



HAL
open science

Asymmetric Multichannel Sampling in Shift Invariant Spaces

Sinuk Kang, K.H. Kwon

► **To cite this version:**

Sinuk Kang, K.H. Kwon. Asymmetric Multichannel Sampling in Shift Invariant Spaces. SAMPTA'09, May 2009, Marseille, France. pp.General session. hal-00453318

HAL Id: hal-00453318

<https://hal.science/hal-00453318>

Submitted on 4 Feb 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Asymmetric Multi-channel Sampling in Shift Invariant Spaces

Sinuk Kang ⁽¹⁾ and K.H. Kwon ⁽¹⁾

(1) KAIST, 335 Gwahangro, Yuseong-gu, Daejeon 305-701, S. Korea.
sukang@kaist.ac.kr, khkwon@kaist.edu

Abstract:

We consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces, while related previous works have supposed that each channel has a symmetric(uniform) sampling rate. Motivated by the fact that shift invariant spaces are isomorphic images of $L^2[0, 2\pi]$, we obtain a sampling expansion in shift invariant spaces by using frame or Riesz basis expansion in $L^2[0, 2\pi]$. The samples in the expansion are expressed in terms of frame coefficients of an appropriate function with respect to a certain frame in $L^2[0, 2\pi]$. The involved reconstruction functions are given explicitly by using the frame operator. We also present relation between asymmetric multi-channel sampling and symmetric one.

1. Introduction

Reconstructing a band-limited signal f from samples which are taken from several channeled versions of f is called multi-channel sampling. The multi-channel sampling method goes back to the work of Shannon [6] and Fogel [2], where the reconstruction of a band-limited signal from samples of the signal and of its derivatives was suggested. Generalized sampling expansion for arbitrary multi-channel sampling was introduced first by Papoulis [5].

Papoulis' result has been extended to a general shift-invariant space [1, 7, 8]. Here, a shift invariant space $V(\phi)$ with a generator $\phi \in L^2(\mathbb{R})$ is defined by the closed subspace of $L^2(\mathbb{R})$ spanned by integer translates $\{\phi(t - n) : n \in \mathbb{Z}\}$ of ϕ . Recently García and Pérez-Villalón [3] derived stable generalized sampling in a shift-invariant space by using some special dual frames in $L^2[0, 1]$.

The previous works related to the multi-channel sampling have assumed that numbers of samples from each channel are uniform, namely, sampling rates of channels are same. In this paper we consider a multi-channel sampling with asymmetric sampling rates in shift invariant spaces. We find an expression for the samples as frame coefficients of an appropriate function in $L^2[0, 2\pi]$ with respect to some particular frame in $L^2[0, 2\pi]$ and present the sufficient and necessary condition under which a sequence of functions of particular form becomes a frame or a Riesz basis for $L^2[0, 2\pi]$. Using isomorphism between a shift invariant space $V(\phi)$ and $L^2[0, 2\pi]$, we derive sampling theory in $V(\phi)$ with some Riesz generator ϕ and find a formula of

reconstruction functions by means of the frame operator. The theory contains both a frame and Riesz basis expansion as sampling formulas.

2. Asymmetric multi-channel sampling

Assume that $\phi(t)$ is everywhere well defined complex valued square integrable function on \mathbb{R} throughout the paper. Moreover, let $\phi(t)$ be a Riesz generator with $C_\phi(t) < \infty$ for any $t \in \mathbb{R}$ so that $V(\phi)$ is an RKHS(see Proposition 2.4 in [4]). We now are given a LTI system $\{L_j[\cdot]\}_{j=1}^N$ whose impulse response is $\{l_j(t) : l_j \in L^2(\mathbb{R}), j = 1, 2, \dots, N\}$. The aim of this paper is to recover any $f(t) \in V(\phi)$ via discrete samples from $\{L_j[f]\}_{j=1}^N$ as

$$f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n) s_{j,n}(t), \quad (1)$$

where $\{s_{j,n}(t) : j = 1, \dots, N \text{ and } n \in \mathbb{Z}\}$ is a frame or a Riesz bases of $V(\phi)$ and $0 \leq \sigma_j < r_j$ with a positive integer r_j for $j \in \{1, 2, \dots, N\}$.

