# Signal-dependent sampling and reconstruction method of signals with time-varying bandwidth

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Institute of Electronics and Computer Science, 14 Dzerbenes str., Riga LV-1006, Latvia.
greitans@edi.lv, shavelis@edi.lv

**Abstract:**
The paper describes the sampling method of nonstationary signals with time-varying spectral bandwidth. The reconstruction procedure exploiting the low-pass filter with time-varying cut-off frequency is derived. The filter application in signal reconstruction from its level-crossing samples is shown. The results of computer simulations are presented.

## 1. Introduction

The spectral characteristics of signals of practical interest often change with time. Generally, a signal with time-varying spectral bandwidth can be approximated with fewer samples per interval using appropriate non-equidistantly spaced samples than using uniform sampling procedure, where the sampling rate is chosen taking into account the highest signal frequency. For example, let us inspect a signal with wide bandwidth regions and narrow spectral bandwidth in the rest of signal observation. It is more efficient to sample the narrow bandwidth regions at a lower rate than the regions, where spectral bandwidth is wide. Solving this problem correctly requires the knowledge of the function of the instantaneous maximum frequency of signal. The paper will show two typical situations. First, information about the time-varying bandwidth is known a priori. In this case the deliberately non-uniform sampling instants can be calculated in advance, and reconstruction is based on application of filter with appropriate time-varying impulse response function. Second, the signal-dependent sampling scheme - level crossing sampling (LCS) is used for analog-to-digital (A/D) conversion. The idea of level-crossing sampling is based on the principle that samples are captured when the input signal crosses predefined levels. Such a sampling strategy has quite long history and is exploited for various applications [1, 2]. It has been shown that LCS has several interesting properties and is more efficient than traditional sampling in many respects [3]. In particular, it can be related to the processing of non-stationary signals, because if a waveform is changing rapidly, the samples are spaced more closely, and conversely – if a signal is varying slowly, the samples are spaced sparsely [4]. This property allows to calculate the estimate of the function of the instantaneous maximum frequency of signal from the positions of samples. In this case to reconstruct the waveform of signal, an additional resampling procedure is needed before the use of time-varying reconstruction filter, which will be described in next section.
Note that in both cases the local sampling density reflects the local bandwidth of the signal, therefore samples are spaced non-uniformly and advanced algorithms are required for digital signal processing.

## 2. Reconstruction of signal with time-varying bandwidth

There are several methods used for reconstruction of non-uniformly sampled band-limited signals. For correct recovery, they typically require that the maximal length of the gaps between the sampling instants does not exceed the Nyquist rate [5]. If the signal is non-stationary with time-varying spectral bandwidth, satisfying globally this requirement is not an appropriate decision, because this provides redundant data. The use of level-crossing sampling scheme can reduce the amount of samples, because the intervals between samples are determined by signal local properties and by the number of quantization levels. The quality of processing can be improved if the recovery procedure takes into account the local bandwidth of the signal [6]. In the following subsections the proposed idea and methods for reconstruction using filters with time-varying bandwidth and for the estimation of local maximum frequency of signal from its level-crossing samples will be discussed.

### 2.1 Idea of signal-dependent reconstruction functions

The sampling theorem states that every bandlimited signal $s(t)$ can be reconstructed from its equidistantly spaced samples if the sampling rate equals or exceeds the Nyquist rate $2F_{max}$, where $F_{max}$ is the maximum frequency in the signal spectrum. The reconstruction in time domain can be expressed as

$$\hat{s}(t) = \sum_{n=0}^{N-1} s(t_n)h(t-t_n), \qquad (1)$$

where $\widehat{s}(t)$ denotes reconstructed signal, $N$ is the number of the original signal samples $s(t_n)$ and $h(t)$ is an appropriate impulse response of the reconstruction filter, classically, sinc-function

$$h_1(t) = \mathrm{sinc}(2\pi F_{max} t) \quad (2)$$

As the sampling instants $t_n = \frac{n}{2F_{max}}$, then the impulse response

$$h_1(t - t_n) = h_1(t, t_n) = \mathrm{sinc}(2\pi F_{max} t - n\pi), \quad (3)$$

where $h_1(t - t_n) = h(t, t_n)$ is written as the function of two arguments. The reconstructed signal becomes

