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The Class of Bandlimited Functions with Unstable Reconstruction under Thresholding

Holger Boche and Ullrich J. Mönich

Technische Universität Berlin, Heinrich-Hertz-Chair for Mobile Communications,
Einsteinufer 25, 10578 Berlin, Germany.
{holger.boche,ullrich.moenich}@mk.tu-berlin.de

Abstract:
The reconstruction of $\mathcal{PW}_\pi^1$-functions by sampling series is not possible in general if the samples are disturbed by the non-linear threshold operator which sets all samples whose absolute value is smaller than some threshold to zero. In this paper we characterize the set of functions for which the sampling series diverges as the threshold goes to zero and show that this set is a residual set.

1. Notation

Before we start our discussion, we introduce some notations and definitions [4]. Let $\hat{f}$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. $L^p(\mathbb{R})$, $1 \leq p < \infty$, is the space of all measurable, pth-power Lebesgue integrable functions on $\mathbb{R}$, with the usual norm $\|\cdot\|_p$, and $L^\infty(\mathbb{R})$ is the space of all measurable functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite.
For $\sigma > 0$ and $1 \leq p \leq \infty$ we denote by $\mathcal{PW}_\sigma^p$ the Paley-Wiener space of functions f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)\,\mathrm{e}^{iz\omega}\ \mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p(-\sigma,\sigma)$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$, $1 \leq p < \infty$, is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi)\int_{-\sigma}^{\sigma}|\hat{f}(\omega)|^p\ \mathrm{d}\omega)^{1/p}$.
Furthermore, we need the threshold operator. For complex numbers $z \in \mathbb{C}$, the threshold operator κ_δ, $\delta > 0$, is defined by

$$\kappa_\delta z = \begin{cases} z & |z| \geq \delta \\ 0 & |z| < \delta. \end{cases}$$

For continuous functions $f : \mathbb{R} \to \mathbb{C}$, we define the threshold operator Θ_δ, $\delta > 0$, pointwise, i.e., $(\Theta_\delta f)(t) = \kappa_\delta f(t)$, $t \in \mathbb{R}$.

2. Motivation

A well known fact [1, 2, 3] about the convergence behavior of the Shannon sampling series for $f \in \mathcal{PW}_\pi^1$ is expressed by the following theorem.

Theorem (Brown). *For all $f \in \mathcal{PW}_\pi^1$ and $T > 0$ fixed we have*

$$\lim_{N\to\infty} \max_{t\in[-T,T]} \left| f(t) - \sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = 0. \tag{1}$$

This theorem plays a fundamental role in applications, because it establishes the uniform convergence on compact subsets of $\mathbb{R}$ for a large class of functions, namely $\mathcal{PW}_\pi^1$, which is the largest space within the scale of Paley-Wiener spaces.
The truncation of the series in (1) is done in the domain of the function f because only the samples $f(k)$, $k = -N, \ldots, N$ are taken into account. In contrast, it is also possible to control the truncation of the series in the codomain of f by considering only the samples $f(k)$, $k \in \mathbb{Z}$, whose absolute value is larger than or equal to some threshold $\delta > 0$. This leads to the approximation formula

$$(A_\delta f)(t) = \sum_{\substack{k=-\infty \\ |f(k)|\geq\delta}}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{2}$$

Since $f \in \mathcal{PW}_\pi^1$ we have $\lim_{t\to\infty} f(t) = 0$ by the Riemann-Lebesgue lemma, and it follows that the series in (2) has only finitely many summands, which implies $A_\delta f \in \mathcal{PW}_\pi^2 \subset \mathcal{PW}_\pi^1$. In general, $A_\delta f$ is only an approximation of f, and we want the function $A_\delta f$ to be close to f if δ is sufficiently small.
The operator A_δ has several properties which complicate its analysis. A_δ, $\delta > 0$, is non-linear. Furthermore, for each $\delta > 0$, the operator $A_\delta : (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1}) \to (\mathcal{PW}_\pi^1, \|\cdot\|_\infty)$ is discontinuous. This implies that $A_\delta : (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1}) \to (\mathcal{PW}_\pi^1, \|\cdot\|_{\mathcal{PW}_\pi^1})$ is discontinuous. For some $f \in \mathcal{PW}_\pi^1$, the operator A_δ is also discontinuous with respect to δ.
Of course (2) can be written as

$$\sum_{k=-\infty}^{\infty} (\Theta_\delta f)(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}, \tag{3}$$

where Θ_δ denotes the threshold operator. Wireless sensor networks are one possible application where the threshold operator Θ_δ and the series (3) are important. The sensors sample some bandlimited signal in space and time and then transmit the samples to the receiver. In order to save energy, it is common to let the sensors transmit only if the absolute value of the current sample exceeds some threshold $\delta > 0$. Thus, the receiver has to reconstruct the signal by using only the samples whose absolute value is larger than or equal to the threshold δ.

In addition to the sensor network scenario, the threshold operator can be used to model non-linearities in many other applications. For example, due to its close relation to the quantization operator, the threshold operator can be employed to analyze the effects of quantization in analog to digital conversion.

3. Problem Formulation and Main Result

Since the series in (2) uses all "important" samples of the function, i.e., all samples that are larger than or equal to δ, one could expect $A_\delta f$ to have an approximation behavior similar to the Shannon sampling series. In particular the approximation error should decrease as the threshold δ goes to zero. But, we will see that $A_\delta f$ exhibits a significantly different behavior.
In this paper we are interested in the structure of the set

$$\mathcal{D}_1 = \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta\to 0} |(A_\delta f)(t)| = \infty \,\forall\, t \in \mathbb{R}\setminus\mathbb{Z}\},$$

i.e., in the structure of the set of functions for which the approximation error $|f(t) - (A_\delta f)(t)|$ grows arbitrarily large for all $t \in \mathbb{R}\setminus\mathbb{Z}$ as $\delta \to 0$.

Remark 1. The analysis of the operator A_δ is difficult because it is non-linear and discontinuous, and therefore the standard theorems of functional analysis, like the Banach-Steinhaus theorem, cannot be used.

For the further discussion we need the following concepts from metric spaces [5]. A subset M of a metric space X is said to be nowhere dense in X if the closure $[M]$ does not contain a non-empty open set of X. M is said to be of the first category (or meager) if M is the countable union of sets each of which is nowhere dense in X. M is said to be of the second category (or nonmeager) if is not of the first category. The complement of a set of the first category is called a residual set. Sets of first category may be considered as "small". According to Baire's theorem [5] we have that in a complete metric space, the residual set is dense and a set of the second category. One property that shows the richness of residual sets is the following: The countable intersection of residual sets is always a residual set. In particular we will use the following fact in our proof. In a complete metric space an open and dense set is a residual set because its complement is nowhere dense.
Theorem 1 will show that the set $\mathcal{D}_1$ it is a residual set. Thus the threshold operator destroys the good reconstruction behavior of the Shannon sampling series for "almost all" functions in $\mathcal{PW}_\pi^1$.

4. Proof of the Main Result

In addition to the threshold operator that sets all samples whose absolute value is smaller than δ to zero, we consider the threshold operator that sets all samples whose absolute value is smaller than or equal to δ to zero. This operator gives rise to the sampling series

$$(\bar{A}_\delta f)(t) := \sum_{\substack{k=-\infty \\ |f(k)|>\delta}}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{4}$$

and the set

$$\mathcal{D}_2 = \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta\to 0} |(\bar{A}_\delta f)(t)| = \infty \,\forall\, t \in \mathbb{R}\setminus\mathbb{Z}\}.$$

Both threshold operators and thus A_δ and $\bar{A}_\delta$ are meaningful in practical applications, and one would expect the difference being not important. However, as we will see, $\bar{A}_\delta$ can be analyzed more easily.
For $\hat{t} \in \mathbb{R}\setminus\mathbb{Z}$ we furthermore define the sets

$$\mathcal{D}_1(\hat{t}) = \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta\to 0} |(A_\delta f)(\hat{t})| = \infty\}$$

and

$$\mathcal{D}_2(\hat{t}) = \{f \in \mathcal{PW}_\pi^1 : \limsup_{\delta\to 0} |(\bar{A}_\delta f)(\hat{t})| = \infty\}.$$

Lemma 1 shows that we do not have to distinguish between the sets D_1 and $D_1(\hat{t})$ and between D_2 and $D_2(\hat{t})$.

