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Global existence vs. blowup for the one dimensional quasilinear Smoluchowski-Poisson system

Tomasz Cieślak* & Philippe Laurençot†

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Abstract

We prove that, unlike in several space dimensions, there is no critical (nonlinear) diffusion coefficient for which solutions to the one dimensional quasilinear Smoluchowski-Poisson equation with small mass exist globally while finite time blowup could occur for solutions with large mass.

1 Introduction

In a previous paper [4] we investigate the influence of the diffusion coefficient a on the life span of solutions to the one dimensional Smoluchowski-Poisson system

$$\partial_t u = \partial_x (a(u)\partial_x u - u\partial_x v) \quad \text{in } (0, \infty) \times (0, 1), \quad (1)$$

$$0 = \partial_x^2 v + u - M \quad \text{in } (0, \infty) \times (0, 1), \quad (2)$$

$$a(u)\partial_x u = \partial_x v = 0 \quad \text{on } (0, \infty) \times \{0, 1\}, \quad (3)$$

$$u(0) = u_0 \geq 0 \quad \text{in } (0, 1), \quad \int_0^1 v(t, x) dx = 0 \quad \text{for any } t \in (0, \infty), \quad (4)$$

where

$$M := \langle u_0 \rangle = \int_0^1 u_0(x) dx$$

denotes the mean value of u_0 , and uncover a fundamental difference with the quasilinear Smoluchowski-Poisson system in higher space dimensions. More precisely, when the space dimension n is greater or equal

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to two, there is a critical diffusion $a_*(r) := (1+r)^{(n-2)/2}$ which separates different behaviours for the quasilinear Smoluchowski-Poisson system. Roughly speaking,

- (a) if the diffusion coefficient a is stronger than a_* (in the sense that $a(r) \geq C(1+r)^\alpha$ for some $\alpha > (n-2)/n$ and $C > 0$), then all solutions exist globally whatever the value of the mass of the initial condition u_0 [5],
- (b) if the diffusion coefficient a is weaker than a_* (in the sense that $a(r) \leq C(1+r)^\alpha$ for some $\alpha < (n-2)/n$ and $C > 0$), then there exists for all $M > 0$ an initial condition u_0 with $\langle u_0 \rangle = M$ for which the corresponding solution to the quasilinear Smoluchowski-Poisson system blows up in finite time (in the sense that $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T$ for some $T \in (0, \infty)$) [3, 5, 7],
- (c) if the diffusion coefficient a behaves as a_* for large values of r , solutions starting from initial data u_0 with small mass $\langle u_0 \rangle$ exist globally while there are initial data with large mass for which the corresponding solution to the quasilinear Smoluchowski-Poisson system blows up in finite time [3, 7].

Observe that, in space dimension $n = 2$, the critical diffusion is constant and a more precise description of the situation (c) is actually available. Namely, when $a \equiv 1$, there is a threshold mass M_* such that, if $\langle u_0 \rangle < M_*$, the corresponding solution is global while, for any $M > M_*$, there are initial data with $\langle u_0 \rangle = M$ for which the corresponding solution blows up in finite time [6, 7, 8]. The threshold mass M_* is known explicitly ($M_* = 4\pi$) but it is worth mentioning that for radially symmetric solutions in a ball, the threshold mass is 8π . Similar results are also available for the quasilinear Smoluchowski-Poisson system in \mathbb{R}^n , $n \geq 2$ [1, 2, 9, 10].

Most surprisingly, the above description fails to be valid in one space dimension and we prove in particular in [4] that all solutions are global for the diffusion $a(r) = (1+r)^{-1}$ though it is a natural candidate to be critical. We actually identify two classes of diffusion coefficients a in [4], one for which all solutions exist globally as in (a) and the other for which there are solutions blowing-up in finite time starting from initial data with an arbitrary positive mass as in (b), but the situation (c) does not seem to occur in one space dimension. The purpose of this note is to show that the dichotomy (a) or (b) can be extended to larger classes of diffusion, thereby extending the analysis performed in [4].

Theorem 1 *Let the diffusion coefficient $a \in \mathcal{C}^1((0, \infty))$ be a positive function.*

(i) *Assume first that $a \in L^1(1, \infty)$ and one of the following assumptions is satisfied, either*

$$\gamma := \sup_{r \in (0,1)} r \int_r^\infty a(s) ds < \infty, \quad (5)$$

or there exist $\vartheta > 0$ and $\alpha \in (\vartheta/(1+\vartheta), 2]$ such that

$$\gamma_\vartheta := \sup_{r \in (0,1)} r^{2+\vartheta} a(r) < \infty \quad \text{and} \quad C_\infty := \sup_{r \geq 1} r^\alpha a(r) < \infty. \quad (6)$$

For any $M > 0$, there exists a positive initial condition $u_0 \in \mathcal{C}([0, 1])$ such that $\langle u_0 \rangle = M$ and the corresponding classical solution to (1)-(4) blows up in finite time.