$$\hat{s}(t) = \sum_{n=0}^{N-1} s(t_n) h_1(t, t_n) \quad (4)$$

If the signal with time-varying frequency bandwidth $f_{max}(t)$ is considered, then the sampling rate of the signal according to Nyquist must be at least $2F_{max}$, where $F_{max} = \max(f_{max}(t))$. In this case any information about the local spectral bandwidth is ignored during the sampling process. To take it into account, it is proposed instead of $h_1(t, t_n)$ to use more general function

$$h_2(t, t_n) = \mathrm{sinc}(\Phi(t) - \Phi(t_n)) = \mathrm{sinc}(\Phi(t) - n\pi), \quad (5)$$

where $\Phi(t) = 2\pi \int_0^t f_{max}(t) dt$ is the phase of the sinusoid, whose frequency changes in time as $f_{max}(t)$, $t \geq 0$ and sampling instants $t_n$ are chosen such that $\Phi(t_n) = n\pi$. If the signal is stationary and band-limited $f_{max}(t) = \mathrm{const} = F_{max}$, Eq. (3) and (5) become equivalent. In case of non-constant $f_{max}(t)$ waveform of the reconstruction function $h_2(t, t_n)$ and the desired sampling instants $t_n$ are determined by $f_{max}(t)$. Samples are spaced non-equidistantly and the mean sampling frequency can be less than it is required by Nyquist criterion, which, in this case, should be satisfied rather in local than in global sense.

### 2.2 Reconstruction algorithm

To reconstruct the non-uniformly sampled signal according to equation (1), the reconstruction procedure involves signal resampling to the equidistantly spaced sampling set $\{t_n\}$ with sampling period $\Delta t = t_n - t_{n-1} = \frac{1}{2F_{max}}$. The estimation of $\hat{s}(t_n)$ is possible according to the simple iterative algorithm [5] the idea of which is to interpolate the sampled band-limited signal $s(t)$ by the sum $\check{s}_{s(t_m)}(t) = \sum_m s(t_m)\psi_m$ and filter it in order to remove high frequencies. Piecewise linear interpolation, which is well suited to level-crossing samples, uses $\psi_m$ consisting of the triangular functions

$$\psi_m(t) = \begin{cases} \frac{t - t_{m-1}}{t_m - t_{m-1}} & \text{for} \quad t_{m-1} \leq t < t_m, \\ \frac{t_{m+1} - t}{t_{m+1} - t_m} & \text{for} \quad t_m \leq t < t_{m+1}, \\ 0 & \text{elsewhere.} \end{cases} \quad (6)$$

It is proved [5] that if the maximum length of the gaps between the sampling instants $\tau_{max} \leq \frac{1}{2F_{max}}$, then every $s(t)$ can be reconstructed from the values $s(t_m)$ of an arbitrary $\tau_{max}$-dense sampling set $\{t_m\}$ iteratively. The recovery algorithm can be written as:

$$\begin{aligned} &\hat{s}_0(t_n) = \check{s}_{s(t_m)}(t_n); \\ &\hat{s}_0(t) = C\left[\hat{s}_0(t_n)\right]; \\ &\hat{s}_i(t_n) = \hat{s}_{i-1}(t_n) + \check{s}_{(s-s_{i-1})(t_m)}(t_n); \\ &\hat{s}_i(t) = C\left[\hat{s}_i(t_n)\right], \end{aligned} \quad (7)$$

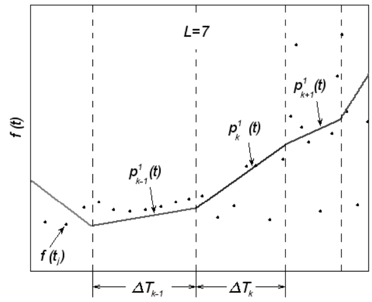


Figure 1: Piecewise polynomial $p_k^1(t)$ approximation.

where $i$ indicates the number of iteration. The linear operator $C$ denotes filtering as the convolution of samples $s(t_n)$ with impulse response $h_1(t, t_n)$ of the filter according to Eq. (4)

$$C\left[s(t_n)\right] = \sum_{n=0}^{N-1} s(t_n) h_1(t, t_n) \quad (8)$$

The sampling of non-stationary signal using level-crossing scheme does not ensure the satisfaction of the requirement $\tau_{max} \leq \frac{1}{2F_{max}}$. Direct application of the above described algorithm leads to a considerable reconstruction error, therefore two substantial enhancements are introduced to the algorithm - performing resampling to the non-equidistantly spaced values and the use of filter with impulse response $h_2(t, t_n)$ instead of classical $h_1(t, t_n)$. The resampling instants $t_n$ are determined by $\Phi(t)$, which depends on $f_{max}(t)$, that in general case is not known in advance. To solve this problem, an algorithm is developed, which estimates the time-varying instantaneous maximum frequency using information about locations of level-crossings.

