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# Double Dirichelet Averages and Complex B-Splines 

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#### Abstract

: A relation between double Dirichlet averages and multivariate complex B-splines is presented. Based on this reationship, a formula for the computation of certain moments of multivariate complex B-splines is derived.


## 1. Introduction

Recently, a new class of B-splines with complex order $z, \operatorname{Re} z>1$, was introduced in [4]. It was shown that complex B-splines generate a multiresolution analysis of $L^{2}(\mathbb{R})$. Unlike the classical cardinal B-splines, complex B-splines $B_{z}$ possess an additional modulation and phase factor in the frequency domain:

$$
\widehat{B}_{z}(\omega)=\widehat{B}_{\operatorname{Re} z}(\omega) e^{i \operatorname{Im} z \ln |\Omega(\omega)|} e^{-\operatorname{Im} z \arg \Omega(\omega)},
$$

where $\Omega(\omega):=\left(1-e^{-i \omega}\right) /(i \omega)$. The existence of these two factors allows the extraction of additional information from sampled data and the manipulation of images.
In [6] and [9], some further properties of complex Bsplines were investigated. In particular, connections between complex derivatives of Riemann-Liouville or Weyl type and Dirichlet averages were exhibited. Whereas in [6] the emphasis was on univariate complex B-splines and their applications to statistical processes, multivariate complex B-splines were defined in [9] using a wellknown geometric formula for classical multivariate Bsplines [7, 10]. It was also shown that Dirichlet averages are especially well-suited to explore the properties of multivariate complex B-splines. Using Dirichlet averages, several classical multivariate B-spline identities were generalized to the complex setting. There also exist interesting relationships between complex B-splines, Dirichlet averages and difference operators, several of which are highlighted in [5].
This short paper presents a generalization of some results found in $[3,12]$ to complex B-splines. For this purpose, the concept of double Dirichlet average [1] was introduced and its definition extended via projective limits to an infinite-dimensional setting suitable for complex Bsplines. Moments of complex B-splines are defined and a formula for their computation in terms of a special double Dirichlet average presented.

## 2. Complex B-Splines

Let $n \in \mathbb{N}$ and let $\triangle^{n}$ denote the standard $n$-simplex in $\mathbb{R}^{n+1}$ :

$$
\begin{gathered}
\Delta^{n}:=\left\{u:=\left(u_{0}, \ldots, u_{n}\right) \in \mathbb{R}^{n+1} \mid u_{j} \geq 0\right. \\
\left.j=0,1, \ldots, n ; \sum_{j=0}^{n} u_{j}=1\right\}
\end{gathered}
$$

The extension of $\triangle^{n}$ to infinite dimensions is done via projective limits. The resulting infinite-dimensional standard simplex is given by

$$
\Delta^{\infty}:=\left\{u:=\left(u_{j}\right)_{j} \in\left(\mathbb{R}_{0}^{+}\right)^{\mathbb{N}_{0}} \mid \sum_{j=0}^{\infty} u_{j}=1\right\}
$$

and endowed with the topology of pointwise convergence, i.e., the weak*-topology. We denote by $\mu_{b}=\lim \mu_{b}^{n}$ the projective limit of Dirichlet measures $\mu_{b}^{n}$ on the $n$ dimensional standard simplex $\triangle^{n}$ with density

$$
\begin{equation*}
\frac{\Gamma\left(b_{0}\right) \cdots \Gamma\left(b_{n}\right)}{\Gamma\left(b_{0}+\cdots+b_{n}\right)} u_{0}^{b_{0}-1} u_{1}^{b_{1}-1} \cdots u_{n}^{b_{n}-1} . \tag{1}
\end{equation*}
$$

Here, $\Gamma: \mathbb{C} \backslash \mathbb{Z}_{0}^{-} \rightarrow \mathbb{C}$ denotes the Euler Gamma function. Let $\mathbb{R}^{+}:=\{x \in \mathbb{R} \mid x>0\}$ and let $\mathbb{C}_{+}:=\{z \in$ $\mathbb{C} \mid \operatorname{Re} z>0\}$.
Definition 1 ([6]). Given a weight vector $b \in \mathbb{C}_{+}^{\mathbb{N}_{0}}$ and an increasing knot sequence $\tau:=\left\{\tau_{k}\right\}_{k} \in \mathbb{R}^{\mathbb{N}_{0}}$ with the property that $\lim _{k \rightarrow \infty} \sqrt[k]{\tau_{k}} \leq \varrho$, for some $\varrho \in[0, e)$, a complex B-spline $B_{z}(\bullet \mid b ; \tau)$ of order $z, \operatorname{Re} z>1$, with weight vector $b$ and knot sequence $\tau$ is a function satisfying

