Daubechies Localization Operator in Bargmann Fock Space and Generating Function of Eigenvalues of Localization Operator

Kunio Yoshino

To cite this version:
Kunio Yoshino. Daubechies Localization Operator in Bargmann Fock Space and Generating Function of Eigenvalues of Localization Operator. SAMPTA’09, May 2009, Marseille, France. pp. General session. hal-00453464

HAL Id: hal-00453464
https://hal.science/hal-00453464
Submitted on 4 Feb 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract:
We will express Daubechies localization operators in Bargmann - Fock space. We will prove that the Hermite functions are eigenfunctions of Daubechies localization operator. By making use of generating function of eigenvalues of Daubechies localization operator, we will show some reconstruction formulas for symbol function of Daubechies localization operator with rotational invariant symbol.

1. Introduction

She obtained following results.

Theorem (Daubechies)(2)
Suppose that symbol function of Daubechies localization operator is rotational invariant. Then
(i) Eigenfunctions of Daubechies localization operator are Hermite functions.
(ii) Eigenvalues are given by Mellin transform of symbol function.

In this paper we realize Daubechies localization opeartor in Bargmann - Fock space. We will consider the eigenvalue problem of Daubechies localization operator in Bargmann - Fock space. By making use of Bargmann - Fock space we will give a new proof of above theorem. We will establish reconstruction formula of symbol function of Daubechies localization operator with rotational invariant symbol by generating function of eigenvalues of Daubechies localization operator. For the simplicity, we will confine ourselves to 1-dimensional case.

2. Bargmann Transform
Put
\[ A(z, x) = \pi^{-1/4} \exp \left\{ -\frac{1}{2} (z^2 + x^2) + \sqrt{2} z \cdot x \right\}, \]
where \( z \in \mathbb{C} \) and \( x \in \mathbb{R} \).

Bargmann transform \( B(\psi) \) is defined as follows:
\[ B(\psi)(z) \overset{def}{=} \int_{\mathbb{R}} \psi(x) A(z, x) dx, \quad (\psi \in L^2(\mathbb{R})). \]

Put
\[ BF = \{ g \in H(\mathbb{C}) : \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^2} d\bar{z} < \infty \} \]
where \( H(\mathbb{C}) \) denotes the space of entire functions in the complex plane.

\( BF \) is called Bargmann-Fock space.

Theorem 1(1)
Bargmann transform is a unitary mapping from \( L^2(\mathbb{R}) \) to Bargmann-Fock space \( BF \).

For the details of Bargmann transform and Bargmann - Fock space, we will refer the reader to [1] and [3].

3. Hermite Functions

Definition 1(1),[3]) Hermite functions \( h_m(x) \) is defined by:
\[ h_m(x) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \exp(x^2/2) \frac{d^m}{dx^m} \exp(-x^2), \quad (m \in \mathbb{N}). \]
Hermite functions has following generating function expansion:
\[ \pi^{-1/4} \exp \left\{ -\frac{1}{2} (z^2 + x^2) + \sqrt{2} z \cdot x \right\} = \sum_{m=0}^{\infty} \frac{z^m}{m!} h_m(x), \quad (z \in \mathbb{C}, x \in \mathbb{R}). \]

We recall some well known facts about Hermite functions.

Proposition 1(1),[3])
Suppose that 

(i) $\{h_m(x)\}_{m=0}^{\infty}$ is complete orthonormal basis in $L^2(\mathbb{R})$.

(ii) $(-\frac{\partial^2}{\partial x^2} + x^2 - 1)h_m(x) = mh_m(x)$.

(iii) $B(h_m)(z) = \frac{z^m}{\sqrt{m!}}$, $(z \in \mathbb{C})$

(iv) $\tilde{g}(h_m)(x) = (-i)^m h_m(x)$,

where $\tilde{g}$ is Fourier transform.

**Proposition 2**([1],[3])

(i) $(B \circ L \circ B^{-1})g(z) = z \frac{\partial}{\partial z}g(z)$.

where $L = -\frac{\partial^2}{\partial x^2} + x^2 - 1$.