Lemma 1. *For all $\hat{t} \in \mathbb{R}\setminus\mathbb{Z}$ we have $D_1 = D_1(\hat{t})$ and $D_2 = D_2(\hat{t})$.*

Proof. The inclusion $D_1 \subset D_1(\hat{t})$ is obvious. It remains to show that $D_1(\hat{t}) \subset D_1$. Let $f \in D_1(\hat{t})$. For all $t_1 \in \mathbb{R}\setminus\mathbb{Z}$ and $\delta > 0$ a short calculation shows that

$$\begin{aligned}&\left|\frac{1}{\sin(\pi t_1)}(A_\delta f)(t_1) - \frac{1}{\sin(\pi\hat{t})}(A_\delta f)(\hat{t})\right| \\ &\le \|f\|_{\mathcal{PW}_\pi^1} \frac{|\hat{t}-t_1|}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{|t_1-k||\hat{t}-k|} = C_1(t_1,\hat{t},f),\end{aligned}$$

where $C_1(t_1,\hat{t},f) < \infty$ is a constant that only depends on t_1, $\hat{t}$, and f. It follows that

$$|(A_\delta f)(t_1)| \ge |(A_\delta f)(\hat{t})| \left|\frac{\sin(\pi t_1)}{\sin(\pi\hat{t})}\right| - C_2(t_1,\hat{t},f). \tag{5}$$

Taking the limit superior on both sides of (5) gives

$$\limsup_{\delta\to 0} |(A_\delta f)(t_1)| = \infty. \tag{6}$$

Since (6) is valid for all $t_1 \in \mathbb{R}\setminus\mathbb{Z}$, it follows that $f \in \mathcal{D}_1$. The same calculation shows that $D_2 = D_2(\hat{t})$. □

According to Lemma 1 it is sufficient to restrict the analysis to the sets $\mathcal{D}_1(\hat{t})$ and $\mathcal{D}_2(\hat{t})$ for some $\hat{t} \in \mathbb{R}\setminus\mathbb{Z}$. Furthermore, we can concentrate on one of both sets, because of the following lemma.

Lemma 2. *We have $\mathcal{D}_1 = \mathcal{D}_2$.*

Proof. Let $f \in \mathcal{D}_2(\hat{t})$ be arbitrary but fixed. By the definition of $\mathcal{D}_2(\hat{t})$, we have $\limsup_{\delta\to 0}|(\bar{A}_\delta f)(\hat{t})| = \infty$. Thus, for every $M > 0$ there exists a $\delta_M > 0$ such that $|(\bar{A}_{\delta_M} f)(\hat{t})| > M$. Let $\mathcal{T}(M) = \{k \in \mathbb{Z} : |f(k)| > \delta_M\}$ and $\underline{f}_M = \min_{k\in\mathcal{T}(M)} |f(k)|$. Then it follows that $\underline{f}_M > \delta_M$. Moreover, for all δ with $\underline{f}_M > \delta > \delta_M$ we have

$$\begin{aligned}(A_\delta f)(\hat{t}) &= \sum_{\substack{k=-\infty \\ |f(k)|\ge\delta}}^{\infty} f(k) \frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)} \\ &= \sum_{\substack{k=-\infty \\ |f(k)|>\delta_M}}^{\infty} f(k) \frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)} = (\bar{A}_{\delta_M} f)(\hat{t}).\end{aligned}$$

Consequently,

$$\sup_{\delta>0}|(A_\delta f)(\hat{t})| > M. \tag{7}$$

Since (7) is valid for all $M > 0$, it follows that $\sup_{\delta>0}|(A_\delta f)(\hat{t})| = \infty$, and, as a consequence, $\limsup_{\delta\to 0}|(A_\delta f)(\hat{t})| = \infty$, because $|(A_\delta f)(\hat{t})| < \infty$ for all $\delta > 0$. This shows that $f \in \mathcal{D}_1(\hat{t})$, which implies that $\mathcal{D}_2(\hat{t}) \subset \mathcal{D}_1(\hat{t})$. The converse $\mathcal{D}_2(\hat{t}) \supset \mathcal{D}_1(\hat{t})$ is shown similarly. Hence $\mathcal{D}_1(\hat{t}) = \mathcal{D}_2(\hat{t})$, and the statement $\mathcal{D}_1 = \mathcal{D}_2$ follows from Lemma 1. □

In order to prove our main result, we need the important Lemma 3.

Lemma 3. *For all $M \in \mathbb{N}$ and $\hat{t} \in \mathbb{R} \setminus \mathbb{Z}$,*

$$\mathcal{D}_2(\hat{t}, M) = \{f \in \mathcal{PW}^1_\pi : \sup_{\delta>0}|(\bar{A}_\delta f)(\hat{t})| > M\}$$

is a residual set.

Proof. Let $M \in \mathbb{N}$ and $\hat{t} \in \mathbb{R} \setminus \mathbb{Z}$ be arbitrary but fixed. First, we show that $\mathcal{D}_2(\hat{t}, M)$ is an open set. Let $f_1 \in \mathcal{D}_2(\hat{t}, M)$ be arbitrary. We have to show that there exists an $\epsilon > 0$ such that, given any $f \in \mathcal{PW}^1_\pi$ with $\|f - f_1\|_{\mathcal{PW}^1_\pi} < \epsilon$, $f \in \mathcal{D}_2(\hat{t}, M)$. By assumption, there exists a $\delta_M > 0$ such that

$$|(\bar{A}_{\delta_M} f_1)(\hat{t})| > M.$$

Furthermore, let $\mathcal{T}(M) = \{k \in \mathbb{Z} : |f_1(k)| > \delta_M\}$ and $\underline{f}_{1,M} = \min_{k\in\mathcal{T}(M)}|f_1(k)|$. Next, we choose $\tilde{\delta}_M = \delta_M + (\underline{f}_{1,M} - \delta_M)/2$. Then we have that

$$\{k \in \mathbb{Z} : |f_1(k)| > \tilde{\delta}_M\} = \mathcal{T}(M). \tag{8}$$

We choose

$$\tilde{\epsilon} < \min\left(\frac{|(\bar{A}_{\tilde{\delta}_M} f_1)(\hat{t})| - M}{|\mathcal{T}(M)|}, \tilde{\delta}_M - \delta_M\right). \tag{9}$$

For all $f \in \mathcal{PW}^1_\pi$ with $\|f_1 - f\|_{\mathcal{PW}^1_\pi} < \tilde{\epsilon}$ we have $|f_1(k) - f(k)| < \tilde{\epsilon}$, $k \in \mathbb{Z}$. It follows, for all $k \in \mathbb{Z}$ with $|f(k)| > \tilde{\delta}_M$, that

$$|f_1(k)| \geq |f(k)| - |f(k) - f_1(k)| > \tilde{\delta}_M - \tilde{\epsilon} > \delta_M,$$

i.e., $k \in \mathcal{T}(M)$. Conversely, $k \in \mathcal{T}(M)$ implies $f_1(k) \geq \underline{f}_{1,M}$, and it follows that

$$\begin{aligned}|f(k)| &\geq |f_1(k)| - |f(k) - f_1(k)| > \underline{f}_{1,M} - \tilde{\epsilon} \\ &> \underline{f}_{1,M} - \tilde{\delta}_M + \delta_M = \tilde{\delta}_M.\end{aligned}$$

Thus we have

$$\{k \in \mathbb{Z} : |f(k)| > \tilde{\delta}_M\} = \mathcal{T}(M). \tag{10}$$

Moreover, using (8) and (10), we obtain that

$$\begin{aligned}&|(\bar{A}_{\tilde{\delta}_M} f)(\hat{t}) - (\bar{A}_{\tilde{\delta}_M} f_1)(\hat{t})| \\ &= \left|\sum_{\substack{k=-\infty \\ |f(k)|>\tilde{\delta}_M}}^{\infty} f(k)\frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)} - \sum_{\substack{k=-\infty \\ |f_1(k)|>\delta_M}}^{\infty} f_1(k)\frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)}\right| \\ &\leq \sum_{k\in\mathcal{T}(M)} |f_1(k) - f(k)|\left|\frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)}\right| \leq \tilde{\epsilon}|\mathcal{T}(M)|\end{aligned}$$

and consequently

$$|(\bar{A}_{\tilde{\delta}_M} f)(\hat{t})| \geq |(\bar{A}_{\tilde{\delta}_M} f_1)(\hat{t})| - \tilde{\epsilon}|\mathcal{T}(M)| > M,$$

where the last inequality is due to (9). Therefore

$$\sup_{\delta>0}|(\bar{A}_\delta f)(\hat{t})| > M,$$

i.e., $f \in \mathcal{D}_2(\hat{t}, M)$, for all $f \in \mathcal{PW}^1_\pi$ with $\|f_1 - f\|_{\mathcal{PW}^1_\pi} < \tilde{\epsilon}$.

Second, we show that $\mathcal{D}_2(\hat{t}, M)$ is dense in $\mathcal{PW}^1_\pi$. Let $f \in \mathcal{PW}^1_\pi$ be arbitrary. We have to show that for every $\epsilon > 0$ there exists a $f_\epsilon \in \mathcal{D}_2(\hat{t}, M)$ such that $\|f - f_\epsilon\|_{\mathcal{PW}^1_\pi} < \epsilon$. Let $\epsilon > 0$ be arbitrary but fixed. Since $\mathcal{PW}^2_\pi$ is dense in $\mathcal{PW}^1_\pi$, there exists a $f^{(1)}_\epsilon \in \mathcal{PW}^2_\pi$ with

$$\|f - f^{(1)}_\epsilon\|_{\mathcal{PW}^1_\pi} < \frac{\epsilon}{3}. \tag{11}$$

Moreover, there exists a $f^{(2)}_\epsilon \in \mathcal{PW}^2_\pi$ such that $f^{(2)}_\epsilon(k) \neq 0$ only for finitely many $k \in \mathbb{Z}$ and

$$\|f^{(1)}_\epsilon - f^{(2)}_\epsilon\|_{\mathcal{PW}^1_\pi} < \frac{\epsilon}{3}. \tag{12}$$

Let N denote the smallest natural number such that $N > \hat{t}$ and $f^{(2)}_\epsilon(k) = 0$ for all $|k| > N$. Furthermore, let $\mathcal{T}_2 = \{k \in \mathbb{Z} : |f^{(2)}_\epsilon(k)| \neq 0\}$ and $\underline{f}^{(2)}_\epsilon = \min_{k\in\mathcal{T}_2}|f^{(2)}_\epsilon(k)|$. For $0 < \eta < 1$ and $L \in \mathbb{N}$, $L > N$, consider the functions h and g defined by

$$h(t, \eta, L) := \sum_{k=-2L+1}^{2L-1} h(k, \eta, L)\frac{\sin(\pi(t-k))}{\pi(t-k)},$$

where

$$h(k, \eta, L) = \begin{cases}(-1)^k(2(1-\eta) + \frac{1-\eta}{L}k), & -2L < k < -L, \\ (-1)^k(1-\eta), & -L \leq k < 0, \\ (-1)^k, & 0 \leq k \leq L, \\ (-1)^k(2 - \frac{1}{L}k), & L < k < 2L,\end{cases}$$

and

$$\begin{aligned}g(t, \eta, L) :=& h(t, \eta, L) - \underbrace{\sum_{k=0}^{N}\frac{(-1)^k\sin(\pi(t-k))}{\pi(t-k)}}_{=:u_1} \\ &- \underbrace{\sum_{k=-N}^{-1}(1-\eta)\frac{(-1)^k\sin(\pi(t-k))}{\pi(t-k)}}_{=:u_2}.\end{aligned}$$

Note that $g(k, \eta, L) = 0$ for $|k| \leq N$. We have

$$\begin{aligned}&\|g(t, \eta, L)\|_{\mathcal{PW}^1_\pi} \\ &\quad\leq \|h(\,\cdot\,, \eta, L)\|_{\mathcal{PW}^1_\pi} + \|u_1\|_{\mathcal{PW}^1_\pi} + \|u_2\|_{\mathcal{PW}^1_\pi}.\end{aligned} \tag{13}$$

The norm $\|u_1\|_{\mathcal{PW}^1_\pi}$ is upper bounded by

$$\|u_1\|_{\mathcal{PW}^1_\pi} < \frac{\pi}{2} + \log(N+1), \tag{14}$$

because

$$\|u_1\|_{\mathcal{PW}^1_\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=0}^{N} \mathrm{e}^{-i\omega k}(-1)^k\right| \,\mathrm{d}\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1-\mathrm{e}^{i\omega(N+1)}}{1-\mathrm{e}^{i\omega}}\right|\mathrm{d}\omega = \frac{1}{\pi}\int_0^{\pi}\left|\frac{\sin(\frac{N+1}{2}\omega)}{\sin(\frac{\omega}{2})}\right|\mathrm{d}\omega$$
$$\le \int_0^{\pi}\frac{\left|\sin(\frac{N+1}{2}\omega)\right|}{\omega}\,\mathrm{d}\omega = \int_0^{N+1}\frac{\left|\sin(\frac{\pi}{2}\omega)\right|}{\omega}\,\mathrm{d}\omega$$
$$\le \int_0^{1}\frac{\sin(\frac{\pi}{2}\omega)}{\omega}\,\mathrm{d}\omega + \int_1^{N+1}\frac{1}{\omega}\,\mathrm{d}\omega < \frac{\pi}{2} + \log(N+1).$$

A similar calculation gives

$$\|u_2\|_{\mathcal{PW}^1_\pi} < \frac{\pi}{2} + \log(N). \tag{15}$$

In addition we have $\|h(\,\cdot\,,0,L)\|_{\mathcal{PW}^1_\pi} \le 3$, which can be proven easily, and $\lim_{\eta\to 0}\|h(\,\cdot\,,\eta,L) - h(\,\cdot\,,0,L)\|_{\mathcal{PW}^1_\pi} = 0$. Therefore, there exists an $0 < \eta_0(L) < 1$ such that

$$\|h(\,\cdot\,,\eta_0(L),L)\|_{\mathcal{PW}^1_\pi} < 4. \tag{16}$$

Combining (13)–(16) gives, that for all $L \in \mathbb{N}$, $L > N$ there exists an $0 < \eta_0(L) < 1$ such that

$$\|g(\,\cdot\,,\eta_0(L),L)\|_{\mathcal{PW}^1_\pi} < 4 + \pi + 3\log(N+1) =: C_3.$$

It is important that the constant C_3 does not depend on L. Next, we analyze

$$G_\epsilon(t,L) = f_\epsilon^{(2)}(t) + \mu g(t,\eta_0(L),L),$$

where $\mu > 0$ is some real number that satisfies $\mu < \min(\epsilon/(3C_3), \underline{f}_\epsilon^{(2)})$. By the choice of μ we have

$$\|f_\epsilon^{(2)} - G_\epsilon(\,\cdot\,,L)\|_{\mathcal{PW}^1_\pi} = \mu C_3 < \frac{\epsilon}{3} \tag{17}$$

for all $L > N$. Combining (11), (12), and (17), we see that

$$\|f - G_\epsilon(\,\cdot\,,L)\|_{\mathcal{PW}^1_\pi} < \epsilon \tag{18}$$

for all $L > N$, i.e., $G_\epsilon(\,\cdot\,,L)$ lies in the ϵ-ball around f. Furthermore, for any $L > N$ we can find a $\delta_0(L)$ that fulfills

$$\max\left((1-\eta_0(L))\mu, \left(1-\frac{1}{L}\right)\mu\right) < \delta_0(L) < \mu.$$

