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► **To cite this version:**

John R. Higgins. Concrete and discrete operator reproducing formulae for abstract Paley–Wiener space. SAMPTA'09, May 2009, Marseille, France. pp.General session. hal-00453459

HAL Id: hal-00453459

<https://hal.science/hal-00453459>

Submitted on 4 Feb 2010

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Concrete and discrete operator reproducing formulae for abstract Paley–Wiener space

J.R. Higgins

I.H.P., 4 rue du Bary, 11250 Montclar, France.
rowlandhiggins@yahoo.com

Abstract:

The classical Paley–Wiener space possesses two reproducing formulae; a ‘concrete’ reproducing equation and a ‘discrete’ analogue, or sampling series, and there is a striking comparison between them. It is shown that such analogies persist in the setting of Paley–Wiener spaces that are more general than the classical case. In fact, there are ‘operator’ versions of the reproducing equation and of the sampling series that are also comparable, not ‘exactly’ but nearly so. Reproducing kernel theory and abstract harmonic analysis are brought together to achieve this, then the special case of multiplier operators with respect to the Fourier transform is considered. The Riesz transforms provide a two-dimensional example, with possibilities of extension to higher dimensions and to further classes of operators.

1. Introduction

It has often been remarked that the classical Paley–Wiener space possesses two reproducing formulae; a ‘concrete’ reproducing equation

$$f(s) = \int_{\mathbb{R}} f(t) \operatorname{sinc}(s-t) dt, \quad (s \in \mathbb{R}), \quad (1)$$

and a ‘discrete’ reproducing equation, or sampling series,

$$f(s) = \sum_{n \in \mathbb{Z}} f(n) \operatorname{sinc}(s-n), \quad (s \in \mathbb{R}), \quad (2)$$

and that there is a striking analogy between the two (see, e.g., [3, p. 58]). Here, $\operatorname{sinc} x$ denotes the standard function $\operatorname{sinc} x := (\sin \pi x)/\pi x$.

The purpose of the present lecture is to point out that concrete and discrete reproducing formulae and analogies between them persist in the setting of Paley–Wiener spaces that are more general than the classical case. It will be shown that for suitably chosen operators there are ‘operator’ versions of the reproducing equation and of the sampling series that are also comparable, in the same way as in the classical case described above.

2. The setting

Abstract theories that lead to reproducing formulae are outlined in §2.1 and §2.2, and are brought together in §2.3.

2.1 The reproducing kernel theory

The basic setting of this paper is that of the reproducing kernel theory of Saitoh [8, Ch. 2, §1]. Very briefly the background is as follows. Let E be an abstract set. For each t belonging to E let K_t belong to H (a separable Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle_H$). Then $k(s, t) := \langle K_t, K_s \rangle_H$ is defined on $E \times E$ and is called the *kernel function* of the map K_t . This kernel function is a *positive matrix* [8, Ch. 2, §2] and as such it determines one and only one Hilbert space for which it is the reproducing kernel. This Hilbert space is denoted by $R(\mathcal{K})$ since it turns out to be the set of images of H under the transformation $(\mathcal{K}g)(t) := \langle g, K_t \rangle_H$, ($g \in H$).

Theorem 1 (Saitoh) *With the notations established above, $R(\mathcal{K})$ (which is now abbreviated to just R) is a Hilbert space which has the reproducing kernel $k(\cdot, \cdot)$, and is uniquely determined by this kernel k . For $f \in R$ there exists $\alpha \in H$ such that*

$$\|f\|_R = \|\mathcal{K}\alpha\|_R \leq \|\alpha\|_H, \quad (3)$$

and there exists a unique member, g say, of the class of all α 's satisfying (3) such that

$$f(t) = \langle g, K_t \rangle_H, \quad (t \in E),$$

and

$$\|f\|_R = \|g\|_H.$$

The reproducing equation for $f \in R$ is

$$f(t) = \langle f, k(\cdot, t) \rangle_R \quad (4)$$

The following theorem is simple but very useful.

Theorem 2 *The convergence of a sequence in the norm of R implies that it converges pointwise over E , and the convergence is uniform over any subset of E on which $k(t, t) = \|k(\cdot, t)\|^2$ is bounded.*

The following Theorem is to be found in [8].