### 2.3 Estimation of instantaneous maximum frequency

The obvious ways to estimate the local bandwidth of the signal is by finding its time-frequency representation (TFR) using, for example, short-time Fourier transform, wavelet transform or Wigner-Ville distribution. These methods are developed for uniformly sampled signals, however, there are some modifications in order to find the TFR of non-uniformly sampled signals [7]. The use of such approach is time consuming, therefore a simpler method is considered that is based on empirical evaluations.

To estimate the function $\widehat{f}_{max}(t)$ from samples $s(t_m)$, starting with the initial index value $m = 0$ two pairs of successive level-crossing samples $s(t_{m'_j}) = s(t_{m'_j+1})$ and $s(t_{m''_j}) = s(t_{m''_j+1})$ are found such that $m''_j > m'_j$ and the difference $m''_j - m'_j$ is minimal. Thereafter the next two pairs are found considering that $m'_{j+1} = m''_j$. For each $j = 1, 2, \ldots$ the value $f(t_j)$ is calculated as

$$f(t_j) = \left(t_{m''_j} + t_{m''_j+1} - t_{m'_j} - t_{m'_j+1}\right)^{-1}, \quad (9)$$

where

$$t_j = \frac{1}{4}\left(t_{m''_j} + t_{m''_j+1} - t_{m'_j} - t_{m'_j+1}\right) \quad (10)$$

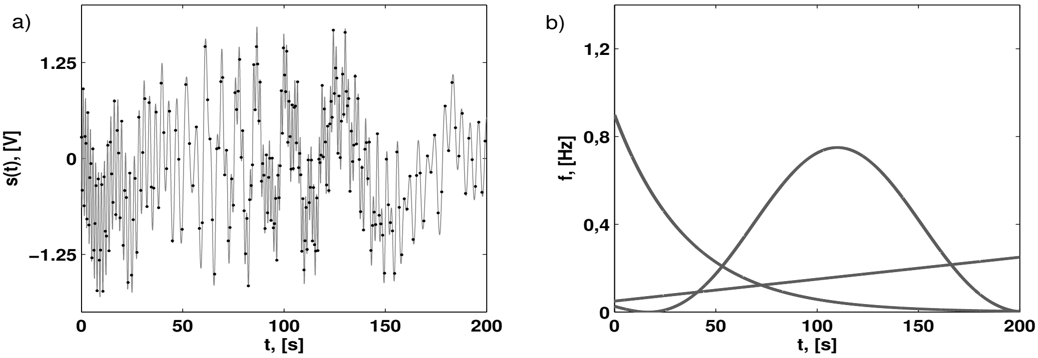


Figure 2: (a) Test signal sampled by $\Phi(t_n) = n\pi$ and (b) frequency traces of its components.

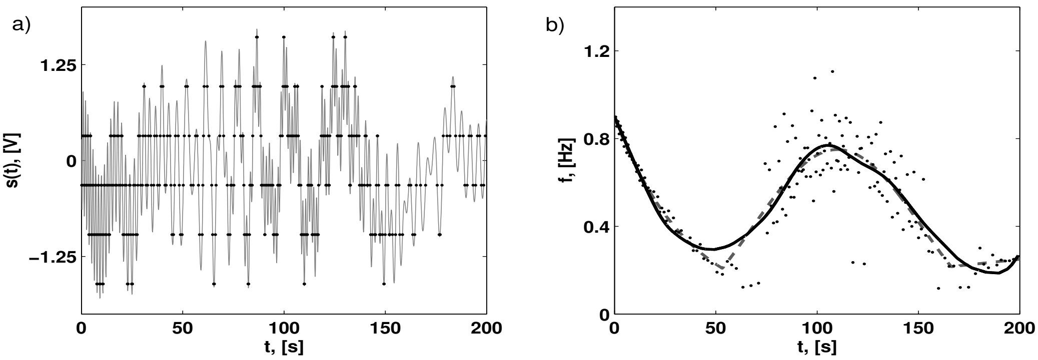


Figure 3: (a) Test signal sampled by level-crossings and (b) estimated instantaneous maximum frequency $\widehat{f}_{max}(t)$ as solid line, true instantaneous maximum frequency as dashed line and $f(t_j)$ as black points.

