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# Asymmetric Multi-channel Sampling in Shift Invariant Spaces

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## Abstract:

We consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces, while related previous works have supposed that each channel has a symmetric(uniform) sampling rate. Motivated by the fact that shift invariant spaces are isomorphic images of  $L^2[0, 2\pi]$ , we obtain a sampling expansion in shift invariant spaces by using frame or Riesz basis expansion in  $L^2[0, 2\pi]$ . The samples in the expansion are expressed in terms of frame coefficients of an appropriate function with respect to a certain frame in  $L^2[0, 2\pi]$ . The involved reconstruction functions are given explicitly by using the frame operator. We also present relation between asymmetric multi-channel sampling and symmetric one.

## 1. Introduction

Reconstructing a band-limited signal  $f$  from samples which are taken from several channeled versions of  $f$  is called multi-channel sampling. The multi-channel sampling method goes back to the work of Shannon [6] and Fogel [2], where the reconstruction of a band-limited signal from samples of the signal and of its derivatives was suggested. Generalized sampling expansion for arbitrary multi-channel sampling was introduced first by Papoulis [5].

Papoulis' result has been extended to a general shift-invariant space [1, 7, 8]. Here, a shift invariant space  $V(\phi)$  with a generator  $\phi \in L^2(\mathbb{R})$  is defined by the closed subspace of  $L^2(\mathbb{R})$  spanned by integer translates  $\{\phi(t - n) : n \in \mathbb{Z}\}$  of  $\phi$ . Recently García and Pérez-Villalón [3] derived stable generalized sampling in a shift-invariant space by using some special dual frames in  $L^2[0, 1]$ .

The previous works related to the multi-channel sampling have assumed that numbers of samples from each channel are uniform, namely, sampling rates of channels are same. In this paper we consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces. We find an expression for the samples as frame coefficients of an appropriate function in  $L^2[0, 2\pi]$  with respect to some particular frame in  $L^2[0, 2\pi]$  and present the sufficient and necessary condition under which a sequence of functions of particular form becomes a frame or a Riesz basis for  $L^2[0, 2\pi]$ . Using isomorphism between a shift invariant space  $V(\phi)$  and  $L^2[0, 2\pi]$ , we derive sampling theory in  $V(\phi)$  with some Riesz generator  $\phi$  and find a formula of

reconstruction functions by means of the frame operator. The theory contains both a frame and Riesz basis expansion as sampling formulas.

## 2. Asymmetric multi-channel sampling

Assume that  $\phi(t)$  is everywhere well defined complex valued square integrable function on  $\mathbb{R}$  throughout the paper. Moreover, let  $\phi(t)$  be a Riesz generator with  $C_\phi(t) < \infty$  for any  $t \in \mathbb{R}$  so that  $V(\phi)$  is an RKHS(see Proposition 2.4 in [4]). We now are given a LTI system  $\{L_j[\cdot]\}_{j=1}^N$  whose impulse response is  $\{l_j(t) : l_j \in L^2(\mathbb{R}), j = 1, 2, \dots, N\}$ . The aim of this paper is to recover any  $f(t) \in V(\phi)$  via discrete samples from  $\{L_j[f]\}_{j=1}^N$  as

$$f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n) s_{j,n}(t), \quad (1)$$

where  $\{s_{j,n}(t) : j = 1, \dots, N \text{ and } n \in \mathbb{Z}\}$  is a frame or a Riesz bases of  $V(\phi)$  and  $0 \leq \sigma_j < r_j$  with a positive integer  $r_j$  for  $j \in \{1, 2, \dots, N\}$ .

### 2.1 An expression for the samples

Define an isomorphism  $J$  from  $L^2[0, 2\pi]$  onto  $V(\phi)$  by

$$JF(t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \langle F(\xi), e^{-in\xi} \rangle \phi(t-n), \quad F(\xi) \in L^2[0, 2\pi].$$

By the isomorphism  $J : L^2[0, 2\pi] \rightarrow V(\phi)$ , the reconstruction formula (1) is equivalent to the following one:

$$F(\xi) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n) S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi], \quad (2)$$

where  $f(t) = JF(t)$  and  $s_{j,n}(t) = JS_{j,n}(t)$ . Notice further that  $L_j f(\sigma_j + r_j n)$  is represented by an inner product of  $F(\xi)$  and some function in  $L^2[0, 2\pi]$ .

