequence we have for $x_r\in\Gamma_r$:

$$P_r(x_r) = -\kappa\,\Delta^{-1}_{\Gamma_r}\mathrm{div}_{\Gamma_r}\left(\mathbf{n}_r(x_r)\times\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\mathbf{j}_r(y_r)d\sigma(y_r)\right) \tag{5.8}$$

and

$$\begin{aligned} Q_r(x_r) &= -\kappa\,(-\Delta^{-1}_{\Gamma_r})\,\mathrm{curl}_{\Gamma_r}\left(\mathbf{n}_r(x_r)\times\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\mathbf{j}_r(y_r)d\sigma(y_r)\right)\\ &\quad -\kappa^{-1}(-\Delta_{\Gamma_r})\,\mathrm{curl}_{\Gamma_r}(-\mathbf{curl}_{\Gamma_r})\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\,\mathrm{div}_{\Gamma_r}\mathbf{j}_r(y_r)d\sigma(y_r),\\ &= \kappa\,\Delta^{-1}_{\Gamma_r}\,\mathrm{curl}_{\Gamma_r}\left(\mathbf{n}_r(x_r)\times\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\mathbf{j}_r(y_r)d\sigma(y_r)\right)\\ &\quad +\kappa^{-1}\int_{\Gamma_r} G(\kappa,|x_r-y_r|)\,\mathrm{div}_{\Gamma_r}\mathbf{j}_r(y_r)d\sigma(y_r). \end{aligned}$$

The operator $\mathbf{P}_r C^r_\kappa\mathbf{P}_r^{-1}$ is defined for $\mathbf{j}=\nabla_\Gamma\,p+\mathbf{curl}_\Gamma\,q\in\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$ by:

$$\mathbf{P}_r C^r_\kappa\mathbf{P}_r^{-1} = \nabla_\Gamma P(r) + \mathbf{curl}_\Gamma\,Q(r),$$

with

$$P(r)(x) = -\kappa\,\left(\tau_r\Delta^{-1}_{\Gamma_r}\mathrm{div}_{\Gamma_r}\,\tau_r^{-1}\right)\left((\tau_r\mathbf{n}_r)(x)\times\tau_r\left\{\int_{\Gamma_r} G(\kappa,|\cdot-y_r|)(\nabla_{\Gamma_r}\tau_r^{-1}p)(y_r)d\sigma(y_r) + \int_{\Gamma_r} G(\kappa,|\cdot-y_r|)(\mathbf{rot}_{\Gamma_r}\tau_r^{-1}q)(y_r)d\sigma(y_r)\right\}(x)\right)$$

and

$$
\begin{aligned}
Q(r)(x) = \quad & \kappa \ \left(\tau_r \Delta_{\Gamma_r}^{-1} \operatorname{curl}_{\Gamma_r} \tau_r^{-1}\right) \Bigg((\tau_r \mathbf{n}_r)(x) \times \tau_r \Bigg\{ \int_{\Gamma_r} G(\kappa, |\cdot - y_r|)(\nabla_{\Gamma_r} \tau_r^{-1} p)(y_r) d\sigma(y_r) \\
& \qquad + \int_{\Gamma_r} G(\kappa, |\cdot - y_r|)(\mathbf{rot}_{\Gamma_r} \tau_r^{-1} q)(y_r) d\sigma(y_r) \Bigg\}(x) \Bigg) \\
& + \kappa^{-1} \tau_r \left(\int_{\Gamma_r} G(\kappa, |\cdot - y_r|)(\Delta_{\Gamma_r} \tau_r^{-1} p)(y_r) d\sigma(y_r) \right)(x).
\end{aligned}
$$

▷ **Integral representation of $\mathbf{P}_r M_\kappa^r \mathbf{P}_r^{-1}$.**
Recall that for all $\mathbf{j}_r \in \mathbf{TH}^{-\frac{1}{2}}(\operatorname{div}_{\Gamma_r}, \Gamma_r)$, the operator M_κ^r is defined by

$$
\begin{aligned}
M_\kappa^r \mathbf{j}_r(x_r) = \quad & \int_{\Gamma_r} \left((\nabla^{x_r} G(\kappa, |x_r - y_r|)) \cdot \mathbf{n}_r(x_r) \right) \mathbf{j}_r(y_r) d\sigma(y_r) \\
& - \int_{\Gamma_r} \nabla^{x_r} G(\kappa, |x_r - y_r|) \left(\mathbf{n}_r(x_r) \cdot \mathbf{j}_r(y_r) \right) d\sigma(y_r).
\end{aligned}
$$

Using the equalities (2.6) and the identity $\mathbf{curl}\,\mathbf{curl} = -\Delta + \nabla \operatorname{div}$, we have

$$
\begin{aligned}
\operatorname{div}_{\Gamma_r} M_\kappa^r \mathbf{j}_r(x_r) = \quad & \mathbf{n}_r(x_r) \cdot \int_{\Gamma_r} \operatorname{curl} \operatorname{curl}^{x_r} \left(G(\kappa, |x_r - y_r|) \mathbf{j}_r(y_r) \right) d\sigma(y_r) \\
= \quad & \kappa^2 \mathbf{n}_r(x_r) \cdot \int_{\Gamma_r} \left(G(\kappa, |x_r - y_r|) \mathbf{j}_r(y_r) \right) d\sigma(y_r) \\
& + \frac{\partial}{\partial \mathbf{n}_r} \int_{\Gamma_r} \left(G(\kappa, |x_r - y_r|) \operatorname{div}_{\Gamma_r} \mathbf{j}_r(y_r) \right) d\sigma(y_r)
\end{aligned}
$$

Proceeding by the same way than with the operator $\mathbf{P}_r C_\kappa^r \mathbf{P}_r^{-1}$, we obtain that the operator $\mathbf{P}_r M_\kappa^r \mathbf{P}_r^{-1}$ is defined for $\mathbf{j} = \nabla_\Gamma \, p + \mathbf{curl}_\Gamma \, q \in \mathbf{TH}^{-\frac{1}{2}}(\operatorname{div}_\Gamma, \Gamma)$ by:

$$
\mathbf{P}_r M_\kappa^r \mathbf{P}_r^{-1} \mathbf{j} = \nabla_\Gamma P'(r) + \mathbf{rot}_\Gamma Q'(r),
$$

with

$$
\begin{aligned}
P'(r)(x) = \quad & \left(\tau_r \Delta_{\Gamma_r}^{-1} \tau_r^{-1}\right) \tau_r \Bigg\{ \kappa^2 \int_{\Gamma_r} \mathbf{n}_r(\,\cdot\,) \cdot \left\{ G(\kappa, |\cdot - y_r|) \, \mathbf{curl}_{\Gamma_r} \, \tau_r^{-1} q(y_r) \right\} d\sigma(y_r) \\
& \qquad + \kappa^2 \int_{\Gamma_r} \mathbf{n}_r(\,\cdot\,) \cdot \left\{ G(\kappa, |\cdot - y_r|) \nabla_{\Gamma_r} \tau_r^{-1} p(y_r) \right\} d\sigma(y_r) \\
& \qquad + \int_{\Gamma_r} \frac{\partial}{\partial \mathbf{n}_r(\,\cdot\,)} G(\kappa, |\cdot - y_r|) (\Delta_{\Gamma_r} \tau_r^{-1} p)(y_r) d\sigma(y_r) \Bigg\}(x),
\end{aligned}
$$