If a single sinusoid is sampled, then $f(t_j) = f(t_{j+1})$ for all $j$ and it equals the frequency of the sinusoid. If the signal consists of more harmonics, then $f(t_j)$ for different $j$ vary around the average value of $\bar{f} = \frac{1}{J}\sum_{j=1}^{J} f(t_j)$, where $J$ is the total number of detected pairs within the observation time of the signal. Experiments show that $\bar{f}$ is close to the frequency of the highest component. Thus, the estimate of function of instantaneous maximum frequency $\widehat{f}_{max}(t)$ can be obtained by $\{f(t_j)\}$ approximation with piecewise polynomials $p_k^r(t)$ of order $r$. By choosing the number $L > 1$ the observation interval of signal is divided into subintervals

$$\Delta T_k : t \in [t_{k,1}; t_{k,2}] \,, \tag{11}$$

where $k = 0, 1, \ldots$ is the number of subinterval and

$$t_{k,1} = \frac{t_{j=kL} + t_{j=kL+1}}{2}, \qquad t_{k,2} = \frac{t_{j=(k+1)L} + t_{j=(k+1)L+1}}{2} \tag{12}$$

For each subinterval $\Delta T_k$ the coefficients $a_{k,r}, a_{k,r-1}, \ldots, a_{k,1}, a_{k,0}$ of polynomial $p_k^r(t) = a_{k,r}t^r + a_{k,r-1}t^{r-1} + \cdots + a_{k,1}t + a_{k,0}$ are found to ensure

$$\begin{gathered}
p_{k-1}^r(t_{k,1})^{(0)} = p_k^r(t_{k,1})^{(0)} \,,\; p_k^r(t_{k,2})^{(0)} = p_{k+1}^r(t_{k,2})^{(0)} \\
p_{k-1}^r(t_{k,1})^{(1)} = p_k^r(t_{k,1})^{(1)} \,,\; p_k^r(t_{k,2})^{(1)} = p_{k+1}^r(t_{k,2})^{(1)} \\
\vdots \\
p_{k-1}^r(t_{k,1})^{(r)} = p_k^r(t_{k,1})^{(r)} \,,\; p_k^r(t_{k,2})^{(r)} = p_{k+1}^r(t_{k,2})^{(r)}
\end{gathered}$$

and the value of expression

$$\sum_{k=0}^{K-1} \sum_{j=kL+1}^{(k+1)L} \left[f(t_j) - p_k^r(t_j)\right]^2 = \min \tag{13}$$

is minimal. The denotation $(\ldots)^{(r)}$ means the derivative of order $r$ and $K$ is the total number of subintervals. After solving the minimization task using the method of least squares, the coefficients of polynomials $p_k^r(t)$ are obtained and the estimate of instantaneous maximum frequency

$$\widehat{f}_{max}(t) = p_k^r(t), \text{ if } t_{k,1} \le t \le t_{k,2} \tag{14}$$

depends on the number $L$ of samples $f(t_j)$ per subinterval. To reduce the dependency the final frequency estimate is obtained by averaging $\widehat{f}_{max}(t)$ calculated for different $L$ values. The example of piecewise polynomial of order $r = 1$ approximation when $L = 7$ is shown in Fig. 1

## 3. Simulation results

The methods described in previous section are applied to reconstruct nonstationary signal from its nonuniform sam-