**Lemma 2.1.1** *Let  $L[\cdot]$  be a LTI system with an impulse response  $l(t) \in L^2(\mathbb{R})$  and  $\psi(t) = L[\phi](t) = (\phi * l)(t)$ .*

(a)  *$L$  is a bounded operator from  $L^2(\mathbb{R})$  into  $L^\infty(\mathbb{R})$ ,  $\|f * l\|_\infty \leq \|f\|_2 \|l\|_2$  and  $Lf(t) \in C_\infty(\mathbb{R})$ ,*

(b)  $\sup_{\mathbb{R}} C_\psi(t) < \infty$ ,

(c) (cf. Lemma 2 in [3]) for any  $f(t) = (\mathbf{c} * \phi)(t)$  with  $\mathbf{c} \in \ell^2$  in  $V(\phi)$ ,  $L[f](t) = (\mathbf{c} * \psi)(t)$  converges absolutely and uniformly on  $\mathbb{R}$ . For any  $f(t) = JF(t) \in V(\phi)$  with  $F(\xi) \in L^2[0, 2\pi]$ ,

$$L[f](t) = \langle F(\xi), \frac{1}{2\pi} \overline{Z_\psi(t, \xi)} \rangle_{L^2[0, 2\pi]}.$$

In particular,

$$L[f](\sigma_j + r_j n) = \langle F(\xi), \frac{1}{2\pi} \overline{Z_\psi(\sigma_j, \xi)} e^{-ir_j n \xi} \rangle_{L^2[0, 2\pi]}. \quad (3)$$

## 2.2 The sampling theorem

For a given LTI system  $\{L_j[\cdot]\}_{j=1}^N$ , let  $L_j \phi(t) = \psi_j(t)$ ,  $1 \leq j \leq N$ . Using equation (3), the expansion (2) is equivalent to

$$F(\xi) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle F(\xi), \frac{1}{2\pi} \overline{Z_{\psi_j}(\sigma_j, \xi)} e^{-ir_j n \xi} \rangle_{L^2[0, 2\pi]} \cdot S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi],$$

where  $f(t) = JF(t)$  and  $s_{j,n}(t) = JS_{j,n}(t)$ .

For convenience, we introduce a few more notations. Let  $g_j(\xi) \in L^2[0, 2\pi]$  for  $1 \leq j \leq N$ ,  $g_{j,m_j}(\xi) := g_j(\xi) e^{ir_j(m_j-1)\xi}$  for  $1 \leq m_j \leq \frac{r}{r_j}$  and

$$G(\xi) = [Dg_{1,1}(\xi), Dg_{1,2}(\xi), \dots, Dg_{1, \frac{r}{r_1}}(\xi), Dg_{2,1}(\xi), \dots, Dg_{N, \frac{r}{r_N}}(\xi)]^T,$$

where  $D$  is a unitary operator from  $L^2[0, 2\pi]$  onto  $L^2(I)^r$  defined by  $(DF)(\xi) = [F(\xi), F(\xi + \frac{2\pi}{r}), \dots, F(\xi + (r-1)\frac{2\pi}{r})]^T$ ,  $F(\xi) \in L^2[0, 2\pi]$ . Note that  $G(\xi)$  is the  $\sum_{j=1}^N \frac{r}{r_j} \times r$  matrix whose entries are in  $L^2[0, \frac{2\pi}{r}]$ . And define  $\lambda_M(\xi)$  (resp.  $\lambda_m(\xi)$ ) as the largest (resp. the smallest) eigenvalue of  $r \times r$  matrix  $G(\xi)^* G(\xi)$ ,  $\beta_G$  as  $\|\lambda_M(\xi)\|_\infty$  and  $\alpha_G$  as  $\|\lambda_m\|_0$ .

**Lemma 2.2.1** Let  $g_j \in L^2[0, 2\pi]$  and  $r_j$  be a positive integer for  $1 \leq j \leq N$ . Define  $r$  as the least common multiplier of  $\{r_j\}_{j=1}^N$ . Then  $\{\overline{g_j(\xi)} e^{-ir_j n \xi} : 1 \leq j \leq N, n \in \mathbb{Z}\}$  is a

- (a) Bessel sequence in  $L^2[0, 2\pi]$  if and only if  $\|\lambda_M(\xi)\|_\infty < \infty$  if and only if  $g_j \in L^\infty[0, 2\pi]$  for  $1 \leq j \leq N$ . In this case, optimal bound is  $\frac{2\pi}{r} \|\lambda_M(\xi)\|_\infty$ ;
- (b) frame of  $L^2[0, 2\pi]$  if and only if  $0 < \|\lambda_m(\xi)\|_0 \leq \|\lambda_M(\xi)\|_\infty < \infty$  so that  $r \leq \sum_{j=1}^N \frac{r}{r_j}$  and optimal bounds are  $\frac{2\pi}{r} \|\lambda_m(\xi)\|_0 \leq \frac{2\pi}{r} \|\lambda_M(\xi)\|_\infty$ ;
- (c) Riesz basis of  $L^2[0, 2\pi]$  if and only if frame of  $L^2[0, 2\pi]$  and  $r = \sum_{j=1}^N \frac{r}{r_j}$ , i.e.,  $1 = \sum_{j=1}^N \frac{1}{r_j}$  if and only if  $g_j(\xi) \in L^\infty[0, 2\pi]$  for  $1 \leq j \leq N$ ,  $1 = \sum_{j=1}^N \frac{1}{r_j}$  and  $|\det G(\xi)| \geq \exists \alpha > 0$  a.e..

