An "algebraic" reconstruction of piecewise-smooth functions from integral measurements

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1. Introduction

This paper presents some results on a well-known problem in Algebraic Signal Sampling and in other areas of applied mathematics: reconstruction of piecewise-smooth functions from their integral measurements (like moments, Fourier coefficients, Radon transform, etc.). Our results concern reconstruction (from the moments) of signals in two specific classes: linear combinations of shifts of a given function, and "piecewise D-finite functions" which satisfy on each continuity interval a linear differential equation with polynomial coefficients.

Let us start with some general remarks and a conjecture. It is well known that the error in the best approximation of a C^k-function f by an N-th degree Fourier polynomial is of order $\frac{C}{N^k}$. The same holds for algebraic polynomial approximation and for other basic approximation tools. However, for f with singularities, in particular, with discontinuities, the error is much larger: its order is only $\frac{C}{\sqrt{N}}$. Considering the so-called Kolmogorow N-width of families of signals with moving discontinuities one can show that *any linear approximation method provides the same order of error, if we do not fix a priori the discontinuities' position* (see [7], Theorem 2.10). Another manifestation of the same problem is the *"Gibbs effect" - a relatively strong oscillation of the approximating function near the discontinuities*. Practically important signals usually do have discontinuities, so the above feature of linear representation methods presents a serious problem in signal reconstruction. In particular, it visibly appears near the edges of images compressed by JPEG, as well as in the noise and low resolution of the CT and MRI images.

Recent non-linear reconstruction methods, in particular, Compressed Sensing ([2, 3]) and Algebraic Sampling ([4, 12, 14, 6, 9]), address this problem in many cases. Both approaches appeal to an a priori information on the character of the signals to be reconstructed, assuming their "simplicity" in one or another sense. Compressed sensing assumes only a sparse representation in a certain (wavelets) basis, and thus it presents a rather general and "universal" approach. Algebraic Sampling usually requires more specific a priori assumptions on the structure of the signals, but it promises a better reconstruction accuracy. In fact, we believe that ultimately the Algebraic Sampling approach has a potential to reconstruct "simple signals with singularities" as good as smooth ones. In particular, the results of [5, 11, 8, 17, 14] strongly support (also apparently do not accurately formulate and prove) the following conjecture:

There is a non-linear algebraic procedure reconstructing any signal in a class of piecewise C^k-functions (of one or several variables) from its first N Fourier coefficients, with the overall accuracy of order $\frac{C}{N^k}$. This includes the discontinuities' positions, as well as the smooth pieces over the continuity domains.

At present there are many approaches available to a robust detection of discontinuities from Fourier data (see [8, 5, 11] and references therein). The remaining problem seems to be an accurate estimate of the accuracy of the solution of the nonlinear systems arising. Our results below can be considered, in particular, as a step in this direction. On the other hand, they have been motivated by the results in [4, 12, 14], and in [9, 6].

2. Linear combinations of shifts of a given function

Reconstruction of this class of signals from sampling has been described in [4, 12]. We study a rather similar problem of reconstruction from the moments. Our method is based on the following approach: we construct convolution kernels dual to the monomials. Applying these kernels, we get a Prony-type system of equations on the shifts and amplitudes.

Let us restate a general reconstruction problem, as it appears in our specific setting. We want to reconstruct signals of the form

$$F(x) = \sum_{i=1}^{N} \sum_{j,l} a_{i,j,l} f_i^{(l)}(x + x^j) \qquad (1)$$

where the f_i's are known functions of $x = (x_1, \ldots, x_d)$, and the form (1) of the signal is known a priori. The parameters $a_{i,j,l}$, $x^j = (x_1^j, \ldots, x_d^j)$ are to be found from a finite number of "measurements", i.e. of linear (usually integral) functionals like polynomial moments, Fourier moments, shifted kernels, evaluation over some grid and more.