(ii) Assume next that $a \notin L^1(1, \infty)$ and consider an initial condition $u_0 \in \mathcal{C}([0, 1])$ such that $u_0 \geq m_0 > 0$ and $\langle u_0 \rangle = M$ for some $M > 0$ and $m_0 \in (0, M)$. Then the corresponding classical solution to (1)-(4) exists globally.

As already mentioned, Theorem 1 extends the results obtained in [4]. More precisely, in [4, Theorem 5], the assertion (ii) of Theorem 1 is proved under the additional assumption that, for each $\varepsilon \in (0, \infty)$, there is $\kappa_\varepsilon > 0$ for which

$$a(r) \leq \varepsilon r a(r) + \frac{\kappa_\varepsilon}{r} \quad \text{for } r \in (0, 1),$$

which roughly means that a cannot have a singularity stronger than $1/r$ near $r = 0$. This assumption turns out to be unnecessary for global existence but nevertheless ensures the global boundedness of the solution in L^∞ . Under the sole assumption of Theorem 1 (ii), our proof does not exclude that solution to (1)-(4) becomes unbounded as $t \rightarrow \infty$. Concerning Theorem 1 (i), it is established in [4, Theorem 10] for $a \in L^1(1, \infty)$ such that there is a concave function B for which

$$0 \leq -rA(r) \leq B(r) \quad \text{with } A(r) = - \int_r^\infty a(s) ds, \quad r \in (0, \infty), \quad (7)$$

$$\lim_{r \rightarrow \infty} \frac{B(r)}{r} = 0. \quad (8)$$

We make this criterion more explicit here by showing that the integrability of a on $(1, \infty)$ and (5) guarantee the existence of a concave function B satisfying (7) and (8), see Lemma 3 below. Let us point out here that the assumption (5) somehow means that a cannot have a singularity stronger than $1/r^2$ near $r = 0$. However, the result remains true if a has an algebraic singularity of higher order near $r = 0$ which is allowed by (6) provided a decays suitably at infinity. Observe that the second condition in (6) is compatible with the integrability of a at infinity as $\vartheta/(1 + \vartheta) < 1$.

Summarizing the outcome of Theorem 1, we realize that, for a given diffusion coefficient a with a singularity weaker than $1/r^2$ near $r = 0$, the integrability or non-integrability of a at infinity completely determines whether we are in the situation **(a)** or **(b)** described above and excludes the situation **(c)**. There is thus no critical diffusion in this class. The same comment applies to the class of diffusion coefficients satisfying (6) with an algebraic singularity stronger than $1/r^2$ near $r = 0$. In particular there is no critical nonlinearity in the class of functions $\mathcal{C}([0, \infty)) \cap \mathcal{C}^1((0, \infty))$.

The paper is organized as follows: in section 2 we recall some statements from [4]. Section 3 is devoted to proving the finite time blowup of solutions to (1)-(4) when $a \in L^1(1, \infty)$. Global existence of solutions for all initial data when a is not integrable at infinity is proved in the last section.

2 Preliminaries.

In this section we summarize some results and methods introduced in [4]. Let $a \in \mathcal{C}^1((0, \infty))$ be a positive function and consider an initial condition $u_0 \in \mathcal{C}([0, 1])$ such that $u_0 \geq m_0 > 0$ and $\langle u_0 \rangle = M$ for some $M > 0$ and $m_0 \in (0, M)$. By [4, Propositions 2 and 3] there is a unique maximal classical solution (u, v) to (1)-(4) defined on $[0, T_{max})$ which satisfies

$$\min_{x \in [0, 1]} u(t, x) > 0, \quad \langle u(t) \rangle := \int_0^1 u(t, x) dx = M, \quad \text{and} \quad \langle v(t) \rangle := \int_0^1 v(t, x) dx = 0 \quad (9)$$

for $t \in (0, T_{max})$. In addition, $T_{max} = \infty$ or $T_{max} < \infty$ with $\|u(t)\|_\infty \rightarrow \infty$ as $t \rightarrow T_{max}$.

We next recall the approach introduced in [4] which will be used herein as well. Owing to the positivity (9) and the regularity of u , the indefinite integral

$$U(t, x) := \int_0^x u(t, z) dz, \quad x \in [0, 1],$$

is a smooth increasing function from $[0, 1]$ onto $[0, M]$ for each $t \in [0, T_{max})$ and has a smooth inverse F defined by

$$U(t, F(t, y)) = y, \quad (t, y) \in [0, T_{max}) \times [0, M]. \quad (10)$$

Introducing $f(t, y) := \partial_y F(t, y)$, we have

$$f(t, y) u(t, F(t, y)) = 1, \quad (t, y) \in [0, T_{max}) \times [0, M], \quad (11)$$

and it follows from (1)-(4) that f solves

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf, \quad (t, y) \in (0, T_{max}) \times (0, M), \quad (12)$$

$$\partial_y f(t, 0) = \partial_y f(t, M) = 0, \quad t \in (0, T_{max}), \quad (13)$$

$$f(0, y) = f_0(y) := \frac{1}{u_0(F(0, y))}, \quad y \in (0, M), \quad (14)$$

where

$$\Psi'(r) := \frac{1}{r^2} a\left(\frac{1}{r}\right) \quad \text{for any } r > 0, \quad \Psi(1) := 0, \quad (15)$$

Moreover the conservation of mass (9) yields

$$\int_0^M f(t, y) dy = F(t, M) - F(t, 0) = 1, \quad t \in [0, T_{max}). \quad (16)$$

At this point, the crucial observation is that, thanks to (11), finite time blowup of u is equivalent to the vanishing (or touch-down) of f . In other words, u exist globally if the minimum of $f(t)$ is positive for each $t > 0$. We refer to [4, Proposition 1] for a more detailed description.