2.1 An expression for the samples

Define an isomorphism J from $L^2[0, 2\pi]$ onto $V(\phi)$ by

$$JF(t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \langle F(\xi), e^{-in\xi} \rangle \phi(t-n), \quad F(\xi) \in L^2[0, 2\pi].$$

By the isomorphism $J : L^2[0, 2\pi] \rightarrow V(\phi)$, the reconstruction formula (1) is equivalent to the following one:

$$F(\xi) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j[f](\sigma_j + r_j n) S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi], \quad (2)$$

where $f(t) = JF(t)$ and $s_{j,n}(t) = JS_{j,n}(t)$. Notice further that $L_j f(\sigma_j + r_j n)$ is represented by an inner product of $F(\xi)$ and some function in $L^2[0, 2\pi]$.

Lemma 2.1.1 *Let $L[\cdot]$ be a LTI system with an impulse response $l(t) \in L^2(\mathbb{R})$ and $\psi(t) = L[\phi](t) = (\phi * l)(t)$.*

(a) *L is a bounded operator from $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$, $\|f * l\|_\infty \leq \|f\|_2 \|l\|_2$ and $Lf(t) \in C_\infty(\mathbb{R})$,*

(b) $\sup_{\mathbb{R}} C_\psi(t) < \infty$,

(c) (cf. Lemma 2 in [3]) for any $f(t) = (\mathbf{c} * \phi)(t)$ with $\mathbf{c} \in \ell^2$ in $V(\phi)$, $L[f](t) = (\mathbf{c} * \psi)(t)$ converges absolutely and uniformly on \mathbb{R} . For any $f(t) = JF(t) \in V(\phi)$ with $F(\xi) \in L^2[0, 2\pi]$,

$$L[f](t) = \langle F(\xi), \frac{1}{2\pi} \overline{Z_\psi(t, \xi)} \rangle_{L^2[0, 2\pi]}.$$

In particular,

$$L[f](\sigma_j + r_j n) = \langle F(\xi), \frac{1}{2\pi} \overline{Z_\psi(\sigma_j, \xi)} e^{-ir_j n \xi} \rangle_{L^2[0, 2\pi]}. \quad (3)$$

2.2 The sampling theorem

For a given LTI system $\{L_j[\cdot]\}_{j=1}^N$, let $L_j \phi(t) = \psi_j(t)$, $1 \leq j \leq N$. Using equation (3), the expansion (2) is equivalent to

$$F(\xi) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} \langle F(\xi), \frac{1}{2\pi} \overline{Z_{\psi_j}(\sigma_j, \xi)} e^{-ir_j n \xi} \rangle_{L^2[0, 2\pi]} \cdot S_{j,n}(\xi), \quad F(\xi) \in L^2[0, 2\pi],$$

where $f(t) = JF(t)$ and $s_{j,n}(t) = JS_{j,n}(t)$.

For convenience, we introduce a few more notations. Let $g_j(\xi) \in L^2[0, 2\pi]$ for $1 \leq j \leq N$, $g_{j,m_j}(\xi) := g_j(\xi) e^{ir_j(m_j-1)\xi}$ for $1 \leq m_j \leq \frac{r}{r_j}$ and

$$G(\xi) = [Dg_{1,1}(\xi), Dg_{1,2}(\xi), \dots, Dg_{1, \frac{r}{r_1}}(\xi), Dg_{2,1}(\xi), \dots, Dg_{N, \frac{r}{r_N}}(\xi)]^T,$$

where D is a unitary operator from $L^2[0, 2\pi]$ onto $L^2(I)^r$ defined by $(DF)(\xi) = [F(\xi), F(\xi + \frac{2\pi}{r}), \dots, F(\xi + (r-1)\frac{2\pi}{r})]^T$, $F(\xi) \in L^2[0, 2\pi]$. Note that $G(\xi)$ is the $\sum_{j=1}^N \frac{r}{r_j} \times r$ matrix whose entries are in $L^2[0, \frac{2\pi}{r}]$. And define $\lambda_M(\xi)$ (resp. $\lambda_m(\xi)$) as the largest (resp. the smallest) eigenvalue of $r \times r$ matrix $G(\xi)^* G(\xi)$, β_G as $\|\lambda_M(\xi)\|_\infty$ and α_G as $\|\lambda_m\|_0$.