In this paper we consider only linear combinations of shifts of one known function f (although the method of "convolution dual" can be extended to several shifted functions and their derivatives - see [16]). First we consider general integral "measurements" and then restrict

ourselves to the moments and Fourier coefficients. In what follows $x = (x_1, \dots, x_d), t = (t_1, \dots, t_d)$, j is a scalar index, while $k = (k_1, \dots, k_d)$, $i = (i_1, \dots, i_d)$ and $n = (n_1, \dots, n_d)$ are multi-indices. Partial ordering of multi-indices is given by $k \le k' \Leftrightarrow k_p \le k'_p,\ p = 1, \dots, d$. So we have

$$F(x) = \sum_{j=1}^{s} a_j f(x + x^j). \tag{2}$$

Let the measurements $\mu_k(F)$ be given by $\mu_k(F) = \int F(t)\varphi_k(t)dt$, for a certain (multi)-sequence of functions $\varphi_k(t),\ k \ge 0 = (0, \dots, 0)$.
Given f and $\varphi = \{\varphi_k(t)\},\ k \ge 0$ we now try to find certain "triangular" linear combinations

$$\psi_k(t) = \sum_{0 \le i \le k} C_{i,k}\varphi_i(t) \tag{3}$$

forming, in a sense, some "f-convolution dual" functions (similar to a bi-orthogonal set of function) with respect to the system $\varphi_k(t)$. More accurately, we require that

$$\int f(t + x)\psi_k(t) = \varphi_k(x). \tag{4}$$

We shall call a sequence $\psi = \{\psi_k(t)\}$ satisfying (3), (4) f - convolution dual to φ. Below we find convolution dual systems to the usual and exponential monomials.
We consider a general problem of finding convolution dual sequences to a given sequence of measurements as an important step in the reconstruction problem. Notice that it can be generalized by dropping the requirement of a specific representation (3): $\psi_k(t) = \sum_{i=0}^{k} C_{i,k}\varphi_i(t)$. Instead we can require only that $\int f(t)\psi_k(t)$ be expressible in terms of the measurements sequence μ_k. Also φ_k in (4) can be replaced by another a priori chosen sequence η_k. This problem leads, in particular, to certain functional equations, satisfied by polynomials and exponents (as well as exponential polynomials and some kinds of elliptic functions).
Now we have the following result:

Theorem 1. *Let a sequence $\psi = \psi_k(t)$ be f-convolution dual to φ. Define M_k by $M_k = \sum_{0 \le i \le k} C_{i,k}\mu_i$. Then the parameters a_j and x^j in (2) satisfy the following system of equations ("generalized Prony system"):*

$$\sum_{j=1}^{s} a_j \varphi_k(x^j) = M_k,\ \ k = 0, \dots. \tag{5}$$

Proof We have $M_k = \sum_{0 \le i \le k} C_{i,k}\mu_i = \int F(t) \sum_{0 \le i \le k} C_{i,k}\varphi_i(t)dt = \int F(t)\psi_k(t) = \sum_{j=1}^{s} a_j \int f(t + x^j)\psi_k(t)dt = \sum_{j=1}^{s} a_j \varphi_k(x^j)$.
In specific examples we can find the minimal number of equations in (5) necessary to uniquely reconstruct the parameters a_j and x^j in (2).

2.1 Reconstruction from moments

We are given a finite number of moments of a signal F as in (2) in the form

$$m_n = \int F(t)t^n dt. \tag{6}$$

So here $\varphi_n(x) = x_1^{n_1} \cdots x_d^{n_d}$ for each multi-index $n = (n_1, \dots, n_d)$. We look for the dual functions ψ_n satisfying the convolution equation

$$\int f(t + x)\psi_n(t)dt = x^n \tag{7}$$

for each multi-index n. To solve this equation we apply Fourier transform to both sides of (7). Assuming that $\hat{f}(\omega) \in C^\infty(\mathbb{R}^d), \hat{f}(0) \ne 0$ we find (see [16]) that there is a unique solution to (7) provided by

$$\varphi_n(x) = \sum_{k \le n} C_{n,k}x^k, \tag{8}$$

where

$$C_{n,k} = \frac{1}{(\sqrt{2\pi})^d} \binom{n}{k} (-i)^{n+k} \left[\frac{\partial^{n-k}}{\partial \omega^{n-k}} \Big|_{\omega=0} \frac{1}{\hat{f}(\omega)} \right].$$

This calculation is symbolic and works for more general cases. The actual calculation in our polynomial case is done using straightforward matrix calculations. We set the generalized polynomial moments as

$$M_n = \sum_{k \le n} C_{n,k}m_k \tag{9}$$

and obtain, as in Theorem 1, the following system of equations:

$$\sum_{j=1}^{s} a_j (x^j)^n = M_n,\ n \ge 0. \tag{10}$$

This system can be solved explicitly in a standard way (see, for example, [13, 4, 15]). In one-dimensional case it goes as follows (see [13]): from (10) we get that for $z = (z_1, \dots, z_d)$ the generalized moments generating function ($d = 1$ yet, notice that the formulas are still multi-dimensional)

$$I(z) = \sum_{n \in \mathbb{N}^d} M_n z^n = \sum_{j=1}^{s} a_j \prod_{l=1}^{d} \frac{1}{1 - x_l^j z_l} \tag{11}$$

is a rational function. Hence its Taylor coefficients satisfy linear recurrence relation, which can be reconstructed through a linear system with the Hankel-type matrix formed by an appropriate number of the moments M_n's. This is, essentially, a procedure of the diagonal Padé approximation for $I(z)$ (see [13]). The parameters a_j, x^j are finally reconstructed as the poles and the residues of $I(z)$. For several variables, although the formulas are the same as above, the generalization of the solution of the Prony system is more involved and should be addressed separately.
In one dimensional case with the derivatives $f^{(l)}$ included we have

$$F(x) = \sum_{j=1}^{s} \sum_{l=0}^{r} a_{j,l} f^{(l)}(x + x^j). \tag{12}$$

The corresponding moment-generating function in this case takes the form

$$I(z) = \sum_{j=1}^{s} \sum_{l=0}^{r} \sum_{q=0}^{l} \binom{l}{q} \frac{(-1)^{q+l} a_{j,l}/(x^j)^l}{(1 - x^j z)^{q+1}}. \tag{13}$$

which is still a rational function (d-dimensional case with derivatives is similar). We would like to stress that in this case the dual polynomials ψ_k are not changed and they are given as in (8). Therefore also the formula for the generalized moments M_n is the same as in (9).

2.2 Fourier case

In the same manner as in section 2.1 we now choose $\varphi_k(x) = e^{ikx}$. We get immediately $\psi_k(x) = \frac{1}{\hat{f}(k)} e^{-ikx}$. Indeed,

$$\int f(t+x)\psi_k(t)dt = \int f(t+x)\frac{1}{\hat{f}(k)}e^{ikt}dt =$$

$$\frac{\hat{f}(k)}{\hat{f}(k)}e^{-ikx} = \varphi_{-k}(x). \qquad (14)$$

Here the triangular system of equations (3) is actually not triangular any more but still since $\psi_k(x) = \frac{1}{\hat{f}(k)}\varphi_{-k}(x)$ we can express the generalized moments through the original ones via $M_k = \frac{1}{\hat{f}(k)}\mu_{-k}[F]$. Now exactly as before we can find a generalized Prony system in the form

$$\frac{1}{\hat{f}(k)}\mu_{-k}[F] = M_k = \sum_j a_j e^{-ikx_j} = \sum_j a_j \rho_j^k \qquad (15)$$

where $\rho_j = e^{-ix_j}$. In this case we get a rational exponential generating function and we can find its poles and residues on the unit complex circle as we did in the polynomial case.

2.3 Further extensions

The approach above can be extended in the following directions: 1. Reconstruction of signals built from several functions or with the addition of dilations also can be investigated (a perturbation approach where the dilations are approximately 1 is studied in [15]). 2. Further study of "convolution duality" can significantly extend the class of signals and measurements allowing for a closed - form signal reconstruction.

3. Reconstruction of piecewise D-finite functions from moments

Let $g : [a, b] \to \mathbb{R}$ consist of $\mathcal{K}+1$ "pieces" $g_0, \dots g_{\mathcal{K}}$ with $\mathcal{K} \geq 0$ jump points

$$a = \xi_0 < \xi_1 \dots < \xi_{\mathcal{K}} < \xi_{\mathcal{K}+1} = b$$

Furthermore, let g satisfy on each continuity interval some linear homogeneous differential equation with polynomial coefficients: $\mathfrak{D}\, g_n \equiv 0,\ n = 0, \dots, \mathcal{K}$ where

$$\mathfrak{D} = \sum_{j=0}^{N}\left(\sum_{i=0}^{k_j} a_{i,j}x^i\right)\frac{d^j}{dx^j} \quad (a_{ij} \in \mathbb{R}) \qquad (16)$$

Each g_n may be therefore written as a linear combination of functions $\{u_i\}_{i=1}^N$ which are a basis for the space $\mathcal{N}_{\mathfrak{D}} = \{f : \mathfrak{D}\, f \equiv 0\}$:

$$g_n(x) = \sum_{i=1}^{N} \alpha_{i,n} u_i(x), \quad n = 0, 1, \dots, \mathcal{K} \qquad (17)$$

We term such functions g "piecewise D-finite". Many real-world signals may be represented as piecewise D-finite functions, in particular: polynomials, trigonometric functions, algebraic functions.
The sequence $\{m_k = m_k(g)\}$ is given by the usual moments

$$m_k(g) = \int_a^b x^k g(x)dx$$

We subsequently formulate the following

Piecewise D-finite Reconstruction Problem. *Given $N, \{k_i\}, \mathcal{K}, a, b$ and the moment sequence $\{m_k\}$ of a piecewise D-finite function g, reconstruct all the parameters $\{a_{i,j}\}, \{\xi_i\}, \{\alpha_{i,n}\}$.*

Below we state some results (see [1] for detailed proofs) which provide *explicit algebraic connections* between the above parameters and the measurements $\{m_k\}$.
The first two theorems assume a single continuity interval (compare [10]).

Theorem 2. *Let $\mathcal{K} = 0$ and $\mathfrak{D}\, g \equiv 0$ with $\mathfrak{D}$ given by* (16)*. Then the moment sequence $\{m_k(g)\}$ satisfies a linear recurrence relation*

$$\left((\mathrm{E} - a\,\mathrm{I})^N(\mathrm{E} - b\,\mathrm{I})^N \cdot \sum_{j=0}^{N}\sum_{i=0}^{k_j} a_{i,j}\Pi^{(i,j)}(k, \mathrm{E})\right)m_k = 0 \qquad (18)$$

where E *is the discrete forward shift operator and $\Pi^{(i,j)}(k, \mathrm{E})$ are monomials in* E *whose coefficients are polynomials in k: $\Pi^{(i,j)}(k, \mathrm{E}) = (-1)^j \frac{(i+k)!}{(i+k-j)!}\,\mathrm{E}^{i-j}$.*

Theorem 3. *Denote*

$$\mathcal{E}(\mathrm{E}) \stackrel{def}{=} (\mathrm{E} - a\,\mathrm{I})^N(\mathrm{E} - b\,\mathrm{I})^N, \qquad v_k^{(i,j)} \stackrel{def}{=} \big(\mathcal{E}(\mathrm{E}) \cdot \Pi^{(i,j)}(k, \mathrm{E})\big)m_k,$$

$$h_j(z) \stackrel{def}{=} \sum_{k=0}^{\infty} v_k^{(0,j)} z^k, \qquad G_j(x) \stackrel{def}{=} \mathcal{E}(x)\frac{d^j}{dx^j}g(x)$$

Assume the conditions of Theorem 2. Then

(1) The vector of the coefficients $\mathbf{a} = (a_{i,j})$ satisfies a linear homogeneous system

$$H\mathbf{a} = \begin{pmatrix} v_0^{(0,0)} & v_0^{(1,0)} & \dots & v_0^{(k_N,N)} \\ v_1^{(0,0)} & v_1^{(1,0)} & \dots & v_1^{(k_N,N)} \\ \vdots & \vdots & \vdots & \vdots \\ v_{\widehat{M}}^{(0,0)} & v_{\widehat{M}}^{(1,0)} & \dots & v_{\widehat{M}}^{(k_N,N)} \end{pmatrix} \begin{pmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{k_N,N} \end{pmatrix} = 0 \qquad (19)$$

for all $\widehat{M} \in \mathbb{N}$.

(2) $v_k^{(i,j)} = m_{i+k}\,(G_j(x))$. Consequently, $h_j(z)$ is the moment generating function of $G_j(x)$.

(3) Denote $p_j(x) = \sum_{i=0}^{k_j} a_{i,j}x^i$. Then the functions $\Phi = \{1, h_0(z), \dots h_N(z)\}$ are polynomially dependent: $\sum_{j=0}^{N} h_j(z)\big(z^{\max k_j}p_j(z^{-1})\big) = Q(z)$ where $Q(z)$ is a polynomial with $\deg Q < \max k_j$. The system of polynomials $\{z^{\max k_j}p_j(z^{-1})\}$ is called the Padé-Hermite form for Φ.

To handle the piecewise case, we represent the jump discontinuities by the step function $\mathcal{H}(x) \stackrel{\text{def}}{=} \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$ and write g as a distribution

$$g(x) = \widetilde{g_0} + \sum_{n=1}^{\mathcal{K}} \widetilde{g_n}(x)\mathcal{H}(x - \xi_n) \tag{20}$$

Theorem 4. *Let $\mathcal{K} > 0$ and let g be as in* (20) *with operator $\mathfrak{D}$ annihilating every piece $\widetilde{g_n}$. Then the operator*

$$\widehat{\mathfrak{D}} \stackrel{def}{=} \left\{ \prod_{n=1}^{\mathcal{K}} (x - \xi_i)^N \, \mathrm{I} \right\} \cdot \mathfrak{D} \tag{21}$$

annihilates the entire g as a distribution. Consequently, conclusions of Theorems 2 and 3 hold with $\mathfrak{D}$ replaced by $\widehat{\mathfrak{D}}$ as in (21).

Proposition 5. *Let $\mathcal{K} \geq 0$ and $\{u_i\}_{i=1}^N$ be a basis for the space $\mathcal{N}_{\mathfrak{D}}$, where $\mathfrak{D}$ annihilates every piece of g. Assume* (17) *and denote $c_{i,k}^n = \int_{\xi_n}^{\xi_{n+1}} x^k u_i(x)$ for $n = 0, \ldots, \mathcal{K}$. A straightforward computation gives $\forall \widetilde{M} \in \mathbb{N}$:*

$$\begin{pmatrix} c_{1,0}^0 & \cdots & c_{N,0}^0 & \cdots & c_{N,0}^{\mathcal{K}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{1,\widetilde{M}}^0 & \cdots & c_{N,\widetilde{M}}^0 & \cdots & c_{N,\widetilde{M}}^{\mathcal{K}} \end{pmatrix} \begin{pmatrix} \alpha_{1,0} \\ \vdots \\ \alpha_{N,0} \\ \vdots \\ \alpha_{N,\mathcal{K}} \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_{\widetilde{M}} \end{pmatrix} \tag{22}$$

The above results can be combined as follows to provide a solution of the Reconstruction Problem:

(a) Let $N, \{k_i\}, \mathcal{K}, a, b$ and $\{m_k(g)\}$ be given. If $\mathcal{K} > 0$, replace $\mathfrak{D}$ (still unknown) with $\widehat{\mathfrak{D}}$ according to (21).

(b) Build the matrix H as in (19). Solve $H\mathbf{a} = 0$ and obtain the operator $\mathfrak{D}^* = \mathfrak{D}_{\mathbf{a}}$ which annihilates g.

(c) If $\mathcal{K} > 0$, factor out all the common roots of the polynomial coefficients of $\mathfrak{D}^*$ with multiplicity N. These are the locations of the jump points $\{\xi_n\}$. The remaining part is the operator $\mathfrak{D}^\dagger$ which annihilates every g_n.

(d) By now $\mathfrak{D}^\dagger$ and $\{\xi_n\}$ are known. So compute the basis for $\mathcal{N}_{\mathfrak{D}^\dagger}$ and solve (22).

The constants $\widehat{M}$ and $\widetilde{M}$ determine the minimal required size of the corresponding linear systems (19) and (22) in order for all the solutions of these systems to be also solutions of the original problem. It can be shown that:

1. There exists no uniform bound on $\widehat{M}$ without any additional information on the nature of the solutions. Explicit bounds may be obtained for simple function classes such as piecewise polynomials of bounded degrees or real algebraic functions.

2. For every specific $\mathfrak{D}$, an explicit bound $\widetilde{M} = \widetilde{M}(\mathfrak{D})$ may be computed for the system (22).

The above algorithm has been tested on exact reconstruction of piecewise polynomials, piecewise sinusoids and rational functions.

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