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Asymmetric Multi-channel Sampling in Shift Invariant Spaces

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Abstract:

We consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces, while related previous works have supposed that each channel has a symmetric(uniform) sampling rate. Motivated by the fact that shift invariant spaces are isomorphic images of $L^2[0, 2\pi]$, we obtain a sampling expansion in shift invariant spaces by using frame or Riesz basis expansion in $L^2[0, 2\pi]$. The samples in the expansion are expressed in terms of frame coefficients of an appropriate function with respect to a certain frame in $L^2[0, 2\pi]$. The involved reconstruction functions are given explicitly by using the frame operator. We also present relation between asymmetric multi-channel sampling and symmetric one.

1. Introduction

Reconstructing a band-limited signal f from samples which are taken from several channeled versions of f is called multi-channel sampling. The multi-channel sampling method goes back to the work of Shannon [6] and Fogel [2], where the reconstruction of a band-limited signal from samples of the signal and of its derivatives was suggested. Generalized sampling expansion for arbitrary multi-channel sampling was introduced first by Papoulis [5].

Papoulis' result has been extended to a general shift-invariant space [1, 7, 8]. Here, a shift invariant space $V(\phi)$ with a generator $\phi \in L^2(\mathbb{R})$ is defined by the closed subspace of $L^2(\mathbb{R})$ spanned by integer translates $\{\phi(t - n) : n \in \mathbb{Z}\}$ of ϕ . Recently García and Pérez-Villalón [3] derived stable generalized sampling in a shift-invariant space by using some special dual frames in $L^2[0, 1]$.

The previous works related to the multi-channel sampling have assumed that numbers of samples from each channel are uniform, namely, sampling rates of channels are same. In this paper we consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces. We find an expression for the samples as frame coefficients of an appropriate function in $L^2[0, 2\pi]$ with respect to some particular frame in $L^2[0, 2\pi]$ and present the sufficient and necessary condition under which a sequence of functions of particular form becomes a frame or a Riesz basis for $L^2[0, 2\pi]$. Using isomorphism between a shift invariant space $V(\phi)$ and $L^2[0, 2\pi]$, we derive sampling theory in $V(\phi)$ with some Riesz generator ϕ and find a formula of

reconstruction functions by means of the frame operator. The theory contains both a frame and Riesz basis expansion as sampling formulas.

2. Asymmetric multi-channel sampling

Assume that $\phi(t)$ is everywhere well defined complex valued square integrable function on \mathbb{R} throughout the paper. Moreover, let $\phi(t)$ be a Riesz generator with $C_\phi(t) < \infty$ for any $t \in \mathbb{R}$ so that $V(\phi)$ is an RKHS (see Proposition 2.4 in [4]). We now are given a LTI system $\{L_j[\cdot]\}_{j=1}^N$ whose impulse response is $\{l_j(t) : l_j \in L^2(\mathbb{R}), j = 1, 2, \dots, N\}$. The aim of this paper is to recover any $f(t) \in V(\phi)$ via discrete samples from $\{L_j[f]\}_{j=1}^N$ as

$$f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n) s_{j,n}(t), \quad (1)$$

where $\{s_{j,n}(t) : j = 1, \dots, N \text{ and } n \in \mathbb{Z}\}$ is a frame or a Riesz bases of $V(\phi)$ and $0 \leq \sigma_j < r_j$ with a positive integer r_j for $j \in \{1, 2, \dots, N\}$.

2.1 An expression for the samples

Define an isomorphism J from $L^2[0, 2\pi]$ onto $V(\phi)$ by

$$JF(t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \langle F(\xi), e^{-in\xi} \rangle \phi(t - n), \quad F(\xi) \in L^2[0, 2\pi].$$

By the isomorphism $J : L^2[0, 2\pi] \rightarrow V(\phi)$, the reconstruction formula (1) is equivalent to the following one:

$$F(\xi) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n) S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi], \quad (2)$$

where $f(t) = JF(t)$ and $s_{j,n}(t) = JS_{j,n}(t)$. Notice further that $L_j f(\sigma_j + r_j n)$ is represented by an inner product of $F(\xi)$ and some function in $L^2[0, 2\pi]$.