(ii) $(B \circ \tilde{g} \circ B^{-1})g(z) = g(-iz)$,

where $\tilde{g}$ is Fourier transform and $g(z) \in BF$.

## 4. Daubechies Localization Operator

Put

$$\phi_{p,q}(x) = \pi^{-1/4} e^{ipx} e^{-(x-q)^2/2}.$$  

$$< \phi_{p,q}, f > = \int_{\mathbb{R}} \phi_{p,q}(x) f(x) dx.$$  

This is so called Short time Fourier transform (or Windowed Fourier transform, or Gabor transform).

**Definition 2**([2])

Suppose that $F(p,q) \in L^1(\mathbb{R}^2)$ and $f(x) \in L^2(\mathbb{R})$.

We put

$$P_F(f)(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} F(p,q) \phi_{p,q}(x) < \phi_{p,q}, f > dpdq,$$  

We call $P_F$ (Daubechies) localization operator $F(p,q)$ is called symbol function.

Daubechies obtained following results.

**Theorem**([2]). Suppose that $F(p,q) \in L^1(\mathbb{R}^2)$ and $F(p,q)$ is rotational invariant function, i.e. $F(p,q) = \tilde{F}(r^2)$, $(r^2 = p^2 + q^2)$.

Then

(i) Hermite functions $h_m(x)$ are eigenfunctions of Daubechies operator $P_F$.

$$P_F(h_m)(x) = \lambda_m h_m(x), \quad (m \in \mathbb{N}),$$

(ii) $\lambda_m = \frac{1}{m!} \int_{0}^{\infty} e^{-s} s^m \tilde{F}(2s) ds$, $(m \in \mathbb{N})$.  

## 5. A Realization of Daubechies Localization Operator in Bargmann Fock space

In this section we will express Daubechies Localization Operator in Bargmann - Fock space.

First we need following lemmas.

**Lemma 1**

$$B(\phi_{p,q})(z) = e^{z w - 1/2 |w|^2 + 1/2 ipq}, \quad (w = \frac{p + iq}{\sqrt{2}})$$

**Lemma 2**([1])

$$g(z) = \frac{1}{2\pi i} \int \int_{\mathbb{C}} e^{-t |z|^2} dt \wedge d\bar{t}, \quad (g \in BF)$$

**Theorem 2** Under the same assumptions in Prop. 3, we have

$$B \circ P_F \circ B^{-1}(g)(z) = \frac{1}{2\pi} \int \int_{\mathbb{C}} F(w, \bar{w}) e^{2\pi w^{-1/2} |w|^2 + 1/2 ipq} g(w) dw \wedge d\bar{w}, \quad (\forall g \in BF)$$

**Proof**

Since Bargmann transform is unitary operator, we have

$$P_F(f)(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} F(p,q) \phi_{p,q}(x) < \phi_{p,q}, f > dpdq,$$

So by lemma 1,

$$B \circ P_F(f)(x) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} F(p,q) B \phi_{p,q}(z) < B \phi_{p,q}, Bf > dpdq,$$

Hence we have

$$B \circ P_F \circ B^{-1}(g)(z) = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} F(p,q) e^{z w - 1/2 |w|^2 + 1/2 ipq} < B \phi_{p,q}, Bf > dpdq,$$

On the other hand

$$< B \phi_{p,q}, f > = \frac{1}{2\pi} \int \int_{\mathbb{R}^2} e^{i w - 1/2 |w|^2 + 1/2 ipq} g(t) e^{-|t|^2} dt df,$$

By Lemma 2,  

$$= e^{-1/2 |w|^2 - 1/2 ipq} g(\bar{w})$$

Thus we obtained our desired result.

**Proposition 3**([2]). Suppose that $F(p,q) \in L^1(\mathbb{R}^2)$ and $F(p,q)$ is rotational invariant function, i.e. $F(p,q) = \tilde{F}(r^2)$, $(r^2 = p^2 + q^2)$.

