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Reconstruction of signals in a shift-invariant space from nonuniform samples

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Abstract- In this paper, we consider the method for reconstructing a signal from a finite number of samples. In shift-invariant space framework, we derive an approximately min-max optimal interpolator to reconstruct a signal on an interval. An effective non-iterative algorithm for signal reconstruction is given also. Numerical examples show the effectiveness of the proposed method.

Index Terms—sampling, signal reconstruction, scaling function, shift-invariant space

I. INTRODUCTION

The problem of signal reconstruction is pervasive in many areas of signal processing, such as in designs of nonuniform antenna arrays, sparse array beamforming, the restoration of signals with missing samples, image acquisition, etc [1-3]. The classical Shannon's sampling theorem was extended theoretically to general shift-invariant subspaces, and various generalized sampling theorems concerning band-limited and nonband-limited signals have been proposed [4-9]. However, the problem of reconstructing a continuous-time signal from its finite number of nonuniform samples is often encountered in practical applications, and truncating infinite reconstruction leads to errors.

Bandlimited and non bandlimited signals are often modeled by shift-invariant spaces. Some authors have proposed interpolation methods and iterative methods for reconstructing signals in shift-invariant spaces in the signal processing literatures[10-14]. A non-iterative reconstruction method is effective to reconstruct continuous-time signals from a finite number of samples by using a suitable interpolator. The Yen interpolator is well known to reconstruct band-limited signals in both minimal energy and least squares senses [13]. Some interpolation methods in shift-invariant spaces were given[9-10,12-14].

In this paper, we are interested in optimally constructing signals in a shift-invariant space from a finite number of nonuniform samples, and develop a new method for reconstructing continuous-time signal on a interval. The upper bound of reconstruction error is derived. We also propose a practical reconstruction algorithm. The method of signal reconstruction can be regarded as a generalization of Yen's in general shift-invariant spaces.

The paper is organized as follows: in Section II we formulate the optimal reconstruction problem in shift-invariant spaces, and Section III derives a new method to reconstruct a signal from a finite number of arbitrarily distributed samples. Section IV propose a

practical algorithm for implementing the optimal signal reconstruction. In Section V, some numerical examples of reconstructing signals demonstrate the effectiveness of the proposed method.

II. THE PROBLEM FORMULATION

Throughout the paper, we focus on one-dimensional signals and denote the space of signals of finite energy on \mathbb{R} by $L^2(\mathbb{R})$. Let $\|f\|^2 = \int_{\mathbb{R}} |f(t)|^2 dt$ be the energy of a signal $f(t) \in L^2(\mathbb{R})$. Given K scaling functions $\varphi_1(t), \varphi_2(t), \dots, \varphi_K(t) \in L^2(\mathbb{R})$, the shift-invariant space $V(\varphi_1, \varphi_2, \dots, \varphi_K)$ is a Hilbert space defined as

$V(\varphi_1, \varphi_2, \dots, \varphi_K) = \text{close}\{f(t) \in L^2(\mathbb{R}) :$

$$f(t) = \sum_{k=1}^K \sum_n c_k(n) \varphi_k(t-n) \\ , (c_k(n)) \in l^2(\mathbb{Z}), k=1, 2, \dots, K\}$$

We assume that $\{\varphi_k(t-n) | 1 \leq k \leq K, n \in \mathbb{Z}\}$ forms a frame of $V(\varphi_1, \varphi_2, \dots, \varphi_K)$, i.e., there exist two constants $A > 0$ and $B < +\infty$ such that

$$A \|f\|^2 \leq \sum_{k=1}^K \sum_n |(f, \varphi_k(t-n))|^2 \leq B \|f\|^2 \quad (1)$$

for any $f \in V(\varphi_1, \varphi_2, \dots, \varphi_K)$.

To make the sampling of functions in $V(\varphi_1, \varphi_2, \dots, \varphi_K)$ well-defined, we additionally assume that there exists a constant C such that

$$\sum_{k=1}^K \sum_n |\varphi_k(t-n)| < C \quad (2)$$

for any $t \in [0, 1]$. To see this, let

$$f(t) = \sum_{k=1}^K \sum_n c_k(n) \varphi_k(t-n) \text{ with } (c_k(n)) \in l^2(\mathbb{Z}).$$

From (2) and (3) it follows that

$$|f(t)| \leq \left(\sum_{k=1}^K \sum_n |\varphi_k(t-n)|^2 \right)^{1/2} \left(\sum_{k=1}^K \sum_n |c_k(n)|^2 \right)^{1/2} \\ \leq C' \|f\|, \quad t \in \mathbb{R} \quad (3)$$

where C' is a constant.