Theorem 3 *With notations as above, let $\{s_n\}$, ($n \in \mathbb{X}$), be points of E such that $\{K_{s_n}\}$ is an orthonormal basis for H . Then the sampling series representation*

$$f(t) = \sum_{n \in \mathbb{X}} f(s_n) k(s_n, t), \quad (5)$$

holds, convergence being in the norm of R ; and then of course Theorem 2 applies.

2.2 Abstract harmonic analysis

A very brief introduction (mostly just notations) to the abstract harmonic analysis that will be needed is now given. All necessary background, and much more, is to be found in [1], [2].

Let G be a locally compact abelian (LCA) group (written additively). Let (t, γ) be a character of G , that is, a continuous homomorphism of G into the circle group. Let $G^\wedge = \Gamma$ denote the group of continuous characters on G , usually called the dual group of G . We assume that Γ has a countable discrete subgroup Λ .

Haar measures on the various groups are normalised in the standard way [1], and this means in particular that there is a measurable transversal (i.e., a complete set of coset representatives) $\Omega \subset \Gamma$ of Γ/Λ , and it has finite Haar measure.

Now

$$\mathfrak{H} = \Lambda^\perp := \{t \in G : (t, \lambda) = 1, (\lambda \in \Lambda)\}.$$

is a subgroup of G and is called the ‘annihilator’ of Λ . We assume that \mathfrak{H} is discrete; it follows that the quotient group Γ/Λ is compact.

The Fourier transform on $L^2(G)$ is defined in the usual way:

$$f^\wedge(\gamma) = (\mathcal{F}f)(\gamma) := \int_G f(t)(t, \gamma) dt,$$

in the L^2 sense, where dt denotes the element of Haar measure on G (likewise, $d\gamma$ denotes the element of Haar measure on Γ). The inverse Fourier transform will be denoted by $^\vee$ or by \mathcal{F}^{-1} .

We shall need the ‘shift’ property of the Fourier transform: $f(\cdot - x)^\wedge(\gamma) = (-x, \gamma)f^\wedge(\gamma)$.

Abstract Paley Wiener space $PW_\Omega(G)$ is defined as follows:

$$PW_\Omega(G) := \{f : f \in L^2(G) \cap C(G), \\ f^\wedge(\gamma) = 0 \text{ (Haar) a.a. } \gamma \notin \Omega\} \quad (6)$$

2.3 Combining harmonic analysis with the reproducing kernel theory

The abstract set E of §2.1 is often taken to be \mathbb{R} or \mathbb{C} . Here, however, we take it to be an LCA group G thus combining two abstract theories, harmonic analysis and the reproducing kernel theory. In the notations of §2.1 and §2.2 we also take $K_t = (t, \cdot)$, $H = L^2(\Omega)$ and $\mathcal{K}g = \mathcal{F}^{-1}g$, $g \in L^2(\Omega)$. Then we have

$$k(s, t) = \int_\Omega (t, \gamma) \overline{(s, \gamma)} d\gamma = \int_\Omega (t - s, \gamma) d\gamma \\ = (\chi_\Omega^\vee)(t - s) =: \varphi_\Omega(t - s), \quad (7)$$

where χ_S denotes the characteristic function of a set S . It does not seem to have been recognised that $\varphi_\Omega(t - s)$ is the reproducing kernel for $PW_\Omega(G)$, and that this allows a

close association between sampling in the harmonic analysis setting and Saitoh’s theory.

The space R of §2.1 is now seen to be the Paley–Wiener space defined in (6), and its reproducing equation is

$$f(t) = \langle f, \varphi_\Omega(t - \cdot) \rangle_{L^2(G)} \quad (8)$$

Klurvánek’s sampling theorem [4, p. 45] is a consequence:

Theorem 4 *Let $f \in PW_\Omega$. With the assumptions of §2.2,*

$$f(t) = \sum_{h \in \mathfrak{H}} f(h) \varphi_\Omega(t - h) \quad (9)$$

in norm, etc., (see Theorem 2).

Our concrete – discrete comparison is between (8) and (9).

3. Operator kernels and operator reproducing formulae

The presence of kernels and reproducing equations associated with operators on a reproducing kernel Hilbert space add greatly to the richness of its structure, as we shall see in this section.