Then

(i) Functions $z^m$ are eigenfunctions of operator $B \circ P_F \circ B^{-1}$.

$$B \circ P_F \circ B^{-1}(z^m) = \lambda_m z^m, \quad (m \in \mathbb{N}),$$

(ii) $\lambda_m = \frac{1}{m!} \int_{0}^{\infty} e^{-s} s^m \tilde{F}(2s) ds$, $(m \in \mathbb{N})$.  

**Proof**
6. An Example of Daubechies Localization Operator

In this section we will consider following special Daubechies localization operators.

Put $F_{a}(p,q) = e^{a} |q^2 + p^2| = e^{a^{-1}} z^2$, \(0 < a < 1\).

Then $\lambda_m = a^{-1} m$, $\Lambda(w) = \frac{a}{1 - aw}$.

$P_{F_{a}}(h_m)(x) = a^{-1} h_m(x)$.

$P_{F_{a}} = \sum_{m=0}^{\infty} a^{m+1} h_m(x) h_m(y)$.

valids in operator sense.

$P_{F_{a}} = \sum_{m=0}^{\infty} a^{m+1} |m > m|$, in Dirac’s Notation.

If $a = 2^{-1}$, this is Schatten decomposition of $P_{F_{a}}$ and $P_{F_{a}}$ is called density operator in quantum statistical mechanics.

**Proposition 5** (Mehler’s formula [3],[5])

$$\sum_{m=0}^{\infty} a^{m+1} h_m(x) h_m(y) = \sqrt{\pi(1 - a^2)} e^{\frac{1}{4a^2}(x+y)^2 + \frac{1}{4a^2}(x-y)^2}, \quad (|a| < 1).$$

**Corollary 3**

(i) $P_{F_{a}}(f)$

$$\int \frac{a}{\sqrt{\pi(1 - a^2)}} e^{\frac{1}{4a^2}(x+y)^2 + \frac{1}{4a^2}(x-y)^2} f(y)dy,$$

$(f \in L^2)$.

(ii) If $a \in \mathbb{C}, |a| < 1$, then $P_{F_{a}} : L^2 \rightarrow L^2$ is bounded linear operator.

**Proof**

If $a \in \{a \in \mathbb{C} : |a| < 1\}$, then real part of $\frac{1 - a}{1 + a} + \frac{1 + a}{1 - a}$ is positive. So $P_{F_{a}}$ is bounded linear operator from $L^2$ to $L^2$. Namely, we obtained analytic continuation of $P_{F_{a}}$ under the condition $(a \in \mathbb{C}, |a| < 1)$.

7. Realization of $P_{F_{a}}$ in Bargmann - Fock space

In this section we will consider $P_{F_{a}}$ in Bargmann - Fock space.

**Proposition 6**

(i) $B \circ P_{F_{a}} \circ B^{-1} = \sum_{m=0}^{\infty} a^{m+1} \frac{z^m}{\sqrt{m!}} \frac{\bar{w}^m}{\sqrt{m!}}$

valids in operator sense.

(ii) $(B \circ P_{F_{a}} \circ B^{-1})(g)(z)$

$$= \frac{ia}{2} \int \int F(2|w|^2) e^{-|w|^2} dw \wedge d\bar{w},$$

$(g \in BF)$

**Proof**

Since $\frac{z^m}{\sqrt{m!}}$, are eigenfunctions of $B \circ P_{F_{a}} \circ B^{-1}$, we have

$B \circ P_{F_{a}} \circ B^{-1} = \sum_{m=0}^{\infty} a^{m+1} \frac{z^m}{\sqrt{m!}} \frac{\bar{w}^m}{\sqrt{m!}}$.