and

$$
\begin{aligned}
Q_r'(x) = \quad & \left(\tau_r \Delta_{\Gamma_r}^{-1} \operatorname{curl}_{\Gamma_r} \tau_r^{-1}\right) \tau_r \left\{ \int_{\Gamma_r} \left(\nabla G(\kappa, |\cdot - y_r|) \cdot \mathbf{n}_r(\cdot)\right) (\mathbf{rot}_{\Gamma_r} \tau_r^{-1} q)(y_r) d\sigma(y_r) \right. \\
& + \int_{\Gamma_r} \left(\left(\nabla G(\kappa, |\cdot - y_r|) \cdot \mathbf{n}_r(\cdot)\right) (\nabla_{\Gamma_r} \tau_r^{-1} p)(y_r) d\sigma(y_r) \right. \\
& - \int_{\Gamma_r} \nabla G(\kappa, |\cdot - y_r|) \left(\mathbf{n}_r(\cdot) \cdot (\mathbf{rot}_{\Gamma_r} \tau_r^{-1} q)(y_r)\right) d\sigma(y_r) \\
& \left. - \int_{\Gamma_r} \nabla G(\kappa, |\cdot - y_r|) \left(\mathbf{n}_r(\cdot) \cdot (\nabla_{\Gamma_r} \tau_r^{-1} p)(y_r)\right) d\sigma(y_r) \right\} (x).
\end{aligned}
$$

These operators are composed of boundary integral operators with weakly singular and pseudo-homogeneous kernels of class -1 and of the surface differential operators defined in section 2. By a change of variables in the integral, we then have to study the differentiability properties of the applications

$$
\begin{array}{rcl}
r & \mapsto & \tau_r \nabla_{\Gamma_r} \tau_r^{-1} \\
r & \mapsto & \tau_r \, \mathbf{curl}_{\Gamma_r} \, \tau_r^{-1} \\
r & \mapsto & \tau_r \operatorname{div}_{\Gamma_r} \tau_r^{-1} \\
r & \mapsto & \tau_r \operatorname{curl}_{\Gamma_r} \tau_r^{-1} \\
r & \mapsto & \tau_r \Delta_{\Gamma_r} \tau_r^{-1}
\end{array}
$$

5.2 Gâteaux differentiability of the surface differential operators

Lemma 5.3 *The application*

$$
\begin{array}{rccc}
G: & B_\epsilon^\infty & \to & \mathscr{L}_c(H^{s+1}(\Gamma), \mathbf{H}^s(\Gamma)) \\
& r & \mapsto & \tau_r \nabla_{\Gamma_r} \tau_r^{-1}
\end{array}
$$

is $\mathscr{C}^\infty$-Gâteaux differentiable and its first derivative at r_0 is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$ by

$$
\frac{\partial G}{\partial r}[r_0, \xi] u = -[G(r_0)\xi] G(r_0) u + (G(r_0) u \cdot [G(r_0)\xi] N(r_0)) \, N(r_0).
$$

Remark 5.4 *Note that we can write $\dfrac{\partial N}{\partial r}[r_0, \xi] = -[G(r_0)\xi] N(r_0)$. Since the first derivative of N and G can be expressed in function of N and G we obtain the Gâteaux derivative of all order iteratively.*

Proof. In accordance to the definition (2.1) and the lemma 4.2, to prove the $\mathscr{C}^\infty$-Gâteaux differentiability of G we have to prove the $\mathscr{C}^\infty$- Gâteaux differentiability of the application

$$
f : r \in B_\epsilon^\infty \mapsto \left\{ u \mapsto \tau_r \left(\widetilde{\nabla \tau_r^{-1} u} \right)_{|\Gamma_r} \right\} \in \mathscr{L}_c(H^{s+1}(\Gamma), \mathbf{H}^s(\Gamma)).
$$

Let $x \in \Gamma$, we have

$$\tau_r\left(\widetilde{\nabla\tau_r^{-1}u}\right)_{|\Gamma_r}(x) = \nabla\left(\widetilde{u}\circ(\mathrm{I}+r)^{-1}\right)_{|\Gamma_r}(x+r(x)) = {}^T(\mathrm{I}+\mathsf{D}r)^{-1}_{|\Gamma_r}(x+r(x))\circ\nabla\widetilde{u}_{|\Gamma}(x),$$

and

$$(\mathrm{I}+\mathsf{D}r)^{-1}_{|\Gamma_r}(x+r(x)) = \left[(\mathrm{I}+\mathsf{D}r)_{|\Gamma}(x)\right]^{-1}.$$

The application $g : r \in B^\infty_\epsilon \mapsto (\mathrm{I}+\mathsf{D}r)_{|\Gamma} \in \mathscr{C}^\infty(\Gamma)$ is continuous, and $\mathscr{C}^\infty$-Gâteaux differentiable. Its first derivative is $\dfrac{\partial}{\partial r}g[0,\xi] = [\mathsf{D}\xi]_{|\Gamma}$ and its higher order derivatives vanish. One can show that the application $h : r \in B^\infty_\epsilon \mapsto \left\{x \mapsto [g(r)(x)]^{-1}\right\} \in \mathscr{C}^\infty(\Gamma)$ is also $\mathscr{C}^\infty$ Gâteaux-differentiable and that we have at r_0 and in the direction ξ:

$$\frac{\partial h}{\partial r}[r_0,\xi] = -h(r_0)\circ\frac{\partial g}{\partial r}[r_0,\xi]\circ h(r_0) = -h(r_0)\circ[\mathsf{D}\xi]_{|\Gamma}\circ h(r_0).$$

and

$$\frac{\partial^n h}{\partial r^n}[r_0,\xi_1,\ldots,\xi_n] = (-1)^n\sum_{\sigma\in\mathscr{S}_n}(\mathrm{I}+\mathsf{D}r_0)^{-1}\circ[\tau_{r_0}\mathsf{D}\tau_{r_0}^{-1}\xi_{\sigma(1)}]\circ\ldots\circ[\tau_{r_0}\mathsf{D}\tau_{r_0}^{-1}\xi_{\sigma(n)}]$$

where $\mathscr{S}_n$ is the permutation groupe of $\{1,\ldots,n\}$. Finally we obtain the $\mathscr{C}^\infty-$ Gâteaux differentiability of f and we have

$$\frac{\partial f}{\partial r}[r_0,\xi]u = -[f(r_0)\xi]f(r_0)u.$$

To obtain the expression of the first derivative of G we have to derive the following expression:

$$\begin{aligned} G(r)u &= (\tau_r\nabla_{\Gamma_r}\tau_r^{-1}u) = \tau_r\nabla\left(\widetilde{\tau_r^{-1}u}\right) - \left(\tau_r\mathbf{n}_r\cdot\left(\tau_r\nabla\left(\widetilde{\tau_r^{-1}u}\right)\right)\right)\tau_r\mathbf{n}_r \\ &= f(r_0)u - (f(r_0)u\cdot N(r_0))\,N(r_0). \end{aligned}$$