An salient property of (1)-(4) is the existence of a Liapunov function [4, Lemma 8] which we recall now:

Lemma 2 *The function*

$$L_1(t) := \frac{1}{2} \int_0^M |\partial_y \Psi(f(t, y))|^2 dy + \int_0^M (\Psi(f(t, y)) - M \Psi_1(f(t, y))) dy$$

is a non-increasing function of time on $[0, T_{max})$, the function Ψ_1 being defined by

$$\Psi_1(1) := 0 \quad \text{and} \quad \Psi_1'(r) := r \Psi'(r) = \frac{1}{r} a\left(\frac{1}{r}\right), \quad r \in (0, \infty). \quad (17)$$

3 Finite time blowup.

In this section we prove the blowup assertion of Theorem 1. To this end we first prove that the condition (5) allows us to construct a concave function B satisfying (7) and (8) so that [4, Theorem 10] can be applied.

Lemma 3 *Let $a \in \mathcal{C}^1((0, \infty))$ be a positive function such that $a \in L^1(1, \infty)$ and (5) holds. Then there exists a concave function $B \in \mathcal{C}([0, \infty))$ such that for all $r \geq 0$*

$$B(r) \geq r \int_r^\infty a(s) ds \quad (18)$$

and

$$\lim_{r \rightarrow \infty} \frac{B(r)}{r} = 0. \quad (19)$$

Proof of Lemma 3. We construct $B : [0, \infty) \rightarrow [0, \infty)$ in the following way: we put

$$b_i := \int_{2^i}^\infty a(s) ds, \quad i \geq 0,$$

and notice that $\{b_i\}_{i \geq 0}$ is a decreasing sequence converging to zero as $i \rightarrow \infty$. We next define

$$B(r) = \begin{cases} b_0 r + \gamma & \text{if } r \in [0, 2], \\ b_i r + \sum_{j=0}^{i-1} (b_j - b_{j+1}) 2^{j+1} + \gamma & \text{if } r \in (2^i, 2^{i+1}] \text{ and } i \geq 1, \end{cases} \quad (20)$$

Clearly, $B \in \mathcal{C}([0, \infty))$ and

$$B'(r) = \begin{cases} b_0 & \text{if } r \in (0, 2), \\ b_i & \text{if } r \in (2^i, 2^{i+1}) \text{ and } i \geq 1. \end{cases} \quad (21)$$

Hence B is concave as a consequence of the fact that the sequence $\{b_i\}_{i \geq 0}$ is decreasing. Furthermore, for $r \in [0, 1]$, we have

$$B(r) \geq \gamma \geq r \int_r^\infty a(s) ds,$$

and for $r \in [2^i, 2^{i+1}]$, $i \geq 0$,

$$B(r) \geq b_i r = r \int_{2^i}^{\infty} a(s) ds \geq r \int_r^{\infty} a(s) ds.$$

Therefore, B satisfies (18).

Finally, let $k \geq 1$. If $i \geq k + 1$ and $r \in (2^i, 2^{i+1}]$, then

$$\begin{aligned} \frac{B(r)}{r} &= b_i + \frac{\gamma}{r} + \sum_{j=0}^{i-1} (b_j - b_{j+1}) \frac{2^{j+1}}{r} \leq b_i + \frac{\gamma}{r} + \sum_{j=k}^{i-1} (b_j - b_{j+1}) + \sum_{j=0}^{k-1} (b_j - b_{j+1}) \frac{2^{j+1}}{r} \\ &\leq b_i + \frac{1}{r} \left(\gamma + 2^k \sum_{j=0}^{k-1} (b_j - b_{j+1}) \right) + (b_k - b_i) \leq b_k + \frac{1}{r} (\gamma + 2^k b_0). \end{aligned}$$

Consequently,

$$\limsup_{r \rightarrow \infty} \frac{B(r)}{r} \leq b_k \text{ for all } k \geq 1.$$

Letting $k \rightarrow \infty$, we obtain (19) since $b_k \rightarrow 0$ as $k \rightarrow \infty$ and Lemma 3 is proved. \square

Proof of Theorem 1 (i), Part 1. When a belongs to $L^1(1, \infty)$ and satisfies (5), it follows from Lemma 3 that the conditions (7) and (8) are satisfied so that Theorem 1 (i) follows from [4, Theorem 10]. \square

