

Significant Reduction of Gibbs' Overshoot with Generalized Sampling Method

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Abstract:

As is well-known, the use of Shannon sampling to interpolate functions with discontinuous jump points leads to the Gibbs' overshoot. In image processing, it can lead to the problem of artifacts close to edges, known as Gibbs ringing. Its amplitude cannot be reduced by increasing the sample density. Here we consider a generalized Shannon sampling method which allows the use of time-varying sample densities so that samples can be taken at a varying rate adapted to the behavior of the function. Using this generalized sampling method to approximate a periodic step function, we observe a strong reduction of Gibbs' overshoot. In a concrete example, the amplitude of the Gibbs' overshoot is reduced by about 70%.

1. Introduction

The Shannon sampling theorem [6] provides the link between continuous and discrete representations of information and has numerous practical uses in communication engineering and signal processing. For a review on Shannon sampling, see [7, 10, 1]. In addition, the Shannon sampling theorem has been used to interpolate samples to approximate a given function.

In the use of Shannon sampling to approximate functions with discontinuous jump points, the well-known Gibbs' overshoot [2, 3] has remained a persistent problem, leading to, e.g., Gibbs ringing in image compression [5]. The clearest example for the Gibbs' phenomenon is the periodic step function $H(t)$, see Figure 1, where $H(t) = 1$ on $(0, \frac{1}{2})$, $H(t) = -1$ on $(\frac{1}{2}, 1)$, $H(t) = 0$ at $t = 0, \frac{1}{2}, 1$, and $H(t)$ has a period $T = 1$.

In Figure 1, $H(t)$ is approximated using Shannon's shifted sinc reconstruction kernel with $N = 24$ sampling points on one periodic interval $[0, 1)$. Samples are denoted by \times in the plot, and the solid line at the top indicates the maximum value of the approximating function, which is 1.0640. Within an error of 0.003, the 6.40% overshoot beyond the maximum amplitude 1 of the step function $H(t)$ can not be further reduced even if we increase the sampling density.

However, using the generalized sampling method [4, 8, 9], which allows the reconstruction of a function on a set of non-equidistant sampling points, chosen adaptively according to the behavior of the function, we show that the

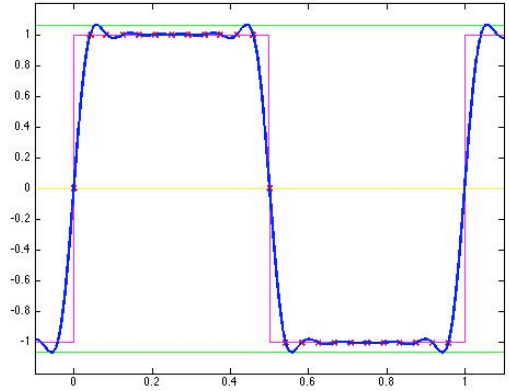


Figure 1: Approximation of the step function by Shannon sampling.

Gibbs overshoot can be strongly reduced. For an example, see Figure 2.

In Figure 2, we use the same number of points $N = 24$ in one period as in the case of Shannon in Figure 1, but we choose the sampling points to match the behaviour of the step function. Intuitively, the jump in the step function contains high frequencies. Thus more samples are taken near the jump points $t = 0, \frac{1}{2}$, and 1. In this example, the maximum value of the approximation is reduced to 1.0074 with an error of 0.0003. This is roughly a 70% reduction of Gibbs' overshoot without increasing the number of samples, but only varying the local sample density.

Figure 3 is a zoom-in of Figure 2 near the jump point. The dashed line on the top indicates the maximum values of the approximating function using the generalized sampling, while the solid line indicates the overshoot in the case of Shannon.