By lemma 4.2 and the chain and product rules we have

$$\begin{aligned} \frac{\partial G}{\partial r}[r_0,\xi] = \; & -[f(r_0)\xi]f(r_0)u + ([f(r_0)\xi]f(r_0)u\cdot N(r_0))\,N(r_0) \\ & + (f(r_0)u\cdot[G(r_0)\xi]N(r_0))\,N(r_0) + (f(r_0)u\cdot N(r_0))\,[G(r_0)\xi]N(r_0) \end{aligned}$$

We had the first two terms in the right handside, it gives :

$$\begin{aligned} \frac{\partial G}{\partial r}[r_0,\xi] = \; & -[G(r_0)\xi]f(r_0)u + (f(r_0)u\cdot N(r_0))\,[G(r_0)\xi]N(r_0) \\ & + (f(r_0)u\cdot[G(r_0)\xi]N(r_0))\,N(r_0) \\ = \; & -[G(r_0)\xi]G(r_0)u + (f(r_0)u\cdot[G(r_0)\xi]N(r_0))\,N(r_0). \end{aligned}$$

To conclude it suffice to note that $(f(r_0)u\cdot[G(r_0)\xi]N(r_0)) = (G(r_0)u\cdot[G(r_0)\xi]N(r_0))$.
■

Lemma 5.5 *The application*

$$\begin{array}{rccc} D: & B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{H}^{s+1}(\Gamma,\mathbb{R}^3), H^s(\Gamma)) \\ & r & \mapsto & \tau_r \operatorname{div}_{\Gamma_r} \tau_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable and its first derivative at r_0 is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$ by

$$\frac{\partial D}{\partial r}[r_0,\xi]\mathbf{u} = -\operatorname{Trace}([G(r_0)\xi][G(r_0)\mathbf{u}]) + ([G(r_0)\mathbf{u}]N(r_0)\cdot[G(r_0)\xi]N(r_0))\,.$$

PROOF. For $\mathbf{u} \in \mathbf{H}^{s+1}(\Gamma,\mathbb{R}^n)$ we have $D(r)\mathbf{u} = \operatorname{Trace}(G(r)\mathbf{u})$. Then we use the differentiation rules.

Remark 5.6 *Since the first derivative of D is composed of G and N and the first derivative of J is composed of J and D, we can obtain an expression of higher order derivatives of the jacobian iteratively.*

■

Lemma 5.7 *The application*

$$\begin{array}{rccc} \mathbf{R}: & B_\epsilon^\infty & \to & \mathscr{L}_c(H^{s+1}(\Gamma), \mathbf{H}^s(\Gamma)) \\ & r & \mapsto & \tau_r \mathbf{curl}_{\Gamma_r} \tau_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable and its first derivative at r_0 is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$ by

$$\frac{\partial \mathbf{R}}{\partial r}[r_0,\xi]u = {}^T[G(r_0)\xi]\mathbf{R}(r_0)u - D(r_0)\xi\cdot\mathbf{R}(r_0)u.$$

PROOF. Let $u \in H^{s+1}(\Gamma)$. By definition, we have $\mathbf{R}(r_0)u = G(r_0)u \times N(r_0)$. By lemmas 4.2 and 5.3 this application is $\mathscr{C}^\infty$ Gâteaux differentiable. We have in r_0 and in the direction $\xi \in \mathscr{C}^\infty(\Gamma,\mathbb{R}^3)$

$$\frac{\partial \mathbf{R}}{\partial r}[r_0,\xi]u = -{}^T[G(r_0)\xi]G(r_0)u \times N(r_0) - G(r_0)u \times [G(r_0)\xi]N(r_0).$$

NB: recall that given a (3×3) matrix A and vectors b and c we have

$$Ab\times c + b\times Ac = \operatorname{Trace}(A)(b\times c) - {}^T A(b\times c).$$

We deduce the expression of the first derivatives with $A = -[G(r_0)\xi]$, $b = G(r_0)u$ et $c = N(r_0)$. ■

Lemma 5.8 *The application*

$$\begin{array}{rccc} R: & B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{H}^{s+1}(\Gamma), H^s(\Gamma)) \\ & r & \mapsto & \tau_r \operatorname{curl}_{\Gamma_r} \tau_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable and its first derivative at r_0 is defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3,\mathbb{R}^3)$ by

$$\frac{\partial R}{\partial r}[r_0,\xi]\mathbf{u} = -\sum_{i=1}^{3}\left(G(r_0)\xi^i \cdot \mathbf{R}(r_0)u_i\right) - D(r_0)\xi \cdot R(r_0)\mathbf{u}$$

where $\mathbf{u} = (u_1,u_2,u_3)$ and $\xi = (\xi^1,\xi^2,\xi^3)$.

Proof. Let $\mathbf{u} \in \mathbf{H}^{s+1}(\Gamma,\mathbb{R}^3)$. By definition of the surface rotational we have

$$R(r_0)\mathbf{u} = -\operatorname{Trace}(\mathbf{R}(r_0)\mathbf{u}).$$

We deduce the $\mathscr{C}^\infty$ differentiability of R and the first derivative in r_0 in the direction ξ is

$$\begin{aligned}\frac{\partial R}{\partial r}[r_0,\xi]\mathbf{u} &= -\operatorname{Trace}\left(\frac{\partial \mathbf{R}}{\partial r}[r_0,\xi]\mathbf{u}\right)\\ &= -\operatorname{Trace}\left({}^T[G(r_0)\xi][\mathbf{R}(r_0)\mathbf{u}]\right) - D(r_0)\xi \cdot \operatorname{Trace}\left(-\mathbf{R}(r_0)\mathbf{u}\right])\\ &= -\sum_{i=1}^{3}\left(G(r_0)\xi_i \cdot \mathbf{R}(r_0)\mathbf{u}_i\right) - D(r_0)\xi \cdot R(r_0)\mathbf{u}.\end{aligned}$$

■

Here again we can obtain higher order derivatives of these operators iteratively.

Remark 5.9 *One can see that we do not need more than the first derivative of the deformations ξ. Thus these results hold true for boundaries and deformations of class $\mathscr{C}^{k+1}$, $k \in \mathbb{N}^*$ with differential operators considered in $\mathscr{L}_c(\mathscr{C}^{k+1}(\Gamma),\mathscr{C}^k(\Gamma))$.*