$$
\begin{equation*}
\int_{\mathbb{R}} B_{z}(t \mid b ; \tau) g^{(z)}(t) d t=\int_{\triangle_{\infty}} g^{(z)}(\tau \cdot u) d \mu_{b}(u) \tag{2}
\end{equation*}
$$

for all $g \in \mathscr{S}(\mathbb{R})$.
Here, $\mathscr{S}(\mathbb{R})$ denotes the space of Schwartz functions on $\mathbb{R}$, and $\tau \cdot u=\sum_{k \in \mathbb{N}_{0}} \tau_{k} u_{k}$ for $u=\left\{u_{k}\right\}_{k \in \mathbb{N}_{0}} \in \triangle^{\infty}$. In addition, we used the Weyl or Riemann-Liouville fractional derivative [8, 11, 13] of complex order $z, \operatorname{Re} z>0$, $W^{z}: \mathscr{S}(\mathbb{R}) \rightarrow \mathscr{S}(\mathbb{R})$, defined by

$$
\left(W^{z} f\right)(x):=\frac{(-1)^{n}}{\Gamma(\nu)} \frac{d^{n}}{d x^{n}} \int_{x}^{\infty}(t-x)^{\nu-1} f(t) d t
$$

with $n=\lceil\operatorname{Re} z\rceil$, and $\nu=n-z$. Here $\lceil\cdot\rceil: \mathbb{R} \rightarrow \mathbb{Z}$, $x \mapsto \min \{n \in \mathbb{Z} \mid n \geq x\}$, denotes the ceiling function. To simplify notation, we write $f^{(z)}$ for $W^{z} f$
It is easy to show that the univariate complex B-spline $B_{z}(t \mid b ; \tau)$ is an element of $L^{2}(\mathbb{R})$ [5].

Remark 2. For finite $\tau=\tau(n)$ and $b=b(n)$ and $z:=$ $n \in \mathbb{N}$, (2) defines also Dirichlet splines if $g$ is chosen in $C^{n}(\mathbb{R})$. For, Dirichlet splines $D_{n}(\cdot \mid b ; \tau)$ of order $n$ are defined as those functions for which

$$
\int_{\mathbb{R}} g^{(n)}(t) D_{n}(t \mid b ; \tau) d t=\int_{\Delta^{n}} g^{(n)}(\tau \cdot u) d \mu_{b}(u)
$$

holds true for $\tau \in \mathbb{R}^{n+1}$ and for all $g \in C^{n}(\mathbb{R})$, and thus for $g \in \mathscr{S}(\mathbb{R})$.
To define a multivariate analogue of the univariate complex B-splines, we proceed as follows. Let $\lambda \in \mathbb{R}^{s} \backslash\{0\}$ be a direction, and let $g: \mathbb{R} \rightarrow \mathbb{C}$ be a function. The ridge function corresponding to $g$ is defined as $g_{\lambda}: \mathbb{R}^{s} \rightarrow \mathbb{C}$,

$$
g_{\lambda}(x)=g(\langle\lambda, x\rangle) \quad \text { for all } x \in \mathbb{R}^{s} .
$$

We denote the canonical inner product in $\mathbb{R}^{s}$ by $\langle\bullet, \bullet\rangle$ and the norm induced by it by $\|\bullet\|$.

Definition 3 ([9]). Let $\tau=\left\{\tau^{n}\right\}_{n \in \mathbb{N}_{0}} \in\left(\mathbb{R}^{s}\right)^{\mathbb{N}_{0}}$ be a sequence of knots in $\mathbb{R}^{s}$ with the property that

$$
\begin{equation*}
\exists \varrho \in[0, e): \limsup _{n \rightarrow \infty} \sqrt[n]{\left\|\tau^{n}\right\|} \leq \varrho \tag{3}
\end{equation*}
$$

The multivariate complex B-spline $\boldsymbol{B}_{z}(\bullet \mid b, \tau): \mathbb{R}^{s} \rightarrow \mathbb{C}$ of order $z, \operatorname{Re} z>1$, with weight vector $b \in \mathbb{C}_{+}^{\mathbb{N}_{o}}$ and knot sequence $\tau$ is defined by means of the identity
$\int_{\mathbb{R}^{s}} g(\langle\lambda, x\rangle) \boldsymbol{B}_{z}(x \mid b, \tau) d x=\int_{\mathbb{R}^{2}} g(t) B_{z}(t \mid b, \lambda \tau) d t$,
where $g \in \mathscr{S}(\mathbb{R})$, and where $\lambda \in \mathbb{R}^{s} \backslash\{0\}$ such that $\lambda \tau:=\left\{\left\langle\lambda, \tau^{n}\right\rangle\right\}_{n \in \mathbb{N}_{0}}$ is separated.