Since $\delta_0(L) < \underline{f}_\epsilon^{(2)}$, by the definition of μ, it follows that

$$(\bar{A}_{\delta_0(L)}G_\epsilon(\,\cdot\,,L))(\hat{t})$$
$$= \sum_{\substack{k=-N\\ |G_\epsilon(k,L)|>\delta_0(L)}}^{N} G_\epsilon(k,L)\frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)}$$
$$+ \sum_{\substack{k=N+1\\ |G_\epsilon(k)|>\delta_0(L)}}^{L} G_\epsilon(k,L)\frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)}$$
$$= \sum_{k=-N}^{N} f_\epsilon^{(2)}(k)\frac{\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)} + \mu\sum_{k=N+1}^{L}\frac{(-1)^k\sin(\pi(\hat{t}-k))}{\pi(\hat{t}-k)}$$
$$= f_\epsilon^{(2)}(\hat{t}) + \mu\frac{\sin(\pi\hat{t})}{\pi}\sum_{k=N}^{L}\frac{1}{\hat{t}-k}.$$

Observing that $N - \hat{t} > 0$, we obtain

$$\left|\sum_{k=N}^{L}\frac{1}{\hat{t}-k}\right| = \sum_{k=0}^{L-N}\frac{1}{k+N-\hat{t}}$$
$$\ge \sum_{k=0}^{L-N}\int_k^{k+1}\frac{1}{\tau+N-\hat{t}}\,\mathrm{d}\tau$$
$$= \int_0^{L-N+1}\frac{1}{\tau+N-\hat{t}}\,\mathrm{d}\tau$$
$$> \log\left(\frac{L-\hat{t}}{N-\hat{t}}\right),$$

and consequently

$$|(\bar{A}_{\delta_0(L)}G_\epsilon(\,\cdot\,,L))(\hat{t})|$$
$$\ge \mu\frac{|\sin(\pi\hat{t})|}{\pi}\log\left(\frac{L-\hat{t}}{N-\hat{t}}\right) - |f_\epsilon^{(2)}(\hat{t})|. \tag{19}$$

The right-hand side of (19) can be made arbitrarily large by choosing L large. Let $L_1 > N$ be the smallest L such that the right hand side of (19) is larger than M. It follows that $f_\epsilon(t) = G_\epsilon(t,L_1)$ is the desired function, because $\sup_{\delta>1}|(\bar{A}_\delta f_\epsilon)(\hat{t})| \ge |(\bar{A}_{\delta_0(L_1)}f_\epsilon)(\hat{t})| > M$, i.e., $f_\epsilon \in \mathcal{D}_2(\hat{t},M)$, and because $\|f - f_\epsilon\|_{\mathcal{PW}^1_\pi} < \epsilon$, according to (18). □

Theorem 1. *$\mathcal{D}_1$ and $\mathcal{D}_2$ are residual sets.*

Proof. Since $\mathcal{D}_2 = \mathcal{D}_1$, by Lemma 2, it is sufficient to show that $\mathcal{D}_2$ is a residual set.
Let $\hat{t} \in \mathbb{R}\setminus\mathbb{Z}$ be arbitrary but fixed. We have

$$\mathcal{D}_2(\hat{t}) = \bigcap_{M\in\mathbb{N}}\mathcal{D}_2(\hat{t},M).$$

From Lemma 3 we know that all $\mathcal{D}_2(\hat{t},M)$, $M \in \mathbb{N}$, are residual sets. It follows that $\mathcal{D}_2(\hat{t})$ is a residual set, because the countable intersection of residual sets is a residual set. The application of Lemma 1 completes the proof. □

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