When $\xi = \mathbf{n}$ we obtain the commutators

$$\begin{aligned}\frac{\partial}{\partial \mathbf{n}}(\nabla_\Gamma u) - \nabla_\Gamma\left(\frac{\partial}{\partial \mathbf{n}}u\right) &= -\mathcal{R}_\Gamma \nabla_\Gamma u\\ \frac{\partial}{\partial \mathbf{n}}(\mathbf{curl}_\Gamma\, u) - \mathbf{curl}_\Gamma\left(\frac{\partial}{\partial \mathbf{n}}u\right) &= \mathcal{R}_\Gamma\, \mathbf{curl}_\Gamma\, u - \mathcal{H}_\Gamma\, \mathbf{curl}_\Gamma\, u\\ \frac{\partial}{\partial \mathbf{n}}(\operatorname{div}_\Gamma \mathbf{u}) - \operatorname{div}_\Gamma\left(\frac{\partial}{\partial \mathbf{n}}\mathbf{u}\right) &= -\operatorname{Trace}(\mathcal{R}_\Gamma[\nabla_\Gamma \mathbf{u}])\\ \frac{\partial}{\partial \mathbf{n}}(\operatorname{curl}_\Gamma \mathbf{u}) - \operatorname{curl}_\Gamma\left(\frac{\partial}{\partial \mathbf{n}}\mathbf{u}\right) &= -\operatorname{Trace}(\mathcal{R}_\Gamma[\mathbf{curl}_\Gamma \mathbf{u}]) - \mathcal{H}_\Gamma \operatorname{curl}_\Gamma \mathbf{u}\end{aligned} \tag{5.9}$$

where $R_\Gamma = [\nabla_\Gamma \mathbf{n}]$ and $\mathcal{H}_\Gamma = \operatorname{div}_\Gamma \mathbf{n}$.

From the precedent results we have:

Lemma 5.10 *The application*

$$\begin{array}{rccc} L: & B_\epsilon^\infty & \to & \mathscr{L}_c(H^{s+2}(\Gamma),H^s(\Gamma))\\ & r & \mapsto & \tau_r \Delta_{\Gamma_r} \tau_r^{-1}\end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable.

PROOF. It suffice to write:

$$\tau_r \Delta_{\Gamma_r} \tau_r^{-1} = (\tau_r \operatorname{div}_{\Gamma_r} \tau_r^{-1})(\tau_r \nabla_{\Gamma_r} \tau_r^{-1}) = -(\tau_r \operatorname{curl}_{\Gamma_r} \tau_r^{-1})(\tau_r \mathbf{curl}_{\Gamma_r} \tau_r^{-1}).$$

The operators $\tau_r \Delta_{\Gamma_r} \tau_r^{-1}$ is composed of operators infinitely Gâteaux differentiable. ■

View the integral representations of the operators $\mathbf{P}_r C_\kappa^r \mathbf{P}_r^{-1}$ and $\mathbf{P}_r M_\kappa^r \mathbf{P}_r^{-1}$ we have to study the Gâteaux differentiability of the applications $r \mapsto \tau_r \Delta_{\Gamma_r}^{-1} \operatorname{curl}_{\Gamma_r} \tau_r^{-1}$ and $r \mapsto \tau_r \Delta_{\Gamma_r}^{-1} \operatorname{curl}_{\Gamma_r} \tau_r^{-1}$. We have seen that for $r \in B_\epsilon^\infty$ the operator $\operatorname{curl}_{\Gamma_r}$ is linear and continuous from $\mathbf{H}^{s+1}(\Gamma_r)$ to $H_*^s(\Gamma_r)$, that the operator $\operatorname{div}_{\Gamma_r}$ is linear and continuous from $\mathbf{TH}^{s+1}(\Gamma_r)$ in $H_*^s(\Gamma_r)$ and that $\Delta_{\Gamma_r}^{-1}$ is defined from $H_*^s(\Gamma_r)$ in $H^{s+2}(\Gamma_r)/\mathbb{R}$. To use the chain rules, it is important to construct derivatives in $r = 0$ between the spaces $\mathbf{H}^{s+1}(\Gamma)$ and $H_*^s(\Gamma)$ for the scalar curl operator, between the spaces $\mathbf{TH}^{s+1}(\Gamma)$ and $H_*^s(\Gamma)$ for the divergence operator and between the spaces $H_*^s(\Gamma)$ and $H^{s+2}(\Gamma)/\mathbb{R}$ for the Laplace-Beltrami operator. As an alternative we use the :

Proposition 5.11 *Let u be a scalar function defined on Γ_r. Then $u_r \in H_*^s(\Gamma_r)$ if and only if $J_r \tau_r u_r = J_r u_r \circ (\mathrm{I} + r) \in H_*^s(\Gamma)$.*

As a consequence the applications $r \mapsto J_r \tau_r \operatorname{curl}_{\Gamma_r} \tau_r^{-1}$ and $r \mapsto J_r \tau_r \operatorname{div}_{\Gamma_r} \pi^{-1}(r)$ are well-defined from $\mathbf{H}^{s+1}(\Gamma)$ and $\mathbf{TH}^{s+1}(\Gamma)$ respectively to $H_*^s(\Gamma)$.

Lemma 5.12 *The applications*

$$\begin{array}{ccc} B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{H}^{s+1}(\Gamma), \mathbf{H}_*^s(\Gamma)) \\ r & \mapsto & J_r \tau_r \operatorname{curl}_{\Gamma_r} \tau_r^{-1} \end{array} \quad \textit{and} \quad \begin{array}{ccc} B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{TH}^{s+1}(\Gamma), \mathbf{H}_*^s(\Gamma)) \\ r & \mapsto & J_r \tau_r \operatorname{div}_{\Gamma_r} \pi^{-1}(r) \end{array}$$

are $\mathscr{C}^\infty$-Gâteaux differentiable end their first derivatives at $r = 0$ defined for $\xi \in \mathscr{C}_b^\infty(\mathbb{R}^3, \mathbb{R}^3)$ by

$$\frac{\partial J_r \tau_r \operatorname{curl}_{\Gamma_r} \tau_r^{-1}}{\partial r}[0, \xi]\mathbf{u} = -\sum_{i=1}^{3} \nabla_\Gamma \xi_i \cdot \mathbf{curl}_\Gamma \mathbf{u}_i.$$

and

$$\begin{aligned} \frac{\partial J_r \tau_r \operatorname{div}_{\Gamma_r} \pi^{-1}(r)}{\partial r}[0, \xi]\mathbf{u} = \; & -\operatorname{Trace}([\nabla_\Gamma \xi]\nabla_\Gamma \mathbf{u}) + \operatorname{div}_\Gamma \xi \operatorname{div}_\Gamma \mathbf{u} + ([\nabla_\Gamma \mathbf{u}]\mathbf{n} \cdot [\nabla_\Gamma \xi]\mathbf{n}) \\ & + (\mathbf{u} \cdot [\nabla_\Gamma \xi]\mathbf{n})\, \mathcal{H}_\Gamma. \end{aligned}$$

PROOF. Let $\mathbf{u} \in \mathbf{TH}^{s+1}(\Gamma)$ We have

$$\frac{\partial \tau_r \pi^{-1}(r)}{\partial r}[0, \xi]\mathbf{u} = (\mathbf{u} \cdot [\nabla_\Gamma \xi]\mathbf{n})\, \mathbf{n}.$$

Next we use the lemma 4.1, 5.5 and 5.8. For $u \in \mathbf{H}^s(\Gamma)$, it is clear that $\sum_{i=1}^3 \nabla_\Gamma \xi_i \cdot \mathbf{curl}_\Gamma \mathbf{u}_i$ is of vanishing mean value since the space $\nabla_\Gamma H^s(\Gamma)$ is orthogonal to $\mathbf{curl}_\Gamma H^s(\Gamma)$ for the

$\mathbf{L}^2$ scalar product. An other argument whithout using the explicit form of the derivatives is : for all $\mathbf{u} \in \mathbf{H}^{s+1}(\Gamma)$, we derive the application

$$r \mapsto \int_\Gamma J_r \tau_r \, \mathrm{curl}_{\Gamma_r} \, \tau_r^{-1} \mathbf{u} \, d\sigma \equiv 0.$$

and for $\mathbf{u} \in \mathbf{TH}^{s+1}(\Gamma)$ we derive the application

$$r \mapsto \int_\Gamma J_r \tau_r \, \mathrm{div}_{\Gamma_r} \, \pi^{-1}(r) \mathbf{u} \, d\sigma \equiv 0.$$

■

Let us note that $u_r \in H^s(\Gamma_r)/\mathbb{R}$ if and only if $\tau_r u_r \in H^s(\Gamma)/\mathbb{R}$.