It is known from [8] that the assumption (3) implies that $V(\varphi_1, \varphi_2, \dots, \varphi_K)$ is a reproducing kernel Hilbert space. For a function $f(t)$ in $V(\varphi_1, \varphi_2, \dots, \varphi_K)$, we adopt the

following interpolator

$$\tilde{f}(t) = \sum_{m=1}^M h_m(t) f(t_m)$$

to reconstruct $f(t)$ on the interval containing sampling instants t_1, \dots, t_M . In order to obtain an optimal interpolator, we discuss the optimization

$$\inf_{\mathbf{h}} \sup_f \frac{|f(t) - \tilde{f}(t)|^2}{\|f\|^2},$$

which yields a min-max type interpolator.

III. DERIVATION OF SIGNAL RECONSTRUCTION

Given M samples of a function $f(t) \in V(\varphi_1, \varphi_2, \dots, \varphi_K)$ at nonuniform instants $t_1, t_2, \dots, t_M \in [a, b]$, let $\tau \in [a, b]$ and

$\tilde{f}(\tau) = \sum_{m=1}^M h_m(\tau) f(t_m)$. The optimal estimation $\tilde{f}(\tau)$ of

$f(\tau)$ is determined by appropriate coefficients $h_m(\tau)$'s. So, we study the following optimization

$$\inf_{\mathbf{h}(\tau)} \sup_{f \in V(\varphi_1, \dots, \varphi_K)} \frac{|f(\tau) - \tilde{f}(\tau)|^2}{\|f\|^2} \quad (4)$$

Letting

$$f(t) = \sum_n \sum_{k=1}^K c_k(n) \varphi_k(t-n), \quad (5)$$

where, $c_k(n) = (f, \tilde{\varphi}_k(t-n))$, $k=1, 2, \dots, K$, $n \in \mathbb{Z}$, and $\{\tilde{\varphi}_k(t-n) | 1 \leq k \leq K, n \in \mathbb{Z}\}$ is the dual frame of $\{\varphi_k(t-n) | 1 \leq k \leq K, n \in \mathbb{Z}\}$. We have

$$\begin{aligned} & |f(\tau) - \tilde{f}(\tau)|^2 \\ &= \left| \sum_n \sum_{k=1}^K c_k(n) [\varphi_k(\tau-n) - \sum_{m=1}^M h_m(\tau) \varphi_k(t_m-n)] \right|^2 \\ &\leq \sum_n \sum_{k=1}^K |c_{k,n}(n)|^2 \sum_n \sum_{k=1}^K |\varphi_k(\tau-n) - \sum_{m=1}^M h_m(\tau) \varphi_k(t_m-n)|^2 \quad (6) \\ &\leq A^{-1} \|f\|^2 \sum_n \sum_{k=1}^K |\varphi_k(\tau-n) - \sum_{m=1}^M h_m(\tau) \varphi_k(t_m-n)|^2. \end{aligned}$$

The above can furthermore be expressed explicitly in vector form as

$$|f(\tau) - \tilde{f}(\tau)|^2 \leq A^{-1} \|f\|^2 \sum_{k=1}^K \left\| \mathbf{e}_{k,\tau} - \sum_{m=1}^M h_m(\tau) \mathbf{e}_{k,m} \right\|^2, \quad (7)$$

where $\mathbf{e}_{k,\tau}^T = (\varphi_k(\tau-n))_n$, $\mathbf{e}_{k,m}^T = (\varphi_k(t_m-n))_n$, $k=1, 2, \dots, K$. So, it follows that

$$\sup_f \frac{|f(\tau) - \tilde{f}(\tau)|^2}{\|f\|^2} \leq A^{-1} \sum_{k=1}^K \left\| \mathbf{e}_{k,\tau} - \sum_{m=1}^M h_m(\tau) \mathbf{e}_{k,m} \right\|^2.$$

It can be seen that minimizing

$$E(\tau) = \sum_{k=1}^K \left\| \mathbf{e}_{k,\tau} - \sum_{m=1}^M h_m(\tau) \mathbf{e}_{k,m} \right\|^2$$

can make $\tilde{f}(\tau)$ approximate $f(t)$ at τ .

