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Abstract

We study in this paper nonlinear subdivision schemes in a multivariate setting allowing arbitrary dilation matrix. We investigate the convergence of such iterative process to some limit function. Our analysis is based on some conditions on the contractivity of the associated scheme for the differences. In particular, we show the regularity of the limit function, in L^p and Sobolev spaces.

Keywords: Nonlinear subdivision scheme, convergence of subdivision schemes, box splines

1. Introduction

Subdivision schemes have been the subject of active research in recent years. In such algorithms, discrete data are recursively generated from coarse to fine by means of local rules. When the local rules are independent of the data, the underlying refinement process is linear. This case is extensively studied in literature. The convergence of this process and the existence of the limit function was studied in [2] and [7] when the scales are dyadic. When the scales are related to a dilation matrix M, the convergence to a limit function in L^p was studied in [8] and generalized to Sobolev spaces in [5] and [13]. In the linear case, the stability is a consequence of the smoothness of the limit function.

The nonlinearity arises naturally when one needs to adapt locally the refinement rules to the data such as in image or geometry processing. Nonlinear subdivision schemes based on dyadic scales were originally introduced by Harten [9][10] through the so-called essentially non-oscillatory (ENO) methods. These methods have recently been adapted to image processing into essentially non-oscillatory edge adapted (ENO-EA) methods. Different versions of ENO methods exist either based on polynomial interpolation as in [6][1] or in a wavelet framework [3], corresponding to interpolatory or non-interpolatory subdivision schemes respectively.

In the present paper, we study nonlinear subdivision schemes associated to dilation matrix M. After recalling the definitions on nonlinear subdivision schemes in that context, we give sufficient conditions for convergence in Sobolev and L^p spaces.

2. General Setting

2.1. Notations

Before we start, let us introduce some notations that will be used throughout the paper. We denote $\#Q$ the cardinal of the set Q. For a multi-index $\mu = (\mu_1, \mu_2, \cdots, \mu_d) \in \mathbb{N}^d$ and a vector $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ we define $|\mu| = \sum_{i=1}^{d} \mu_i$, $\mu! = \prod_{i=1}^{d} \mu_i!$ and $x^\mu = \prod_{i=1}^{d} {x_i}^{\mu_i}$.

For two multi-index $m, \mu \in \mathbb{N}^d$ we also define

$$\begin{pmatrix} \mu \\ m \end{pmatrix} = \begin{pmatrix} \mu_1 \\ m_1 \end{pmatrix} \cdots \begin{pmatrix} \mu_d \\ m_d \end{pmatrix}.$$

Let $\ell(\mathbb{Z}^d)$ be the space of all sequences indexed by $\mathbb{Z}^d$. The subspace of bounded sequences is denoted by $\ell^\infty(\mathbb{Z}^d)$ and $\|u\|_{\ell^\infty(\mathbb{Z}^d)}$ is the supremum of $\{|u_k| : k \in \mathbb{Z}^d\}$. We denote $\ell^0(\mathbb{Z}^d)$ the subspace of all sequences with finite support (i.e. the number of non-zero components of a sequence is finite). As usual, let $\ell^p(\mathbb{Z}^d)$ be the Banach space of sequences u on $\mathbb{Z}^d$ such that $\|u\|_{\ell^p(\mathbb{Z}^d)} < \infty$, where

$$\|u\|_{\ell^p(\mathbb{Z}^d)} := \left(\sum_{k \in \mathbb{Z}^d} |u_k|^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty.$$

As in the discrete case, we denote by $L^p(\mathbb{R}^d)$ the space of all measurable functions f such that $\|f\|_{L^p(\mathbb{R}^d)} < \infty$, where

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

and $\|f\|_{L^\infty(\mathbb{R}^d)}$ is the essential supremum of $|f|$ on $\mathbb{R}^d$.

Let $\mu \in \mathbb{N}^d$ be a multi-index, we define ∇^μ the difference operator $\nabla_1^{\mu_1} \cdots \nabla_d^{\mu_d}$, where $\nabla_j^{\mu_j}$ is the μ_jth difference operator with respect to the

jth coordinate of the canonical basis. We define D^μ as $D_1^{\mu_1} \cdots D_d^{\mu_d}$, where D_j is the differential operator with respect to the jth coordinate of the canonical basis. Similarly, for a vector $x \in \mathbb{R}^d$ the differential operator with respect to x is denoted by D_x.

A matrix M is called a dilation matrix if it has integer entries and if $\lim_{n\to\infty} M^{-n} = 0$. In the following, the invertible dilation matrix is always denoted by M and m stands for $|det(M)|$.

For a dilation matrix M and any arbitrary function Φ we put $\Phi_{j,k}(x) = \Phi(M^j x - k)$.

We also recall that a compactly supported function Φ is called L^p-stable if there exist two constants $C_1, C_2 > 0$ satisfying

$$C_1 \|c\|_{\ell^p(\mathbb{Z}^d)} \leq \| \sum_{k\in\mathbb{Z}^d} c_k \Phi(x-k) \|_{L^p(\mathbb{R}^d)} \leq C_2 \|c\|_{\ell^p(\mathbb{Z}^d)}.$$

Finally, for two positive quantities A and B depending on a set of parameters, the relation $A \lesssim B$ implies the existence of a positive constant C, independent of the parameters, such that $A \leq CB$. Also $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2.2. Local, Bounded and Data Dependent Subdivision Operators, Uniform Convergence Definition

In the sequel, we will consider the general class of local, bounded and data dependent subdivision operators which are defined as follows:

Definition 1. *For $v \in \ell^\infty(\mathbb{Z}^d)$, a local, bounded and data dependent subdivision operator is defined by*

$$S(v)w_k = \sum_{l\in\mathbb{Z}^d} a_{k-Ml}(v) w_l, \tag{1}$$

for any w in $\ell^\infty(\mathbb{Z}^d)$ and where the real coefficients $a_{k-Ml}(v) \in \mathbb{R}$ are such that

$$a_{k-Ml}(v) = 0, \quad if \quad \|k - Ml\|_{\ell^\infty(\mathbb{Z}^d)} > K \tag{2}$$

for a fixed constant K. The coefficients $a_k(v)$ are assumed to be uniformly bounded by a constant C, i.e. there is $C > 0$ independent of v such that:

$$|a_k(v)| \leq C.$$

Note that the definition of the coefficients depends on some sequence v, while $S(v)$ acts on the sequence w.

Note also that, from (1) *and* (2) *the new defined value $S(v)w_k$ depends only on those values l satisfying $\|k - Ml\|_{\ell^\infty(\mathbb{Z}^d)} > K$. The subdivision operator in this sense is local.*

To simplify, in what follows a data dependent subdivision operator is an operator in the sense of Definition 1. With this definition, the associated subdivision scheme is the recursive action of the data dependent rule $Sv = S(v)v$ on an initial set of data v^0, according to:

$$v^j = Sv^{j-1} = S(v^{j-1})v^{j-1}, \ j \geq 1. \tag{3}$$

2.3. Polynomial Reproduction for Data Dependent Subdivision Operators

The study of the convergence of data dependent subdivision operators will involve the polynomial reproduction property. We recall the definition of the space $\mathbb{P}_N$ of polynomials of total degree N:

$$\mathbb{P}_N := \{P; P(x) = \sum_{|\mu| \leq N} a_\mu x^\mu\}.$$

With these notations, the polynomial reproduction properties read:

Definition 2. *Let $N \geq 0$ be a fixed integer.*

1. *The data dependent subdivision operator S has the property of reproduction of polynomials of total degree N if for all $u \in \ell^\infty(\mathbb{Z}^d)$ and $P \in \mathbb{P}_N$ there exists $\tilde{P} \in \mathbb{P}_N$ with $P - \tilde{P} \in \mathbb{P}_{N-1}$ such that $S(u)p = \tilde{p}$ where p and $\tilde{p}$ are defined by $p_k = P(k)$ and $\tilde{p}_k = \tilde{P}(M^{-1}k)$.*
2. *The data dependent subdivision operator S has the property of exact reproduction of polynomials of total degree N if for all $u \in \ell^\infty(\mathbb{Z}^d)$ and $P \in \mathbb{P}_N$, $S(u)p = \tilde{p}$ where p and $\tilde{p}$ are defined by $p_k = P(k)$ and $\tilde{p}_k = P(M^{-1}k)$.*

Remark:

The case $N = 0$ is the so-called "constant reproduction property". For a data dependent subdivision operator defined as in (1), the constant reproduction property reads $\sum_{k \in \mathbb{Z}^d} a_{k-Ml}(v) = 1$, for all $v \in \ell^\infty(\mathbb{Z}^d)$.