Lemma 5.13 *The application*

$$\begin{array}{ccc} B_\epsilon^\infty & \to & \mathscr{L}_c(H_*^{s+2}(\Gamma), H^s(\Gamma)/\mathbb{R}) \\ r & \mapsto & \tau_r \Delta_{\Gamma_r}^{-1} \tau_r^{-1} J_r^{-1} \end{array}$$

is $\mathscr{C}^\infty$-Gâteaux differentiable.

Proof. We have seen in section 2, that the Laplace-Beltrami operator is invertible from $H^{s+2}(\Gamma_r)/\mathbb{R}$ to $H_*^s(\Gamma_r)$. As a consequence $J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1}$ is invertible from $H^{s+2}(\Gamma)/\mathbb{R}$ to $H_*^s(\Gamma)$. By lemma 3.5 we deduce that $r \mapsto \tau_r \Delta_{\Gamma_r}^{-1} \tau_r^{-1} J_r^{-1}$ is $\mathscr{C}^\infty$-Gâteaux differentiable and that we have

$$\frac{\partial \, \tau_r \Delta_{\Gamma_r}^{-1} \tau_r^{-1} J_r^{-1}}{\partial r}[0,\xi] = \, -\Delta_\Gamma^{-1} \circ \left(\frac{\partial \, J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1}}{\partial r}[0,\xi] \right) \circ \Delta_\Gamma^{-1}.$$

■

Now we have all the tools to establish the differentiability properties of the electromagnetic boundary integral operators and then of the solution to the dielectric scattering problem.

5.3 Shape derivatives of the solution to the dielectric problem

For more simplicity in the writing we use the following notations :

$$\Psi_{E_\kappa}(r) = \Psi_{E_\kappa}^r \mathbf{P}_r^{-1}, \; \Psi_{M_\kappa}(r) = \Psi_{M_\kappa}^r \mathbf{P}_r^{-1}, \; C_\kappa(r) = \mathbf{P}_r C_\kappa^r \mathbf{P}_r^{-1}, \text{ et } M_\kappa(r) = \mathbf{P}_r M_\kappa^r \mathbf{P}_r^{-1}.$$

Theorem 5.14 *The applications*

$$\begin{array}{ccc} B_\epsilon^\infty & \to & \mathscr{L}_c(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma), \mathbf{H}(\mathbf{curl}, K_p)) \\ r & \mapsto & \Psi_{E_\kappa}(r) \\ r & \mapsto & \Psi_{M_\kappa}(r) \end{array}$$

are infinitely Gâteaux differentiable. Moreover, their first derivative at $r = 0$ can be extended in linear an bounded operators from $\mathbf{TH}^{\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ to $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}_{loc}(\mathbf{curl}, \Omega^c)$ and given $\mathbf{j} \in \mathbf{TH}^{\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ the potentials $\frac{\partial \Psi_{E_\kappa}}{\partial r}[0, \xi]\mathbf{j}$ et $\frac{\partial \Psi_{M_\kappa}}{\partial r}[0, \xi]\mathbf{j}$ satisfy the Maxwell's equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2 \mathbf{u} = 0$$

in Ω and Ω^c and the Silver-Müller condition.

Proof. Let $\mathbf{j} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ and $\nabla_\Gamma p + \mathbf{curl}_\Gamma\, q$ its Helmholtz decomposition. Recall that $\Psi_{E_\kappa}(r)\mathbf{j}$ and $\Psi_{M_\kappa}(r)\mathbf{j}$ can be written:

$$\begin{aligned}\Psi_{E_\kappa}(r)\mathbf{j} =& \kappa \Psi^r_\kappa \tau_r^{-1}(\tau_r {\mathbf{P}_\mathbf{r}}^{-1}\mathbf{j}) - \kappa^{-1}\nabla \Psi^r_\kappa \tau_r^{-1}(\tau_r \Delta_{\Gamma_r} \tau_r^{-1} p),\\ \Psi_{E_\kappa}(r)\mathbf{j} =& \,\mathrm{curl}\,\psi^r_\kappa \tau_r^{-1}(\tau_r {\mathbf{P}_\mathbf{r}}^{-1}\mathbf{j}).\end{aligned}$$

By composition of differentiable applications, we deduce that $r \mapsto \Psi_{E_\kappa}(r)$ and $r \mapsto \Psi_{M_\kappa}(r)$ are infinitely Gâteaux differentiable far from the boundary and that their first derivatives are continuous from $\mathbf{TH}^{\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ to $\mathbf{L}^2(\Omega) \cup \mathbf{L}^2_{loc}(\Omega^c)$. Recall that we have,

$$\mathrm{curl}\,\Psi_{E_\kappa}(r)\mathbf{j} = \kappa \Psi_{M_\kappa}(r)\mathbf{j} \text{ and } \mathrm{curl}\,\Psi_{M_\kappa}(r)\mathbf{j} = \kappa \Psi_{E_\kappa}(r)\mathbf{j}.$$

Far from the boundary we can invert the differentiation with respect to x and the derivation with respect to r and it gives:

$$\mathbf{curl}\,\frac{\partial \Psi_{E_\kappa}}{\partial r}[0, \xi]\mathbf{j} = \kappa \frac{\partial \Psi_{M_\kappa}}{\partial r}[0, \xi]\mathbf{j} \text{ et } \mathbf{curl}\,\frac{\partial \Psi_{M_\kappa}}{\partial r}[0, \xi]\mathbf{j} = \kappa \frac{\partial \Psi}{\partial r}[0, \xi]\mathbf{j}.$$

It follows that $\frac{\partial \Psi_{E_\kappa}}{\partial r}[0, \xi]\mathbf{j}$ and $\frac{\partial \Psi_{M_\kappa}}{\partial r}[0, \xi]\mathbf{j}$ are in $\mathbf{H}(\mathbf{curl}, \Omega) \cup \mathbf{H}_{loc}(\mathbf{curl}, \Omega^c)$ and that they satisfy the Maxwell equations and the Silver-Müller condition. ■