As consequence of the fact that $B_{z}(\bullet \mid b ; \tau) \in L^{2}(\mathbb{R})$, one obtains from the above definition that $\boldsymbol{B}_{z}(\bullet \mid b, \tau) \in$ $L^{2}\left(\mathbb{R}^{s}\right)$ [5]. Moreover, it follows from the HermiteGenocchi formula for the univariate complex B-splines $B_{z}(\bullet \mid b, \lambda \tau)$ and (4), that $\boldsymbol{B}_{z}(x \mid b, \tau)=0$, when $x \notin$ $[\tau]$, the convex hull of $\tau$.

## 3. Dirichlet Averages

Let $\Omega$ to be a nonempty open convex set in $\mathbb{C}^{s}, s \in \mathbb{N}$, and let $b \in \mathbb{C}_{+}^{\mathbb{N}_{0}}$. Let $f \in \mathscr{S}(\Omega):=\mathscr{S}(\Omega, \mathbb{C})$ be a measurable function. For $\tau \in \Omega^{\mathbb{N}_{0}} \subset\left(\mathbb{C}^{s}\right)^{\mathbb{N}_{0}}$ and $u \in \triangle^{\infty}$, define $\tau \cdot u$ to be the bilinear mapping $(\tau, u) \mapsto \sum_{i=1}^{\infty} u_{i} \tau^{i}$. The infinite sum exists if there exists a $\varrho \in[0, e)$ so that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \sqrt[n]{\left\|\tau^{n}\right\|} \leq \varrho \tag{5}
\end{equation*}
$$

Here, $\|\cdot\|$ now denotes the canonical Euclidean norm on $\mathbb{C}^{s}$. (See also [6].)

Definition 4. Let $f: \Omega \subset \mathbb{C}^{s} \rightarrow \mathbb{C}$ be a measurable function. The Dirichlet average $F: \mathbb{C}_{+}^{\mathbb{N}_{0}} \times \Omega^{\mathbb{N}_{0}} \rightarrow \mathbb{C}$ over $\Delta^{\infty}$ is defined by

$$
F(b ; \tau):=\int_{\Delta \infty} f(\tau \cdot u) d \mu_{b}(u)
$$

where $\mu_{b}=\lim _{\rightleftarrows} \mu_{b}^{n}$ is the projective limit of Dirichlet measures on the $n$-dimensional standard simplex $\triangle^{n}$.

We remark that the Dirichlet average is holomorphic in $b \in\left(\mathbb{C}_{+}\right)^{\mathbb{N}_{0}}$ when $f \in C(\Omega, \mathbb{C})$ for every fixed $\tau \in \Omega^{\mathbb{N}_{0}}$. (See [2] for the finite-dimensional case and [9] for the infinite-dimensional setting.)

Definition 5. [1] Let $f: \Omega \subset \mathbb{C} \rightarrow \mathbb{C}$ be continuous. Let $b \in \mathbb{C}_{+}^{k+1}$ and $\beta \in \mathbb{C}_{+}^{\varkappa+1}$. Suppose that for fixed $k, \varkappa \in \mathbb{N}, X \in \mathbb{C}^{(k+1) \times(\varkappa+1)}$ and that the convex hull [ $X$ ] of $X$ is contained in $\Omega$. Then the double Dirichlet average of $f$ is defined by

$$
\mathscr{F}(b ; X ; \beta):=\int_{\triangle^{k}} \int_{\Delta^{\kappa}} f(u \cdot X v) d \mu_{b}^{k}(u) d \nu_{\beta}^{\varkappa}(v),
$$

where $u \cdot X v:=\sum_{i=0}^{k} \sum_{j=0}^{\varkappa} u_{i} X_{i j} v_{j}$.
Note that $\mathscr{F}(b ; X ; \beta)$ is holomorphic on $\Omega$ in the elements of $b, \beta$, and $X$.
We again use projective limits to extend the notion of double Dirichlet average to an infinite-dimenional setting. To this end, let $u, v \in \triangle^{\infty}$ and let $\mu_{b}=\lim \mu_{b}^{n}$ and $\nu_{\beta}=\lim \nu_{\beta}^{n}$ be the projective limits of Dirichlet measures $\mu_{b}^{n}$ and $\nu_{\beta}^{n}$ of the form (1) on the $n$-dimensional standard simplex, where $b, \beta \in \mathbb{C}_{+}^{\mathbb{N}_{0}}$. Now suppose that $X \in \mathbb{C}^{\mathbb{N}_{0} \times \mathbb{N}_{0}}$ is a infinite matrix with the property that $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty}\left|X_{i j}\right|$ converges. Let

$$
u \cdot X v:=\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{i} X_{i j} v_{j}
$$