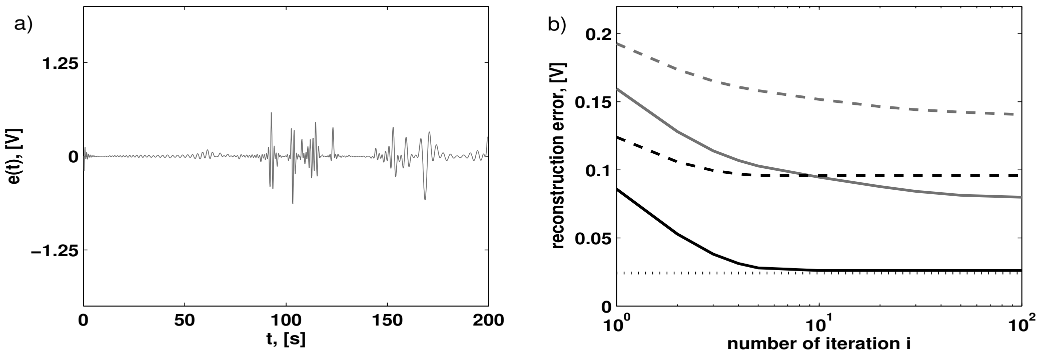


Figure 4: (a) The difference between original and recovered signal from its 349 level-crossing samples after 10 iterations and (b) reconstruction error (solid lines - reconstruction from level-crossings using $h_2(t, t_n)$, dashed lines - reconstruction from level-crossings using $h_1(t, t_n)$, dotted line - reconstruction from samples obtained by $\Phi(t_n) = n\pi$).

ples $s(t_n)$ obtained in two different ways. The first one is when $f_{max}(t)$ is given and sampling instants $t_n$ satisfy $\Phi(t_n) = n\pi$ (Fig. 2). The second way is by level-crossing sampling and $f_{max}(t)$ is not known in advance (Fig. 3).
In the first case 239 nonequidistantly spaced samples were obtained during 200 seconds of the test signal, which consists of three sinusoids with constant amplitudes and time-varying frequencies as shown in Fig. 2b. As the reconstructed signal according to Eq. (4) using $h_2(t, t_n)$ differs insignificantly from the original one, it is not illustrated here. In order to obtain similar result in uniform sampling case, at least 360 samples would be required since the maximum frequency of the signal is $F_{max} = 0.9$ Hz.
In the level-crossing sampling case 349 samples were captured using 6 quantization levels (Fig. 3a). To recover the signal the first task was to find the values $f(t_j)$ according to Eq. (9) in order to estimate the instantaneous maximum frequency (14). In Fig. 3b $f(t_j)$ are shown as black points, true $f_{max}(t)$ as dashed line and calculated $\widehat{f}_{max}(t)$ as solid line. The similarity between frequency traces is obvious. The second step was to recover the original signal according to Eq. (7) using level-crossing samples and estimated $\widehat{f}_{max}(t)$. The difference signal $e_i(t) = s(t) - s_i(t)$ after 10 iterations $i = 10$ is illustrated in Fig. 4a. The reconstruction error $\sqrt{\frac{1}{T}\int_0^T e_i(t)^2 dt}$ reduces as the number of iterations $i$ increases. It is shown in Fig. 4b as a grey solid line. The grey dashed line corresponds to reconstruction error, when instead of time-varying bandwidth filter $h_2(t, t_n)$ the filter with constant cut-off frequency of $F_{max} = 0.9$ Hz and impulse response $h_1(t, t_n)$ is used. In this case the achieved result is not so good as the reconstruction quality remains only in intervals, where the sampling density is sufficient. The reconstruction error can be reduced by decreasing the distance between quantization levels giving 437 level-crossing samples. It is shown in Fig. 4b as black solid and dashed lines. The dotted line corresponds to the first case when $f_{max}(t)$ is given and sampling instants $t_n$ satisfy $\Phi(t_n) = n\pi$.

## 4. Conclusions

The proposed approach for non-stationary signal processing uses signal dependent techniques: level crossing sampling for data acquisition and filtering with time-varying bandwidth for signal reconstruction. The information carried by level-crossing samples is employed in two ways – time instants of samples are used to estimate the instantaneous maximum frequency of the signal, while the amplitude values of samples are used in reconstruction algorithm. The reconstruction procedure is based on the use of iterative filtering with time-varying bandwidth filter. The enhancement of classical signal reconstruction approach is made by introducing signal-dependent, ”non-stationary” impulse response and resampling to the corresponding, nonuniform sampling set.
Speech signal processing can be quoted as one of the potential application areas of the proposed algorithm. The level-crossing sampling technique reduces the number of samples and leads to effective signal coding approaches.

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