To minimize $E(\tau)$, we further express $E(\tau)$ as

$$\begin{aligned} E(\tau) &= \sum_{k=1}^K \left\| \mathbf{e}_{k,\tau} - \sum_{m=1}^M h_m(\tau) \mathbf{e}_{k,m} \right\|^2 = \\ &= \sum_{k=1}^K \left\| \mathbf{e}_{k,\tau} \right\|^2 - 2 \sum_{k=1}^K \mathbf{e}_{k,\tau}^T \mathbf{A}_k \mathbf{H}(\tau) + \mathbf{H}^T(\tau) \left(\sum_{k=1}^K \mathbf{A}_k^T \mathbf{A}_k \right) \mathbf{H}(\tau). \end{aligned}$$

where $\mathbf{H}(\tau) = (h_1(\tau), h_2(\tau), \dots, h_M(\tau))^T$ and $\mathbf{A}_k = (\mathbf{e}_{k,1}, \mathbf{e}_{k,2}, \dots, \mathbf{e}_{k,M})$ for $k=1, 2, \dots, K$. It is followed that

$$\frac{\partial E(\tau)}{\partial \mathbf{H}(\tau)} = -2 \sum_{k=1}^K \mathbf{A}_k^T \mathbf{e}_{k,\tau} + 2 \sum_{k=1}^K \mathbf{A}_k^T \mathbf{A}_k \mathbf{H}(\tau). \quad (8)$$

Therefore, solving $\frac{\partial E(\tau)}{\partial \mathbf{H}(\tau)} = 0$ yields that

$$\mathbf{H}(\tau) = \left(\sum_{k=1}^K \mathbf{A}_k^T \mathbf{A}_k \right)^{-1} \sum_{k=1}^K \mathbf{A}_k^T \mathbf{e}_{k,\tau}. \quad (9)$$

and the minimal error between $f(\tau)$ and $\tilde{f}(\tau)$ is given by

$$\begin{aligned} |f(\tau) - \tilde{f}(\tau)| &\leq r(\tau) = \\ &= \sqrt{A^{-1}} \|f\| \left[\sum_{k=1}^K \left\| \mathbf{e}_{k,\tau} - \mathbf{A}_k \left(\sum_{l=1}^K \mathbf{A}_l^T \mathbf{A}_l \right)^{-1} \sum_{m=1}^K \mathbf{A}_m^T \mathbf{e}_{m,\tau} \right\|^2 \right]^{1/2}. \quad (10) \end{aligned}$$

Note that the matrix $\sum_{k=1}^K \mathbf{A}_k^T \mathbf{A}_k$ is invertible when the samples $\{f(t_m)\}_{m=1}^M$ is independent, that is, there doesn't exist one sample in $\{f(t_m)\}_{m=1}^M$ which can be expressed by the others for each $f \in V(\varphi_1, \dots, \varphi_M)$.

Letting $\mathbf{A}_k = (a_{ij}^{(k)})$ with $a_{ij}^{(k)} = \varphi_k(t_j - i)$

and $\mathbf{X} = (x_{ij}) = \left(\sum_{k=1}^K \mathbf{A}_k^T \mathbf{A}_k \right)^{-1}$, we then rewrite the optimal interpolating vector as

$$\begin{aligned} \mathbf{H}(\tau) &= (h_1(\tau), h_2(\tau), \dots, h_M(\tau))^T = \mathbf{X} \sum_{k=1}^K \mathbf{A}_k^T \mathbf{e}_{k,\tau} = \\ &= \sum_{k=1}^K \mathbf{X} \left(\sum_n \varphi_k(t_1-n) \varphi_k(\tau-n), \dots, \sum_n \varphi_k(t_M-n) \varphi_k(\tau-n) \right)^T \\ &= \left(\sum_{k=1}^K \sum_{l=1}^M \sum_n x_{kl} \varphi_k(t_l-n) \varphi_k(\tau-n), \dots, \right. \\ &\quad \left. \sum_{k=1}^K \sum_{l=1}^M \sum_n x_{Ml} \varphi_k(t_l-n) \varphi_k(\tau-n) \right)^T. \end{aligned} \quad (11)$$

So, the optimal reconstruction of $f(t)$ can be expressed as

$$\begin{aligned} \tilde{f}(t) &= \sum_{m=1}^M f(t_m) h_m(t) \\ &= \sum_{m=1}^M f(t_m) \left[\sum_{k=1}^K \sum_{l=1}^M \sum_n x_{ml} \varphi_k(t_l-n) \varphi_k(t-n) \right]. \end{aligned} \quad (12)$$