To handle the other case, we proceed in a different way by showing an upper bound for the function f defined in section 2. We first observe that the function Ψ defined in (15) satisfies

$$\Psi(r) = \int_1^{\infty} \frac{1}{s^2} a\left(\frac{1}{s}\right) ds = \int_{1/r}^1 a(s) ds, \quad r \in (0, \infty),$$

so that, if $a \in L^1(1, \infty)$, $\Psi(r)$ has a finite limit $\Psi(0) := -\|a\|_{L^1(1, \infty)}$ as $r \rightarrow 0$. We then define

$$\tilde{\Psi}(r) := \Psi(r) - \Psi(0) = \int_0^r \frac{1}{s^2} a\left(\frac{1}{s}\right) ds = \int_{1/r}^{\infty} a(s) ds, \quad r \in (0, \infty). \quad (22)$$

Lemma 4 *Let $a \in \mathcal{C}^1((0, \infty))$ be a positive function such that $a \in L^1(1, \infty)$. There exists a positive constant $\mu_M > 0$ depending only on M and a such that, for any non-negative function $g \in H^1(0, M)$ satisfying $\|g\|_{L^1(0, M)} = 1$, we have*

$$\|\tilde{\Psi}(g)\|_{L^\infty(0, M)}^2 \leq 32M \mathcal{L}_1(g) + \mu_M, \quad (23)$$

with

$$\mathcal{L}_1(g) := \frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0, M)}^2 + \int_0^M (\Psi(g) - M\Psi_1(g))(y) dy, \quad (24)$$

the functions Ψ and Ψ_1 being defined in (15) and (17), respectively.

Proof of Lemma 4. We set $G := \|g\|_{L^\infty(0,M)}$ which is finite owing to the continuous embedding of $H^1(0, M)$ in $L^\infty(0, M)$. Assume first that $G > 1$. Then, for $y \in (0, M)$ and $z \in (0, M)$, we have

$$\tilde{\Psi}(g(y)) = \tilde{\Psi}(g(z)) + \int_z^y \partial_x \tilde{\Psi}(g(x)) \, dx \leq \tilde{\Psi}(g(z)) + M^{1/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}.$$

Integrating the above inequality over $(0, M)$ with respect to z gives

$$\begin{aligned} M\tilde{\Psi}(g(y)) &\leq \int_0^M \tilde{\Psi}(g(z)) \, dz + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\leq \int_0^M \mathbf{1}_{[0,2/M]}(g(z)) \tilde{\Psi}(g(z)) \, dz + \int_0^M \mathbf{1}_{(2/M,\infty)}(g(z)) \tilde{\Psi}(g(z)) \, dz + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\leq M\tilde{\Psi}\left(\frac{2}{M}\right) + \frac{M\tilde{\Psi}(G)}{2} \int_0^M \mathbf{1}_{(2/M,\infty)}(g(z))g(z) \, dz + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\leq M\tilde{\Psi}\left(\frac{2}{M}\right) + \frac{M\tilde{\Psi}(G)}{2} + M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}, \end{aligned}$$

where we have used the property $\|g\|_{L^1(0,M)} = 1$ to obtain the last inequality. Taking the supremum over $y \in (0, M)$ and using the monotonicity and non-negativity of $\tilde{\Psi}$, we deduce that

$$\tilde{\Psi}(G) \leq 2\tilde{\Psi}\left(\frac{2}{M}\right) + 2M^{1/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}. \quad (25)$$

We next observe that the integrability of a at infinity also ensures that $\Psi_1(0) > -\infty$, so that $\tilde{\Psi}_1 := \Psi_1 - \Psi_1(0)$ is well-defined and satisfies

$$\tilde{\Psi}_1(r) = \int_0^r s\Psi'(s) \, ds \leq r\tilde{\Psi}(r), \quad r \in (0, \infty). \quad (26)$$

Since $\|g\|_{L^1(0,M)} = 1$, it follows from (25) and (26) that

$$\int_0^M \tilde{\Psi}_1(g) \, dy \leq \int_0^M g\tilde{\Psi}(g) \, dy \leq \tilde{\Psi}(G) \int_0^M g \, dy \leq 2\tilde{\Psi}\left(\frac{2}{M}\right) + 2M^{1/2} \|\partial_y \Psi(g)\|_{L^2(0,M)}. \quad (27)$$

We next infer from (27) and the non-negativity of $\tilde{\Psi}$ that

$$\begin{aligned} \mathcal{L}_1(g) &\geq \frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + \int_0^M \tilde{\Psi}(g) \, dy + M\Psi(0) - M \int_0^M \tilde{\Psi}_1(g) \, dy \\ &\geq \frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right) - 2M^{3/2} \|\partial_y \Psi(g)\|_{L^2(0,M)} \\ &\geq \frac{1}{4} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 + \left(\frac{1}{2} \|\partial_y \Psi(g)\|_{L^2(0,M)} - 2M^{3/2}\right)^2 - 4M^3 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right) \\ &\geq \frac{1}{4} \|\partial_y \Psi(g)\|_{L^2(0,M)}^2 - 4M^3 + M\Psi(0) - 2M\tilde{\Psi}\left(\frac{2}{M}\right), \end{aligned}$$

whence

$$\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 \leq 4\mathcal{L}_1(g) + 16M^3 - 4M\Psi(0) + 8M\tilde{\Psi}\left(\frac{2}{M}\right).$$