Lemma 2.2.1 Let $g_j \in L^2[0, 2\pi]$ and r_j be a positive integer for $1 \leq j \leq N$. Define r as the least common multiplier of $\{r_j\}_{j=1}^N$. Then $\{\overline{g_j(\xi)} e^{-ir_j n \xi} : 1 \leq j \leq N, n \in \mathbb{Z}\}$ is a

- (a) Bessel sequence in $L^2[0, 2\pi]$ if and only if $\|\lambda_M(\xi)\|_\infty < \infty$ if and only if $g_j \in L^\infty[0, 2\pi]$ for $1 \leq j \leq N$. In this case, optimal bound is $\frac{2\pi}{r} \|\lambda_M(\xi)\|_\infty$;
- (b) frame of $L^2[0, 2\pi]$ if and only if $0 < \|\lambda_m(\xi)\|_0 \leq \|\lambda_M(\xi)\|_\infty < \infty$ so that $r \leq \sum_{j=1}^N \frac{r}{r_j}$ and optimal bounds are $\frac{2\pi}{r} \|\lambda_m(\xi)\|_0 \leq \frac{2\pi}{r} \|\lambda_M(\xi)\|_\infty$;
- (c) Riesz basis of $L^2[0, 2\pi]$ if and only if frame of $L^2[0, 2\pi]$ and $r = \sum_{j=1}^N \frac{r}{r_j}$, i.e., $1 = \sum_{j=1}^N \frac{1}{r_j}$ if and only if $g_j(\xi) \in L^\infty[0, 2\pi]$ for $1 \leq j \leq N$, $1 = \sum_{j=1}^N \frac{1}{r_j}$ and $|\det G(\xi)| \geq \exists \alpha > 0$ a.e..

Appealing to the setting $g_j(\xi) = \frac{1}{2\pi} \overline{Z_{\psi_j}(\sigma_j, \xi)}$ for $1 \leq j \leq N$, we have

Theorem 2.2.2 Let $\phi(t)$ be a Riesz generator with $C_\phi(t) < \infty$, $t \in \mathbb{R}$ and $\{L_j[\cdot]\}_{j=1}^N$ be LTI systems with an impulse response $\{l_j(t)\}_{j=1}^N \in L^2(\mathbb{R})$. Let $\{\psi_j(t) = (\phi * l_j)(t)\}_{j=1}^N$, $r_j \geq 1$ an integer and $0 \leq \sigma_j < r_j$.

- (a) If $0 < \alpha_G \leq \beta_G < \infty$, i.e., $0 < \alpha_G$ and $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq N$, then there is a frame $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ of $V(\phi)$ for which

$$f(t) = \sum_{j=1}^N \sum_{n \in \mathbb{Z}} L_j f(\sigma_j + r_j n) s_{j,n}(t), \quad f(t) \in V(\phi). \quad (4)$$

- (b) Assume that $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq N$. Then there is a frame $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ of $V(\phi)$ for which (4) holds if and only if $0 < \alpha_G$.

- (c) Assume that $Z_{\psi_j}(\sigma_j, \xi) \in L^\infty[0, 2\pi]$, $1 \leq j \leq N$. Then there is a Riesz basis $\{s_{j,n}(t) : 1 \leq j \leq N, n \in \mathbb{Z}\}$ of $V(\phi)$ for which (4) holds if and only if $0 < \alpha_G$ and $1 = \sum_{j=1}^N \frac{1}{r_j}$.