We recall that the operator $C_\kappa(r)$ admit the following representation :

$$C_\kappa(r) = \mathbf{P_r} C^r_\kappa {\mathbf{P_r}}^{-1}\mathbf{j} = \nabla_\Gamma P(r) + \mathbf{curl}_\Gamma\, Q(r), \tag{5.10}$$

where

$$P(r) = -\kappa\, (J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1})^{-1} (J_r \tau_r \,\mathrm{curl}_{\Gamma_r}\, \tau_r^{-1}) \left(\tau_r V^r_\kappa \tau_r^{-1}\right) \left[\left(\tau_r \nabla_{\Gamma_r} \tau_r^{-1} p\right) + \left(\tau_r\, \mathbf{curl}_{\Gamma_r}\, \tau_r^{-1} q\right)\right]$$

and $Q(r) =$

$$\begin{aligned}&-\kappa\, (J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1})^{-1} (J_r \tau_r\, \mathrm{div}_{\Gamma_r}\, \pi^{-1}(r)) \pi(r) \left(\tau_r V^r_\kappa \tau_r^{-1}\right) \left[\left(\tau_r \nabla_{\Gamma_r} \tau_r^{-1} p\right) + \left(\tau_r\, \mathbf{curl}_{\Gamma_r}\, \tau_r^{-1} q\right)\right]\\ &+\kappa^{-1} \left(\tau_r V^r_\kappa \tau_r^{-1}\right) \left(\tau_r \Delta_{\Gamma_r} \tau_r^{-1} p\right).\end{aligned}$$

Remark 5.15 *Let $\mathbf{j} \in \mathbf{TH}^s(\mathrm{div}_\Gamma, \Gamma)$ and $\nabla_\Gamma\ p + \mathbf{curl}_\Gamma\ q$ its helmholtz decomposition. We want to derive:*

$$\begin{aligned}\mathbf{P}_r C^r_\kappa \mathbf{P}_r^{-1}\mathbf{j} &= \mathbf{P}_r C^r_\kappa (\nabla_{\Gamma_r} \tau_r^{-1} p + \mathbf{curl}_{\Gamma_r}\, \tau_r^{-1} q)\\ &= \mathbf{P}_r (\nabla_{\Gamma_r} P_r + \mathbf{curl}_{\Gamma_r}\, Q_r)\\ &= \nabla_\Gamma P(r) + \mathbf{curl}_\Gamma\, Q(r).\end{aligned}$$

We have:

$$\frac{\partial \mathbf{P_r} C_\kappa^r \mathbf{P_r}^{-1}\mathbf{j}}{\partial r}[0,\xi] = \nabla_\Gamma \frac{\partial P}{\partial r}[0,\xi] + \mathbf{rot}_\Gamma \frac{\partial Q}{\partial r}[0,\xi].$$

The derivative with respect to r à r de $\mathbf{P_r} C_\kappa^r \mathbf{P_r}^{-1}\mathbf{j}$ is given by the derivatives of the functions $P(r) = \tau_r(P_r)$ and of $Q(r) = \tau_r(Q_r)$.
We also have

$$\frac{\partial\, \pi(r) f(r)}{\partial r}[0,\xi] = \pi(0) \frac{\partial\, f(r)}{\partial r}[0,\xi]$$

By composition of infinite differentiable applications we obtain the

Theorem 5.16 *The application:*

$$\begin{array}{lcl} B_\epsilon^\infty & \to & \mathscr{L}_c\left(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma), \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)\right) \\ r & \mapsto & \mathbf{P}_r C_\kappa^r \mathbf{P}_r^{-1} \end{array}$$

is infinitely Gâteaux differentiable.

Recall that the operator $\mathbf{P}_r M_\kappa^r \mathbf{P_r}^{-1}$ admit the following representation :

$$\mathbf{P_r} M_\kappa^r \mathbf{P_r}^{-1}\mathbf{j} = \nabla_\Gamma P'(r) + \mathbf{curl}_\Gamma\, Q'(r),$$

where

$$\begin{array}{rl} P'(r) = & \left(J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1}\right)^{-1} (\kappa^2 J_r \tau_r \mathbf{n}_r \cdot (\tau_r V_\kappa^r \tau_r^{-1}) \left[\left(\tau_r \nabla_{\Gamma_r} \tau_r^{-1} p\right) + \left(\tau_r\, \mathbf{curl}_{\Gamma_r}\, \tau_r^{-1} q\right)\right] \\ & + \left(J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1}\right)^{-1} (J_r \tau_r D_\kappa^r \tau_r^{-1})(\tau_r \Delta_{\Gamma_r} \tau_r^{-1} p) \end{array}$$

and $Q'(r) =$

$$\left(J_r \tau_r \Delta_{\Gamma_r} \tau_r^{-1}\right)^{-1} (J_r \tau_r\, \mathrm{curl}_{\Gamma_r}\, \tau_r^{-1})(\tau_r (B_\kappa^r - D_\kappa^r)\tau_r^{-1}) \left[\left(\tau_r \nabla_{\Gamma_r} \tau_r^{-1} p\right) + \left(\tau_r\, \mathbf{curl}_{\Gamma_r}\, \tau_r^{-1} q\right)\right]$$

with

$$\begin{array}{rl} \tau_r B_k^r \mathbf{P}_r^{-1}\mathbf{j} = & \tau_r \left\{ \displaystyle\int_{\Gamma_r} \nabla G(\kappa, |\cdot - y_r|) \left(\mathbf{n}_r(\cdot) \cdot (\nabla_{\Gamma_r} \tau_r^{-1} p)(y_r)\right) d\sigma(y_r) \right. \\ & \left. + \displaystyle\int_{\Gamma_r} \nabla G(\kappa, |\cdot - y_r|) \left(\mathbf{n}_r(\cdot) \cdot (\mathbf{curl}_{\Gamma_r}\, \tau_r^{-1} q)(y_r)\right) d\sigma(y_r) \right\} \Big\}. \end{array}$$

Theorem 5.17 *The application:*

$$\begin{array}{lcl} B_\epsilon^\infty & \to & \mathscr{L}_c\left(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma), \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)\right) \\ r & \mapsto & \mathbf{P}_r M_\kappa \mathbf{P}_r^{-1} \end{array}$$

is infinitely Gâteaux differentiable and the Gâteaux derivatives have the same regularity than M_κ so that it is compact.

PROOF. By composition of infinite differentiable applications it remains to prove the infinite Gâteaux differentiability of the application

$$\begin{aligned} B_\delta &\to \mathscr{L}_c\left(\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma),\mathbf{H}^{\frac{1}{2}}(\Gamma)\right)\\ r &\mapsto \tau_r B_\kappa^r \mathbf{P}_r^{-1}. \end{aligned}$$