Suppose that $\Omega \subset \mathbb{C}$ contains the convex hull $[X]$ of $X$ and that $f: \Omega \rightarrow \mathbb{C}$ is continuous. The double Dirichlet average of $f$ over $\triangle^{\infty}$ is then given by

$$
\begin{equation*}
\mathscr{F}(b ; X ; \beta):=\int_{\Delta^{\infty}} \int_{\Delta_{\infty}} f(u \cdot X v) d \mu_{b}(u) d \nu_{\beta}(v) . \tag{6}
\end{equation*}
$$

(We use the same symbol for the (double) Dirichlet average over $\triangle^{\infty}$ and its finite-dimensional projections $\triangle^{n}$.) It is easy to show that

$$
\begin{equation*}
\mathscr{F}(b ; X ; \beta)=\int_{\triangle_{\infty}} F(\beta ; u X) d \mu_{b}(u) \tag{7}
\end{equation*}
$$

where $u X:=\left\{\left\langle u, X_{j}\right\rangle\right\}_{j \in \mathbb{N}_{0}}$, with $X_{j}$ denoting the $j$ column of $X$.
We note that $\mathscr{F}(b ; X ; \beta)$ is holomorphic in the elements of $b, \beta$, and $X$ over $\triangle^{\infty}$.
For $z \in \mathbb{C}_{+}$, we define
$\mathscr{F}^{(z)}(b ; X ; \beta):=\int_{\Delta_{\infty}} \int_{\Delta_{\infty}} f^{(z)}(u \cdot X v) d \mu_{b}(u) d \nu_{\beta}(v)$.
(See also [9] for the case of a single Dirichlet average.)

## 4. Double Dirichlet Averages and Complex B-Splines

Assume now that the matrix $X$ is real-valued and of the form $X_{i j}=0$, for $i \geq s$ and all $j \in \mathbb{N}_{0}$, some $s \in \mathbb{N}$. In other words, $X \in \mathbb{R}^{s \times \mathbb{N}_{o}}$.

Theorem 6. Suppose that $\beta \in \mathbb{R}_{+}^{\infty}$ and that $\operatorname{Re} z>1$. Let $b:=\left(b_{0}, b_{1}, \ldots, b_{s-1}\right) \in \mathbb{R}^{s}$ be such that $\sum_{i=0}^{s-1} b_{i} \notin$ $-\mathbb{N}_{0}$. Assume that $f \in \mathscr{S}\left(\mathbb{R}^{+}\right)$. Further assume that uX is separated for all $u \in \triangle^{s-1}$. Then

$$
\mathscr{F}^{(z)}(b ; X ; \beta)=\int_{\mathbb{R}^{s}} \boldsymbol{B}_{z}(x \mid \beta, X) F^{(z)}(b ; x) d x .
$$

Proof. We prove the formula first for $b \in \mathbb{R}_{+}^{s}$. To this end, we identify $u=\left(u_{0}, u_{1}, \ldots, u_{s-1}, 0,0, \ldots\right) \in \Delta^{\infty}$ with $\left(u_{0}, u_{1}, \ldots, u_{s-1}\right) \in \triangle^{s-1}$. By the Hermite-Genocchi formula for complex B-splines (see [6] and to some extend [9]), we have that

$$
\begin{aligned}
F^{(z)}(\beta ; u X) & =\int_{\triangle^{\infty}} f^{(z)}\left(u^{\prime} \cdot u X\right) d \mu_{\beta}\left(u^{\prime}\right) \\
& =\int_{\mathbb{R}} f^{(z)}(t) B_{z}(t \mid \beta, u X) d t
\end{aligned}
$$

Substituting this expression into (7) and using (4) gives

$$
\begin{aligned}
& \mathscr{F}^{(z)}(b ; X ; \beta)= \\
& \quad \int_{\Delta \infty} \int_{\mathbb{R}^{s}} f^{(z)}(\langle u, x\rangle) \boldsymbol{B}_{z}(x \mid \beta, u X) d x d \mu_{b}(u) .
\end{aligned}
$$

Interchanging the order of integration yields the statement for $b \in \mathbb{R}_{+}^{s}$. To obtain the general case $b \in \mathbb{R}^{s}$, we note that by Theorem 6.3-7 in [2], the Dirichlet average $F$ can be holomorphically continued in the $b$-parameters provided $\sum_{i=0}^{s-1} b_{i} \notin-\mathbb{N}_{0}$.