3.1 Operator kernels and operator reproducing equations

Let R be the separable Hilbert space of functions defined on E with reproducing kernel $k(s, t)$, as we have discussed it in §2.1. Let \mathcal{B} be a bijection on R , and let \mathcal{B}^* denote the adjoint operator. The action of \mathcal{B} on R is governed by the action of \mathcal{B}^* on the reproducing kernel k , because for $f \in R$,

$$(\mathcal{B}f)(t) = \langle \mathcal{B}f, k(\cdot, t) \rangle_R = \langle f, \mathcal{B}^*k(\cdot, t) \rangle_R. \quad (10)$$

See, e.g., [5].

Definition 1 *The kernel*

$$h(s, t) := (\mathcal{B}^*k(\cdot, t))(s), \quad s, t \in E$$

will be called the operator kernel of \mathcal{B} .

In this notation (10) is

$$(\mathcal{B}f)(t) = \langle f, h(\cdot, t) \rangle. \quad (11)$$

Now from Definition 1 above, $((\mathcal{B}^*)^{-1}h(\cdot, t))(s) = k(s, t)$, so that, using the ordinary reproducing formula (4), we have

$$f(t) = \langle f, k(\cdot, t) \rangle = \langle f, (\mathcal{B}^*)^{-1}h(\cdot, t) \rangle \\ = \langle ((\mathcal{B}^*)^{-1})^* f, h(\cdot, t) \rangle.$$

Now using standard properties of operators and their adjoints (e.g., [6, p. 202]) we can summarise these calculations as:

$$f(t) = \langle (\mathcal{B}^{-1}f)(\cdot), h(\cdot, t) \rangle. \quad (12)$$

This formula tells us that f can be reproduced, not from its own values as in the ordinary reproducing kernel theory, but from the result of acting on it with an operator. We can call this an *operator reproducing equation* in analogy with the ordinary reproducing equation (4).

Similar formulae for \mathcal{B}^* can be obtained in the same way. First, we make the following

Definition 2

$$h^*(s, t) := \overline{h(t, s)} \quad ((t, s) \in E \times E)$$

will be called the adjoint operator kernel of \mathcal{B} .

Kernels and their adjoints occur in important areas of study such as the theory of integral equations (see, e.g., [6, p. 170] for basic information). We shall find series expansions for such kernels and identify the action of h^* explicitly in Theorem 5 below.

First, let $\{\varphi_n\}$, $n \in \mathbb{X}$, be an orthonormal basis for R .

Lemma 1

$$h(s, t) = \sum_{n \in \mathbb{X}} \overline{(\mathcal{B}\varphi_n)(t)} \varphi_n(s), \quad (s, t \in E). \quad (13)$$

Convergence is in the norm of R for each $t \in E$, and the pointwise convergence is governed by Theorem 2.

Proof The coefficients for the expansion of $h(\cdot, t)$, t fixed, in the basis $\{\varphi_n\}$ are

$$\langle h(\cdot, t), \varphi_n \rangle = \overline{\langle \varphi_n, h(\cdot, t) \rangle} = \overline{(\mathcal{B}\varphi_n)(t)}$$

by (11), thus (13) is obtained. \square

It will be recalled that if we put

$$\begin{cases} \mathcal{B}\varphi_n = \psi_n \\ (\mathcal{B}^*)^{-1}\varphi_n = \psi_n^*, \end{cases} \quad (14)$$

then $\{\psi_n\}$ is a Riesz basis for R with dual basis $\{\psi_n^*\}$. In this notation (13) can be written

$$h(s, t) = \sum_{n \in \mathbb{X}} \overline{\psi_n(t)} \varphi_n(s). \quad (15)$$

Hence by Definition 2 we have

$$h^*(s, t) = \sum_{n \in \mathbb{X}} \overline{\varphi_n(t)} \psi_n(s). \quad (16)$$

in the norm of R for each $t \in E$.

By uniqueness the coefficients $\{\varphi_n(t)\}$ are such that

$$\varphi_n(t) = \overline{\langle h^*(\cdot, t), \psi_n^* \rangle} = \langle (\mathcal{B}^*)^{-1}\varphi_n, h^*(\cdot, t) \rangle, \quad (17)$$

Since this relationship is true for every member φ_n of a basis for R , it holds for every $f \in R$ by the usual density argument. This argument runs as follows:

Let $\sum_N c_n \varphi_n$ be the N th partial sum of the expansion for f in the basis φ_n . Then taking linear combinations in (17),

$$\sum_N c_n \varphi_n(t) = \langle (\mathcal{B}^*)^{-1} \sum_N c_n \varphi_n, h^*(\cdot, t) \rangle. \quad (18)$$

Consider

$$f(t) - \langle (\mathcal{B}^*)^{-1} f, h^*(\cdot, t) \rangle. \quad (19)$$

Put $f(t) - \sum_N c_n \varphi_n(t) = F_N(t)$. Now inserting the right and left hand sides of (18) we find from (19) that

$$|f(t) - \langle (\mathcal{B}^*)^{-1} f, h^*(\cdot, t) \rangle| \quad (20)$$

$$= |F_N(t) - \langle (\mathcal{B}^*)^{-1} (f - \sum_N c_n \varphi_n), h^*(\cdot, t) \rangle|$$

$$\leq |F_N(t)| + |\langle (\mathcal{B}^*)^{-1} (f - \sum_N c_n \varphi_n), h^*(\cdot, t) \rangle|$$

$$\leq |F_N(t)| + \|(\mathcal{B}^*)^{-1} (f - \sum_N c_n \varphi_n)\| \|h^*(\cdot, t)\|$$

$$\leq |F_N(t)| + B \|f - \sum_N c_n \varphi_n\| \|h^*(\cdot, t)\| \quad (21)$$

for a constant B which is consequent upon the fact that, since \mathcal{B} is bounded, \mathcal{B}^* is bounded and by Banach's 'bounded inverse' theorem $(\mathcal{B}^*)^{-1}$ is bounded.

Now N can be made to approach ∞ . Since $F_N(t)$ converges to 0 both in norm and pointwise on E (see Theorem 2), the expression in (21) approaches 0 for each fixed $t \in E$. Finally, from (20) we obtain the following

Theorem 5 Let R , \mathcal{B} and E be as above. Then we have the adjoint operator reproducing formula

$$f(t) = \langle (\mathcal{B}^*)^{-1} f, h^*(\cdot, t) \rangle.$$

This shows the basic property of h^* ; it reproduces f from $(\mathcal{B}^*)^{-1} f$.

3.2 Operator sampling series

There are connections here to the theory of single channel sampling (see, e.g., [3, Ch. 12]), but the present approach is much more general.

In order to match the operator reproducing equation (12) with a discrete analogue, some further assumption will have to be made. In fact we shall assume the existence of a sequence $(s_n) \subset E$, $n \in \mathbb{X}$ such that $\{h(s_n, t)\}$ is an orthogonal basis for R with normalising factors ν_n , so that $\{\nu_n h(s_n, t)\}$ is orthonormal. This can sometimes be traced back to the condition that $\{K_{s_n}\}$ be an orthogonal basis for \mathcal{H} . Again, we could assume that $\{h(s_n, t)\}$ is just a basis for R , or just a frame. However, weaker assumptions demand more technicalities and we will not pursue this kind of generality here.

Let $f \in R$. Its expansion in our assumed orthonormal basis is

$$f(t) = \sum_{n \in \mathbb{X}} c_n \nu_n \overline{h(s_n, t)} \quad (22)$$

where

$$c_n = \langle f, \nu_n \overline{h(s_n, \cdot)} \rangle = \overline{\nu_n} \langle f, h^*(\cdot, s_n) \rangle = \overline{\nu_n} (\mathcal{B}^* f)(s_n)$$

by Theorem 5. So (22) is

$$f(t) = \sum_{n \in \mathbb{X}} |\nu_n|^2 (\mathcal{B}^* f)(s_n) \overline{h(s_n, t)}. \quad (23)$$

Then (12) and (23) are concrete – discrete analogues of each other.

4. Multiplier operators with respect to the Fourier transform

Take E to be an LCA group G with dual Γ (for notations and references, see §2.2), and let R be a Paley–Wiener space PW_Ω . Let $\mu(\gamma)$ be a non-nul complex valued function on Γ such that

$$\begin{cases} 0 < \alpha \leq |\mu(\gamma)| \leq \beta < \infty, & (\text{Haar a.a. } \gamma \in \Omega; \\ \mu(\gamma) = 0, & \gamma \notin \Omega. \end{cases} \quad (24)$$

Let \mathcal{M} denote the operation of multiplication by $\chi_\Omega(\gamma)\mu(\gamma)$.

Definition 3 Let $f \in PW_\Omega$. The operator \mathcal{T} is defined by

$$(\mathcal{T}f)(s) := (\mathcal{F}^{-1}\mathcal{M}\mathcal{F}f)(s)$$

Lemma 2 The operator \mathcal{T} of Definition 3 is a bijection on PW_Ω

Proof Clearly \mathcal{T} is linear. Furthermore it is one-to-one, since the null space of \mathcal{T} is

$$\{f : \mathcal{T}f = \theta\} = \{f : \mu(\gamma)f^\wedge(\gamma) = \theta\}$$

which implies that $f = \theta$.

Again, \mathcal{T} is ‘‘onto’’. Let $g \in PW_\Omega$. Then if \mathcal{M}^{-1} denotes multiplication by $[\mu(\gamma)]^{-1}$, $f = \mathcal{F}^{-1}\mathcal{M}^{-1}\mathcal{F}g \in PW_\Omega$. Then from Definition 3, $\mathcal{T}f = g$.

The boundedness of \mathcal{T} follows from two applications of Plancherel’s Theorem. Indeed, let $f \in PW_\Omega$. Then

$$\begin{aligned} \|\mathcal{T}f\|_{L^2(G)} &= \|\mathcal{F}^{-1}\mathcal{M}\mathcal{F}f\|_{L^2(G)} = \|\mathcal{M}\mathcal{F}f\|_{L^2(\Gamma)} \\ &\leq |\mu| \|\mathcal{F}f\|_{L^2(\Gamma)} = |\mu| \|f\|_{L^2(G)}. \end{aligned}$$

□

4.1 The operator kernel for \mathcal{T}

First we need to know the adjoint \mathcal{T}^* . Let $f_1, f_2 \in PW_\Omega$. The defining equation is

$$\langle \mathcal{T}f_1, f_2 \rangle = \langle f_1, \mathcal{T}^*f_2 \rangle.$$

Suppose that \mathcal{T}^* is of the same form as \mathcal{T} of Definition 3, that is,

$$\mathcal{T}^*f = \mathcal{F}^{-1}\mathcal{M}^*\mathcal{F}f, \quad (25)$$

where \mathcal{M}^* denotes multiplication by the multiplier μ^* which is to be determined.

In the integral notation, and using the ‘hat’ notation for the Fourier transform, the criterion is:

$$\int_G (\mu(\cdot)f_1^\wedge(\cdot))^\vee(t) \overline{f_2(t)} dt = \int_G f_1(t) \overline{(\mu^*(\cdot)f_2^\wedge(\cdot))^\vee(t)} dt.$$

By Plancherel’s theorem this is:

$$\int_\Gamma \mu(\gamma)f_1^\wedge(\gamma) \overline{f_2^\wedge(\gamma)} d\gamma = \int_\Gamma f_1^\wedge(\gamma) \overline{\mu^*(\gamma)f_2^\wedge(\gamma)} d\gamma,$$

from which we may choose $\mu^*(\gamma) = \overline{\mu(\gamma)}$.

It may be noted that \mathcal{T} is self-adjoint if μ is real-valued.

It is now evident that the assumption (25) leads to

$$(\mathcal{T}^*f)(s) = \int_\Gamma \overline{\mu(\gamma)} f^\wedge(\gamma)(s, \gamma) d\gamma \quad (26)$$

The operator kernel for \mathcal{T} can now be calculated. From Definition 1 and (7) we have

$$h(s, t) = (\mathcal{T}^*\varphi_\Omega(\cdot - t))(s), \quad s, t \in G.$$

Therefore from (26), and using the ‘shift’ property of the Fourier transform,

$$\begin{aligned} h(s, t) &= \int_\Gamma \overline{\mu(\gamma)} (\chi_\Omega^\vee(\cdot - t))^\wedge(\gamma)(s, \gamma) d\gamma \\ &= \int_\Gamma \overline{\mu(\gamma)}(-t, \gamma) \chi_\Omega(\gamma)(s, \gamma) d\gamma \\ &= \int_\Omega \overline{\mu(\gamma)}(s - t, \gamma) d\gamma \\ &= \overline{\mu(\cdot)^\vee}(s - t). \end{aligned}$$