The function $(x,y-x)\mapsto \nabla G(\kappa,|x-y|)$ is pseudo-homogeneous of class 0. We then have to prove that for any fixed $(x,y)\in(\Gamma\times\Gamma)^*$ and any function $p\in H^{\frac{3}{2}}(\Gamma)$ the Gâteaux derivatives of

$$r\mapsto(\tau_r\mathbf{n}_r)(x)\cdot\left(\tau_r\nabla_{\Gamma_r}\tau_r^{-1}p\right)(y)$$

behave as $|x-y|^2$ when $x-y$ tends to zero. To do so, either we write

$$(\tau_r\mathbf{n}_r)(x)\cdot\left(\tau_r\nabla_{\Gamma_r}\tau_r^{-1}p\right)(y)=\left((\tau_r\mathbf{n}_r)(x)-(\tau_r\mathbf{n}_r)(y)\right)\cdot\left(\tau_r\nabla_{\Gamma_r}\tau_r^{-1}p\right)(y)$$

or we use lemmas 4.2 and 2.1. ■

Theorem 5.18 *Assume that :*
1) $\mathbf{E}^{inc}\in\mathbf{H}^1_{loc}(\mathbf{curl},\mathbb{R}^3)$ *and*
2) the applications

$$\begin{aligned} B_\epsilon^\infty &\to \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)\\ r &\mapsto \mathbf{P}_r\left(\mathbf{n}_r\times\mathbf{E}^{inc}_{|\Gamma_r}\right)\\ r &\mapsto \mathbf{P}_r\left(\mathbf{n}_r\times\left(\mathbf{curl}\,\mathbf{E}^{inc}\right)_{|\Gamma_r}\right) \end{aligned}$$

are Gâteaux differentiable at $r=0$. Then the application mapping r onto the solution $\mathscr{E}(r)=\mathbf{E}(\Omega_r)\in\mathbf{H}(\mathbf{curl},\Omega)\cup\mathbf{H}_{loc}(\mathbf{curl},\Omega^c)$ to the scattering problem the obstacle Ω_r is Gâteaux differentiable at $r=0$.

PROOF. By composition of differentiable applications. We write for the exterior field $\mathscr{E}^s$:

$$\begin{aligned} \frac{\partial\mathscr{E}^s}{\partial r}[0,\xi] = &\left(-\frac{\partial\Psi_{E_{\kappa_e}}}{\partial r}[0,\xi]-i\eta\frac{\partial\Psi_{M_{\kappa_e}}}{\partial r}[0,\xi]C_0-i\eta\Psi_{M_{\kappa_e}}\frac{\partial C_0}{\partial r}[0,\xi]\right)\mathbf{j}\\ &+(-\Psi_{E_{\kappa_e}}-i\eta\Psi_{M_{\kappa_e}}C_0^*)\mathbf{S}^{-1}\left(-\frac{\partial\mathbf{S}}{\partial r}[0,\xi]\mathbf{j}\right)\\ &+(-\Psi_{E_{\kappa_e}}-i\eta\Psi_{M_{\kappa_e}}C_0^*)\mathbf{S}^{-1}\left(-\rho\frac{\partial M_{\kappa_i}}{\partial r}[0,\xi]\gamma_D\mathbf{E}^{inc}-\frac{\partial C_{\kappa_i}}{\partial r}[0,\xi]\gamma_{N_{\kappa_e}}\mathbf{E}^{inc}\right)\\ &+(-\Psi_{E_{\kappa_e}}-i\eta\Psi_{M_{\kappa_e}}C_0^*)\mathbf{S}^{-1}\left(-\rho\left(\frac{1}{2}+M_{\kappa_i}\right)\frac{\partial\mathbf{P}_r\gamma_D^r\mathbf{E}^{inc}}{\partial r}[0,\xi]\right)\\ &+(-\Psi_{E_{\kappa_e}}-i\eta\Psi_{M_{\kappa_e}}C_0^*)\mathbf{S}^{-1}\left(-C_{\kappa_i}\frac{\partial\mathbf{P}_r\gamma_{N_{\kappa_e}}^r\mathbf{E}^{inc}}{\partial r}[0,\xi]\right). \end{aligned}$$

The condition 1) guarantees that the solution $\mathbf{j} \in \mathbf{TH}^{\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ so that the first term in the right handside are in $\mathbf{H}_{loc}(\mathbf{curl}, \Omega^c)$ and the second condition guarantees that the last two term is in $\mathbf{H}_{loc}(\mathbf{curl}, \Omega^c)$. Of the same for the interior field we write:

$$\begin{aligned}\frac{\partial \mathscr{E}^i}{\partial r}[0,\xi] = &-\frac{1}{\rho}\frac{\partial \Psi_{E_{\kappa_i}}}{\partial r}[0,\xi]\gamma^c_{N_{\kappa e}}\left(\mathbf{E}^s + \mathbf{E}^{inc}\right) - \frac{\partial \Psi_{M_{\kappa_i}}}{\partial r}[0,\xi]\gamma^c_D\left(\mathbf{E}^s + \mathbf{E}^{inc}\right)\\ &-\frac{1}{\rho}\Psi_{E_{\kappa_i}}\frac{\partial \mathbf{P}_r\gamma^r_{N_{\kappa e}}\left(\mathscr{E}^s(r) + \mathbf{E}^{inc}\right)}{\partial r}[0,\xi] - \Psi_{M_{\kappa_i}}\frac{\partial \mathbf{P}_r\gamma^r_D\left(\mathscr{E}^s(r) + \mathbf{E}^{inc}\right)}{\partial r}[0,\xi]\end{aligned}$$

The condition 1) guarantees that $\gamma^c_{N_{\kappa e}}\left(\mathbf{E}^s + \mathbf{E}^{inc}\right)$ and $\gamma^c_D\left(\mathbf{E}^s + \mathbf{E}^{inc}\right)$ are in $\mathbf{TH}^{\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ so that the first two terms are in $\mathbf{H}(\mathbf{curl}, \Omega)$ and the second condition guarantees that the last two term is in $\mathbf{H}(\mathbf{curl}, \Omega)$. ■

Theorem 5.19 *The application mapping r to the far field pattern $\mathbf{E}^\infty(\Omega_r) \in \mathbf{T}\mathscr{C}^\infty(S^2)$ of the solution to the scattering problem the obstacle Ω_r is $\mathscr{C}^\infty$-Gâteaux differentiable.*

5.4 Characterisation of the first derivative

The following theorem give a caracterisation of the first Gâteaux derivative of $r \mapsto \mathscr{E}(r)$ in $r = 0$.

Theorem 5.20 *Under the hypothesis of theorem 5.18 the first derivative at $r = 0$ in the direction ξ solve the following scattering problem :*

$$\begin{cases}\mathbf{curl}\,\mathbf{curl}\,\dfrac{\partial \mathscr{E}^i}{\partial r}[0,\xi] - \kappa_i^2\dfrac{\partial \mathscr{E}^i}{\partial r}[0,\xi] = 0\\ \mathbf{curl}\,\mathbf{curl}\,\dfrac{\partial \mathscr{E}^s}{\partial r}[0,\xi] - \kappa_e^2\dfrac{\partial \mathscr{E}^s}{\partial r}[0,\xi] = 0\end{cases} \tag{5.11}$$

with the boundary conditions :

$$\begin{cases}\mathbf{n}\times\dfrac{\partial \mathscr{E}^i}{\partial r}[0,\xi] - \mathbf{n}\times\dfrac{\partial \mathscr{E}^s}{\partial r}[0,\xi] = g_D\\ \mu_i^{-1}\mathbf{n}\times\mathbf{curl}\,\dfrac{\partial \mathscr{E}^i}{\partial r}[0,\xi] - \mu_e^{-1}\mathbf{n}\times\mathbf{curl}\,\dfrac{\partial \mathscr{E}^s}{\partial r}[0,\xi] = g_N,\end{cases} \tag{5.12}$$