Remark 7. Theorem 6 extends Theorem 6.1 in [12] to complex B-splines and the $\triangle^{\infty}$-setting.

## 5. Moments of Complex B-Splines

Following [2], we define the $R$-hypergeometric function $R_{a}(b ; \tau): \mathbb{R}_{+}^{s} \times \Omega^{s} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
R_{a}(b ; \tau):=\int_{\Delta^{s-1}}(\tau \cdot u)^{a} d \mu_{b}^{s-1}(u) \tag{8}
\end{equation*}
$$

where $\Omega:=H, H$ a half-plane in $\mathbb{C} \backslash\{0\}$, if $a \in \mathbb{C} \backslash \mathbb{N}$, and $\Omega:=\mathbb{C}$, if $a \in \mathbb{N}$. It can be shown (see [2]) that $R_{-a}$, $a \in \mathbb{C}_{+}$, has a holomorphic continuation in $\tau$ to $\mathbb{C}_{0}$, where $\mathbb{C}_{0}:=\{\zeta \in \mathbb{C} \mid-\pi<\arg \zeta<\pi\}$.
Taking in the definition of the double Dirichlet average (6) for $f$ the real-valued function $t \mapsto t^{-c}$, where $c:=$ $\sum_{i=0}^{s-1} b_{i}$, the resulting double Dirichlet average is denoted by $\mathscr{R}_{-c}(b ; X ; \beta)$ and generalizes power functions. The corresponding single Dirichlet average $R_{-c}(b ; x)$, where $x=\left(x_{0}, \ldots, x_{s-1}\right)$, is given by

$$
\begin{equation*}
R_{-c}(b ; x)=\prod_{i=0}^{s-1} x_{i}^{-b_{i}}, \quad x \notin[X] . \tag{9}
\end{equation*}
$$

(See, [2], (6.6-5).)
Now, let $p=\left(p_{0}, p_{1}, \ldots, p_{s-1}\right) \in \mathbb{R}^{s}, s \in \mathbb{N}$, be a multiindex all of whose components satisfy $p_{i}<-\frac{1}{2}$. The moment $\mathrm{M}_{|p|}^{(z)}(b ; X)$ of order $|p|:=\sum_{i=1}^{s} p_{i}$ of the complex B-spline $\boldsymbol{B}_{z}(\bullet \mid \beta, X)$ is defined by

$$
\mathrm{M}_{|p|}^{(z)}(b ; X):=\int_{\mathbb{R}^{s}} x^{p} \boldsymbol{B}_{z}(x \mid \beta, X) d x
$$

Note that since $\boldsymbol{B}_{z}(\bullet \mid \beta, X) \in L^{2}\left(\mathbb{R}^{s}\right)$ and $\boldsymbol{B}_{z}(\bullet \mid$ $\beta, X)=0$, for $x \notin[X]$, an easy application of the Cauchy-Schwartz inequality shows that the above integral exists provided the multi-index $p$ satisfies the aforementioned condition on its components.
Using a result from [8], namely Property 2.5 (b), and requiring that $\operatorname{Re} z<\operatorname{Re} c$, we substitute the function $f:=\frac{\Gamma(c-z)}{\Gamma(c)}(\bullet)^{-(c-z)}$ into (8) to obtain

$$
R_{-(c-z)}^{(z)}(b ; x)=R_{-c}(b ; x)=\prod_{i=0}^{s-1} x_{i}^{b_{i}}
$$

The above considerations together with Theorem 6 immediately yield the next result.

Corollary 8. Suppose that $\beta \in \mathbb{R}_{+}^{\infty}$ and that $\operatorname{Re} z>1$. Let $b:=\left(b_{0}, b_{1}, \ldots, b_{s-1}\right) \in\left(-\infty,-\frac{1}{2}\right)^{s}$ be such that $c:=\sum_{i=0}^{s-1} b_{i} \notin-\mathbb{N}_{0}$. Moreover, suppose that $\operatorname{Re} z<$ Rec. Then

$$
\begin{equation*}
\mathrm{M}_{-c}^{(z)}(b ; X)=\mathscr{R}_{-(c-z)}^{(z)}(b ; X ; \beta) \tag{10}
\end{equation*}
$$

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