2. Generalized Shannon Sampling Method

The generalized Shannon sampling theory considered here was not specifically developed for the application of reducing Gibbs' phenomena. It was originally motivated by some fundamental physics problem in quantum gravity [4] and was introduced to engineering for spaces of functions with a new notion of time-varying Nyquist rate [8, 9]. The starting observation is that each set of Nyquist sampling points in Shannon sampling turns to be the eigen-

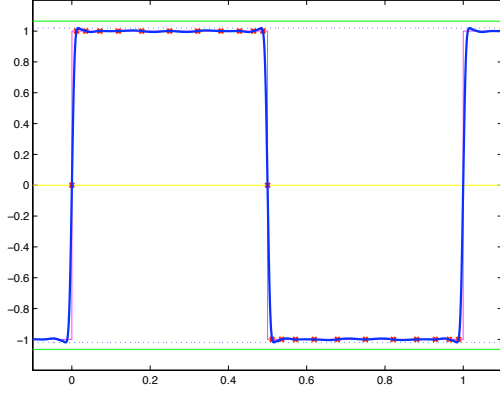


Figure 2: Approximating the step function by the generalized sampling method with non-equidistant sampling points..

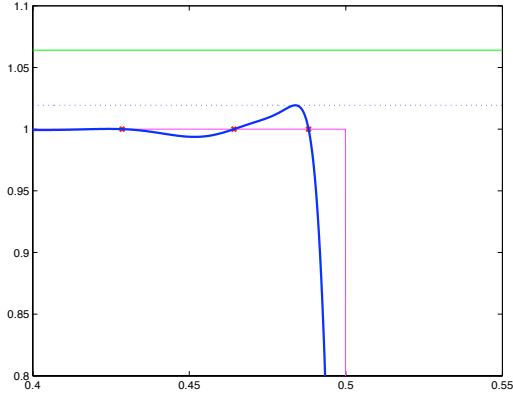


Figure 3: This is a zoom-in of Figure 2 near the jump point..

values of one of the self-adjoint extensions of a particular simply symmetric multiplication operator T with deficiency indices $(1, 1)$, and the shifted sinc kernels are the corresponding eigenfunctions. The Shannon sampling theorem is the special case when the self-adjoint extensions of T have equidistant eigenvalues. By considering a generic such symmetric operator T , one obtains a generalized sampling method. We can not cover the mathematical derivations of the new generalized sampling method here, but we will review the key features of the generalization along with a comparison to the Shannon sampling theorem.

The Shannon sampling theorem states that if a function $\phi(t)$ is in the space of Ω -bandlimited functions, i.e., $\phi(t)$ has a frequency upper bound Ω , then $\phi(t)$ can be perfectly reconstructed from its sample values $\{\phi(t_n)\}_n$ taken on a set of sampling points $\{t_n\}_n$ with an equidistant spacing $t_{n+1} - t_n = 1/(2\Omega)$ via:

$$\phi(t) = \sum_{n=-\infty}^{\infty} G(t, t_n) \phi(t_n) \quad (1)$$

The function $G(t, t_n)$ is the so-called reconstruction kernel, which is the shifted sinc function $\text{sinc}(2\Omega(t - t_n))$. The frequency upper bound Ω is called the bandwidth, and the sampling rate $1/(2\Omega)$ is the Nyquist sampling rate.

2.1 One-Parameter Family of Sampling Lattices

We will call a set of Nyquist sampling points $\{t_n\}_n$ a sampling lattice. The Shannon sampling theorem only specifies the constant spacing between adjacent points in one lattice, but it does not specify an initial sampling point. Therefore, we can parameterize all possible sampling lattices as:

$$t_n(\theta) = \frac{n + \theta}{2\Omega}, \quad 0 \leq \theta < 1 \quad (2)$$

Hence the Shannon sampling method possesses a natural one-parameter family of sampling lattices, and any function in the function space can be perfectly reconstructed from its values on any fixed lattice via Eq. (1).

The generalized sampling method also possesses an analogous one-parameter family of sampling lattices, but the points in each lattice are generally non-equidistant now. To distinguish from the case of Shannon, we use a different parameter α in $\{t_n(\alpha)\}_n$, $0 \leq \alpha < 1$, and assume that $\{t_n(\alpha)\}_n$ are differentiable with respect to the parameter α :

$$t'_n(\alpha) = \frac{dt_n(\alpha)}{d\alpha}$$

Shannon's family of sampling lattices $\{t_n(\theta)\}_n$ can be generated by a single number, namely, the constant bandwidth Ω . It is so simple because the function space in the case of Shannon has a constant bandwidth Ω . However, in the generalization, since we have a time-varying 'bandwidth', in the sense of Nyquist lattices with non-equidistant points, more specification is required. The entire family of sampling lattices is now generated from the knowledge of a given lattice, say $\{t_n(0)\}$, and a set of corresponding derivatives $\{t'_n(0)\}_n$ by solving for $t = t_n(\alpha)$ in:

$$\sum_m \frac{t'_m(0)}{t - t_m(0)} = \pi \cot(\pi\alpha) \quad (3)$$

This equation implies that one sampling lattice and the corresponding derivatives are enough to determine the entire family of sampling lattices, and hence the reconstruction kernel and the function space. This is important for practical purposes, because one usually takes samples of a given signal on only one lattice.

The family of sampling lattices $\{t_n(\alpha)\}_n$ in the generalization shares many important properties of the uniform lattices $\{t_n(\theta)\}$ of Shannon: as the parameter α (or β in the case of Shannon) increases from 0 to 1, the sampling lattices specified by the parameter move to the right on the real line simultaneously and continuously with the following continuity condition:

$$t_n(1) := \lim_{\alpha \rightarrow 1^-} t_n(\alpha) = t_{n+1}(0), \quad t'_n(1) = t'_{n+1}(0) \quad (4)$$

Hence, together, these sampling points in all lattices again cover the real line exactly once. Namely, for any $t \in \mathbb{R}$, there exists a unique integer n and a unique α in $[0, 1)$ such that $t = t_n(\alpha)$.

2.2 The Generalized Reconstruction Kernel

From the theory of self-adjoint extensions, if on each fixed but arbitrary lattice $\{t_n(\alpha)\}$, α fixed, we let $t_n = t_n(\alpha)$,

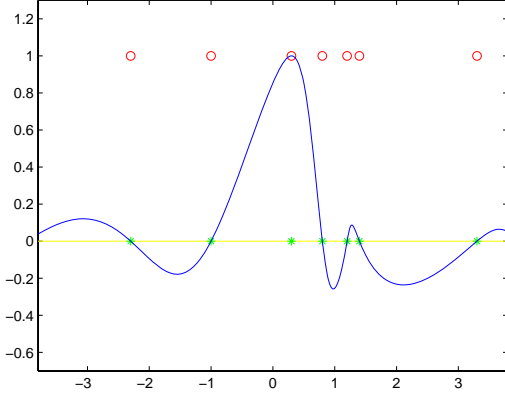


Figure 4: An example of generalized sinc function (or reconstruction kernel) on an arbitrary non-equidistant sampling lattice. The stars on the real line indicate the points in an arbitrary non-equidistant sampling lattice, and the circles denote the same set of points with an amplitude 1.

$t'_n = t'_n(\alpha)$, then the reconstruction kernel in the generalized sampling theorem reads:

$$G(t, t_n) = (-1)^{z(t, t_n)} \frac{\sqrt{t'_n}}{|t - t_n|} \left(\sum_m \frac{t'_m}{(t - t_m)^2} \right)^{-1/2} \quad (5)$$

where $z(t, t_n)$ is the number of the sampling points $\{t_m\}_m$ between t and t_n exclusively.

As functions in t , for each fixed α , the set of functions

$$\left\{ g_n^{(\alpha)}(t) = G(t, t_n(\alpha)) \right\}_n \quad (6)$$

forms a basis of the function space. Thus, indeed, in the generalized sampling theorem, every function in the function space specified by the family of sampling lattices can be expanded using these basis functions.

These continuous functions in Eq. (6) have analogous properties to the shifted sinc function of Shannon: they interpolate all the points in the lattice specified by α

$$g_n^{(\alpha)}(t_m(\alpha)) = G(t_m(\alpha), t_n(\alpha)) = \delta_{mn}$$

and their maximum values are all 1 at the sampling points about which they are ‘centered’. This is important for the stability of reconstruction. We will refer to these basis functions as generalized sinc functions. See Figure 4 for an example.