It then follows from (25) and the above inequality that

$$\begin{aligned} \tilde{\Psi}(G)^2 &\leq 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + 8M\|\partial_y \Psi(g)\|_{L^2(0,M)}^2 \\ &\leq 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + 32M\mathcal{L}_1(g) + 128M^4 - 32M^2\Psi(0) + 64M^2\tilde{\Psi}\left(\frac{2}{M}\right) \\ &\leq 32M\mathcal{L}_1(g) + \mu_M, \end{aligned}$$

with

$$\mu_M := 1 + 128M^4 - 32M^2\Psi(0) + 64M^2\tilde{\Psi}\left(\frac{2}{M}\right) + 8\tilde{\Psi}\left(\frac{2}{M}\right)^2 + \Psi(0)^2 - 32M\Psi(0).$$

We have thus shown Lemma 4 when $G = \|g\|_{L^\infty(0,M)} > 1$. To complete the proof, we finally consider the case $G \in [0, 1]$ and notice that, in that case,

$$0 \leq \tilde{\Psi}(G) \leq -\Psi(0) \quad \text{and} \quad \mathcal{L}_1(g) \geq \int_0^M \tilde{\Psi}(g) dy + M\Psi(0) \geq M\Psi(0),$$

since $\Psi_1 \leq 0$ in $(0, 1)$ and $\tilde{\Psi} \geq 0$. Consequently,

$$\tilde{\Psi}(G)^2 \leq \Psi(0)^2 = 32M\Psi(0) + \Psi(0)^2 - 32M\Psi(0) \leq 32M\mathcal{L}_1(g) + \mu_M,$$

and the proof of Lemma 4 is complete. \square

As an obvious consequence of Lemmas 2 and 4 we have the following result:

Corollary 5 *Let $a \in \mathcal{C}^1((0, \infty))$ be a positive function such that $a \in L^1(1, \infty)$. For $t \in [0, T_{max})$ and $y \in [0, M]$, we have*

$$0 \leq \tilde{\Psi}(f(t, y)) \leq (32M \max\{\mathcal{L}_1(f_0), 0\} + \mu_M)^{1/2}.$$

Proof of Corollary 5. Clearly $\mathcal{L}_1(f(t)) = L_1(t) \leq L_1(0) = \mathcal{L}_1(f_0) \leq \max\{\mathcal{L}_1(f_0), 0\}$ for $t \in [0, T_{max})$ by Lemma 2 and Corollary 5 readily follows from Lemma 4. \square

Remark 6 *Corollary 5 provides an L^∞ -bound on f only if $\Psi(r) \rightarrow \infty$ as $r \rightarrow \infty$, that is, if $a \notin L^1(0, 1)$. In that case, it gives a positive lower bound for u by (11).*

We next turn to the proof of the second part of Theorem 1 for which we develop further the arguments from [4, Theorem 10].

Proof of Theorem 1 (i), Part 2. Assume now that $a \in L^1(1, \infty)$ and satisfies (6). We fix $M > 0$, $q > 2$, and $\varepsilon_M \in (0, 1)$ such that

$$q > \max \left\{ 3 + \vartheta, \frac{5 + 3\vartheta}{\alpha(\vartheta + 1) - \vartheta} \right\} \quad \text{and} \quad \frac{q(q+1)}{M^2} \int_{1/\varepsilon_M}^{\infty} a(s) ds \leq \frac{1}{2}, \quad (28)$$

the existence of ε_M being guaranteed by the integrability of a at infinity.

For

$$\delta \in \left(0, \min \left\{ 1, 2M, (2M)^{-1/q} \right\} \right), \quad (29)$$

we put

$$f_0(y) := \frac{2(1 - M\delta^q)}{\delta^2} (\delta - y)_+ + \delta^q \geq \delta^q > 0, \quad y \in [0, M]. \quad (30)$$

Then

$$\int_0^M f_0(y) dy = 1, \quad \|f_0\|_{L^\infty(0, M)} = \frac{2(1 - M\delta^q)}{\delta} + \delta^q \leq \frac{2}{\delta}. \quad (31)$$

Introducing next

$$m_q(t) := \int_0^M y^q f(t, y) dy, \quad t \in [0, T_{max}],$$

we have

$$m_q(0) = \left(\frac{2(1 - M\delta^q)}{(q+1)(q+2)} + \frac{M^{q+1}}{q+1} \right) \delta^q \leq C_1 \delta^q \quad \text{with} \quad C_1 := \left(\frac{2 + (q+2)M^{q+1}}{(q+1)(q+2)} \right). \quad (32)$$

It follows from (12), (13), and the non-negativity of $\tilde{\Psi}$ that

$$\begin{aligned} \frac{dm_q}{dt} &= -q \int_0^M y^{q-1} \partial_y \tilde{\Psi}(f) dy + Mm_q - \frac{M^{q+1}}{q+1}, \\ \frac{dm_q}{dt} &\leq q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f) dy + Mm_q - \frac{M^{q+1}}{q+1}. \end{aligned} \quad (33)$$