Hence

$$\overline{h(s, t)} = \overline{\overline{\mu(\cdot)^\vee}(s - t)} = \mu^\wedge(s - t).$$

Now (12) becomes

$$f(t) = \langle (\mathcal{T}^{-1}f)(\cdot), \overline{\mu}^\vee(\cdot - t) \rangle, \quad (27)$$

and (23) becomes

$$f(t) = \sum_{n \in \mathbb{X}} |\nu_n|^2 (\mathcal{T}^*f)(s_n) \mu^\wedge(s_n - t). \quad (28)$$

5. Examples

Example 1 The classical case

Naturally, we expect to recover the case of the classical reproducing equation and sampling formula as special cases of the theory. To do this we pick $G = \mathbb{R}$, $\Omega = [-\pi, \pi]$, $\mathcal{T} = \mathcal{I} = \mathcal{T}^* = \mathcal{T}^{-1}$ and $\mu = \chi_{[-\pi, \pi]}(y)$. Therefore we have

$$\mu^\vee(s - t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{i(s-t)y} dy = \sqrt{2\pi} \operatorname{sinc}(s - t).$$

Here and in subsequent Examples the choice of Haar measure on G , Γ , etc., accounts for apparent anomalies in the normalising constants in the formulae (e.g., Haar measure on \mathbb{R} is taken to be $(2\pi)^{-1/2}$ times Lebesgue measure. See [2, p. 257]). With these choices, (27) becomes (1).

The classical sampling series (2) now follows the textbook proof. Since $\{e^{-iny}/\sqrt{2\pi} : n \in \mathbb{Z}\}$ is an orthonormal (ON) basis of $L^2(-\pi, \pi)$, Plancherel’s theorem shows that the inverse Fourier transforms $\{\operatorname{sinc}(n - t) : n \in \mathbb{Z}\}$ form an orthonormal basis of $PW_{[-\pi, \pi]}$. Coefficients in the expansion of f in this basis are obtained from (1) and so, with $s_n = n$, our choices for \mathcal{T} and μ show that (28) becomes (2).

Example 2 The Hilbert transform

Another well-known example illustrates the present theory; a member of $PW_{[-\pi, \pi]}$ can be sampled and reconstructed from samples of its Hilbert transform (see, e.g., [3, p. 126] and references there). This idea can be fitted into the theme of the present lecture by taking $G = \mathbb{R}$, $\Omega = [-\pi, \pi]$, $\mathcal{T} = \mathcal{H} := \mathcal{F}^{-1}\mathcal{M}\mathcal{F}$ where \mathcal{M} denotes multiplication by $-i \operatorname{sgn}(y)$. \mathcal{H} is the Hilbert transform on $PW_{[-\pi, \pi]}$.

For (27) we need

$$\begin{aligned}\bar{\mu}^\vee(s-t) &= \frac{1}{\sqrt{2\pi}} \frac{i}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \operatorname{sgn}(y) e^{i(s-t)y} dy \\ &= -\operatorname{sinc} \frac{1}{2}(s-t) \sin \frac{\pi}{2}(s-t)\end{aligned}\quad (29)$$

after a simple calculation. Also we have $\mathcal{H}^{-1} = -\mathcal{H} = \mathcal{H}^*$, therefore (27) is

$$f(t) = - \int_{\mathbb{R}} (\mathcal{H}f)(\tau) \operatorname{sinc} \frac{1}{2}(\tau-t) \sin \frac{\pi}{2}(\tau-t) d\tau. \quad (30)$$