Let us take a look at the case $K=1$ in detail. Given M samples of $x(t) \in V(\varphi)$ at instants t_1, t_2, \dots, t_M for a proper scaling function $\varphi(t) \in L^2(\mathbb{R})$, we can express the optimal interpolating vector as $\mathbf{H}(\tau) = (A^T A)^{-1} A^T \mathbf{e}_\tau$,

where $A = (\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_M)$ and $\mathbf{e}_\tau = (\varphi(\tau - n))_n^T$, $\mathbf{e}_m = (\varphi(t_m - n))_n^T$ for $m = 1, 2, \dots, M$. It is easy to see that the optimal interpolating vector $\mathbf{H}(\tau)$ is exactly the orthogonal projection of vector \mathbf{e}_τ onto the subspace spanned by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_M$ and hence from (12) $\tilde{x}(t) \in V(\varphi)$ with $x(t_m) = \tilde{x}(t_m)$ for $m = 1, 2, \dots, M$. This implies that the reconstructed signal best fits the sampling data. Especially, for $\sigma > 0$ and $\varphi(t) = \text{sinc}(\sigma t)$, the optimal reconstruction $\tilde{x}(t)$ of $x(t) \in V(\varphi)$ from samples $x(t_1), x(t_2), \dots, x(t_M)$ is also band-limited to σ with $x(t_m) = \tilde{x}(t_m)$ for $m = 1, 2, \dots, M$. It is easy to show that the interpolator obtained here is just Yen's for band-limited signals.

IV. ALGORITHM AND DISCUSSION

In the previous section we have derived an interpolator for signal reconstruction. However, because computing the min-max interpolator requires calculating the inverse of a matrix with possibly larger dimension, the reconstruction formula (12) would be unfeasible when the number of samples is much large. To circumvent this problem, we reshape (12) as

$$\tilde{f}(\tau) = \sum_n \left(\sum_{l=1}^M \sum_{k=1}^K \sum_{m=1}^M f(t_m) x_{ml} \varphi_k(t_l - n) \right) \varphi_k(\tau - n). \quad (13)$$

From this, a non-iterative reconstruction algorithm can be given as follows.

Algorithm:

- (1) Let $\mathbf{f} = (f(t_1), f(t_2), \dots, f(t_M))^T$,
 $\mathbf{E}_k(t) = (\dots, \varphi_k(t - n), \varphi_k(t - n + 1), \dots)^T$,
 $\mathbf{A}_k = (b_{mn}^{(k)})$ with $b_{mn}^{(k)} = \varphi_k(t_m - n)$ for
 $i = 1, 2, \dots, M$ and $k = 1, 2, \dots, K$;
- (2) Compute $\mathbf{T} = \sum_{k=1}^K \mathbf{A}_k \mathbf{A}_k^T$
- (3) Solve $\mathbf{T}\mathbf{h} = \mathbf{f}$;
- (4) $\tilde{f}(t) = \mathbf{h}^T \sum_{k=1}^K \mathbf{A}_k \mathbf{E}_k(t)$.

The most crucial step is solving the equation system of equations $\mathbf{T}\mathbf{h} = \mathbf{f}$. This can be done effectively by computing the Cholesky factorization of the matrix \mathbf{T} when \mathbf{T} is invertible. In fact, the cholesky factorization of \mathbf{T} gives a upper triangular matrix \mathbf{S} such that $\mathbf{T} = \mathbf{S}^T \mathbf{S}$. Then the solution of the system $\mathbf{T}\mathbf{h} = \mathbf{f}$ can be obtained by sequentially solving the systems $\mathbf{S}^T \mathbf{b} = \mathbf{f}$ and $\mathbf{S}\mathbf{h} = \mathbf{b}$ by Gaussian elimination. This procedure is faster and more robust than directly computing the inverse of \mathbf{T} . When \mathbf{T} is not invertible, the equation can be solved effectively by the least squares method.

Note that although the proposed method has no restriction on sampling locations, the obtained

reconstruction error is strongly related to the sampling pattern. As pointed in [18], the reconstruction errors are smaller in the neighborhood of the sampling instants. So, the quality of reconstruction should be evaluated in a pointwise manner. From (10) we know that the min-max reconstruction error is pointwise upper-bounded by

$$r(\tau) \leq A^{-1/2} \|\mathbf{f}\| \left[\sum_{k=1}^K \|\mathbf{e}_{k,\tau} - \mathbf{A}_k \left(\sum_{l=1}^K \mathbf{A}_l^T \mathbf{A}_l \right)^{-1} \sum_{m=1}^K \mathbf{A}_m^T \mathbf{e}_{m,\tau}\|^2 \right]^{1/2} \quad (14)$$

and it can estimate the quality of reconstruction when the sampling instants are known.

V. DEMONSTRATIVE EXAMPLES

Some numerical examples are given to demonstrate the performance of the proposed method, where signals are selected randomly in shift-invariant subspaces, and the sampling instants are generated by adding random perturbation, distributed uniformly in the interval $[-u, u]$ for $u > 0$, to each equally-spaced sampling instant, i.e., the sampling instants are $mT + u_m$, where u_m randomly distributes in $[-u, u]$ for $m = 1, 2, \dots, M$.

Example 1 For the first example, we choose arbitrarily a signal of band $[-\pi, \pi]$. We reconstruct it on $[0, 40]$ from 42 samples. The average sampling period is $T = 0.995$ s and $u = 0.7T$. It is clear that the average sampling period is almost critical. The reconstructed signal, its reconstruction from its nonuniform samples and the errors between the original signal and its reconstructions were plotted in Fig.1. From this experiment, it can be seen that under such a relaxed condition, the reconstruction of a signal is quite satisfying.

Example 2 For non band-limited signals, we choose the cubic spline [19] as a scaling function, and randomly choose a signal in the shift-invariant space. The average sampling period is $T = 1.0$ s and the maximum of irregular perturbation is $u = 0.5T$. The signal to be reconstructed, its reconstruction and the reconstruction error (in dB) were plotted in Fig.2, respectively. Note that, although the sampling density is much lower than that estimated in [5](see the examples therein for details), the quality of signal reconstruction is still considerably high. In addition, although the cubic spline is supported compactly, the method given in [10] could not be used in this case because the maximal gap between adjacent sampling instants is too large. In fact, the local reconstruction methods in [10] required the condition that the maximal gap of the sampling instants must be less 1 and the number of the samples must be bigger than the length of the reconstruction interval, but our method doesn't rely on any sampling condition. In contrast to the results in [10], we also give another example to show the performance of the proposed method in Fig.3. In this example, we chose the same scaling function, a Gaussian function, and the similar sampling condition as in [10].