We shall now estimate the integral on the right-hand side of (33): to this end, we split the domain of integration into three parts which we handle differently. As a preliminary step, we notice that, by (6),

$$\Psi'(r) \leq \gamma_\vartheta r^\vartheta \quad \text{and} \quad \Psi(r) \leq \frac{\gamma_\vartheta}{\vartheta+1} r^{\vartheta+1} \leq \gamma_\vartheta r^{\vartheta+1}, \quad r \geq 1. \quad (34)$$

We next define

$$K_0 := (32M \max \{ \mathcal{L}_1(f_0), 0 \} + \mu_M)^{1/2(2+\vartheta)} > 1,$$

and consider $(t, y) \in [0, T_{max}] \times [0, M]$.

- If $f(t, y) \in (0, \varepsilon_M]$, it follows from (28) and the monotonicity of $\tilde{\Psi}$ that

$$\tilde{\Psi}(f(t, y)) \leq \tilde{\Psi}(\varepsilon_M) = \int_{1/\varepsilon_M}^{\infty} a(s) ds \leq \frac{M^2}{2q(q+1)}. \quad (35)$$

- If $f(t, y) \in (\varepsilon_M, K_0)$, then (34) and the monotonicity of $\tilde{\Psi}$ yield

$$\tilde{\Psi}(f(t, y)) = \frac{\tilde{\Psi}(f(t, y))}{f(t, y)} f(t, y) \leq \frac{\Psi(K_0) - \Psi(0)}{\varepsilon_M} f(t, y) \leq \frac{\gamma_\vartheta K_0^{\vartheta+1} - \Psi(0)}{\varepsilon_M} f(t, y). \quad (36)$$

- If $f(t, y) \geq K_0$, Corollary 5 ensures that

$$\tilde{\Psi}(f(t, y)) = \frac{\tilde{\Psi}(f(t, y))}{f(t, y)} f(t, y) \leq \frac{K_0^{\vartheta+2}}{K_0} f(t, y) \leq K_0^{\vartheta+1} f(t, y). \quad (37)$$

Consequently, recalling that $K_0 > 1$ and $\Psi(0) < 0$, we deduce from (33) and (35)-(37) that

$$\begin{aligned} \frac{dm_q}{dt} &\leq q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f) \mathbf{1}_{(0, \varepsilon_M]}(f) dy + q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f) \mathbf{1}_{(\varepsilon_M, K_0)}(f) dy \\ &+ q(q-1) \int_0^M y^{q-2} \tilde{\Psi}(f) \mathbf{1}_{[K_0, \infty)}(f) dy + Mm_q - \frac{M^{q+1}}{q+1} \\ &\leq \frac{(q-1)M^2}{2(q+1)} \int_0^M y^{q-2} dy + q(q-1) \frac{\gamma_\vartheta K_0^{\vartheta+1} - \Psi(0)}{\varepsilon_M} \int_0^M y^{q-2} f dy \\ &+ q(q-1)K_0^{\vartheta+1} \int_0^M y^{q-2} f dy + Mm_q - \frac{M^{q+1}}{q+1} \\ &\leq C_2 K_0^{\vartheta+1} \int_0^M y^{q-2} f dy + Mm_q - \frac{M^{q+1}}{2(q+1)}, \end{aligned}$$

with $C_2 := q(q-1)(\gamma_\vartheta - \Psi(0) + \varepsilon_M)/\varepsilon_M$. We next use Hölder's inequality and (16) to conclude that

$$\frac{dm_q}{dt} \leq C_2 K_0^{\vartheta+1} m_q^{(q-2)/q} + Mm_q - \frac{M^{q+1}}{2(q+1)}. \quad (38)$$

It remains to estimate K_0 and in fact $\mathcal{L}_1(f_0)$. Since Ψ is negative on $(0, 1)$ and Ψ_1 is bounded from below by $\Psi_1(0)$, it follows from (29) and (30) that

$$\begin{aligned} \mathcal{L}_1(f_0) &\leq \frac{2}{\delta^4} (1 - M\delta^q)^2 \int_0^\delta |\Psi'(f_0)|^2 dy + \int_0^\delta \Psi(f_0) dy - M^2\Psi_1(0) \\ &\leq \frac{2}{\delta^4} \int_0^\delta |\Psi'(f_0)|^2 dy + \int_0^\delta \Psi(f_0) dy - M^2\Psi_1(0). \end{aligned}$$

On the one hand, we infer from (31), (34), and the monotonicity of Ψ that

$$\int_0^\delta \Psi(f_0) dy \leq \delta \Psi\left(\frac{2}{\delta}\right) \leq \gamma_\vartheta 2^{\vartheta+1} \delta^{-\vartheta}.$$