For (28) we need to find $\{s_n\}$ such that $\{\mu^\wedge(s_n-t)\}$, $n \in \mathbb{Z}$, is an ON basis of PW_π . We can start with the ON basis $\{e^{iny}/\sqrt{2\pi}\}$, ($n \in \mathbb{Z}$), of $L^2(-\pi, \pi)$, then multiply each member by $-i \operatorname{sgn}(y)$. The result is again an ON basis, as a consequence of $|-i \operatorname{sgn}(y)| = 1$ a.e. on $[-\pi, \pi]$. The inverse Fourier transform of a typical one of these basis elements is

$$\frac{-i}{2\pi} \mathcal{F}^{-1}(\operatorname{sgn}(\cdot) e^{-in\cdot})(t) = -\operatorname{sinc} \frac{1}{2}(n-t) \sin \frac{\pi}{2}(n-t)$$

by the same calculation as in (29). But, taking account of Haar measure, this also gives $\mu^\wedge(n-t)$. Hence (28) becomes

$$f(t) = \sum_{n \in \mathbb{Z}} (\mathcal{H}f)(n) \operatorname{sinc} \frac{1}{2}(n-t) \sin \frac{\pi}{2}(n-t) \quad (31)$$

Our concrete – discrete comparison is between (30) and (31).

Example 3 The Riesz transforms

For background on the Riesz transforms see [9, p. 223] Take G to be \mathbb{R}^d , ($d \in \mathbb{N}$). Let $\mathbf{t} = (t_1, \dots, t_d)$ and let $\mathbf{y} = (y_1, \dots, y_d)$ etc. Let the scalar product in \mathbb{R}^d be denoted by $\langle \cdot, \cdot \rangle$.

Definition 4 Let $f \in L^2(\mathbb{R}^d)$, and define

$$\mathcal{M}_j f := \mathcal{F}^{-1} \mathcal{M}_j \mathcal{F} f, \quad j = 1, \dots, d, \quad (32)$$

\mathcal{M}_j denoting multiplication by $-iy_j/|\mathbf{y}| \chi_{[-\pi, \pi]^d}(\mathbf{y})$.

We note that this multiplier is not bounded away from zero when $d \geq 2$ and $\mathbf{y} \in [-\pi, \pi]^d$ and therefore does not always satisfy the criterion (24). However, it is possible to define operators involving the Riesz transforms which do satisfy the criterion (24).

First we consider the case $d = 2$, and define the operator

$$\mathcal{R} := \mathcal{R}_1 + i\mathcal{R}_2 \quad (33)$$

acting on $PW_{[-\pi, \pi]^2}$. Its multiplier is

$$m(\mathbf{y}) := (-i) \left(\frac{y_1}{|\mathbf{y}|} + i \frac{y_2}{|\mathbf{y}|} \right)$$

and clearly we have $|m(\mathbf{y})| = 1$ a.e. Hence m satisfies the criterion (24) with respect to two-dimensional Lebesgue measure. Now (27) becomes

$$f(\mathbf{t}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\mathcal{R}^{-1}f)(\mathbf{s}) \bar{m}^\vee(\mathbf{s}-\mathbf{t}) d\mathbf{s}. \quad (34)$$

Since the multiplier is of unit modulus, a two dimensional version of the construction that we used in the previous example shows that

$$\left\{ \frac{-i}{2\pi} \left(\frac{y_1}{|\mathbf{y}|} + i \frac{y_2}{|\mathbf{y}|} \right) e^{-i\langle \mathbf{k}, \mathbf{y} \rangle} \right\}, \quad (\mathbf{k} \in \mathbb{Z}^2),$$

is an ON basis of $L^2([-\pi, \pi]^2)$. Then (28) becomes

$$f(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{Z}^2} (R^* f)(\mathbf{k}) m^\wedge(\mathbf{k}-\mathbf{t}). \quad (35)$$

The comparison for this example lies between the concrete (34) and the discrete (35).

Other combinations of the Riesz transforms are possible, in two and higher dimensions, whose multipliers satisfy (24) but are not always of unit modulus.

6. Conclusions

The multiplier transforms treated in this study form a rather restricted class of operators; nevertheless, the methods can be used in connection with the very important Riesz transforms. It remains to investigate extensions to other types of operator. Likely candidates are, for example, multiplier transforms with less restrictive conditions on the multiplier, the singular integral operators of Calderón–Zygmund type (a class containing the Riesz transforms, see, e.g., [9, Ch. VI]), and operators of the Hankel and Toeplitz type (e.g., [7]).

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