Lemma 2.1.1 *Let $L[\cdot]$ be a LTI system with an impulse response $l(t) \in L^2(\mathbb{R})$ and $\psi(t) = L[\phi](t) = (\phi * l)(t)$.*

(a) *L is a bounded operator from $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$, $\|f * l\|_\infty \leq \|f\|_2 \|l\|_2$ and $Lf(t) \in C_\infty(\mathbb{R})$,*

(b) *$\sup_{\mathbb{R}} C_\psi(t) < \infty$,*

(c) (cf. Lemma 2 in [3]) for any $f(t) = (\mathbf{c} * \phi)(t)$ with $\mathbf{c} \in \ell^2$ in $V(\phi)$, $L[f](t) = (\mathbf{c} * \psi)(t)$ converges absolutely and uniformly on \mathbb{R} . For any $f(t) = JF(t) \in V(\phi)$ with $F(\xi) \in L^2[0, 2\pi]$,

$$L[f](t) = \langle F(\xi), \frac{1}{2\pi} \overline{Z_\psi(t, \xi)} \rangle_{L^2[0, 2\pi]}.$$

In particular,

$$L[f](\sigma_j + r_j n) = \langle F(\xi), \frac{1}{2\pi} \overline{Z_\psi(\sigma_j, \xi)} e^{-ir_j n \xi} \rangle_{L^2[0, 2\pi]}. \quad (3)$$

2.2 The sampling theorem

For a given LTI system $\{L_j[\cdot]\}_{j=1}^N$, let $L_j \phi(t) = \psi_j(t)$, $1 \leq j \leq N$. Using equation (3), the expansion (2) is equivalent to

$$F(\xi) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle F(\xi), \frac{1}{2\pi} \overline{Z_{\psi_j}(\sigma_j, \xi)} e^{-ir_j n \xi} \rangle_{L^2[0, 2\pi]} \cdot S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi],$$

where $f(t) = JF(t)$ and $s_{j,n}(t) = JS_{j,n}(t)$.

For convenience, we introduce a few more notations. Let $g_j(\xi) \in L^2[0, 2\pi]$ for $1 \leq j \leq N$, $g_{j,m_j}(\xi) := g_j(\xi) e^{ir_j(m_j-1)\xi}$ for $1 \leq m_j \leq \frac{r}{r_j}$ and

$$G(\xi) = [Dg_{1,1}(\xi), Dg_{1,2}(\xi), \dots, Dg_{1,\frac{r}{r_1}}(\xi), Dg_{2,1}(\xi), \dots, Dg_{N,\frac{r}{r_N}}(\xi)]^T,$$

where D is a unitary operator from $L^2[0, 2\pi]$ onto $L^2(I)^r$ defined by $(DF)(\xi) = [F(\xi), F(\xi + \frac{2\pi}{r}), \dots, F(\xi + (r-1)\frac{2\pi}{r})]^T$, $F(\xi) \in L^2[0, 2\pi]$. Note that $G(\xi)$ is the $\sum_{j=1}^N \frac{r}{r_j} \times r$ matrix whose entries are in $L^2[0, \frac{2\pi}{r}]$. And define $\lambda_M(\xi)$ (resp. $\lambda_m(\xi)$) as the largest (resp. the smallest) eigenvalue of $r \times r$ matrix $G(\xi)^* G(\xi)$, β_G as $\|\lambda_M(\xi)\|_\infty$ and α_G as $\|\lambda_m(\xi)\|_0$.

Lemma 2.2.1 Let $g_j \in L^2[0, 2\pi]$ and r_j be a positive integer for $1 \leq j \leq N$. Define r as the least common multiplier of $\{r_j\}_{j=1}^N$. Then $\{\overline{g_j(\xi)} e^{-ir_j n \xi} : 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a

- (a) Bessel sequence in $L^2[0, 2\pi]$ if and only if $\|\lambda_M(\xi)\|_\infty < \infty$ if and only if $g_j \in L^\infty[0, 2\pi]$ for $1 \leq j \leq N$. In this case, optimal bound is $\frac{2\pi}{r} \|\lambda_M(\xi)\|_\infty$;
- (b) frame of $L^2[0, 2\pi]$ if and only if $0 < \|\lambda_m(\xi)\|_0 \leq \|\lambda_M(\xi)\|_\infty < \infty$ so that $r \leq \sum_{j=1}^N \frac{r}{r_j}$ and optimal bounds are $\frac{2\pi}{r} \|\lambda_m(\xi)\|_0 \leq \frac{2\pi}{r} \|\lambda_M(\xi)\|_\infty$;
- (c) Riesz basis of $L^2[0, 2\pi]$ if and only if frame of $L^2[0, 2\pi]$ and $r = \sum_{j=1}^N \frac{r}{r_j}$, i.e., $1 = \sum_{j=1}^N \frac{1}{r_j}$ if and only if $g_j(\xi) \in L^\infty[0, 2\pi]$ for $1 \leq j \leq N$, $1 = \sum_{j=1}^N \frac{1}{r_j}$ and $|\det G(\xi)| \geq \exists \alpha > 0$ a.e..