On the other hand, we have

$$\begin{aligned} f_0(y) &\geq 1 \quad \text{for } y \in [0, y_\delta] \quad \text{with } y_\delta := \delta - \frac{1 - \delta^q}{2(1 - M\delta^q)} \delta^2 > 0, \\ f_0(y) &\in [\delta^q, 1] \quad \text{for } y \in [y_\delta, \delta], \end{aligned}$$

so that, if $y \in [0, y_\delta]$,

$$\Psi'(f_0(y))^2 \leq \gamma_\delta^2 f_0(y)^{2\vartheta} \leq \gamma_\vartheta 4^\vartheta \delta^{-2\vartheta}$$

by (31) and (34), while, if $y \in (y_\delta, \delta]$,

$$\Psi'(f_0(y))^2 \leq \frac{1}{f_0(y)^4} a \left(\frac{1}{f_0(y)} \right)^2 \leq C_\infty^2 f_0(y)^{2(\alpha-2)} \leq C_\infty^2 \delta^{-2q(2-\alpha)}$$

by (6) since $\alpha \leq 2$. Therefore,

$$\begin{aligned} \mathcal{L}_1(f_0) &\leq \frac{2}{\delta^4} \left[\int_0^{y_\delta} \gamma_\vartheta 4^\vartheta \delta^{-2\vartheta} dy + \int_{y_\delta}^\delta C_\infty^2 \delta^{-2q(2-\alpha)} dy \right] + \gamma_\vartheta 2^{\vartheta+1} \delta^{-\vartheta} - M^2 \Psi_1(0) \\ &\leq \gamma_\vartheta 4^{\vartheta+1} \delta^{-3-2\vartheta} + C_\infty^2 \frac{1 - \delta^q}{2(1 - M\delta^q)} \delta^{-2-2q(2-\alpha)} + \gamma_\vartheta 2^{\vartheta+1} \delta^{-\vartheta} - M^2 \Psi_1(0) \\ &\leq \gamma_\vartheta 4^{\vartheta+1} \delta^{-2(2+\vartheta)} + C_\infty^2 \delta^{-2-2q(2-\alpha)} + \gamma_\vartheta 2^{\vartheta+1} \delta^{-\vartheta} - M^2 \Psi_1(0) \\ &\leq C_3 \left(\delta^{-2(2+\vartheta)} + \delta^{-2-2q(2-\alpha)} \right) \end{aligned}$$

with $C_3 := \gamma_\vartheta 4^{\vartheta+2} + C_\infty^2 - M^2 \Psi_1(0)$. Therefore,

$$K_0^{\vartheta+1} \leq C_4 \left(\delta^{-(\vartheta+1)} + \delta^{-(\vartheta+1)(1+q(2-\alpha))/(\vartheta+2)} \right) \quad (39)$$

for some constant $C_4 > 0$ depending only on M and a .

Combining (38) and (39) yields

$$\frac{dm_q}{dt} \leq \Lambda_\delta(m_q) := C_5 \left(\delta^{-(\vartheta+1)} + \delta^{-(\vartheta+1)(1+q(2-\alpha))/(\vartheta+2)} \right) m_q^{(q-2)/q} + M m_q - \frac{M^{q+1}}{2(q+1)} \quad (40)$$

for $t \in [0, T_{max})$ and some constant $C_5 > 0$ depending only on M and a . At this point, we note that the monotonicity of Λ_δ and (40) imply that $\Lambda_\delta(m_q(t)) \leq \Lambda_\delta(m_q(0))$ for $t \in [0, T_{max})$ if $\Lambda_\delta(m_q(0)) < 0$, the latter condition being satisfied for δ small enough as

$$\Lambda_\delta(m_q(0)) \leq C_1^{(q-2)/q} C_5 \left(\delta^{q-3-\vartheta} + \delta^{(q(\alpha(\vartheta+1)-\vartheta)-3\vartheta-5)/(\vartheta+2)} \right) + M C_1 \delta^q - \frac{M^{q+1}}{2(q+1)}$$

by (28) and (32).

Summarizing, we have shown that, if δ satisfies (29) and

$$C_1^{(q-2)/q} C_5 \left(\delta^{q-3-\vartheta} + \delta^{(q(\alpha(\vartheta+1)-\vartheta)-3\vartheta-5)/(\vartheta+2)} \right) + MC_1 \delta^q < \frac{M^{q+1}}{2(q+1)}, \quad (41)$$

we have

$$\frac{dm_q}{dt}(t) \leq \Lambda_\delta(m_q(t)) \leq \Lambda_\delta(m_q(0)) < 0, \quad t \in [0, T_{max}),$$

an inequality which can only be true on a finite time interval owing to the non-negativity of m_q . Therefore, $T_{max} < \infty$ in that case and, for any $M > 0$, we have found an initial condition u_0 given by (10), (11), and (30) (for δ small enough according to the above analysis) such that $\langle u_0 \rangle = M$ and the first component u of the corresponding solution to (1)-(4) blows up in finite time. \square

4 Global existence.

The proof of Theorem 1 (ii) also relies on the study of the function L_1 defined in Lemma 2. For that purpose, we first recall another property from [4]. We define the function E_1 by

$$E_1(h) := \frac{1}{2} \|\partial_y h\|_2^2 + \int_0^M \mathbf{1}_{(-\infty, 0)}(h(y)) h(y) dy, \quad h \in H^1(0, M), \quad (42)$$

for which we have the following lower bound.