3. Definition of Schemes for the Differences

Another ingredient for our study is the schemes for the differences associated to the data dependent subdivision operator. The existence of schemes for the differences is obtained by using the polynomial reproduction property of the data dependent subdivision operator.

Let us denote $\Delta^l = (\nabla^\mu, |\mu| = l)$ and then state the following result on the existence of schemes for the differences:

Proposition 1. *Let S be a data dependent subdivision operator which reproduces polynomials up to total degree N. Then for $1 \leq l \leq N + 1$ there*

exists a data dependent subdivision rule S_l with the property that for all v,w in $\ell^\infty(\mathbb{Z}^d)$,

$$\Delta^l S(v)w := S_l(v)\Delta^l w$$

PROOF:

Let l be an integer such that $1 \leq l \leq N+1$. By using the definition of ∇^μ with $|\mu| = l$, we write:

$$\nabla^\mu S(v)w_k = \nabla_1^{\mu_1} \cdots \nabla_d^{\mu_d} S(v)w_k.$$

From the definition of $S(v)w$ we infer that

$$\nabla^\mu S(v)w_k = \sum_{m_1,\cdots,m_d=0}^{\max(\mu_1,\cdots,\mu_d)} (-1)^l \begin{pmatrix} \mu \\ m \end{pmatrix} \sum_{p\in\mathbb{Z}^d} a_{k-m\cdot e-Mp}(v)w_p,$$

where we have used the notation $m \cdot e = m_1 e_1 + \cdots + m_d e_d$. Straightforward computations give

$$\begin{aligned} \nabla^\mu S(v)w_k &= \sum_{p\in\mathbb{Z}^d} w_p \sum_{m_1,\cdots,m_d=0}^{\max(\mu_1,\cdots,\mu_d)} (-1)^l \begin{pmatrix} \mu \\ m \end{pmatrix} a_{k-m\cdot e-Mp}(v) \\ &= \sum_{p\in\mathbb{Z}^d} w_p f_{k,p}(v,\mu). \end{aligned} \tag{4}$$

Let us clarify the definition of $f_{k,p}(v,\mu)$. Since the data dependent subdivision operator is local we have $a_{k-Mp}(v) = 0$ for any data $v \in \ell^\infty(\mathbb{Z}^d)$ and any index k such that $\|k - Mp\|_{\ell^\infty(\mathbb{Z}^d)} > K$. Now by putting $k = \varepsilon + Mn$, we get that $f_{k,p}(v,\mu)$ is defined for p in the set

$$V^\mu(k) := \left\{p : \|n - p + M^{-1}(\varepsilon - m\cdot e)\|_\infty \leq K\|M^{-1}\|_\infty,\ 0 \leq m_i \leq \mu_i \forall i\right\}$$

Then, we define $V(k) := \{p : \|k - Mp\|_\infty \leq K\}$. Since the data dependent subdivision scheme reproduces polynomials up to total degree N, we have

for any $|\nu| = r \leq N$:

$$\sum_{p \in V(k)} a_{k-Mp}(v) p^\nu = P_\nu(k) \quad \text{for all } k \in \mathbb{Z}^d, \tag{5}$$

where P_ν is a polynomial of total degree r. By tacking the differences of order $|\nu'| = r+1$ in (5) we get

$$\sum_{p \in V^{\nu'}(k)} f_{k,p}(v,\nu') p^\nu \quad = \quad 0.$$

Note that the above equality is true for any ν such that $|\nu| = r$. We deduce that $\left(f_{k,p}(v,\nu')\right)_k \in \mathbb{Z}^d$ is orthogonal to $\left(p^q\right)_{p \in V^{\nu'}(k)}$ where $|q| \leq r$. Note that $\left\{ \left(\nabla^\nu \delta_{n-\beta}\right)_{n \in V^{\nu'}(k)}, |\nu| = r+1, \beta \in \mathbb{Z}^d \right\}$ spans $\left(p^q\right)_{p \in V^{\nu'}(k)}$ and we may thus write for any $p \in V^{\nu'}(k)$:

$$f_{k,p}(v,\nu') = \sum_{|\nu|=r+1} \sum_{r \in \mathbb{Z}^d} c^\mu_{k,r}(v) \nabla^\mu \delta_{p-r}.$$

Now, by using (4) we obtain for any $|\mu| \leq N+1$:

$$\begin{aligned} \nabla^\mu S(v) w_k &= \sum_{p \in V^\mu(k)} w_p \sum_{|\nu|=l} \sum_{r \in \mathbb{Z}^d} c^\nu_{k,r}(v) \nabla^\nu \delta_{p-r} \\ &= \sum_{p \in V^\mu(k)} \sum_{|\nu|=l} c^\nu_{k,r}(v) \nabla^\nu w_p \end{aligned}$$

If we now make μ vary, we obtain the desired relation. □

Now that we have proved the existence of schemes for the differences, we introduce the notion of joint spectral radius for these schemes, which is a generalization of the one dimensional case which can be found in [15].

Definition 3. *Let $S(v) : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$ be a data dependent subdivision operator such that the difference operators $S_l(v) : \left(\ell^p(\mathbb{Z}^d)\right)^{q_l} \to \left(\ell^p(\mathbb{Z}^d)\right)^{q_l}$,*

with $q_l = \#\{\mu, |\mu| = l\}$ *exists for* $l \leq N+1$. *Then, to each operator* S_l, $l = 0, \cdots, N+1$ *(putting* $S_0 = S$*) we can associate the joint spectral radius given by*

$$\rho_{p,l}(S) := \inf_{j \geq 1} \|(S_l)^j\|^{\frac{1}{j}}_{\ell^p(\mathbb{Z}^d)^{q_l}}.$$

In other words, $\rho_p(S)$ is the infimum of all $\rho > 0$ such that for all $v \in \ell^p(\mathbb{Z}^d)$, one has

$$\|\Delta^l S^j v\|_{\ell^p(\mathbb{Z}^d)^{q_l}} \lesssim \rho^j \|\Delta^l v\|_{\ell^p(\mathbb{Z}^d)^{q_l}}, \tag{6}$$

for all $j \geq 0$.

Remark: Let us define a set of vectors $\{x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}\}$ such that $[x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}]\mathbb{Z}^n = \mathbb{Z}^d$, $n \geq d$ (i.e. a set such that the linear combinations of its elements with coefficients in $\mathbb{Z}$ spans $\mathbb{Z}^d$). We use the bold notation in the definition of the set so as to avoid the confusion with the coordinates of vector x. Then, consider the differences in the directions $x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}$. One can show that there exists a scheme for that differences which we call $\tilde{S}_l$ for $l \leq N+1$ provided the data dependent subdivision operator reproduces polynomials up to degree N (the proof is similar to that using the canonical directions). If we denote by $\tilde{\Delta}^l$ the difference operator of order l in the directions $x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}$, one can see that $\|\tilde{\Delta}^l v\|_{\ell^p(\mathbb{Z}^d)^{\tilde{q}_l}} \sim \|\Delta^l v\|_{\ell^p(\mathbb{Z}^d)^{q_l}}$ for all v in $\ell^p(\mathbb{Z}^d)$ and where $\tilde{q}_l = \#\{\mu, |\mu| = l, \mu = (\mu_i)_{i=1,\cdots,n}\}$.Then, following (6), one can deduce that the joint spectral radius of $\tilde{S}_l$ is the same as that of S_l.

4. Convergence in L^p spaces

In the following, we study the convergence of data dependent subdivision schemes in L^p which corresponds to the following definition:

Definition 4. *The subdivision scheme $v^j = Sv^{j-1}$ converges in $L^p(\mathbb{R}^d)$, if for every set of initial control points $v^0 \in \ell^p(\mathbb{Z}^d)$, there exists a non-trivial function v in $L^p(\mathbb{R}^d)$, called the limit function, such that*

$$\lim_{j\to\infty} \|v_j - v\|_{L^p(\mathbb{R}^d)} = 0.$$

where $v_j(x) = \sum_{k\in\mathbb{Z}^d} v_k^j \phi_{j,k}(x)$ with $\phi(x) = \prod_{i=1}^{d} \max(0, 1 - |x_i|)$.