**Proposition 7** Suppose that $|a| < 1, (a \in \mathbb{C})$. Then we have

$$(B \circ P_{F_{a}} \circ B^{-1})(g)(z) = ag(az), \quad (g \in BF).$$
As seen in Proposition 8, these operators are obtained as
\[
(B \circ P_{F_a} \circ B^{-1})(g)(z) = (2\pi i)^{-1} \int g(t) \Lambda \left( \frac{z}{t} \right) \frac{dt}{t}
\]
\[
= (2\pi i)^{-1} \int g(t) \frac{a}{t-a} dt = ag(az).
\]

Proposition 8 For \( f \in L^2 \), we have
\[
(i) \lim_{a \to -i} P_{F_a}(f) = f,
\]
\[
(ii) \lim_{a \to -i} P_{F_a}(f) = (-i)\mathfrak{F}f,
\]
\[
(iii) \lim_{a \to -i} P_{F_a}(f) = i\mathfrak{F}^{-1}f,
\]
where \( \mathfrak{F} \) is Fourier transform.

(Proof) By Prop.7, we have
\[
(B \circ P_{F_a} \circ B^{-1})(g)(z) = ag(az), \quad (g \in BF).
\]
\[
(i) \lim_{a \to -i} \lambda^m \left( B \circ P_{F_a} \circ B^{-1} \right)(g) = \lim_{a \to -i} ag(az) = g(z).
\]
This means that \( \lim_{a \to -i} P_{F_a} = \text{Identity operator} \).
\[
(ii) \lim_{a \to -i} \lambda^m \left( B \circ P_{F_a} \circ B^{-1} \right)(g) = \lim_{a \to -i} ag(az) = (-i)g(-iz).
\]
By (ii) in Proposition 2, this means that \( \lim_{a \to -i} P_{F_a} = (-i)\mathfrak{F} \).
Proof of (iii) is same as that of (ii).

Proposition 9
\[
(i) \quad G = \{ P_{F_a} : a \in \mathbb{C}, |a| < 1 \} \cup \{ I_d \}
\]
is semigroup.
\[
(ii) \quad P_{F_a} \circ P_{F_b} = P_{F_{ab}}.
\]

(Proof) By Proposition 7,
\[
(B \circ P_{F_a} \circ B^{-1})(g)(z) = ag(az), \quad (g \in BF)
\]
So, we have
\[
(B \circ P_{F_a} \circ P_{F_b} \circ B^{-1})(g)(z) = bag(baz)
\]
Hence we have
\[
P_{F_a} \circ P_{F_b} = P_{F_{ab}}.
\]
In these cases, \( F_a(p,q) \notin L^1 \). But these operators still define bounded operators from \( L^2 \) to \( L^2 \).
As seen in Proposition 8, these operators are obtained as limit of \( P_{F_a}, (F_a \in L^1) \).

8. Reconstruction formulas

We assume that \( F(p,q) \) is rotational invariant \( L^1 \) function. Namely, \( F(p,q) = F(\sqrt{p^2+q^2}) \).

In section 5, we introduced following generating function:
\[
\Lambda(w) = \sum_{m=0}^{\infty} \lambda_m w^m
\]
\( \Lambda(w) \) is called generating function for eigenvalues of \( P_{F_a} \).
Now we consider following formal power series :
\[
\sum_{m=0}^{\infty} m! \lambda_m t^{-m-1}
\]
In general this series is divergent series. We put
\[
G(t) = \int_0^\infty \frac{\hat{F}(2s)e^{-s}}{t-s} ds, \quad (t \in \mathbb{C}\setminus[0,\infty]).
\]
We have

Proposition 10\[\text{[8]}\]
Formal power series
\[
\sum_{m=0}^{\infty} m! \lambda_m t^{-m-1}
\]
is an asymptotic expansion of \( G(t) \).

Remark \( \Lambda(w) \) is the Borel transform of formal power series \( \sum_{m=0}^{\infty} m! \lambda_m t^{-m-1} \).
Since \( G(t) \) is Hilbert transform of \( \hat{F}(2s)e^{-s} \), we have

Theorem 5
\[
\hat{F}(2s) = e^s \lim_{t \to -\infty} \frac{1}{2\pi i} \left( G(s+it) - G(s-it) \right)
\]
We also have

Theorem 6\[\text{[8]}\]
\[
\hat{F}(2s) = (2\pi)^{-n} e^s \mathfrak{F}(\Lambda(iv))(s),
\]
valids in distribution sense.
where \( \mathfrak{F} \) is Fourier transform.

References: