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Abstract:

The harmonizable Piranashvili – type stochastic processes are approximated by a finite time shifted average sampling sum. Truncation error upper bound is established; various consequences and special cases are discussed.

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1. Introduction and preparation

Given a probability space $\big(\Omega, \mathfrak{F}, \mathsf{P}\big)$ and the related Hilbert–space $L_2(\Omega) := \{X\colon \mathsf{E}|X|^2 < \infty\}$. Let us consider a non–stationary, centered stochastic $L_2(\Omega)$–process $\xi\colon \mathbb{R}\times\Omega \mapsto \mathbb{R}$ having covariance function (associated to some domain $\Lambda \subseteq \mathbb{R}$ with some sigma–algebra $\sigma(\Lambda)$) in the form:

$$B(t,s) = \int_\Lambda\int_\Lambda f(t,\lambda) f^*(s,\mu) F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu), \qquad (1)$$

with analytical exponentially bounded kernel function $f(t,\lambda)$, while F_ξ is a positive definite measure on $\mathbb{R}^2$ provided the total variation $\|F_\xi\|(\Lambda,\Lambda)$ of the spectral distribution function F_ξ such that satisfies

$$\|F_\xi\|(\Lambda,\Lambda) = \int_\Lambda\int_\Lambda \big|F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu)\big| < \infty.$$

(We mention that the sample function $\xi(t) \equiv \xi(t,\omega_0)$ and $f(t,\lambda)$ possess the same exponential types [1, Theorem 4], [11, Theorem 3]). Then, by the Karhunen–Cramér theorem the process $\xi(t)$ has the spectral representation as a Lebesgue integral

$$\xi(t) = \int_\Lambda f(t,\lambda) Z_\xi(\mathrm{d}\lambda); \qquad (2)$$

in (1) and (2)

$$F_\xi(S_1,S_2) = \mathsf{E} Z_\xi(S_1) Z_\xi^*(S_2) \qquad S_1, S_2 \subseteq \sigma(\Lambda).$$

Such a process will be called Piranashvili process in the sequel [11], [12].

Being $f(t,\lambda)$ entire, it possesses the Maclaurin expansion $f(t,\lambda) = \sum_{n=0}^\infty f^{(n)}(0,\lambda)t^n/n!$. Put

$$\gamma := \sup_\Lambda c(\lambda) = \sup_\Lambda \overline{\lim_n} \sqrt[n]{|f^{(n)}(0,\lambda)|} < \infty\,. \qquad (3)$$

As the exponential type of $f(t,\lambda)$ is equal to γ, for all $w > \gamma$ there holds

$$\xi(t) = \sum_{n\in\mathbb{Z}} \xi\Big(\frac{n\pi}{w}\Big) \frac{\sin(wt - n\pi)}{wt - n\pi}, \qquad (4)$$

uniformly in the mean square and in the almost sure sense [11, Theorem 1]. This result we call Whittaker–Kotel'nikov–Shannon (WKS) stochastic sampling theorem [12].

Specifying $F_\xi(x,y) = \delta_{xy} F_\xi(x)$ in (1) we conclude the Karhunen–representation of the covariance function

$$B(t,s) = \int_\Lambda f(t,\lambda) f^*(s,\lambda) F_\xi(\mathrm{d}\lambda).$$

Also, putting $f(t,\lambda) = e^{it\lambda}$ in (1) one gets the Loève-representation:

$$B(t,s) = \int_\Lambda\int_\Lambda e^{i(t\lambda - s\mu)} F_\xi(\mathrm{d}\lambda, \mathrm{d}\mu).$$

Here is $c(\lambda) = |\lambda|$. Therefore, WKS–formula (4) holds for all $w > \gamma = \sup|\Lambda|$. Then, the Karhunen process with the Fourier kernel $f(t,\lambda) = e^{it\lambda}$ we recognize as the weakly stationary stochastic process having covariance

$$B(\tau) = \int_\Lambda e^{i\tau\lambda} F_\xi(\mathrm{d}\lambda), \qquad \tau = t - s.$$

Deeper insight into different kind harmonizabilities present [5, 13, 14] and the related references therein. Finally, using $\Lambda = [-w, w]$ for some finite w in this considerations, we get the band–limited variants of the same kind processes.

By physical and applications reasons the measured samples in practice may not be the exact values of the measured process $\xi(t)$, or its covariance $B(t,s)$ itself, near to the sample time t_n, but only the local average of the signal ξ near to t_n. So, the measured sample values will be

$$\langle \xi, u_n\rangle_U = \int_U \xi(x) u_n(x)\mathrm{d}x, \quad U = \mathrm{supp}(u_n) \qquad (5)$$

for a sequence $\mathbf{u} := \big(u_n(t)\big)_{n\in\mathbb{Z}}$ of non–negative, normalized, that is $\langle 1, u_n\rangle \equiv 1$, averaging functions such that

$$\mathrm{supp}(u_n) \subseteq \big[t_n - \sigma_n', t_n + \sigma_n''\big]\,. \tag{6}$$

The local averaging method was introduced by Gröchenig [2] and developed by Butzer and Lei. Recently Sun and Zhou gave some results in this direction, while the stochastic counterpart of this average sampling was intensively studied in the last three–four years by He, Song, Sun, Yang and Zhu in a set of articles [15], [16] and their references therein; see for example the exhaustive references list in [4]. The listed, recently considered stochastic average sampling results are restricted to weakly stationary stochastic processes, while the approximation average sampling sums are used around the origin.

Our intentions are to extend these results to time shifted average sampling, considered for the very wide class of Piranashvili processes.

2. Time shifted average sampling

Now, instead to follow the approach used in [16] we take time shifted [7], [8] finite average sampling sum in approximating the initial stochastic signal ξ. First, we consider weighted average over $\mathfrak{J}_n(t) := \big[n\pi/w - \sigma_n'(t), n\pi/w + \sigma_n''(t)\big]$ for the measured value of $\xi(t)$ at $n\pi/w$, $n \in \mathbb{I}_N(t)$ where

$$\mathbb{I}_N(t) := \{n \in \mathbb{Z}\colon |tw/\pi - n| \le N\}, \qquad N \in \mathbb{N}.$$

Let N_t be the integer nearest to tw/π.

By obvious reasons we restrict the study to

$$\sigma := \max_{\mathbb{I}_N(t)} \sup_{\mathbb{R}} \max\big(\sigma_n'(t), \sigma_n''(t)\big) \le \frac{\pi}{2w}\,.$$

Let us define the time shifted average sampling approximation sum in the form

$$\mathcal{A}_{\mathbf{u}}(\xi;t) = \sum_{\mathbb{Z}} \langle \xi, u_n\rangle_{\mathfrak{J}_n(t)} \cdot \frac{\sin(wt - n\pi)}{wt - n\pi}\,,$$

and its truncated variant

$$\mathcal{A}_{\mathbf{u},N}(\xi;t) = \sum_{\mathbb{I}_N(t)} \langle \xi, u_n\rangle_{\mathfrak{J}_n(t)} \cdot \frac{\sin(wt - n\pi)}{wt - n\pi}\,.$$

One defines mean–square, time shifted, average sampling truncation error $\mathfrak{T}_{\mathbf{u},N}(\xi;t) := \mathsf{E}\big|\xi(t) - \mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2$. Now, we are interested in some reasonably simple efficient mean square truncation error upper bound appearing in the approximation $\xi(t) \approx \mathcal{A}_{\mathbf{u},N}(\xi;t)$.

Let us introduce some auxiliary results. As N_x stands for the integer nearest to xw/π, $x \in \mathbb{R}$, let

$$\Gamma_N(x) := \Big\{z \in \mathbb{C}\colon |z - N_x| \le \big(N + \tfrac{1}{2}\big)\tfrac{\pi}{w}\Big\}, \quad N \in \mathbb{N}.$$

In what follows denote $\mathrm{int}(R)$ the interior of some R, while the series

$$\lambda(q) := \sum_{n=1}^{\infty} \frac{1}{(2n-1)^q}$$

stands for the Dirichlet lambda function.

Theorem 1 *Let $f(z)$ be entire, bounded on the real axis and exponentially bounded having type $\gamma < w$. Denote*

$$L_f := \sup_{\mathbb{R}} \big|f(x)\big|, \quad L_0(z) := \frac{2wL_f|\sin(wz)|}{\pi(w-\gamma)\big(1 - e^{-\pi}\big)}\,.$$

Then for all $z \in \mathrm{int}\big(\Gamma_N(x)\big)$ and $N \in \mathbb{N}$ enough large it holds

$$\Big|\sum_{\mathbb{Z}\setminus\mathbb{I}_N(x)} f\Big(n\frac{\pi}{w}\Big)\frac{\sin(wz - n\pi)}{wz - n\pi}\Big| \le \frac{L_0(z)e^{-(N+1/2)\pi(w-\gamma)/w}}{(N+1/2)\big|1 - \frac{|z-N_x|w}{(N+1/2)\pi}\big|} < \frac{L_0(z)}{N}\,. \tag{7}$$

The proving method is contour integration, following Piranashvili's traces [11]. Denote here and in what follows

$$Y_N(\xi;t) := \sum_{\mathbb{I}_N(t)} \xi\Big(\frac{n\pi}{w}\Big)\frac{\sin(wt - n\pi)}{wt - n\pi}$$

the time shifted truncated WKS restoration sum.

By simple use of (1), (2) and the Theorem 1 one deduces the following modest generalization of [11, Theorem 2] to time shifted case of sampling restoration procedure.

Theorem 2 *Let $\xi(t)$ be a Piranashvili process with exponentially bounded kernel function $f(t,\lambda)$ and let*

$$\widetilde{L}_f := \sup_{\mathbb{R}}\sup_{\Lambda}|f(t,\lambda)|,$$
$$\widetilde{L}_0(t) := \frac{2\widetilde{L}_f w\,|\sin(wt)|}{\pi(w-\gamma)\big(1-e^{-\pi}\big)}\,. \tag{8}$$

Then for all $t \in \mathrm{int}\big(\Gamma_N(t)\big)$, we have

$$\mathsf{E}\big|\xi(t) - Y_N(\xi;t)\big|^2 < \frac{\widetilde{L}_0^2(t)}{N^2}\,\|F_\xi\|(\Lambda,\Lambda)\,. \tag{9}$$

Remark 1 *Let us point out that the straightforward consequence of* (9) *is not only the exact L_2–restoration of the initial Piranashvili–type harmonizable process ξ by a sequence of approximants $Y_N(\xi;t)$ when $N \to \infty$, but since*

$$\mathsf{E}\big|\xi(t) - Y_N(\xi;t)\big|^2 = \mathcal{O}\big(N^{-2}\big)\,,$$

the perfect reconstruction is possible in the a.s. *sense as well (by the celebrated Borel–Cantelli Lemma).*

Second, the first order difference $\Delta_{x,y}B$ [3] of $B(t,s)$ on the plane satisfies

$$\begin{aligned}\big(\Delta_{x,y}B\big)(t,s) &= B(t+x,s+y) - B(t+x,s)\\ &\quad - B(t,s+y) + B(t,s)\\ &= \int_0^x\int_0^y \frac{\partial^2}{\partial u\partial v}B\big(t+u,s+v\big)\,\mathrm{d}v\mathrm{d}u\,.\end{aligned} \tag{10}$$

Theorem 3 *Let $\xi(t)$ be a Piranashvili process with the covariance $B(t,t) \in C^2(\mathbb{R})$. Let (p,q) be a conjugated Hölder pair of exponents:*

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad p > 1\,.$$

Then we have

$$\mathsf{E}\big|Y_N(\xi;t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2 \le \frac{\mathsf{C}_q\,\pi^2}{4w^2}\sup_{\mathbb{R}}\big|B''(t,t)\big|\cdot(2N+1)^{2/p}\,, \qquad (11)$$

where

$$\mathsf{C}_q=\Big(1+\frac{2^{q+1}\,|\sin(wt)|^q}{\pi^q}\,\lambda(q)\Big)^{2/q}. \qquad (12)$$

PROOF. Having on mind (1), the properties of averaging functions sequence $\mathbf{u}$ and (10), we clearly derive

$$\begin{aligned}
&\mathsf{E}\big|Y_N(\xi;t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2\\
&=\mathsf{E}\Big|\sum_{\mathbb{I}_N(t)}\langle\xi\big(\tfrac{n\pi}{w}\big)-\xi(x),u_n\rangle_{\mathfrak{I}_n(t)}\cdot\frac{\sin(wt-n\pi)}{wt-n\pi}\Big|^2\\
&=\sum_{\mathbb{I}_N^2(t)}\int_{-\sigma_n'(t)}^{\sigma_n''(t)}\int_{-\sigma_m'(t)}^{\sigma_m''(t)}u_n(x+n\tfrac{\pi}{w})u_m(y+m\tfrac{\pi}{w})\\
&\quad\cdot\int_0^x\int_0^y\frac{\partial^2}{\partial u\partial v}B\big(u+n\tfrac{\pi}{w},v+m\tfrac{\pi}{v}\big)\mathrm{d}v\mathrm{d}u\\
&\quad\cdot\frac{\sin(wt-n\pi)}{wt-n\pi}\,\frac{\sin(wt-m\pi)}{wt-m\pi}\\
&\le\sum_{\mathbb{I}_N^2(t)}\Big|\frac{\sin(wt-n\pi)}{wt-n\pi}\Big|\,\Big|\frac{\sin(wt-m\pi)}{wt-m\pi}\Big|\\
&\quad\cdot\sup_{x,y\le\sigma}\Big|\int_0^x\int_0^y\frac{\partial^2}{\partial u\partial v}B\big(u+n\tfrac{\pi}{w},v+m\tfrac{\pi}{v}\big)\mathrm{d}v\mathrm{d}u\Big|
\end{aligned}$$

being $\mathbf{u}$ normalized. For the sake of brevity let us denote $H_\sigma(n,m)$ the sup–term in the last display. Then, by the Hölder inequality with conjugate exponents $p,q;\,p>1$, we get

$$\mathsf{E}\big|Y_N(\xi;t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2 \le\Big\{\sum_{\mathbb{I}_N^2(t)}H_\sigma^p(n,m)\Big\}^{1/p}\Big\{\sum_{\mathbb{I}_N(t)}\Big|\frac{\sin(wt-n\pi)}{wt-n\pi}\Big|^q\Big\}^{2/q}.$$

It is not hard to see that for all $n,m\in\mathbb{I}_N(t)$ there holds

$$\begin{aligned}
H_\sigma(n,m)&\le\sigma^2\sup_{\mathbb{R}^2}\Big|\frac{\partial^2B(t,s)}{\partial t\partial s}\Big|\\
&\le\frac{\pi^2}{4w^2}\sup_{\mathbb{R}^2}\Big|\frac{\partial^2B(t,s)}{\partial t\partial s}\Big|\,.
\end{aligned}$$

Applying now the Cauchy–Bunyakovsky–Schwarz inequality to the covariance ∂^2B, we deduce

$$\begin{aligned}
\sup_{\mathbb{R}^2}\Big|\frac{\partial^2B(t,s)}{\partial t\partial s}\Big|&\le\sup_{\mathbb{R}}\Big|\frac{\partial^2B(t,t)}{\partial t^2}\Big|\\
&=\sup_{\mathbb{R}}|B''(t,t)|\,.
\end{aligned}$$

It remains to evaluate the sum of qth power of the sinc–functions. As

$$\frac{\sin(wt-N_t\pi)}{wt-N_t\pi}\le1$$

we conclude

$$\begin{aligned}
\sum_{\mathbb{I}_N(t)}\Big|\frac{\sin(wt-n\pi)}{wt-n\pi}\Big|^q&\le1+C\sum_{n=1}^N\Big\{\frac{1}{(n-\Delta)^q}+\frac{1}{(n+\Delta)^q}\Big\}\\
&<1+2C\sum_{n=1}^\infty\frac{1}{(n-1/2)^q}\\
&<1+2^{q+1}C\,\lambda(q)\,,
\end{aligned}$$

where

$$C=\frac{|\sin(wt)|^q}{\pi^q}\,.$$

Collecting all these estimates, we deduce (11). □

3. Main result

We are ready to formulate our upper bound result for the mean square, time shifted average sampling truncation error $\mathfrak{T}_{\mathbf{u},N}(\xi;t)$. The almost sure sense restoration procedure has been treated too.

As we use average sampling sum $\mathcal{A}_{\mathbf{u},N}(\xi;t)$ instead of $Y_N(\xi;t)$ to obtain asymptotically vanishing $\mathfrak{T}_{\mathbf{u},N}(\xi;t)$, it is not enough letting $N\to\infty$ as in Remark 1. For average sampling we need additional conditions upon w or σ to guarantee smaller average intervals for larger/denser sampling grids.

Theorem 4 *Assume the conditions of* Theorems 2 *and* 3 *have been fulfilled. Then, we have*

$$\mathfrak{T}_{\mathbf{u},N}(\xi;t)\le\frac{2\widetilde{L}_0^2(t)}{N^2}\,\|F_\xi\|(\Lambda,\Lambda) +\frac{\mathsf{C}_q\,\pi^2}{2w^2}\sup_{\mathbb{R}}\big|B''(t,t)\big|\cdot(2N+1)^{2/p}\,, \qquad (13)$$

where $\widetilde{L}_0,\mathsf{C}_q$ are described by (8), (11) *respectively. Moreover, when $w=\mathcal{O}\big(N^{1/2+1/p+\varepsilon}\big)$, $\varepsilon>0$, we have*

$$\mathsf{P}\big\{\lim_{N\to\infty}\mathcal{A}_{\mathbf{u},N}(\xi;t)=\xi(t)\big\}=1 \qquad (14)$$

for all $t\in\mathbb{R}$.

PROOF. By direct calculation we deduce

$$\begin{aligned}
\mathfrak{T}_{\mathbf{u},N}(\xi;t)&=\mathsf{E}\big|\xi(t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2\\
&=\mathsf{E}\big|\xi(t)-Y_N(\xi;t)+Y_N(\xi;t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2\\
&\le2\mathsf{E}\big|\xi(t)-Y_N(\xi;t)\big|^2\\
&\quad+2\mathsf{E}\big|Y_N(\xi;t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|^2\,.
\end{aligned}$$

Now, we get the asserted upper bound by (9) and (11).

To derive (14), we apply the Chebyshev inequality to evaluate the probability

$$\mathsf{P}_N:=\mathsf{P}\big\{\big|\xi(t)-\mathcal{A}_{\mathbf{u},N}(\xi;t)\big|\ge\eta\big\}\le\eta^{-2}\mathfrak{T}_{\mathbf{u},N}(\xi;t)\,.$$

Accordingly, since $\widetilde{L}_0(t)=\mathcal{O}(1)$ as $N\to\infty$, we have

$$\sum_{\mathbb{N}}\mathsf{P}_N\le K\sum_{\mathbb{N}}\Big(\frac{1}{N^2}+\frac{(2N+1)^{2/p}}{w^2}\Big)<\infty,$$

K being a suitable absolute constant. Therefore, by the Borel–Cantelli Lemma, the the a.s. convergence result (14) holds true. $\square$

Remark 2 Theorem 4 *ensures the perfect time shifted average sampling restoration in the mean square sense when* $w = \mathcal{O}\big(N^{1/p+\varepsilon}\big)$, $\varepsilon > 0$*:*

$$\lim_{N\to\infty} \mathfrak{T}_{\mathbf{u},N}(\xi;t) = 0\,.$$

The a.s. sense restoration (14) *requires stronger assumption, it holds when* $w = \mathcal{O}\big(N^{1/2+1/p+\varepsilon}\big)$.

Remark 3 *In both cases we use the so called* approximate sampling procedure*, that is, when in the restoration procedure* $w \to \infty$ *in some fashion. The consequence of these results is that we have to restrict ourselves to the case* $\Lambda = \mathbb{R}$*, such that we recognize as the non–bandlimited Piranashvili type harmonizable process case.*

The importance of approximate sampling procedures for investigations of aliasing errors in sampling restorations and different conditions on joint asymptotic behaviour of N *and* w *have been discussed in detail in* [7].

4. Conclusions

We have analyzed upper bounds on truncation error for time shifted average sampling restorations in the stochastic initial signal case. The convergence of the truncation error to zero was discussed. However, certain new questions immediately arise:

- to derive sharp upper bounds in Theorems 3 and 4;
- to obtain new results for L_p–processes using recent deterministic findings [9], [10];
- to obtain similar results for irregular/nonuniform sampling restoration using methods exposed in [6] and [10].

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