Lemma 7 [4, Lemma 9] *For $M > 0$, we have*

$$E_1(h) \geq \frac{1}{4} \|\partial_y h\|_2^2 - M^3 - M \left| \Psi \left(\frac{1}{M} \right) \right|, \quad (43)$$

and

$$\|h\|_1 \leq M^{3/2} \|\partial_y h\|_2 + M \left| \Psi \left(\frac{1}{M} \right) \right| \quad (44)$$

for every $h \in H^1(0, M)$ satisfying

$$\int_0^M \Psi^{-1}(h)(y) dy = 1. \quad (45)$$

We now show that the non-integrability of a at infinity allows us to show that $T_{max} = \infty$. To this end, we use the alternative formulation (12)-(14) as in [4] and prove that f cannot vanish in finite time.

Proof of Theorem 1 (ii). Owing to (14) and the assumptions made on u_0 , we have

$$0 < f_0(y) \leq \frac{1}{m_0}, \quad y \in [0, M].$$

Introducing $\Sigma(t) := M^{-1} + e^{Mt} (m_0^{-1} - M^{-1})$ for $t \geq 0$ we have

$$\partial_t \Sigma - \partial_y^2 \Psi(\Sigma) - M\Sigma + 1 = M \left(\Sigma - \frac{1}{M} \right) - M\Sigma + 1 = 0,$$

$$\Sigma(0) = \frac{1}{m_0} \geq f_0(y), \quad y \in (0, M),$$

and the comparison principle warrants that

$$f(t, y) \leq \Sigma(t), \quad (t, y) \in [0, T_{max}) \times [0, M]. \quad (46)$$

We now follow the strategy of the proof of [4, Theorem 5] and first use the properties of Ψ , Ψ_1 , and (46) to estimate the function L_1 defined in Lemma 2 from below. Indeed, since $\Psi \geq 0$ on $(1, \infty)$ and $\Psi_1 \leq 0$ on $(0, 1)$ we arrive at

$$\begin{aligned} L_1(0) \geq L_1(t) &= \frac{1}{2} \|\partial_y \Psi(f(t))\|_{L^2(0, M)}^2 + \int_0^M \mathbf{1}_{(0, 1)}(f(t, y)) (\Psi - M\Psi_1)(f(t, y)) dy \\ &+ \int_0^M \mathbf{1}_{(1, \infty)}(f(t, y)) (\Psi - M\Psi_1)(f(t, y)) dy \\ &\geq \frac{1}{2} \|\partial_y \Psi(f(t))\|_{L^2(0, M)}^2 + \int_0^M \mathbf{1}_{(-\infty, 0)}(\Psi(f(t, y))) \Psi(f(t, y)) dy \\ &- M \int_0^M \mathbf{1}_{(1, \infty)}(f(t, y)) \Psi_1(f(t, y)) dy \\ &\geq E_1(\Psi(t)) - M^2 \Psi_1(\Sigma(t)), \end{aligned}$$

where E_1 is defined in (42) and we have used (46) to obtain the last inequality. Next, by Lemma 7 and (16), we have

$$L_1(0) \geq \frac{1}{4} \|\partial_y \Psi(f(t))\|_{L^2(0, M)}^2 - M^3 - M \left| \Psi \left(\frac{1}{M} \right) \right| - M^2 \Psi_1(\Sigma(t)),$$

whence

$$\frac{1}{4} \|\partial_y \Psi(f(t))\|_{L^2(0, M)}^2 \leq L_1(0) + M^3 + M \left| \Psi \left(\frac{1}{M} \right) \right| + M^2 \Psi_1(\Sigma(t)). \quad (47)$$

Using again Lemma 7, we have

$$\begin{aligned} \|\Psi(f(t))\|_{L^1(0, M)} &\leq M^{3/2} \|\partial_y \Psi(f(t))\|_{L^2(0, M)} + M \left| \Psi \left(\frac{1}{M} \right) \right| \\ &\leq 2M^{3/2} \left(L_1(0) + M^3 + M \left| \Psi \left(\frac{1}{M} \right) \right| + M^2 \Psi_1(\Sigma(t)) \right)^{1/2} + M \left| \Psi \left(\frac{1}{M} \right) \right|. \end{aligned}$$

Combining the previous inequality with (47) and the Poincaré inequality leads us to the bound

$$\|\Psi(f(t))\|_{H^1(0, M)} \leq C_6(T), \quad t \in [0, T] \cap [0, T_{max}), \quad (48)$$

for all $T > 0$. Together with the continuous embedding of $H^1(0, M)$ in $L^\infty(0, M)$, (48) gives

$$-C_7(T) \leq \Psi(f(t, y)) \leq C_7(T), \quad (t, y) \in ([0, T] \cap [0, T_{max})) \times [0, M].$$

Since

$$\lim_{r \rightarrow 0} \Psi(r) = -\infty$$

due to $a \notin L^1(1, \infty)$, the above lower bound on $\Psi(f)$ ensures that $f(t)$ cannot vanish in finite time, from which Theorem 1 (ii) follows as already discussed in section 2. \square

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