4.1. Convergence in the Linear Case

When S is independent of v, the rule (1) defines a linear subdivision scheme:

$$Sv_k = \sum_{l\in\mathbb{Z}^d} a_{k-Ml} v_l.$$

If the linear subdivision scheme converges for any $v \in \ell^p(\mathbb{Z}^d)$ to some function in $L^p(\mathbb{R}^d)$ and if there exists v^0 such that $\lim_{j\to+\infty} v^j \neq 0$, then $\{a_k, k \in \mathbb{Z}^d\}$ determines a unique continuous compactly supported function Φ satisfying

$$\Phi(x) = \sum_{k\in\mathbb{Z}^d} a_k \Phi(Mx - k) \text{ and } \sum_{k\in\mathbb{Z}^d} \Phi(x - k) = 1.$$

Moreover, $v(x) = \sum_{k\in\mathbb{Z}^d} v_k^0 \Phi(x - k)$.

4.2. Convergence of Nonlinear Subdivision Schemes in L^p Spaces

In the sequel, we give a sufficient condition for the convergence of nonlinear subdivision schemes in $L^p(\mathbb{R}^d)$. This result will be a generalization of the existing result in the linear context established in [8] and only uses the operator S_1.

Theorem 1. *Let S be a data dependent subdivision operator that reproduces the constants. If $\rho_{p,1}(S) < m^{\frac{1}{p}}$, then Sv^j converges to a L^p limit function.*

PROOF: Let us consider

$$v_j(x) := \sum_{k\in\mathbb{Z}^d} v_k^j \phi_{j,k}(x), \tag{7}$$

where $\phi(x) = \prod_{i=1}^{d} \max(0, 1-|x_i|)$ is the hat function. With this choice, one can easily check that $\sum_{k\in\mathbb{Z}^d} \phi(x-k) = 1$. Let $\rho_{p,1}(S) < \rho < m^{\frac{1}{p}}$, it follows that

$$\|\Delta^1 v^j\|_{(\ell^p(\mathbb{Z}^d))^d} \lesssim \rho^j \|\Delta^1 v^0\|_{(\ell^p(\mathbb{Z}^d))^d},$$

since $q_1 = d$. We now show that the sequence v_j is a Cauchy sequence in L^p:

$$\begin{aligned} v_{j+1}(x) - v_j(x) &= \sum_{k\in\mathbb{Z}^d} v_k^{j+1}\phi_{j+1,k}(x) - \sum_{p\in\mathbb{Z}^d} v_p^j \phi_{j,p}(x) \\ &= \sum_{k\in\mathbb{Z}^d}\sum_{p\in\mathbb{Z}^d} (v_k^{j+1} - v_p^j)\phi_{j+1,k}(x)\phi_{j,p}(x) \end{aligned}$$

where we have used $\sum_{k\in\mathbb{Z}^d} \phi(\cdot - k) = 1$. Now, since the subdivision operator reproduces the constants:

$$v_{j+1}(x) - v_j(x) = \sum_{p\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} a_{k-Ml}(v^j)(v_l^j - v_p^j)\phi_{j+1,k}(x)\phi_{j,p}(x).$$

Note that

$$\sum_{l\in\mathbb{Z}^d} a_{k-Ml}(v^j)(v_l^j - v_p^j) = \sum_{l\in\mathbb{Z}^d} a_{k-Ml}(v^j)v_l^j - v_p^j = \sum_{l\in\mathbb{Z}^d} (a_{k-Ml}(v^j) - \delta_{p-l})v_l^j.$$

Since $\sum_{l\in\mathbb{Z}^d} a_{k-Ml}(v^j) - \delta_{p-l} = 0$, $\left\{\nabla_i \delta_{l-\beta}, l \in \{F(k)\cup\{p\}\}, \beta\in\mathbb{Z}^d, i = 1,\cdots,d\right\}$ spans $(a_{k-Ml} - \delta_{p-l})_{l\in\{F(k)\cup\{p\}\}}$. This enables us to write:

$$v_{j+1}(x) - v_j(x) = \sum_{p\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\sum_{l\in V(k)\bigcup\{p\}}\sum_{i=1}^{d} d_{k,p,l}^i \nabla_i v_l^j \phi_{j+1,k}(x)\phi_{j,p}(x).$$

Since $|\sum_{k\in\mathbb{Z}^d}\phi_{j+1,k}(x)| = 1$ following the same argument as in Theorem 3.2 of [8], we may write:

$$\begin{aligned}\|v_{j+1}-v_j\|_{L^p(\mathbb{R}^d)} &\lesssim m^{-\frac{j}{p}}\max_{1\le i\le d}\|\nabla_i v^j\|_{\ell^p(\mathbb{Z}^d)}\\ &\lesssim m^{-\frac{j}{p}}\|\Delta^1 v^j\|_{\ell^p(\mathbb{Z}^d)}\\ &\sim (\frac{\rho}{m^{\frac{1}{p}}})^j\|\Delta^1 v^0\|_{\ell^p(\mathbb{Z}^d)} \qquad (8)\end{aligned}$$

which proves that v_j converges in L^p, since $\rho < m^{\frac{1}{p}}$. Note that, for $p=\infty$, we obtain that the limit function is continuous. □

Furthermore, the above proof is valid for any function Φ_0 satisfying the property of partition of unity when $p=\infty$. In general, we could show, following Theorem 3.4 of [8], that the limit function in L^p is independent of the choice of a continuous and compactly supported Φ_0.

4.3. Uniform Convergence of the Subdivision Schemes to C^s functions ($s<1$)

We are now ready to establish a sufficient condition for the C^s smoothness of the limit function with $s<1$.

Theorem 2. *Let $S(v)$ be a data dependent subdivision operator which reproduces the constants. If the scheme for the differences satisfies $\rho_{p,1}(S) < m^{-s+\frac{1}{p}}$, for some $0<s<1$ then Sv^j is convergent in L^p and the limit function is C^s .*

PROOF: First, the convergence in L^p is a consequence of $\rho_{p,1}(S) < m^{\frac{1}{p}}$. In order to prove that the limit function v be in C^s, it suffices to evaluate

$|v(x)-v(y)|$ for $\|x-y\|_\infty \leq 1$. Let j be such that $m^{-j-1} \leq \|x-y\|_\infty \leq m^{-j}$. We then write :

$$\begin{aligned} |v(x)-v(y)| &\leq |v(x)-v_j(x)| + |v(y)-v_j(y)| + |v_j(x)-v_j(y)| \\ &\leq 2\|v-v_j\|_{L^\infty(\mathbb{R}^d)} + |v_j(x)-v_j(y)| \end{aligned}$$

Note that (8) implies that $\|v-v_j\|_{L^\infty(\mathbb{R}^d)} \lesssim \rho^j \|\Delta^1 v^0\|_{\ell^\infty(\mathbb{Z}^d)}$. Since v_j is absolutely continuous, it is almost everywhere differentiable, so putting $y = x + M^{-j}h$, with $h = (h_i)_{i=1,\cdots,d}$ satisfying $\|h\|_\infty \leq 1$ we get:

$$\begin{aligned} |v_j(x+M^{-j}h) - v_j(x)| &\leq |v_j(x+M^{-j}h) - v_j(x+M^{-j}(h-h_de_d))| \\ &\quad + |v_j(x+M^{-j}(h-h_de_d)) - v_j(x+M^{-j}(h-h_de_d-h_{d-1}e_{d-1}))| \\ &\quad + \cdots + |v_j(x+M^{-j}(h_1e_1)) - v_j(x)| \end{aligned}$$

Then, using a Taylor expansion we remark that, there exists $\theta \in]-h_d, h_d[$ such that:

$$|v_j(x+M^{-j}h) - v_j(x+M^{-j}(h-h_de_d))| = \sum_{k\in\mathbb{Z}^d} v_k^j h_d D_d\phi(M^jx+h-k+\theta_de_d)$$

If we denote $\Psi_d(x) = \Phi(x_1)\cdots\Phi(x_{d-1})\Psi(x_d)$, where Ψ is the characteristic function of $[0,1]$ and $\Phi(x_i) = \max(0, 1-|x_i|)$, we may write:

$$|v_j(x+M^{-j}h) - v_j(x+M^{-j}(h-h_de_d))| \sim \sum_{k\in\mathbb{Z}^d} \nabla_d v_k^j h_d \Psi_d(y)$$

where $y_i = (M^{-j}x+h-k+\theta_de_d)_i$ if $i < d$ and $y_d = 2(M^{-j}x+h+\theta_de_d)_d - k_d$ (we have used the fact that the differential of the hat function Φ is the Haar wavelet). Iterating the procedure for other differences in the sum, we get:

$$|v_j(x+M^{-j}h) - v_j(x)| \lesssim \sum_{i=1}^d \|\nabla_i v^j\|_{\ell^\infty(\mathbb{Z}^d)} \lesssim \|\Delta^1 v^j\|_{\ell^\infty(\mathbb{Z}^d)}.$$

Combining these results we may finally write:

$$\begin{aligned} |v(x)-v(y)| &\leq |v(x)-v_j(x)|+|v(y)-v_j(y)|+|v_j(x)-v_j(y)| \\ &\leq 2\|v-v_j\|_{L^\infty(\mathbb{R}^d)}+|v_j(x)-v_j(y)| \\ &\lesssim \rho^j\|\Delta^1 v^0\|_{\ell^\infty(\mathbb{Z}^d)}+\|\Delta^1 v^j\|_{\ell^\infty(\mathbb{Z}^d)} \\ &\lesssim \rho^j\|\Delta^1 v^0\|_{\ell^\infty(\mathbb{Z}^d)} \lesssim \|x-y\|_\infty^s \end{aligned}$$

with $s < -\log(\rho_{\infty,1})/\log m$. □

5. Examples of Bidimensional Subdivision Schemes

In the first part of this section, we construct an interpolatory subdivision scheme having the dilation matrix the hexagonal matrix which is:

$$M = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix},$$

For the hexagonal dilation matrix, the coset vectors are $\varepsilon_0 = (0,0)^T, \varepsilon_1 = (1,0)^T, \varepsilon_2 = (1,-1)^T, \varepsilon_3 = (2,-1)^T$. The coset vector $\varepsilon_i, i = 0, \cdots, 3$ of M defines a partition of $\mathbb{Z}^2$ as follows:

$$\mathbb{Z}^2 = \bigcup_{i=0}^{3} \left\{ Mk + \varepsilon_i, k \in \mathbb{Z}^2 \right\}.$$

The discrete data at the level j, v^j is defined on the grid $\Gamma^j = M^{-j}\mathbb{Z}^2$, the value v_k^j is then associated to the location $M^{-j}k$. We now define our bidimensional interpolatory subdivision scheme based on the data dependent subdivision operator which acts from the coarse grid Γ^{j-1} to the fine grid grid Γ^j. To this end, we will compute v^j at the different coset points on the fine grid Γ^j using the existing values v^{j-1} of the coarse grid Γ^{j-1}, as follows:

for the first coset vector $\varepsilon_0 = (0,0)^T$ we simply put $v^j_{Mk+\varepsilon_0} = v^{j-1}_k$, for the coset vectors ε_i, $i = 1, \cdots, 3$. the value $v^j_{Mk+\varepsilon_i}$, $i = 1, \cdots, 3$ is defined by affine interpolation of the values on the coarse grid. To do so, we define four different stencils on Γ^{j-1} as follows:

$$\begin{aligned}
V^{j,1}_k &= \{M^{-j+1}k, M^{-j+1}(k+e_1), M^{-j+1}(k+e_2)\},\\
V^{j,2}_k &= \{M^{-j+1}k, M^{-j+1}(k+e_2), M^{-j+1}(k+e_1+e_2)\},\\
W^{j,1}_k &= \{M^{-j+1}(k+e_1), M^{-j+1}(k+e_2), M^{-j+1}(k+e_1+e_2)\},\\
W^2_k &= \{M^{-j+1}k, M^{-j+1}(k+e_1), M^{-j+1}(k+e_1+e_2)\}.
\end{aligned}$$

We determine to which stencils each point of Γ^j belongs to, and we then define the prediction as its barycentric coordinates. Since we use an affine interpolant we have:

$$v^j_{Mk} = v^{j-1}_k \text{ and } v^j_{Mk+\varepsilon_1} = \frac{1}{2}v^{j-1}_k + \frac{1}{2}v^{j-1}_{k+e_1}. \tag{9}$$

To compute the rules for the coset point ε_2, V^1_k or V^2_k can be used leading respectively to:

$$\begin{aligned}
v^{j,1}_{Mk+\varepsilon_2} &= \frac{1}{4}v^{j-1}_{k+e_1} + \frac{1}{2}v^{j-1}_{k+e_2} + \frac{1}{4}v^{j-1}_k\\
v^{j,2}_{Mk+\varepsilon_2} &= \frac{1}{2}v^{j-1}_k + \frac{1}{4}v^{j-1}_{k+e_2} + \frac{1}{4}v^{j-1}_{k+e_1+e_2}.
\end{aligned} \tag{10}$$

When one considers the rules for the coset point ε_3, W^1_k or W^2_k can be used leading respectively to:

$$\begin{aligned}
v^{j,1}_{Mk+\varepsilon_3} &= \frac{1}{4}v^{j-1}_{k+e_2} + \frac{1}{4}v^{j-1}_{k+e_1+e_2} + \frac{1}{2}v^{j-1}_{k+e_1}\\
v^{j,2}_{Mk+\varepsilon_3} &= \frac{1}{4}v^{j-1}_k + \frac{1}{4}v^{j-1}_{k+e_1} + \frac{1}{2}v^{j-1}_{k+e_1+e_2}.
\end{aligned} \tag{11}$$

This nonlinear scheme is converging in L^∞ since we have the following result:

Proposition 2. *The prediction defined by (9), (10), (11) satisfies:*

$$\|\Delta^1 v^j_{M.+\varepsilon_i}\|_{\ell^\infty(\mathbb{Z}^d)} \leq \frac{3}{4}\|\Delta^1 v^{j-1}\|_{\ell^\infty(\mathbb{Z}^d)}$$

We do not detail the proof here but the result is obtained by computing the differences in the canonical directions at each coset points. If we then use Theorem 2 we can find the regularity of the corresponding limit function is C^s with $s < -\frac{\log(3/4)}{\log(4)} \approx 0.207$.

In the second pat of the section, we build an example of bidimensional subdivision scheme based on the same philosophy but this time using as dilation matrix the quincunx matrix defined by:

$$M = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix},$$

whose coset vectors are $\varepsilon_0 = (0,0)^T$ and $\varepsilon_1 = (0,1)^T$. Note that $a_{0,0} = 1$ and since the nonlinear subdivision operator reproduces the constants we have $\sum_i a_{Mi+\varepsilon} = 1$ for all coset vectors ε. To build the subdivision operator, we consider the subdivision rules based on interpolation by of first degree polynomials on the grid Γ^{j-1}. $v^j_{Mk+\epsilon_1}$ corresponds to a point inside the cell delimited by $M^{-j+1}\{k, k+e_1, k+e_2, k+e_1+e_2\}$. There are four potential stencils, leading in this case only to two subdivision rules:

$$\hat{v}^{j,1}_{Mk+\epsilon_1} = \tfrac{1}{2}(v^{j-1}_k + v^{j-1}_{k+e_1+e_2}) \tag{12}$$

$$\hat{v}^{j,2}_{Mk+\epsilon_1} = \tfrac{1}{2}(v^{j-1}_{k+e_1} + v^{j-1}_{k+e_2}) \tag{13}$$

Note also, that since the scheme is interpolatory we have the relation: $v^j_{Mk} = v^{j-1}_k$. Let us now prove a contraction property for the above scheme.

Proposition 3. *The nonlinear subdivision scheme defined by (12) and (13) satisfies the following property:*

1. *when* $k = Mk'$*:*

$$\begin{aligned} \|v^{j,1}_{M.+\varepsilon_1} - v^j_{M.}\|_{\ell^\infty(\mathbb{Z}^d)} &\leq \frac{1}{2}\|\Delta^1 v^{j-2}_{\cdot}\|_{\ell^\infty(\mathbb{Z}^d)} \\ \|v^{j,2}_{M.+\varepsilon_1} - v^j_{M.}\|_{\ell^\infty(\mathbb{Z}^d)} &\leq \|\Delta^1 v^{j-2}_{\cdot}\|_{\ell^\infty(\mathbb{Z}^d)} \end{aligned}$$

2. *when* $k = Mk' + \varepsilon_1$*, we can show that:*

$$\begin{aligned} \|v^{j,2}_{M.+\varepsilon_1} - v^j_{M.}\|_{\ell^\infty(\mathbb{Z}^d)} &\leq \frac{1}{2}\|\Delta^1 v^{j-2}_{\cdot}\|_{\ell^\infty(\mathbb{Z}^d)} \\ \|v^{j,1}_{M.+\varepsilon_1} - v^j_{M.}\|_{\ell^\infty(\mathbb{Z}^d)} &\leq \|\Delta^1 v^{j-2}_{\cdot}\|_{\ell^\infty(\mathbb{Z}^d)} \end{aligned}$$

The proof of this theorem is obtained computing all the potential differences. This theorem shows that the nonlinear subdivision scheme converges in L^∞ since $\rho_{1,\infty}(S) < 1$.

6. Convergence in Sobolev Spaces

In this section, we extend the result established in [13] on the convergence of linear subdivision scheme to our nonlinear setting. We will first recall the notion of convergence in Sobolev spaces in the linear case. Following [12] Theorem 4.2, when $\Phi_0(x) = \sum_{k\in\mathbb{Z}^d} a_k \Phi_0(Mx - k)$ is L^p-stable, the so-called "moment condition of order $k+1$ for a" is equivalent to the polynomial reproduction property of polynomial of total degree k for the subdivision scheme associated to a. In what follows, we will say that Φ_0 reproduces polynomial of total degree k. When the subdivision associated to a exactly reproduces polynomials, we will say that Φ_0 exactly reproduces polynomials. We then have the following definition for the convergence of subdivision schemes in Sobolev spaces in the linear case [13]:

Definition 5. *We say that $v^j = Sv^{j-1}$ converges in the Sobolev space $W_N^k(\mathbb{R}^d)$ if there exists a function v in $W_N^k(\mathbb{R}^d)$ satisfying:*

$$\lim_{j\to+\infty} \|v_j - v\|_{W_N^k(\mathbb{R}^d)} = 0$$

where v is in $W_N^k(\mathbb{R}^d)$, and $v_j = \sum_{k\in\mathbb{Z}^d} v_k^j \Phi_0(M^j x - k)$ for any Φ_0 reproducing polynomials of total degree k.

We are going to see that in the nonlinear case, to ensure the convergence we are obliged to make a restriction on the choice of Φ_0. We will first give some results when the matrix M is an isotropic dilation matrix, we will also emphasize a particular class of isotropic matrices, very useful in image processing.

6.1. Definitions and Preliminary Results

Definition 6. *We say that a matrix M is isotropic if it is similar to the diagonal matrix $diag(\sigma_1, \ldots, \sigma_d)$, i.e. there exists an invertible matrix Λ such that*

$$M = \Lambda^{-1} diag(\sigma_1, \ldots, \sigma_d)\Lambda,$$

with $|\sigma_1| = \ldots = |\sigma_d|$ being the eigenvalues of matrix M.

Evidently, for an isotropic matrix holds $|\sigma_1| = \ldots = |\sigma_d| = \sigma = m^{\frac{1}{d}}$. Moreover, for any given norm in $\mathbb{R}^d$, any integer n and any $v \in \mathbb{R}^d$ we have

$$\sigma^n \|u\| \lesssim \|M^n u\| \lesssim \sigma^n \|u\|.$$

A particular class of isotropic matrices is when there exists a set $\tilde{e}_1, \tilde{e}_2, \cdots, \tilde{e}_q$ such that:

$$M\tilde{e}_i = \lambda_i \tilde{e}_{\gamma(i)} \tag{14}$$

where γ is a permutation of $\{1, \cdots, q\}$. Such matrices are particular cases of isotropic matrices since $M^q = \lambda I$ where I is the identity matrix and where $\lambda = \prod_{i=1}^{d} \lambda_i$. For instance, when $d = 2$, the quincunx (resp. hexagonal) matrix satisfies $M^2 = 2I$ (resp. $M^2 = 4I$).

We establish the following property on joint spectral radii that will be useful when dealing with the convergence in Sobolev spaces.

Proposition 4. *Assume that S reproduces polynomials up to total degree N. Then,*

$$\rho_{p,n+1}(S) \geq \frac{1}{\|M\|_\infty} \rho_{p,n}(S),$$

for all $n = 0, \ldots, N$.

Remark: If M is an isotropic matrix and S reproduces polynomials up to total degree N, then

$$\rho_{p,n+1}(S) \geq \sigma^{-1} \rho_{p,n}(S),$$

for all $n = 0, \ldots, N$.

Proof: It is enough to prove

$$\rho_{p,1}(S) \geq \frac{1}{\|M\|_\infty} \rho_p(S).$$

According to the definition of spectral radius there exists $\rho > \rho_{p,1}(S)$ such that for any u^0

$$\|S_1(u^{j-1}) \ldots S_1(u^0) \nabla u\|_{\ell^p(\mathbb{Z}^d)} \lesssim \rho^j \|\nabla u\|_{(\ell^p(\mathbb{Z}^d)}.$$

Using the notation $\omega^j := S(u^{j-1}) \cdot \ldots \cdot S(u^0) u$ we obtain

$$\|\nabla \omega^j\|_{\ell^p(\mathbb{Z}^d)} \lesssim \rho^j \|\nabla u\|_{\ell^p(\mathbb{Z}^d)}.$$

Since

$$\omega_l^j = \sum_n A_{l,n}^j u_n,$$

where

$$A_{l,n}^j = \sum_{l_1,\ldots,l_{j-1}} a_{l-Ml_{j-1}}(u^{j-1}) a_{l_{j-1}-Ml_{j-2}} \cdot \ldots \cdot a_{l_1-Mn}(u^0).$$

We can write down the ℓ^p-norm as follows:

$$\|\omega^j\|_{\ell^p(\mathbb{Z}^d)}^p = \sum_{k\in\mathbb{Z}^d} \sum_{i=1}^{m^j} |\omega_{M^jk+\varepsilon_i^j}^j|^p,$$

where $\{\varepsilon_i^j\}_{i=1}^{m^j}$ are the representatives of cosets of M^j. First note that:

$$\|k-n\|_\infty \le \|k-n+M^{-j}\varepsilon_i^j\|_\infty + \|M^{-j}\varepsilon_i^j\|_\infty.$$

Note that $M^{-j}\varepsilon_i^j$ belongs to the unit square so that $\|M^{-j}\varepsilon_i^j\|_\infty \le K_1$. When $A_{M^jk+\varepsilon_i^j,n}^j \neq 0$, one can prove that there exists $K_2 > 0$ such that

$$\|k-n+M^{-j}\varepsilon_i^j\|_\infty \le K_2,$$

the proof being similar to that of Lemma 2 in [11]. From these inequalities it follows that if $A_{M^jk+\varepsilon_i^j,n}^j \neq 0$ there exists $K_3 > 0$ such that

$$\|k-n\|_\infty \le K_3,$$

that is, for a fixed k, the values of ω_l^j for $l \in \{M^jk+\varepsilon_i^j\}_{i=1}^{m^j}$ depend only on u_n with $n : \{\|k-n\|_\infty \le K_3\}$.

Let us now fix k and define $\tilde{u}$ such that

$$\tilde{u}_l = \begin{cases} u_l, & \text{if } \|k-l\|_\infty \le K_3; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tilde{\omega}^j := S(u^{j-1}) \cdot \ldots \cdot S(u^0)\tilde{u}$, then

$$\tilde{\omega}_l^j = \begin{cases} \omega_l^j, & \text{if } l \in \{M^j k + \varepsilon_i^j\}_{i=1}^{m^j}; \\ 0, & \text{if } \|k - M^{-j} l\|_\infty \geq K_4, \end{cases}$$

since if $A_{l,n}^j \neq 0$, then

$$\|k - M^{-j} l\|_\infty \leq \|k - n\|_\infty + \|n - M^{-j} l\|_\infty \leq K_3 + K_2 := K_4.$$

Moreover, from $\|k - M^{-j} l\|_\infty \leq K_4$, it follows that $\|M^j k - l\|_\infty \leq K_4 \|M^j\|_\infty$. Taking all this into account, we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \sum_{l \in \{M^j k + \varepsilon_i^j\}} |\omega_l^j|^p &= \sum_{k \in \mathbb{Z}^d} \sum_{l \in \{M^j k + \varepsilon_i^j\}} |\tilde{\omega}_l^j|^p \leq \sum_{k \in \mathbb{Z}^d} \sum_{\|M^j k - l\|_\infty \leq K_4 \|M^j\|_\infty} |\tilde{\omega}_l^j|^p \\ &\lesssim \|M\|_\infty^j \|\Delta^1 \tilde{\omega}_l^j\|_{\ell^p(\mathbb{Z}^d)} \lesssim (\|M\|_\infty \rho)^j \|\Delta^1 \tilde{u}\|_{\ell^p(\mathbb{Z}^d)}. \end{aligned}$$

That is, $\|\omega^j\|_{\ell^p(\mathbb{Z}^d)} \lesssim (\|M\|_\infty \rho)^j \|u\|_{\ell^p(\mathbb{Z}^d)}$, consequently $\rho_p(S) \lesssim \|M\|_\infty \rho$. Now, if $\rho \to \rho_{p,1}(S)$ we get $\rho_p(S) \leq \|M\|_\infty \rho_{p,1}(S)$. □

6.2. Convergence in Sobolev Spaces When M is Isotropic

First, Let us recall that the Sobolev norm on $W_N^p(\mathbb{R}^d)$ is defined by:

$$\|f\|_{W_N^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\mu| \leq N} \|D^\mu f\|_{L^p(\mathbb{R}^d)}. \tag{15}$$

If one considers a set $x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}$ such that $[x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}]\mathbb{Z}^n = \mathbb{Z}^d$, an equivalent norm is given by:

$$\|f\|_{W_N^p(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + \sum_{|\mu| \leq N} \|\tilde{D}^\mu f\|_{L^p(\mathbb{R}^d)}. \tag{16}$$

where $\tilde{D}^\mu = D_{x_{\mathbf{1}}}^{\mu_1} \cdots D_{x_{\mathbf{n}}}^{\mu_n}$.

We then enounce a convergence theorem for general isotropic matrix M:

Theorem 3. *Let S be a data dependent nonlinear subdivision scheme which exactly reproduces polynomials up to total degree $N-1$, then the subdivision scheme Sv^j converges in $W_N^p(\mathbb{R}^d)$, provided Φ_0 is compactly supported and exactly reproduces polynomials up to total degree $N-1$ and*

$$\rho_{p,N}(S) < m^{\frac{1}{p}-\frac{s}{d}} \text{ for some } s > N. \tag{17}$$

PROOF: Note that because of Proposition 4, the hypotheses of Theorem 3 imply that $\rho_{p,k}(S) < m^{\frac{1}{d}}\rho_{p,k+1}(S) < m^{\frac{1}{p}-\frac{s-1}{d}}$, which means that (17) is also true for $k < N$. Let us now show that v_j is a Cauchy sequence in L^p. To do so, let us define

$$q_j(x) = \sum_{l=1}^{d} \lambda_{j,l} x_l,$$

where $\Lambda = (\lambda_{j,l})$ is defined in (6). For a multi-index $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d$ let

$$q_\mu(x) = q_1^{\mu_1}(x) \ldots q_d^{\mu_d}(x).$$

Since Λ is invertible, the set $\{q_\mu : |\mu| = N\}$ forms a basis of the space of all polynomials of exact degree N, which proves that

$$\|D^\mu(v_{j+1} - v_j)\|_{L^p(\mathbb{R}^d)} \sim \|q_\mu(D)(v_{j+1} - v_j)\|_{L^p(\mathbb{R}^d)}$$

Now, we use the fact that, since M is isotropic, $q_\mu(D)(f(M^j x)) = \sigma^{j\mu}(q_\mu(D)f)(M^j x)$ where $\tilde{\sigma}^\mu = \prod_{i=1}^{d} \sigma_i^{\mu_i}$ ([5]). We can thus write:

$$q_\mu(D)(v_{j+1} - v_j) = q_\mu(D)\left(\sum_{l\in\mathbb{Z}^d} v_l^{j+1}\Phi_0(M^{j+1}x - l) - \sum_{l\in\mathbb{Z}^d} v_l^j \Phi_0(M^j x - l)\right).$$

We use now the scaling equation of Φ_0 to get

$$\begin{aligned} q_\mu(D)(v_{j+1}-v_j) &= q_\mu(D)\left(\sum_{l\in\mathbb{Z}^d} v_l^{j+1}\Phi_0(M^{j+1}x-l) - \sum_{l\in\mathbb{Z}^d}\sum_{r\in\mathbb{Z}^d} v_r^j g_{l-Mr}\Phi_0(M^{j+1}x-l)\right) \\ &= \sum_{l\in\mathbb{Z}^d}(v_l^{j+1} - \sum_{r\in\mathbb{Z}^d} v_r^j g_{k-Mr}) q_\mu(D)\left(\Phi_0(M^{j+1}x-l)\right) \\ &= \sum_{l\in\mathbb{Z}^d}\sum_{r\in\mathbb{Z}^d}(a_{l-Mr}(v^j)-g_{l-Mr})v_r^j\tilde{\sigma}^{\mu(j+1)}(q_\mu(D)\Phi_0)(M^{j+1}x-l). \end{aligned}$$

Since S and Φ_0 exactly reproduce polynomials up to total degree $N-1$, we have for $|\mu|\leq N-1$:

$$\sum_{r\in\mathbb{Z}^d}(a_{l-Mr}(v^j)-g_{l-Mr})r^\mu = 0.$$

Remark that $g_{l-Mr}=0$ for $\|l-Mr\|>\tilde{K}$ since Φ_0 is compactly supported. Since $\left\{\nabla^\nu\delta_{l-\beta}, |\nu|=N, r\in F(l)=\left\{\|l-Mr\|\leq\max(K,\tilde{K})\right\}, \beta\in\mathbb{Z}^d\right\}$ spans $(a_{l-Mr}(v^j)-g_{l-Mr})_{r\in F(l)}$, we deduce:

$$q_\mu(D)(v_{j+1}-v_j) = \sum_{l\in\mathbb{Z}^d}\sum_{r\in F(l)}\sum_{|\nu|=N} c_r^\nu(v^j)\nabla^\nu v_r^j\tilde{\sigma}^{\mu(j+1)}(q_\mu(D)\Phi_0)(M^{j+1}x-l),$$

Consequently,

$$\|q_\mu(D)(v_{j+1}-v_j)\|_{L^p(\mathbb{R}^d)} \lesssim \sigma^{(j+1)N}m^{-(j+1)/p}(\rho_{p,N}(S))^j\|\Delta^N v^0\|_{(\ell^p(\mathbb{Z}^d))^{q_N}}$$

Since $\rho_{p,N}(S)<m^{1/p-s/d}$, with $s>N$ we obtain

$$\|q_\mu(D)(v_{j+1}-v_j)\|_{L^p(\mathbb{R}^d)} \lesssim \sigma^{j(N-s)}\|\Delta^N v^0\|_{(\ell^p(\mathbb{Z}^d))^{q_N}}.$$

From this we deduce that $\|q_\mu(D)(v_{j+1}-v_j)\|_{L^p(\mathbb{R}^d)}$ tends to 0 with j. Making μ vary, we deduce the convergence in $W_N^p(\mathbb{R}^d)$ □

We now show that when the matrix M satisfies (14) and when Φ_0 is a box spline satisfying certain properties, the limit function is in $W_N^p(\mathbb{R}^d)$. Before

that, we need to recall the definition of box splines and some properties that we will use. Let us define a set of n vectors, not necessarily distinct:

$$X_n = \{x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}\} \subset \mathbb{Z}^d \setminus \{0\}.$$

We assume that d vectors of X_n are linearly independent. Let us rearrange the family X_n such that $X_d = \{x_{\mathbf{1}}, \cdots, x_{\mathbf{d}}\}$ are linearly independent. We denote by $[x_{\mathbf{1}}, \cdots, x_{\mathbf{d}}][0,1[^d$ the collection of linear combinations $\sum_{i=1}^{d} \lambda_i x_{\mathbf{i}}$ with $\lambda_i \in [0,1[$. Then, we define multivariate box splines as follows [4][16]:

$$\begin{aligned} \beta_0(x, X_d) &= \begin{cases} \frac{1}{|\det(x_{\mathbf{1}}, \cdots, x_{\mathbf{d}})|} \text{ if } x \in [x_{\mathbf{1}}, \cdots, x_{\mathbf{d}}][0,1[^d \\ 0 \text{ otherwise} \end{cases} \\ \beta_0(x, X_k) &= \int_0^1 \beta_0(x - t x_{\mathbf{k}}, X_{k-1}) dt, \quad n \geq k > d. \end{aligned} \tag{18}$$

One can check by induction that the support of $\beta_0(x, X_n)$ is $[x_{\mathbf{1}}, x_{\mathbf{2}}, \cdots, x_{\mathbf{n}}][0,1]^n$. The regularity of box splines is then given by the following theorem [16]:

Proposition 5. *$\beta_0(x, X_n)$ is r times continuously differentiable if all subsets of X_n obtained by deleting $r+1$ vectors spans $\mathbb{R}^d$.*

We recall a property on the directional derivatives of box splines, which we use in the convergence theorem that follows:

Proposition 6. *Assume that $X_n \setminus x_{\mathbf{r}}$ spans $\mathbb{R}^d$, and consider the following box spline function $s(x) = \sum_{k \in \mathbb{Z}^d} c_k \beta_0(x-k, X_n)$ then the directional derivative of s in the direction $x_{\mathbf{r}}$ reads:*

$$D_{x_{\mathbf{r}}} s(x) = \sum_{k \in \mathbb{Z}^d} \nabla_{x_{\mathbf{r}}} c_k \beta_0(x - k, X_n \setminus x_{\mathbf{r}}).$$

We will also need the property of polynomial reproduction which is [16]:

Proposition 7. *If $\beta_0(x, X_n)$ is r times continuously differentiable then, for any polynomial $c(x)$ of total degree $d \leq r+1$,*

$$p(x) = \sum_{i \in \mathbb{Z}^d} c(i)\beta_0(x-i, X_n) \tag{19}$$

is a polynomial with total degree d, with the same leading coefficients (i.e. the coefficients corresponding to degree d). Conversely, for any polynomial p, it satisfies (19) with c being a polynomial having the same leading coefficients as p.

Theorem 4. *Let S be a data dependent nonlinear subdivision scheme which reproduces polynomials up to total degree $N-1$ and assume that M satisfies relation (14), then the subdivision scheme Sv^j converges in $W_N^p(\mathbb{R}^d)$, if when $N \geq 2$, Φ_0 is a C^{N-2} box spline generated by $x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}$ satisfying $\Phi_0(x) = \sum_k g_k \Phi_0(Mx-k)$ and if $N = 1$ $\Phi_0(x) = \sum_k g_k \Phi_0(Mx-k)$ and $\sum_{k \in \mathbb{Z}^d} \Phi_0(x-k) = 1$ and if*

$$\rho_{p,N}(S) < m^{\frac{1}{p}-\frac{s}{d}} \text{ for some } s > N. \tag{20}$$

PROOF: We here prove the case $N \geq 2$, the case $N = 1$ can be proved similarly. First note that since $\Phi_0(x)$ is a C^{N-2} box spline, we can write for any polynomial p of total degree $N-1$ at most:

$$\begin{aligned} p(M^{-1}x) &= \sum_{i \in \mathbb{Z}^d} \tilde{p}(i)\Phi_0(M^{-1}x - i, X_n) \\ &= \sum_{q \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} g_{q-Mi}\tilde{p}(i)\Phi_0(x-q, X_n) \end{aligned}$$

Using Proposition 7 we get p and $\tilde{p}$ have the same leading coefficients, and that $\sum_{i \in \mathbb{Z}^d} g_{q-Mi}\tilde{p}(i)$ is a polynomial evaluated in $M^{-1}i$ having the same leading

coefficients as p. That is to say the subdivision scheme $(Sv^j)_q = \sum_{i\in\mathbb{Z}^d} g_{q-Mi}v_i^j$ reproduces polynomials up to degree $N-1$.

As already noticed, the joint spectral radius of difference operator is independent of the choice of the directions $x_{\mathbf{1}}, \cdots, x_{\mathbf{n}}$ that spans $\mathbb{Z}^d$. Furthermore, it is shown in [14], that the existence of a scaling equation for Φ_0 implies that the vectors $x_{\mathbf{i}}$, $\mathbf{i} = \mathbf{1}, \cdots, \mathbf{n}$ satisfy a relation of type (14). We consider such a set $\{x_{\mathbf{i}}\}_{\mathbf{i}=\mathbf{1},\cdots,\mathbf{n}}$ and then define $\Phi_0(x) = \beta_0(x, Y_N)$ the box spline associated to the set

$$Y_N := \left\{ \overbrace{x_{\mathbf{1}}, \cdots, x_{\mathbf{1}}}^{N}, \cdots, \overbrace{x_{\mathbf{n}}, \cdots, x_{\mathbf{n}}}^{N} \right\}.$$

which is C^{N-2} by definition. We then define the differential operator $\tilde{D}^{\mu}_{M^{-j}} := \tilde{D}^{\mu_1}_{M^{-j}x_{\mathbf{1}}} \cdots \tilde{D}^{\mu_n}_{M^{-j}x_{\mathbf{n}}}$. We will use the characterization (16) of Sobolev spaces therefore $\mu = (\mu_i)_{i=1,\cdots,n}$. For any $|\mu| \leq N$ we may write:

$$\begin{aligned}\tilde{D}^{\mu}_{M^{-j-1}}(v_{j+1}(x) - v_j(x)) &= \sum_{k\in\mathbb{Z}^d} v_k^{j+1}(\tilde{D}^{\mu}\beta_0)(M^{j+1}x - k, Y_N) \\ &\quad - \sum_{p\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d} v_i^j g_{p-Mi}(\tilde{D}^{\mu}\beta_0)(M^{j+1}x - p, Y_N),\end{aligned}$$

using the scaling property satisfied by β_0. Then, we get:

$$\begin{aligned}\tilde{D}^{\mu}_{M^{-j-1}}(v_{j+1}(x) - v_j(x)) &= \sum_{k\in\mathbb{Z}^d}\sum_{i\in\mathbb{Z}^d}(a_{k-Mi}(v^j) - g_{k-Mi})v_i^j(\tilde{D}^{\mu}\beta_0)(M^{j+1}x - k, Y_N) \\ &= \sum_{k\in\mathbb{Z}^d}\tilde{\nabla}^{\mu}\Big(\sum_{i\in\mathbb{Z}^d}(a_{k-Mi}(v^j) - g_{k-Mi})v_i^j\Big)\beta_0(M^{j+1}x - k, Y_N^{\mu})\end{aligned}$$

where Y_N^{μ} is obtained by removing μ_i vector $x_{\mathbf{i}}$, $i = 1, \cdots, d$ to Y_N and $\tilde{\nabla}^{\mu} = \left(\nabla^{\mu_i}_{x_{\mathbf{i}}}\right)_{i=1,\cdots,n}$. As both $a_{k-M.}(v^j)$ and $g_{k-M.}$ reproduce polynomials up

to total degree $N-1$, there exist a finite sequence $c_{k,p}$ such that:

$$\tilde{\nabla}^{\mu}(\sum_{i\in\mathbb{Z}^d}(a_{k-Mi}(v^j)-g_{k-Mi})v_i^j) \;=\; \sum_{p\in V(k)\bigcup\tilde{V}(k)}\sum_{|\nu|=|\mu|}c_{k,p}(\nu)\tilde{\nabla}^{\nu}v_p^j,$$

where $\tilde{V}(k)=\{i,\|k-Mi\|\leq\tilde{K}\}$, where $g_{k-Mi}=0$ if $\|k-Mi\|>\tilde{K}$. We finally deduce that:

$$\tilde{D}^{\mu}_{M^{-j-1}}(v_{j+1}(x)-v_j(x))=\sum_{k\in\mathbb{Z}^d}\sum_{p\in V(k)\bigcup\tilde{V}(k)}\sum_{|\nu|=|\mu|}c_{k,p}(\nu)\tilde{\nabla}^{\nu}v_p^j\beta_0(M^{j+1}x-k,Y_N^{\mu}).$$

From this, we conclude that:

$$\|\tilde{D}^{\mu}_{M^{-j-1}}(v_{j+1}(x)-v_j(x))\|_{L^p(\mathbb{R}^d)}\;\lesssim\;\rho_{p,|\mu|}(S)^j m^{-\frac{j+1}{p}}\|\tilde{\Delta}^{|\mu|}v_0^j\|_{(\ell^p(\mathbb{Z}^d))^{\tilde{q}_{|\mu|}}}.$$

Now, consider a sufficiently differentiable function f and remark that $D_{M^{-j-1}x_{\mathbf{1}}}f(x)=(Df)(x).M^{-j-1}x_{\mathbf{1}}$, where Df is the differential of the function f. We also note that $M^q=\lambda I$ which implies that $\lambda=\sigma^q$ and we then put $j+1=q\times\lfloor\frac{j+1}{q}\rfloor+r$ with $r<q$ and where $\lfloor.\rfloor$ denotes the integer part. From this we may write:

$$D_{M^{-j-1}x_{\mathbf{1}}}f(x)=\sigma^{-q\lfloor\frac{j+1}{q}\rfloor}(Df)(x).M^{-r}x_{\mathbf{1}}$$

and then

$$D_{M^{-j-1}x_{\mathbf{1}}}f(x)\sim\sigma^{-q\lfloor\frac{j+1}{q}\rfloor}(Df)(x).x_{r_j}$$

where r_j depends on j. Making the same reasoning for any order μ of differentiation and any direction $x_{\mathbf{i}}$, we get, in L^p:

$$\|(\tilde{D}^{\mu}_{M^{-j-1}}f)(x)\|_{L^p(\mathbb{R}^d)}\sim\sigma^{-q|\mu|\lfloor\frac{j+1}{q}\rfloor}\|(\tilde{D}^{\mu}f)(x)\|_{L^p(\mathbb{R}^d)}.$$

We may thus conclude that

$$\begin{aligned}\|\tilde{D}^{\mu}(v_{j+1}(x)-v_j(x))\|_{L^p(\mathbb{R}^d)} \;&\sim\; \|\tilde{D}^{\mu}_{M^{-j-1}}(v_{j+1}(x)-v_j(x))\|_{L^p(\mathbb{R}^d)}\sigma^{q|\mu|(\times\lfloor\frac{j+1}{q}\rfloor)|}\\ &\lesssim\; \rho_{p,|\mu|}(S)^j m^{-\frac{j+1}{p}}\sigma^{q|\mu|\lfloor\frac{j+1}{q}\rfloor}\|\tilde{\Delta}^{|\mu|}v^0\|_{(\ell^p(\mathbb{Z}^d))^{\tilde{q}_{|\mu|}}}.\end{aligned}$$

To state the above result, we have used the fact that the joint spectral radius is independent of the directions used for its computation. Since we have the hypothesis that $\rho_{p,|\mu|}(S) \leq m^{\frac{1}{p}-\frac{s}{d}}$ for $s > |\mu|$, we get that

$$\|\tilde{D}^{\mu}(v_{j+1}(x) - v_j(x))\|_{L^p(\mathbb{R}^d)} \lesssim \sigma^{(|\mu|-s)j} \|\tilde{\Delta}^{|\mu|} v^0\|_{(\ell^p(\mathbb{Z}^d))^{\tilde{q}_{|\mu|}}},$$

which tends to zero with j, and thus the limit function is in $W_N^p(\mathbb{R}^d)$ ■.
A comparison between Theorem 3 and 4 shows that when the subdivision scheme reproduces exactly polynomials, which is the case of interpolatory subdivision schemes, the convergence is ensured provided Φ_0 also exactly reproduces polynomials. When the subdivision scheme only reproduces polynomial the convergence is ensured provided that Φ_0 is a box spline. Note also that the condition on the joint spectral radius is the same. We are currently investigating illustrative examples which involve the adaptation of the local averaging subdivision scheme proposed in [6] to our non-separable context.

7. Conclusion

We have addressed the issue of the definition of nonlinear subdivision schemes associated to isotropic dilation matrix M. After the definition of the convergence concept of such operators, we have studied the convergence of these subdivision schemes in L^p and in Sobolev spaces. Based on the study of the joint spectral radius of these operators, we have exhibited sufficient conditions for the convergence of the proposed subdivision schemes. This study has also brought into light the importance of an appropriate choice of Φ_0 to define the limit function. In that context, box splines functions have shown to be a very interesting tool.

[1] F. Arandiga,A. Cohen,R. Donat, N. Dyn and B. Matei *Approximation of Piecewise Smooth Functions and Images by edge-adapted (ENO-EA) multiresolution Techniques*, ACHA 24, 225-250, 2008.

[2] Cavaretta, A.S., W. Dahmen and C.A. Michelli *Stationary Subdivision*, Memoirs of Amer. Math. Soc.,Volume 93.,1991.

[3] T. F. Chan, H.M. Zhou, *ENO-Wavelet Transforms for Piecewise Smooth Functions*, SIAM J. Numer. Anal. 40 (2002) 1369-1404.

[4] C. K. Chui, *Multivariate Splines*, SIAM, Philadelphia, Pennsylvania, 1988.

[5] D. R. Chen, R. Q. Jia, S.D. Riemenschneider, *Convergence of Vector Subdivision Schemes in Sobolev Spaces*, Appl. Comput. Harmon. Anal. 12 (2002) 128-149.

[6] A. Cohen, N. Dyn and B. Matei *Quasi-linear Subdivision Schemes with Applications to ENO Interpolation*, Appl. Comput. Harmon. Anal. 15 (2003) 89-116.

[7] N. Dyn, *Subdivision Schemes in computer aided geometric design*, Advances in Numerical AnalysisII., Subdivision algorithms and radial functions, W.A. Light (ed.), Oxford University Press, 1992, pp. 36-104.

[8] B. Han and R. Jia, R. Multivariate Refinement Equation and Convergence of Subdivision Schemes, SIAM J. Math. Anal. 29 (1998) 1177-1199.

[9] A. Harten, *Discrete Multiresolution Analysis and Generalized Wavelets*, J. Appl. Num. Math. 12 (1993) 153-193.

[10] A. Harten, B. Enquist, S. Osher and S. Chakravarthy, *Uniformly high order accurate essentially non-oscillatory schemes III*, J. Comput. Phys. 71 (1987) 231-303.

[11] T.N.T Goodman, C.A. Michelli, and J. D. Ward, *Spectral Radius formulas for subdivision operators*, Recent Advances in Wavelet Analysis",(L.L. Schumaker and G. Webb, Eds.), Academic Press, San Diego, 1994, pp. 335-360.

[12] R.Q. Jia, *Characterisation of Smoothness of Multivariate Refinable Functions in Sobolev Spaces*, Transactions of the AMrican Mathematical Society, 351 (1999), 4089-4112.

[13] R.Q. Jia, Q. Jiang, S.L. Lee, *Convergence of Cascade Algorithms in Sobolev Spaces and Integrals of Wavelets*, Numer. Math. 91 (2002) 453-473.

[14] S. Meignen, *Application of the Convergence of the Control Net of Box Splines to Scale-Space Filtering*, IEEE Transactions on Image Processing 16 (2007) 2842-2848.

[15] S. Harizanov and P. Oswald, *Stability of Nonlinear Subdivision and Multiscale Transforms*, Constr. Approx., to appear.

[16] H. Prautsch, W. Boehm, M. Paluszny, *Bézier and B-spline Techniques*, Springer, 2002.