where

$$\begin{aligned}g_D = &-(\xi\cdot\mathbf{n})\,\mathbf{n}\times\frac{\partial}{\partial\mathbf{n}}\left(\mathbf{E}^i - \mathbf{E}^s - \mathbf{E}^{inc}\right)\\ &+\mathbf{curl}_\Gamma(\xi\cdot\mathbf{n})\,\mathbf{n}\cdot\left(\mathbf{E}^i - \mathbf{E}^s - \mathbf{E}^{inc}\right),\end{aligned}$$

and

$$\begin{aligned}g_N = &-(\xi\cdot\mathbf{n})\,\mathbf{n}\times\frac{\partial}{\partial\mathbf{n}}\left(\mu_i^{-1}\,\mathbf{curl}\,\mathbf{E}^i - \mu_e^{-1}\,\mathbf{curl}\,\mathbf{E}^s - \mu_e^{-1}\,\mathbf{curl}\,\mathbf{E}^{inc}\right)\\ &+\mathbf{curl}_\Gamma(\xi\cdot\mathbf{n})\,\mathbf{n}\cdot\left(\mu_i^{-1}\,\mathbf{curl}\,\mathbf{E}^i - \mu_e^{-1}\,\mathbf{curl}\,\mathbf{E}^s - \mu_e^{-1}\,\mathbf{curl}\,\mathbf{E}^{inc}\right).\end{aligned}$$

and where $\dfrac{\partial \mathscr{E}^s}{\partial r}[0,\xi]$ satisfies the Silver-Müller condition.

PROOF. We have shown in the previous paragraph that the potential operators and their Gâteaux derivatives satisfy the Maxwell' equations and the Silver-Müller condition. It remains to compute the boundary conditions. We could use the integral representation as Potthast did but it would need to write too long formula. For $x \in \Gamma$ we derive in $r = 0$ the expression:

$$\mathbf{n}_r(x + r(x)) \times \left(\mathscr{E}^i(r)(x + r(x)) - \mathscr{E}^s(r)(x + r(x)) - \mathbf{E}^{inc}(x + r(x))\right) = 0. \tag{5.13}$$

It gives in the direction ξ:

$$\begin{aligned} 0 = \quad & \frac{\partial \tau_r \mathbf{n}_r}{\partial r}[0,\xi](x) \times \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right) \\ & + \mathbf{n}(x) \times \left(\frac{\partial \mathscr{E}^i}{\partial r}[0,\xi](x) - \frac{\partial \mathscr{E}^s}{\partial r}[0,\xi](x)\right) \\ & + \mathbf{n} \times \left(\xi(x) \cdot \nabla \left(\mathbf{E}^i - \mathbf{E}^s - \mathbf{E}^{inc}\right)\right). \end{aligned}$$

We recall that $\dfrac{\partial \tau_r \mathbf{n}_r}{\partial r}[0,\xi](x) = -\left[\nabla_\Gamma \xi\right] \mathbf{n}$ and we use

$$\nabla u = \nabla_\Gamma u + \left(\frac{\partial u}{\partial \mathbf{n}}\right) \mathbf{n}.$$

We obtain :

$$\begin{aligned} \mathbf{n}(x) \times \left(\frac{\partial \mathscr{E}^i}{\partial r}[0,\xi](x) - \frac{\partial \mathscr{E}^s}{\partial r}[0,\xi](x)\right) = \quad & \left[\nabla_\Gamma \xi\right] \mathbf{n} \times \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right) \\ & - \mathbf{n} \times \left(\xi(x) \cdot \nabla_\Gamma \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right)\right) \\ & - (\xi \cdot \mathbf{n}) \mathbf{n} \times \frac{\partial}{\partial \mathbf{n}} \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right). \end{aligned}$$

Since the tangential component of $\mathbf{E}^i - \mathbf{E}^s - \mathbf{E}^{inc}$ vanish we have:

$$\left(\xi(x) \cdot \nabla_\Gamma \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right)\right) = \left(\left[{}^T \nabla_\Gamma \mathbf{n}\right] \xi\right) \left(\mathbf{n} \cdot \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right)\right)$$

and

$$\left[\nabla_\Gamma \xi\right] \mathbf{n} \times \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right) = \left(\left[\nabla_\Gamma \xi\right] \mathbf{n}\right) \times \mathbf{n} \left(\mathbf{E}^i(x) - \mathbf{E}^s(x) - \mathbf{E}^{inc}(x)\right) \cdot \mathbf{n}.$$

For regular surface we have $\nabla_\Gamma \mathbf{n} = {}^T \nabla_\Gamma \mathbf{n}$ and

$$\left(\left[\nabla_\Gamma \xi\right] \mathbf{n}\right) \times \mathbf{n} - \mathbf{n} \times \left(\left[{}^T \nabla_\Gamma \mathbf{n}\right] \xi\right) = \mathbf{curl}_\Gamma \left(\xi \cdot \mathbf{n}\right).$$

We deduce the first boundary conditions. The second boundary condition corresponds to the same computation with the magnetic fields. ■

Using the commutators (5.9) one can verify that the solution of this problem is in $\mathbf{H}_{loc}(\mathbf{curl}, \Omega)$ since the trace

$$\mathbf{u} \in \mathbf{H}^1(\mathbf{curl}, \Omega) \mapsto \mathbf{n} \times \frac{\partial}{\partial \mathbf{n}} \mathbf{u} \in \mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

is linear and continuous.

Conclusion

In this paper we have presented a complete shape differentiability analysis of the solution to the dielectric scattering problem using the boundary integral equation approach. These results can be extended to many others electromagnetic boundary value problems. Thanks to the numerous computations of Gâteaux derivatives we obtain two alternatives to compute the first shape derivative of the solution : either we derive the integral representation or we solve the new boundary value problem associated to the shape derivatives with boundary integral equation method. Whereas this last alternative needs boundaries of class $\mathscr{C}^2$ at least since it appears any derivatives of the normal vector, many results in this paper are still available for Lipschitz domains as for example the computations of the Gâteaux derivatives of all the surface differential operators of order 1 with deformations of class $\mathscr{C}^1$ only and other functionals viewed in section 4. One can find in the litterature, the theory of pseudo-differential operators on Lipschitz domain [34], it remains to find the optimal regularity of the deformations in order that this integral operators are still Gâteaux differentiable. According to the Helmholtz decomposition we have on Lipschitz domain:

$$\mathbf{TH}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma) = \nabla_\Gamma\left(\mathcal{H}(\Gamma)\right) \bigoplus \mathbf{curl}_\Gamma\left(H^{\frac{1}{2}}(\Gamma)/\mathbb{R}\right).$$

where

$$\mathcal{H}(\Gamma) = \{u \in H^1(\Gamma)\backslash\mathbb{R};\ \Delta_\Gamma \in H_*^{-\frac{1}{2}}(\Gamma)\}.$$

If we want to extend the result to Lipschitz domain, we have to construct another invertible operator between $\mathcal{H}(\Gamma_r)$ and $\mathcal{H}(\Gamma)$.

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