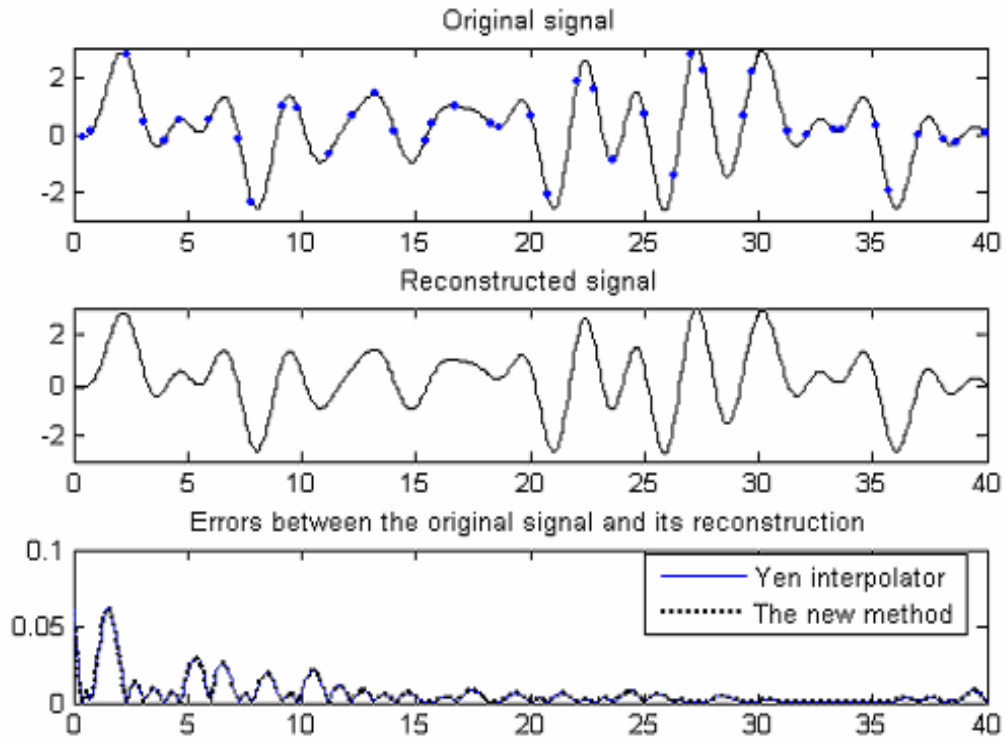


Fig. 1 Top : original signal and sampling points marked by dots; middle: reconstructed signal obtained by the proposed method; bottom: normalized errors between the original signal the its reconstruction obtained by the proposed method and Yen interpolator.

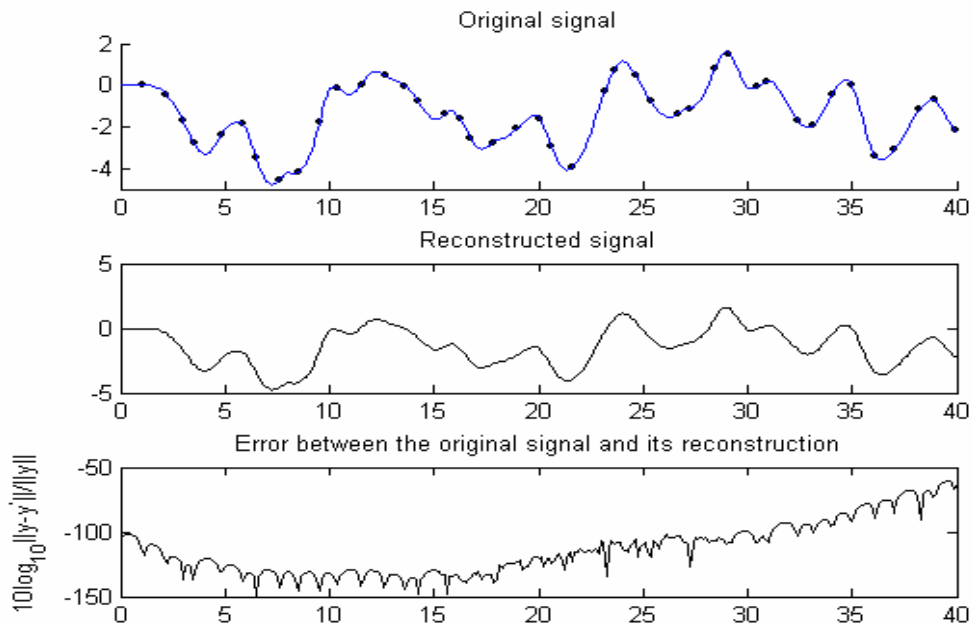


Fig. 2 original signal with sampling points marked by stars, reconstructed signal obtained by the proposed method, normalized error(in dB) between the original signal the its reconstruction.

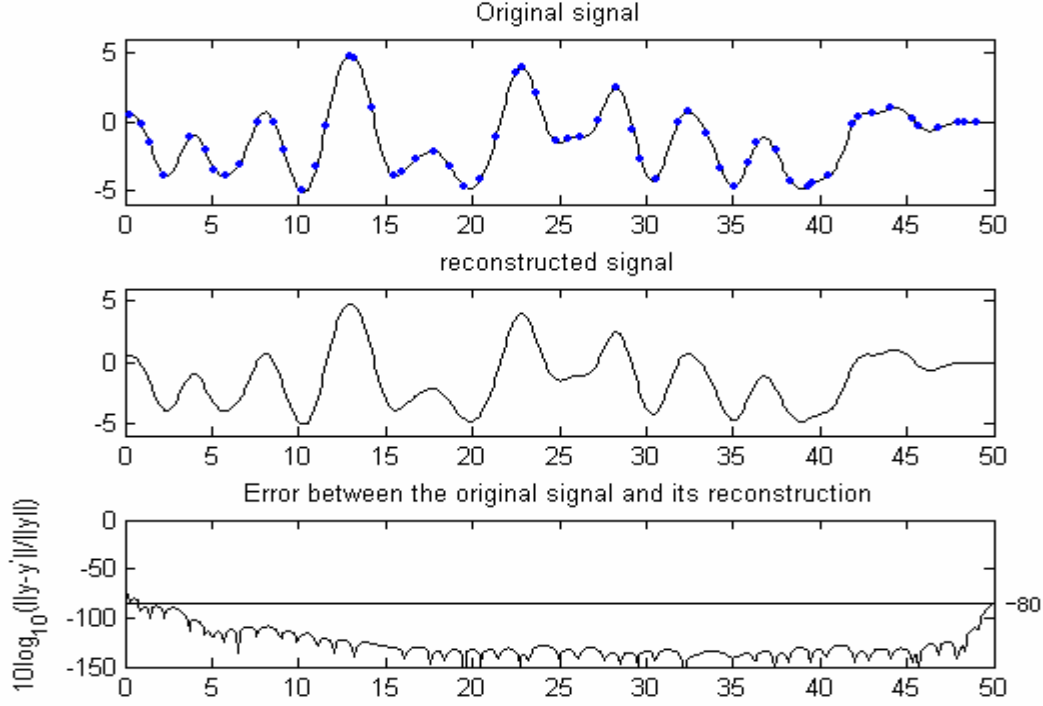


Fig. 3 original signal with sampling points marked by stars, reconstructed signal obtained by the proposed method, normalized error(in dB) between the original signal the its reconstruction with scaling function $\varphi(t) = e^{-t^2/2\sigma^2}$, $\sigma = 0.81$, and sampling density 0.85.

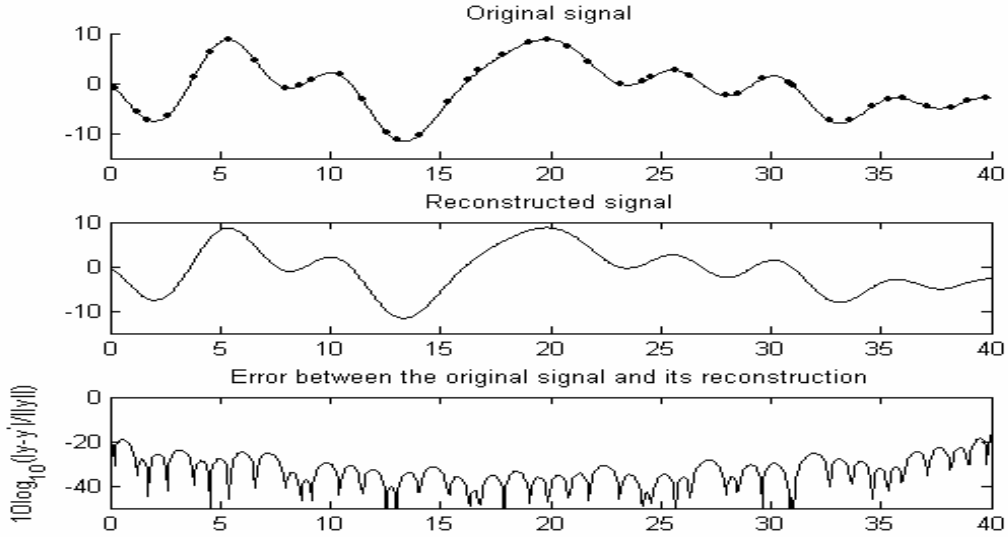


Fig. 4 original signal with sampling points marked by stars, reconstructed signal obtained by the proposed method, normalized error(in dB) between the original signal the its reconstruction.

Example 3. Finally, we select two functions $\varphi_1(t) = a_1 e^{-t^2/4}$ and $\varphi_2(t) = a_2 (t + t^3) e^{-t^2/4}$ as scaling functions, where a_1 and a_2 are normalized constants. Here the average sampling period is $T = 0.8$ s and the maximum of irregular perturbation $u = 0.6T$. The

simulation results were showed in Fig.4. This example also indicates the feasibility of the proposed method for signal reconstruction in a shift-invariant spaced with several scaling functions

VI. CONCLUSION

The proposed method of reconstructing a signal from its finite nonuniform samples has the following advantages: (a) the method doesn't require the usual hypotheses on the maximal gap between adjacent sampling instants and the compactness of the scaling functions of the shift-invariant space as in the literature, and therefore can be applied in various shift-invariant spaces with sampling locations distributed arbitrarily. (b) The reconstruction error function as sensitivity functions [17] can measure the quality of the reconstruction prior to the practical implementation. (c) the method can be used effectively in a multi-wavelet space and can be extended straightforward to two-dimensional spaces. However, our method does not incorporate the case when samples are noisy, which we will investigate in future.

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