It is important to recall that each set of basis functions $\{g_n^{(\alpha)}(t)\}_n$ specified by α spans the same function space. This property is remarkable since as in Figure 4, the shape of the generalized sinc functions is quite non-trivial.

To recover the Shannon sampling theorem as a special case, we choose any uniform sampling lattice $\{t_n\}_n$ with $t_{n+1} - t_n = \frac{1}{2\Omega}$ for all n , together with constant derivatives $t'_n = C$. Then the reconstruction kernel in (5) simplifies to the sinc kernel $\text{sinc}(2\Omega(t - t_n))$, by using the following trigonometric identity:

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{+\infty} \frac{1}{(z - k)^2} \quad (7)$$

2.3 Interpolation Strategy

To approximate a given function, depending on the behavior of the function, one must select a sampling lattice for interpolation. Arising from the theory of self-adjoint extensions, the chosen lattice $\{t_n\}_n$ must have a minimum and maximum spacing, namely, there must exist positive real numbers δ_{\min} and Δ_{\max} such that:

$$0 < \delta_{\min} \leq \Delta t_n = t_{n+1} - t_n \leq \Delta_{\max} \quad \text{for all } n \quad (8)$$

From Section 2.1 Eq. (3), we know that one also needs a set of corresponding derivatives to apply the generalized sampling method. So the question is, for a given lattice $\{t_n\}_n$, what is a suitable choice of the set of corresponding derivatives $\{t'_n\}_n$?

To this end, we notice that the derivative $t'_n(\alpha)$ is the velocity with which the sampling points $t_n(\alpha)$ are moving to the right along the real line for increasing α at $t = t_n(\alpha)$. Hence, a good candidate for t'_n is the distance travelled in one period of α , which is the spacing between two adjacent points $\Delta t_n = t_{n+1} - t_n$. For symmetry, we set t'_n to be the average distance between t_n to its previous and successive points:

$$t'_n = \frac{1}{2} (\Delta t_n + \Delta t_{n-1}) = \frac{1}{2} (t_{n+1} - t_{n-1}) \quad (9)$$

Here a constant prefactor for the derivatives on a fixed lattice does not matter because the reconstruction kernel is independent of a scalar multiplication of the derivatives: in (5), the prefactor in $\sqrt{t'_n}$ -term will cancel out the one in t'_m on the numerator inside the series.

With this set of initial data $\{t_n\}_n$ and $\{t'_n\}_n$, we have an explicit expression of the reconstruction kernel (5). Hence we can construct the interpolating function $\phi(t)$ through all the sample points $\{(t_n, \phi(t_n))\}_n$ using the reconstruction formula (1).

3. Reduction of Gibbs’ Overshoot

3.1 Reconstruction of Periodic Functions

The clearest example to demonstrate the reduction of Gibbs’ overshoot using the generalized sampling method is the periodic step function $H(t)$. One of the reasons for choosing a periodic function is that the infinite summations in the both reconstruction kernel (5) and the reconstruction formula (1) will simplify to a finite sum. Hence, we eliminate the truncation error in the summation.

To this end, assume that the function $\phi(t)$ has a period of T , and we take N sampling points on one period $[0, T)$, which are denoted by $\{\tau_1, \tau_2, \dots, \tau_N\} \subseteq [0, T)$. Hence, all the sampling points are

$$t_{nN+k} = nT + \tau_k, \quad 1 \leq k \leq N, n \in \mathbb{N} \quad (10)$$

and from the periodicity, we have

$$t'_{nN+k} = t'_k, \quad \phi(t_{nN+k}) = \phi(t_k) \quad (11)$$

After a lengthy calculation, the reconstruction kernel (5)

on this periodic lattice now reads:

$$G(t, t_{nN+K}) = \frac{(-1)^{z(t, t_{nN+K})} \sqrt{t'_k}}{|t - t_{nN+K}|} \left(\sum_{l=1}^N t'_l \sin^{-2} \left(\frac{\pi}{T} (t - \tau_l) \right) \right)^{-1/2} \quad (12)$$