In all cases, sampling series (4) converges in $L^2(\mathbb{R})$, absolutely on \mathbb{R} and uniformly on any subset of \mathbb{R} on which $C_\phi(t)$ is bounded.

Remark 2.2.3 Asymmetric multi-channel sampling series with LTI system $\{L_j[\cdot]\}_{j=1}^N$ whose impulse response is $\{l_j(t)\}_{j=1}^N$ can be considered as symmetric multi-channel sampling series with LTI system $\{\tilde{L}_{j,m_j}[\cdot]\}_{j=1, m_j=1}^{N, \frac{r}{r_j}}$ with impulse response $\{\tilde{l}_{j,m_j}(t)\}_{j=1, m_j=1}^{N, \frac{r}{r_j}}$, where $\tilde{l}_{j,m_j}(t) = l_j(r_j(m_j-1) + t)$.

2.3 Reconstruction functions

Let S be a frame operator with frame $\{\overline{g_j(\xi)} e^{-ir_j n \xi}\}_{j,n}$. For any $F(\xi) \in L^2[0, 2\pi]$,

$$SF(\xi) = \sum_{j=1}^N \sum_{m_j=1}^{\frac{r}{r_j}} \overline{g_j(\xi)} e^{-ir_j(m_j-1)\xi} \cdot \frac{2\pi}{r} g_{j,m_j}(\xi)^T DF(\xi)$$

so that

$$DSF(\xi) = \frac{2\pi}{r} G^* G(\xi) DF(\xi).$$

Then, from Lemma 2.2.1 (b), there exists $(G^* G)^{-1}(\xi)$ a.e. such that

$$D(S^{-1}(\overline{g_j(\xi)} e^{-ir_j n \xi})) = \frac{r}{2\pi} (G^* G)^{-1}(\xi) D(\overline{g_j(\xi)} e^{-ir_j n \xi})$$

for $1 \leq j \leq N$ and $n \in \mathbb{Z}$. Hence,

$$\{s_{j,n}\}_{j,n} = \left\{ \frac{r}{2\pi} JD^{-1}[(G^* G)^{-1}(\xi) D(\overline{g_j(\xi)} e^{-ir_j n \xi})] \right\}_{j,n}.$$

Remark 2.3.1 One sufficient condition under which $\{s_{j,n}\}_{j,n}$ is translates of a single function in $L^2[0, 2\pi]$ is that r divides r_j for all $1 \leq j \leq N$. Since r is the least common multiplier of $\{r_j\}_{j=1}^N$, the condition holds if and only if $r = r_j$ for all $1 \leq j \leq N$.

References:

- [1] I. Djokovic, P. P. Vaidyanathan, Generalized sampling theorems in multiresolution subspaces, *IEEE Trans. Signal Process.*, 45:583-599, 1997.
- [2] L. J. Fogel, A note on the sampling theorem, *IRE Tran. Infor. Theory IT-1*:47-48, 1995.
- [3] A. G. García and G. Pérez-Villarón, Dual frames in $L^2(0, 1)$ connected with generalized sampling in shift-invariant spaces, *Appl. Comput. Harmon. Anal.*, 20:422-433, 2006.
- [4] J. M. Kim, K. H. Kwon, Sampling expansion in shift invariant spaces, *Intern. J Wavelets, Multiresolution and Inform. Processing*, 6(2):223-248, 2008.
- [5] A. Papoulis, Generalized sampling expansion, *IEEE Trans. Circuits Systems*, 24(11), 652-654, 1977.
- [6] C. E. Shannon, Communication in the presence of noise, *Proc. IRE*, 37:10-21, 1949.
- [7] M. Unser, J. Zerubia, Generalized sampling: Stability and performance analysis, *IEEE trans. Signal Process.*, 45(12):2941-2950, 1997.
- [8] M. Unser, J. Zerubia, A generalized sampling theory without band-limiting constraints, *IEEE Trans. Circuits Syst. 2*, 45(8):959-969, 1998.