Appealing to the setting  $g_j(\xi) = \frac{1}{2\pi} \overline{Z_{\psi_j}(\sigma_j, \xi)}$  for  $1 \leq j \leq N$ , we have

**Theorem 2.2.2** Let  $\phi(t)$  be a Riesz generator with  $C_\phi(t) < \infty$ ,  $t \in \mathbb{R}$  and  $\{L_j[\cdot]\}_{j=1}^N$  be LTI systems with an impulse response  $\{l_j(t)\}_{j=1}^N \in L^2(\mathbb{R})$ . Let  $\{\psi_j(t) = (\phi * l_j)(t)\}_{j=1}^N$ ,  $r_j \geq 1$  an integer and  $0 \leq \sigma_j < r_j$ .

- (a) If  $0 < \alpha_G \leq \beta_G < \infty$ , i.e.,  $0 < \alpha_G$  and  $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$ ,  $1 \leq j \leq N$ , then there is a frame  $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$  of  $V(\phi)$  for which

$$f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j f(\sigma_j + r_j n) s_{j,n}(t), \quad f(t) \in V(\phi). \quad (4)$$

- (b) Assume that  $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$ ,  $1 \leq j \leq N$ . Then there is a frame  $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$  of  $V(\phi)$  for which (4) holds if and only if  $0 < \alpha_G$ .

- (c) Assume that  $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$ ,  $1 \leq j \leq N$ . Then there is a Riesz basis  $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$  of  $V(\phi)$  for which (4) holds if and only if  $0 < \alpha_G$  and  $1 = \sum_{j=1}^N \frac{1}{r_j}$ .

In all cases, sampling series (4) converges in  $L^2(\mathbb{R})$ , absolutely on  $\mathbb{R}$  and uniformly on any subset of  $\mathbb{R}$  on which  $C_\phi(t)$  is bounded.

**Remark 2.2.3** Asymmetric multi-channel sampling series with LTI system  $\{L_j[\cdot]\}_{j=1}^N$  whose impulse response is  $\{l_j(t)\}_{j=1}^N$  can be considered as symmetric multi-channel sampling series with LTI system  $\{\tilde{L}_{j,m_j}[\cdot]\}_{j=1, m_j=1}^{N, \frac{r}{r_j}}$  with impulse response  $\{\tilde{l}_{j,m_j}(t)\}_{j=1, m_j=1}^{N, \frac{r}{r_j}}$ , where  $\tilde{l}_{j,m_j}(t) = l_j(r_j(m_j-1) + t)$ .

## 2.3 Reconstruction functions

Let  $S$  be a frame operator with frame  $\{\overline{g_j(\xi)} e^{-ir_j n \xi}\}_{j,n}$ . For any  $F(\xi) \in L^2[0, 2\pi]$ ,

$$SF(\xi) = \sum_{j=1}^N \sum_{m_j=1}^{\frac{r}{r_j}} \overline{g_j(\xi)} e^{-ir_j(m_j-1)\xi} \cdot \frac{2\pi}{r} g_{j,m_j}(\xi)^T DF(\xi)$$

so that

$$DSF(\xi) = \frac{2\pi}{r} G^* G(\xi) DF(\xi).$$

Then, from Lemma 2.2.1 (b), there exists  $(G^* G)^{-1}(\xi)$  a.e. such that

$$D(S^{-1}(\overline{g_j(\xi)} e^{-ir_j n \xi})) = \frac{r}{2\pi} (G^* G)^{-1}(\xi) D(\overline{g_j(\xi)} e^{-ir_j n \xi})$$

for  $1 \leq j \leq N$  and  $n \in \mathbb{Z}$ . Hence,

$$\{s_{j,n}\}_{j,n} = \left\{ \frac{r}{2\pi} JD^{-1}[(G^* G)^{-1}(\xi) D(\overline{g_j(\xi)} e^{-ir_j n \xi})] \right\}_{j,n}.$$

**Remark 2.3.1** One sufficient condition under which  $\{s_{j,n}\}_{j,n}$  is translates of a single function in  $L^2[0, 2\pi]$  is that  $r$  divides  $r_j$  for all  $1 \leq j \leq N$ . Since  $r$  is the least common multiplier of  $\{r_j\}_{j=1}^N$ , the condition holds if and only if  $r = r_j$  for all  $1 \leq j \leq N$ .

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