Appealing to the setting $g_j(\xi) = \frac{1}{2\pi} Z_{\psi_j}(\sigma_j, \xi)$ for $1 \leq j \leq N$, we have

Theorem 2.2.2 Let $\phi(t)$ be a Riesz generator with $C_\phi(t) < \infty$, $t \in \mathbb{R}$ and $\{L_j[\cdot]\}_{j=1}^N$ be LTI systems with an impulse response $\{l_j(t)\}_{j=1}^N \in L^2(\mathbb{R})$. Let $\{\psi_j(t) = (\phi * l_j)(t)\}_{j=1}^N$, $r_j \geq 1$ an integer and $0 \leq \sigma_j < r_j$.

- (a) If $0 < \alpha_G \leq \beta_G < \infty$, i.e., $0 < \alpha_G$ and $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq N$, then there is a frame $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ of $V(\phi)$ for which

$$f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j f(\sigma_j + r_j n) s_{j,n}(t), \quad f(t) \in V(\phi). \quad (4)$$

- (b) Assume that $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq N$. Then there is a frame $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ of $V(\phi)$ for which (4) holds if and only if $0 < \alpha_G$.

- (c) Assume that $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq N$. Then there is a Riesz basis $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ of $V(\phi)$ for which (4) holds if and only if $0 < \alpha_G$ and $1 = \sum_{j=1}^N \frac{1}{r_j}$.

In all cases, sampling series (4) converges in $L^2(\mathbb{R})$, absolutely on \mathbb{R} and uniformly on any subset of \mathbb{R} on which $C_\phi(t)$ is bounded.

Remark 2.2.3 Asymmetric multi-channel sampling series with LTI system $\{L_j[\cdot]\}_{j=1}^N$ whose impulse response is $\{l_j(t)\}_{j=1}^N$ can be considered as symmetric multi-channel sampling series with LTI system $\{\tilde{L}_{j,m_j}[\cdot]\}_{j=1, m_j=1}^{N, \frac{r}{r_j}}$ with impulse response $\{\tilde{l}_{j,m_j}(t)\}_{j=1, m_j=1}^{N, \frac{r}{r_j}}$, where $\tilde{l}_{j,m_j}(t) = l_j(r_j(m_j-1) + t)$.

2.3 Reconstruction functions

Let S be a frame operator with frame $\{\overline{g_j(\xi)} e^{-ir_j n \xi}\}_{j,n}$. For any $F(\xi) \in L^2[0, 2\pi]$,

$$SF(\xi) = \sum_{j=1}^N \sum_{m_j=1}^{\frac{r}{r_j}} \overline{g_j(\xi)} e^{-ir_j(m_j-1)\xi} \cdot \frac{2\pi}{r} g_{j,m}(\xi)^T DF(\xi)$$

so that

$$DSF(\xi) = \frac{2\pi}{r} G^* G(\xi) DF(\xi).$$

Then, from Lemma 2.2.1 (b), there exists $(G^* G)^{-1}(\xi)$ a.e. such that

$$D(S^{-1}(\overline{g_j(\xi)} e^{-ir_j n \xi})) = \frac{r}{2\pi} (G^* G)^{-1}(\xi) D(\overline{g_j(\xi)} e^{-ir_j n \xi})$$

for $1 \leq j \leq N$ and $n \in \mathbb{Z}$. Hence,

$$\{s_{j,n}\}_{j,n} = \left\{ \frac{r}{2\pi} JD^{-1}[(G^* G)^{-1}(\xi) D(\overline{g_j(\xi)} e^{-ir_j n \xi})] \right\}_{j,n}.$$

Remark 2.3.1 One sufficient condition under which $\{s_{j,n}\}_{j,n}$ is translates of a single function in $L^2[0, 2\pi]$ is that r divides r_j for all $1 \leq j \leq N$. Since r is the least common multiplier of $\{r_j\}_{j=1}^N$, the condition holds if and only if $r = r_j$ for all $1 \leq j \leq N$.

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