and the reconstruction formula (1) of the T -periodic function $\phi(t)$ reads:

$$\phi(t) = \sum_{k=1}^N (-1)^{z(t, t_{nN+k})} \sqrt{t'_k} \cot \left(\frac{\pi}{T} (t - \tau_k) \right) \left(\sum_{l=1}^N t'_l \sin^{-2} \left(\frac{\pi}{T} (t - \tau_l) \right) \right)^{-1/2} \phi(t_k) \quad (13)$$

As discussed in Section 2.3, for using the formulae (12) and (13) to approximate a periodic step function $H(t)$, the only task now left is to find a sampling lattice adapted to the behavior of $H(t)$. With a periodic lattice (10), we only need to pick up a finite number N of them on $[0, T)$.

3.2 Approximating a Periodic Step Function

Before we discuss how to determine a set of non-equidistant sampling points, let us first consider why the uniform lattices of Shannon do not work very well. Intuitively, because of the sudden change in the amplitude of a step function $H(t)$ at its jump points $t = 0, \frac{1}{2}$ and 1 , the function can be considered to suddenly oscillate at an “infinite” frequency in a sufficiently small neighborhoods at the jump points, namely to have an ‘infinite’ bandwidth at $t = 0, \frac{1}{2}$ and 1 . Recall that the constant Nyquist spacing $1/(2\Omega)$ in the case of Shannon is inversely proportional to the bandwidth Ω . A uniform lattice implies uniform bandwidth. Intuitively, the uniform lattice in the case of Shannon is therefore not matched with the increase of bandwidth in the small neighborhoods of jump points.

We therefore choose N sampling points with non-equidistant spacings so that the smallest spacing (the highest bandwidth) occurs near the jump points at $t = 0, \frac{1}{2}, 1$, and the spacing gradually increases away from the jump points (the bandwidth decreases). We used the easiest such increasing change in spacing, which is linear.

Due to the symmetry of the jump points at $t = 0, \frac{1}{2}, 1$, we divide one period $[0, 1)$ into four equal subintervals with length $\frac{1}{4}$. On the first subinterval, $[0, \frac{1}{4})$, we choose K points so that their adjacent spacing is linearly increasing. Let δ be the linear increment in spacing, then

$$\begin{aligned} \tau_1 &= 0, \tau_2 = \delta, \tau_3 = 3\delta, \dots \\ \tau_K &= \frac{1}{2} K(K-1)\delta \end{aligned} \quad (14)$$

The $(K+1)^{\text{st}}$ point is $\frac{1}{4}$. The sampling points on $(\frac{1}{4}, \frac{1}{2})$ are a mirror image of the points on $[0, \frac{1}{4})$ with respect to $t = \frac{1}{4}$, and the points on $(\frac{1}{2}, 1)$ repeat the ones on $[0, \frac{1}{2})$. Therefore, we have in total $N = 4K$ points on $[0, 1)$.

The approximation in Figure 2 is obtained in this way with $K = 6$. Hence it has the same total number of sampling points ($N = 24$) on $[0, 1)$ as in Figure 1. Its maximum

amplitude is 1.0193, which is a significant reduction compared to the maximum amplitude 1.0640 in Gibbs’ overshoot (Figure 1).

4. Outlook

The question arises how far one can ultimately reduce the Gibbs’ overshoot? Is the linear change in sampling spacing, as in Eq. (14), the optimal lattice spacing to match the behavior of a step function? This question will be addressed in a longer following-up paper, in which we will pursue an analytical optimization of the Gibbs’ overshoot reduction.

To this end, the fact that the closed form of the reconstruction kernel (12) is available in the case of periodic functions has an important advantage: it in effect reduces infinitely many points to a set of finitely many points. We can then analytically study the behavior of the constructed approximating functions. Eventually, we hope such an analytical study can lead us to the ultimately goal, which is to provide solution to design optimally adapted lattices for arbitrary given functions.

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