HAL
open science

Multivariate Complex BSplines, Dirichlet Averages and Difference Operators

Brigitte Forster, Peter Massopust

To cite this version:

Brigitte Forster, Peter Massopust. Multivariate Complex BSplines, Dirichlet Averages and Difference Operators. SAMPTA'09, May 2009, Marseille, France. pp.General session. hal-00453453

HAL Id: hal-00453453

https://hal.science/hal-00453453v1

Submitted on 4 Feb 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Multivariate Complex B-Splines, Dirichlet Averages and Difference Operators

Brigitte Forster [1,2] and Peter Massopust [2,1]

(1) Zentrum Mathematik, M6, Technische Universität München, Germany
(2) Institut für Biomathematik und Biometrie, Helmholtz Zentrum München, Germany
forster@ma.tum.de, massopust@ma.tum.de

Abstract:

For the Schoenberg B-splines, interesting relations between their functional representation, Dirichlet averages and difference operators are known. We use these relations to extend the B-splines to an arbitrary (infinite) sequence of knots and to higher dimensions. A new Fourier domain representation of the multidimensional complex B-spline is given.

1. Complex B-Splines

Complex B-splines are a natural extension of the classical Curry-Schoenberg B-splines [2] and the fractional splines first investigated in [16]. The complex B-splines $B_z : \mathbb{R} \to \mathbb{C}$ are defined in Fourier domain as

$$\mathcal{F}(B_z)(\omega) = \int_{\mathbb{R}} B_z(t) e^{-i\omega t}\, dt = \left(\frac{1 - e^{-i\omega}}{i\omega}\right)^z$$

for $\operatorname{Re} z > 1$. They are well-defined, because of $\{\frac{1-e^{-i\omega}}{i\omega} \mid \omega \in \mathbb{R}\} \cap \{y \in \mathbb{R} \mid y < 0\} = \emptyset$ they live on the main branch of the complex logarithm. Complex B-splines are elements of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. They have several interesting basic properties, which are discussed in [5]. Let $\operatorname{Re} z, \operatorname{Re} z_1, \operatorname{Re} z_2 > 1$.

- Complex B-splines B_z are piecewise polynomials of complex degree.
- Smoothness and decay:
  - $B_z \in W_2^r(\mathbb{R})$ for $r < \operatorname{Re} z - \frac{1}{2}$. Here $W_2^r(\mathbb{R})$ denotes the Sobolev space with respect to the L^2-Norm and with weight $(1 + |x|^2)^r$.
  - $B_z(x) = \mathcal{O}(x^{-m})$ for $m < \operatorname{Re} z + 1$, $|x| \to \infty$.
- Recursion formula: $B_{z_1} * B_{z_2} = B_{z_1+z_2}$.
- Complex B-splines are scaling functions and generate multiresolution analyses and wavelets.
- But in general, they don't have compact support.
- Last but not least: They relate difference and differential operators.

In this paper, we take closer look at this last relation and the respective multivariate setting. To this end, we will consider the known relations between classical B-splines, difference operators and Dirichlet averages.

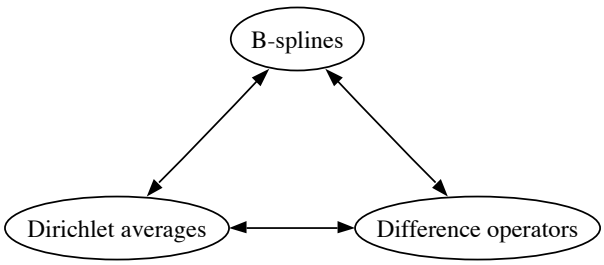

Figure 1: Relations between classical B-splines, difference operators and Dirichlet averages.

2. Representation in time-domain

We defined complex B-splines in Fourier domain, and Fourier inversion shows that these functions are piecewise polynomials of complex degree:

Proposition 1. [5] Complex B-splines have a time-domain representation of the form

$$B_z(t) = \frac{1}{\Gamma(z)} \sum_{k \geq 0} (-1)^k \binom{z}{k} (t-k)_+^{z-1},$$

pointwise for all $t \in \mathbb{R}$ and in $L^2(\mathbb{R})$-norm. Here,

$$t_+^z = \begin{cases} t^z = e^{z \ln t}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

is the truncated power function, and $\Gamma : \mathbb{C} \setminus \mathbb{Z}_0^- \to \mathbb{C}$ denotes the Euler Gamma function.

Compare: The cardinal B-spline B_n, $n \in \mathbb{N}$, has the similar representation

$$\begin{aligned} B_n(t) &= \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (t-k)_+^{n-1} \\ &= \frac{1}{\Gamma(n)} \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} (t-k)_+^{n-1}. \end{aligned}$$

3. Relations to Difference Operators

It is well-known that in the construction of the Curry-Schoenberg B-splines difference operators are deeply involved. The same is true for complex B-splines. To establish the corresponding relation, let us first recall the definition of the backward difference operator ∇.

Let $g : \mathbb{R} \to \mathbb{C}$ be a function. Then the backward difference operator $\nabla = \nabla^1$ is recursively defined as follows:

$$\begin{aligned} \nabla g(t) &= g(t) - g(t-1), \\ \nabla^{n+1} g(t) &= \nabla(\nabla^n g(t)) \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

This definition yields the explicit representation

$$\nabla^n g(t) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k g(t-k).$$

For the cardinal B-splines B_n we can write:

$$\begin{aligned} B_n(t) &= \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (t-k)_+^{n-1} \\ &= \frac{1}{(n-1)!} \nabla^n t_+^{n-1}. \end{aligned}$$

In comparison: For the complex B-splines, we have an analog representation:

$$B_z(t) = \frac{1}{\Gamma(z)} \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} (t-k)_+^{z-1}, \quad \operatorname{Re} z \geq 1.$$

This invites to define a complex difference operator:

Definition 2. [5, 6] The difference operator ∇^z of complex order z is defined as

$$\nabla^z g(t) := \sum_{k=0}^{\infty} (-1)^k \binom{z}{k} g(t-k), \quad z \in \mathbb{C}, \quad \operatorname{Re} z \geq 1.$$

Hence a second time domain representation of the complex B-spline is

$$B_z(t) = \frac{1}{\Gamma(z)} \nabla^z t_+^{z-1}.$$

In a similar way, we can establish a relation to divided differences. Recall that for a knot sequences $\{t_0, \ldots, t_n\} \subset \mathbb{R}$, $n \geq 1$, divided differences are recursively defined as follows. Let $g : \mathbb{R} \to \mathbb{C}$ be some function.

$$\begin{aligned} [t_0]g &= g(t_0), \\ [t_0, \ldots, t_n]g &= \frac{[t_0, \ldots, t_{n-1}]g - [t_1, \ldots, t_n]g}{t_0 - t_n} \\ &= \sum_{j=0}^{n} \frac{g(t_j)}{\prod_{l \neq j} (t_j - t_l)}. \end{aligned}$$

For the cardinal B-spline,

$$\begin{aligned} B_n(t) &= \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (t-k)_+^{n-1} \\ &= n \sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!} (t-k)_+^{n-1} \\ &= (-1)^n n \sum_{k=0}^{n} \frac{(t-k)_+^{n-1}}{\prod_{l \neq k} (k-l)} \\ &= (-1)^n n [0, 1, \ldots, n](t - \bullet)_+^{n-1}. \end{aligned}$$

(The factor $(-1)^n$ is due to our representation of the cardinal B-spline via backward difference operators.)
The same ideas give rise to the definition of complex divided differences.

Definition 3. Let $g : \mathbb{R} \to \mathbb{C}$ be some function. We define the complex divided differences for the knot sequence $\mathbb{N}_0$ via

$$[z; \mathbb{N}_0]g := \sum_{k \geq 0} (-1)^k \frac{g(k)}{\Gamma(z-k+1)\Gamma(k+1)}.$$

Then the complex B-spline can be written as

$$B_z(t) = z[z, \mathbb{N}_0](t - \bullet)_+^{z-1}.$$

Comparing "old" and "new" divided difference operator for $z = n \in \mathbb{N}$, yields

$$(-1)^n [0, 1, \ldots n] = [n, \mathbb{N}_0].$$

Proposition 4. [6, 7] Let $\operatorname{Re} z > 0$ and $g \in \mathcal{S}(\mathbb{R}^+)$. Then

$$[z; \mathbb{N}_0]g = \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, dt,$$

where $g^{(z)} = W^z g$ is the complex Weyl derivative:
For $n = \lceil \operatorname{Re} z \rceil$, $\nu = n - z$,

$$W^z g(t) = (-1)^n \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(\nu)} \int_t^{\infty} (x-t)^{\nu-1} g(x)\, dx \right].$$

Sketch of proof:

$$\begin{aligned} & \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, dt \\ = \; & \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} z[z, \mathbb{N}_0](t - \bullet)_+^{z-1} W^z g(t)\, dt \\ = \; & [z, \mathbb{N}_0] \frac{1}{\Gamma(z)} \int_{\bullet}^{\infty} (t - \bullet)_+^{z-1} W^z g(t)\, dt \\ = \; & [z, \mathbb{N}_0] W^{-z} W^z g \quad = \quad [z, \mathbb{N}_0] g. \end{aligned}$$

Here, $W^{-z} f = \frac{1}{\Gamma(z)} \int_{\bullet}^{\infty} (t - \bullet)_+^{z-1} f(t)\, dt$ is the complex Weyl integral of the function f, i.e., the inverse operator of W^z. □

Now we are able to establish a first relation between divided difference operators and Dirichlet averages.

Proposition 5. (Generalized Hermite-Genocchi-Formula: Divided Differences and Dirichlet Averages) [6, 7]
Let Δ^∞ be the infinite-dimensional simplex

$$\Delta^\infty := \{u := (u_j) \in (\mathbb{R}_0^+)^{\mathbb{N}_0} | \sum_{j=0}^{\infty} u_j = 1\} = \varprojlim \Delta^n,$$

defined as the projective limit of the finite dimensional simplices Δ^n, and let μ_e^∞ be the generalized Dirichlet measure defined by the projective limit

$$\mu_e^\infty = \varprojlim \Gamma(n+1) \lambda^n,$$

where λ^n the Lebesgue measure on Δ^n. Then

$$\begin{aligned} [z, \mathbb{N}_0]g &= \frac{1}{\Gamma(z+1)} \int_{\Delta^\infty} g^{(z)}(\mathbb{N}_0 \cdot u) d\mu_e^\infty(u) \\ &= \frac{1}{\Gamma(z+1)} \int_{\mathbb{R}} B_z(t) g^{(z)}(t)\, dt \end{aligned}$$

for all real-analytic $g \in \mathcal{S}(\mathbb{R}^+)$.

Up to now we have considered complex B-splines with knot sequence $\mathbb{N}_0$ and derived from there new difference operators and finally the relation to Dirichlet averages, just as indicated in the diagram in Fig. 1:

B-splines $\rightarrow$ Difference operators $\rightarrow$ Dirichlet averages.

Our next step will consist of generalizing the setting with appropriate weights in travelling through the diagram another way round: Dirichlet averages for other knot sequences τ and with weights $\rightarrow$ Generalized B-splines with knot sequence $\tau \rightarrow$ Difference operators.

4. Splines and Dirichlet Averages

Let $b \in \mathbb{R}_+^\infty$ be a weight vector and $\tau = \{t_k\}_{k\in\mathbb{N}_0} \in \mathbb{R}_+^{\mathbb{N}_0}$ an increasing sequence of knots with $\limsup_{k\to\infty} \sqrt[k]{t_k} \leq \rho < e$.

Definition 6. A complex B-spline $B_z(\bullet \mid b;\tau)$ with weight vector b and knot sequence τ is a function satisfying

$$\int_{\mathbb{R}} B_z(t \mid b;\tau) g^{(z)}(t)\, dt = \int_{\Delta^\infty} g^{(z)}(\tau \cdot u) d\mu_b^\infty(u) \quad (1)$$

for all real-analytic $g \in \mathcal{S}(\mathbb{R}^+)$. Here, $\mu_b^\infty = \varprojlim \mu_b^n$ is the projective limit of Dirichlet measures with densities

$$\frac{\Gamma(b_0)\ldots\Gamma(b_n)}{\Gamma(b_0+\ldots+b_n)} u_0^{b_0-1} u_1^{b_1-1} \ldots u_n^{b_n-1}.$$

Since both W^z and W^{-z} are linear operators mapping $\mathcal{S}(\mathbb{R}^+)$ into itself [11, 15] and since the real-analytic functions in $\mathcal{S}(\mathbb{R}^+)$ are dense in $\mathcal{S}(\mathbb{R}^+)$ [13], (1) holds for all $g \in \mathcal{S}(\mathbb{R}^+)$. Moreover, since $\mathcal{S}(\mathbb{R}^+)$ is dense in $L^2(\mathbb{R}^+)$, we deduce that $B_z(\bullet \mid b,\tau) \in L^2(\mathbb{R}^+)$.
Equation (1) means, we define the weighted version of the complex B-spline in a weak sense via Dirichlet averages. Referring again to the diagram in Fig. 1, we now move from the generalized B-splines to generalized divided differences.

Definition 7. For knot sequences $\tau \in \mathbb{R}_+^{\mathbb{N}_0}$ and weight vectors $b \in \mathbb{R}_+^\infty$ as above, we define the generalized complex divided differences $[z;\tau]_b$ as follows. Let $g : \mathbb{R} \to \mathbb{C}$ be some function.

$$[z;\tau]_b g := \frac{1}{\Gamma(z)} \int_{\mathbb{R}} B_z(t|b;\tau) g^{(z)}(t)\, dt$$

for all $g \in \mathcal{S}(\mathbb{R})$.

This definition is compatible with the usual Dirichlet splines. In fact, for all finite $\tau = \tau(n) \in \mathbb{R}_+^{n+1}$ and $b = b(n) \in \mathbb{R}_+^{n+1}$, and for $z = n \in \mathbb{N}_0$ the Dirichlet spline $D_n(\bullet|b;\tau)$ of order n is defined by

$$\begin{aligned} \int_{\mathbb{R}} g^{(n)}(t) D_n(t|b;\tau)\, dt &= \int_{\Delta^n} g^{(n)}(\tau \cdot u)\, d\mu_b^n(u) \\ &= G^{(n)}(b;\tau) \end{aligned}$$

for all $g \in C^n(\mathbb{R})$. Here, G is the Dirichlet average of g:

$$G(b;\tau) = \int_{\Delta^n} g(\tau \cdot u)\, d\mu_b^n(u).$$

5. Multivariate Complex B-Splines

To define complex B-splines in a multivariate setting, we consider ridge functions and define multivariate B-splines on their basis. Then, we walk again through the diagram in Fig. 1: Multivariate B-splines $\rightarrow$ Multivariate difference operators. Results on Dirichlet averages yield new recurrence relations for multivariate B-splines: Dirichlet averages $\rightarrow$ B-splines.
Note that the approach via ridge functions had already let to an extension of the Curry-Schoenberg-splines to a multivariate setting, e.g. [3, 4, 10, 12]. However, some of these approaches have certain restrictions on the knots and none of them considers complex splines.
Given $\lambda \in \mathbb{R}^s \backslash \{0\}$, a direction, and $g : \mathbb{R} \to \mathbb{C}$ a function. The ridge function g_λ corresponding to g is defined via

$$g_\lambda : \mathbb{R}^s \to \mathbb{C}, \quad g_\lambda(x) = g(\langle \lambda, x\rangle) \quad \text{for all } x \in \mathbb{R}^s.$$

Definition 8. [9] Let $\boldsymbol{\tau} = \{\tau^n\}_{n\in\mathbb{N}_0} \in (\mathbb{R}^s)^{\mathbb{N}_0}$ a sequence of knots in $\mathbb{R}^s$ with $\limsup_{n\to\infty} \sqrt[n]{\|\tau^n\|} \leq \rho < e$. The multivariate complex B-spline $\boldsymbol{B}_z(\bullet|b;\boldsymbol{\tau})$ with weights $b \in \mathbb{C}_+^{\mathbb{N}_0}$ and knots $\boldsymbol{\tau}$ is defined on ridge functions via

$$\int_{\mathbb{R}^s} g(\langle \lambda, x\rangle) \boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})\, dx = \int_{\mathbb{R}} g(t) B_z(t \mid b;\lambda\boldsymbol{\tau})\, dt, \quad (2)$$

where $g \in \mathcal{S}(\mathbb{R}^+)$ and $\lambda \in \mathbb{R}^s \setminus \{0\}$, such that $\lambda\boldsymbol{\tau} = \{\langle \lambda, \tau^n\rangle\}_{n\in\mathbb{N}_0}$ is separated.

Since ridge functions are dense in $L^2(\mathbb{R}^s)$ [14], we deduce that $\boldsymbol{B}_z(\bullet \mid b;\boldsymbol{\tau}) \in L^2((\mathbb{R}^+)^s)$.

Example 9. (Divided differences in the multivariate case) Given $b = e := (1,1,1,\ldots)$. Then for all $g \in \mathcal{S}(\mathbb{R}^\infty)$:

$$\begin{aligned} [z;\boldsymbol{\tau}]_e g_\lambda &= [z;\boldsymbol{\tau}] g_\lambda = [z;\boldsymbol{\tau}] g(\langle \lambda, \bullet\rangle) \\ &= \frac{1}{\Gamma(z)} \int_{\mathbb{R}^s} g^{(z)}(\langle \lambda, x\rangle) \boldsymbol{B}_z(x \mid e;\boldsymbol{\tau})\, dx \\ &= \frac{1}{\Gamma(z)} \int_{\mathbb{R}} g^{(z)}(t) B_z(t \mid e;\lambda\boldsymbol{\tau})\, dt = [z;\lambda\boldsymbol{\tau}] g. \end{aligned}$$

for all $\lambda \in \mathbb{R}^s$ such that $\lambda\boldsymbol{\tau}$ is separated.

Example 10. (Multivariate cardinal B-splines) For $n \in \mathbb{N}$ and a finite sequence of knots $\boldsymbol{\tau} = \{\tau^0, \tau^1, \ldots, \tau^n\}$:

$$\begin{aligned} & [\tau^0, \ldots, \tau^n] g_\lambda := [n;\boldsymbol{\tau}] g(\langle \lambda, \bullet\rangle) \\ = & \frac{1}{n!} \int_{\mathbb{R}^s} g^{(n)}(\langle \lambda, x\rangle) \boldsymbol{B}_n(x \mid e;\boldsymbol{\tau})\, dx \\ = & \frac{1}{n!} \int_{\mathbb{R}} g^{(n)}(t) B_n(t \mid e;\lambda\boldsymbol{\tau})\, dt \\ = & [n;\lambda\boldsymbol{\tau}] g = \sum_{j=0}^n \frac{g(\langle \lambda, \tau^j\rangle)}{\prod_{l\neq j} \langle \lambda, \tau^j - \tau^l\rangle}. \end{aligned}$$

Given a sequence of knots $\boldsymbol{\tau} \subset \mathbb{R}^s$ and a weight vector b as above. In addition, let $b \in l^1(\mathbb{N}_0)$ such that $\|b\|_1 =: c$. Then the Dirichlet averages of $g^{(z)} \in \mathcal{D}(\mathbb{R})$ and $g_j^{(z+1)} := (\langle \lambda, \tau^j\rangle - \bullet) g^{(z+1)}$, $j \in \mathbb{N}_0$, satisfy:

$$(c-1) G^{(z)}(b;\lambda\boldsymbol{\tau}) = (c-1) G^{(z)}(b-e_j;\lambda\boldsymbol{\tau}) + G_j^{(1+z)}(b;\lambda\boldsymbol{\tau}).$$

For the finite dimensional case see [1, 12]. These and other relations of similar type on Dirichlet averages yield new results for multivariate complex B-splines. As a example, we state:

Proposition 11. [9] Under the above conditions, for all $j \in \mathbb{N}_0$:

$$\begin{aligned} & (c-1)\int_{\mathbb{R}^s} g_\lambda^{(z)}(x)\boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})dx = \\ = \quad & (c-1)\int_{\mathbb{R}^s} g_\lambda^{(z)}(x)\boldsymbol{B}_z(x \mid b-e_j;\boldsymbol{\tau})dx \\ + \quad & \int_{\mathbb{R}^s} \langle \lambda, \tau^j - x\rangle\, g_\lambda^{(1+z)}(x)\boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})dx. \end{aligned}$$

More relations of this type are given in [8].

6. Fourier representation of multivariate complex B-splines

We saw above that both the univariate and the multivariate complex B-splines are L^2-functions: $B_z(\bullet \mid b;\tau) \in L^2(\mathbb{R}^+)$ and $\boldsymbol{B}_z(\bullet \mid b;\boldsymbol{\tau}) \in L^2((\mathbb{R}^+)^s)$ Therefore, we can apply the Plancherel transform to both functions and consider their frequency spectrum.
Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_s) \in \mathbb{R}^s$ and let $\lambda \in \mathbb{R}^s$, $\|\lambda\| = 1$, be the direction of $\boldsymbol{\omega}$, i.e., $\boldsymbol{\omega} = \omega\lambda$ for some $\omega \geq 0$. For the Fourier transform of the generalized complex B-spline we have for $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$:

$$\begin{aligned} & \widehat{B}_z(\omega \mid b;\lambda\boldsymbol{\tau}) = \\ = \quad & \int_{\mathbb{R}} e^{-i\omega t} B_z(t \mid b;\lambda\boldsymbol{\tau})\, dt \\ = \quad & \int_{\mathbb{R}^s} e^{-i\omega\langle\lambda,x\rangle}\boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})\, dx \\ = \quad & \int_{\mathbb{R}^s} e^{-i\omega(\lambda_1 x_1+\ldots+\lambda_s x_s)}\boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})\, dx \\ = \quad & \int_{\mathbb{R}^s} e^{-i(\omega_1 x_1+\ldots+\omega_s x_s)}\boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})\, dx \\ = \quad & \int_{\mathbb{R}^s} e^{-i\langle\boldsymbol{\omega},\boldsymbol{x}\rangle}\boldsymbol{B}_z(x \mid b;\boldsymbol{\tau})\, dx \\ = \quad & \widehat{\boldsymbol{B}}_z(\boldsymbol{\omega} \mid b;\boldsymbol{\tau}) \quad = \quad \widehat{\boldsymbol{B}}_z(\omega\lambda \mid b;\boldsymbol{\tau}). \end{aligned}$$

This shows that the frequency spectrum of the multivariate complex B-spline along directions λ is given by the spectrum of the univariate spline with knots projected onto these λ.

7. Summary

Complex B-splines allow to define difference and divided difference operators of complex order for arbitrary knots and weights. Via their relation to Dirichlet averages and Dirichlet splines, they can be extended to higher dimensions via ridge functions. The Fourier transform of the univariate and multivariate complex B-spline are also related on ridges.

8. Acknowledgments

This work was partially supported by the grant MEXT-CT-2004-013477, Acronym MAMEBIA, of the European Commission.

References:

[1] B. C. Carlson. B-Splines, hypergeometric functions and Dirichlet averages. *J. Approx. Th.*, 67:311–325, 1991.

[2] H. B. Curry and I. J. Schoenberg. On spline distributions and their limits: The Pólya distribution functions. *Bulletin of the AMS*, 53(7–12):1114, 1947. Abstract.

[3] W. Dahmen and C. A. Micchelli. Statistical Encounters with B-Splines. *Contemporary Mathematics*, 59:17–48, 1986.

[4] C. de Boor. Splines as linear combinations of B-splines. In G. G. Lorentz et al., editor, *Approximation Theory II*, pages 1–47. Academic Press, 1976.

[5] B. Forster, T. Blu, and M. Unser. Complex B-splines. *Appl. Comp. Harmon. Anal.*, 20:281–282, 2006.

[6] B. Forster and P. Massopust. Statistical encounters with complex B-Splines. to appear in Constructive Approximation.

[7] B. Forster and P. Massopust. Some remarks about the connection between fractional divided differences, fractional B-Splines, and the Hermite-Genocchi formula. *International Journal of Wavelets, Multiresolution and Information Processing*, 6(2):279–290, 2008.

[8] P. Massopust. Double Dirichlet averages and complex B-splines. Submitted to SAMPTA 2009.

[9] P. Massopust and B. Forster. Multivariate complex B-splines and Dirichlet averages. Submitted to Journal of Approximation Theory.

[10] C. A. Micchelli. A constructive approach to Kergin interpolation in $\mathbb{R}^k$: Multivariate B-splines and Lagrange interpolation. *Rocky Mt. J. Math.*, 10(3):485–497, 1980.

[11] K. S. Miller and B. Ross. *An introduction to the fractional calculus and fractional differential equations*. Wiley, 1993.

[12] E. Neuman and P. J. Van Fleet. Moments of Dirichlet splines and their applications to hypergeometric functions. *Journal of Computational and Applied Mathematics*, 53:225–241, 1994.

[13] O. V. Odinokov. Spectral analysis in certain spaces of entire functions of exponential type and its applications. *Izv. Math.*, 64(4):777–786, 2000.

[14] A. Pinkus. Approximating by ridge functions. In A. Le Méhauté, C. Rabut, and L. L. Schumaker, editors, *Surface Fitting and Multiresolution Methods*, pages 1–14. Vanderbilt University Press, 1997.

[15] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives*. Gordon and Breach Science Publishers, Minsk, Belarus, 1987.

[16] M. Unser and T. Blu. Fractional splines and wavelets. *SIAM Review*, 42(1):43–67, March 2000.