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Walid Al-Kawarit, Francois Couchot

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COMPARISON OF SOME PURITIES, FLATNESSES AND INJECTIVITIES

WALID AL-KAWARIT AND FRANÇOIS COUCHOT

ABSTRACT. In this paper, we compare (n, m) -purities for different pairs of positive integers (n, m) . When R is a commutative ring, these purities are not equivalent if R doesn't satisfy the following property: there exists a positive integer p such that, for each maximal ideal P , every finitely generated ideal of R_P is p -generated. When this property holds, then the (n, m) -purity and the (n, m') -purity are equivalent if m and m' are integers $\geq np$. These results are obtained by a generalization of Warfield's methods. There are also some interesting results when R is a semiperfect strongly π -regular ring. We also compare (n, m) -flatnesses and (n, m) -injectivities for different pairs of positive integers (n, m) . In particular, if R is right perfect and right self $(\aleph_0, 1)$ -injective, then each $(1, 1)$ -flat right R -module is projective. In several cases, for each positive integer p , all (n, p) -flatnesses are equivalent. But there are some examples where the $(1, p)$ -flatness is not equivalent to the $(1, p + 1)$ -flatness.

All rings in this paper are associative with unity, and all modules are unital. Let n and m be two positive integers. A right R -module M is said to be (n, m) -presented if it is the factor module of a free right module of rank n modulo a m -generated submodule. A short exact sequence (Σ) of left R -modules is called (n, m) -pure if it remains exact when tensoring it with any (n, m) -presented right module. We say that (Σ) is (\aleph_0, m) -pure exact (respectively (n, \aleph_0) -pure exact) if, for each positive integer n (respectively m) (Σ) is (n, m) -pure exact. Let us observe that the $(1, 1)$ -pure exact sequences are exactly the RD-exact sequences (see [11]) and the (\aleph_0, \aleph_0) -exact sequences are the pure-exact sequences in the Cohn's sense. Similar results as in the classical theories of purity hold, with similar proofs. In particular, a left R -module is (n, m) -pure-projective if and only if it is a summand of a direct sum of (m, n) -presented left modules, and each left R -module has a (n, m) -pure-injective hull which is unique up to an isomorphism.

In this paper, we compare (n, m) -purities for different pairs of positive integers (n, m) . When R is commutative, we shall see that some of these purities are equivalent only if R satisfies the following property: there exists a positive integer p such that, for each maximal ideal P , every finitely generated ideal of R_P is p -generated. When this property holds, then (n, m) -purity and (n, m') -purity are equivalent if m and m' are integers $\geq np$. These results are obtained by using the following: if R is a local commutative ring for which there exists a $(p + 1)$ -generated ideal, where p is a positive integer, then, for each positive integer n , for each integer m , $n(p - 1) + 1 \leq m \leq np + 1$, there exists a (n, m) -presented R -module whose

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endomorphism ring is local. It is a generalization of the Warfield's construction of indecomposable finitely presented modules when R is not a valuation ring.

When R is semiperfect and strongly π -regular, we show that there exists an integer $m > 0$ such that, for any integer $n > 0$, each (n, m) -pure exact sequence of right modules is (n, \aleph_0) -pure exact if and only if there exists an integer $p > 0$ such that every finitely generated left ideal is p -generated.

As in [16] we define (n, m) -flat modules and (n, m) -injective modules. We also compare (n, m) -flatnesses and (n, m) -injectivities for different pairs of positive integers (n, m) . In particular, if R is a right perfect ring which is right self $(\aleph_0, 1)$ -injective, then each $(1, 1)$ -flat right R -module is projective. For many classes of rings, for each positive integer p , we show that the $(1, p)$ -flatness implies the (\aleph_0, p) -flatness, but we have no general result. If R is a local commutative ring with a non finitely generated maximal ideal P satisfying $P^2 = 0$, then for each positive integer p , there exists an R -module which is (\aleph_0, p) -flat (resp, (\aleph_0, p) -injective) and which is not $(1, p+1)$ -flat (resp, $(1, p+1)$ -injective).

As in [16] we define left (n, m) -coherent rings. When R is a commutative locally perfect ring which is $(1, 1)$ -coherent and self $(1, 1)$ -injective, we show that R is an IF-ring, each $(1, 1)$ -flat R -module is flat and each $(1, 1)$ -injective R -module is FP-injective. For other classes of rings, for each positive integer p , we show that the left $(1, p)$ -coherence implies the left (\aleph_0, p) -coherence, but we have no general result. If $R = V[[X]]$, the power series ring over a valuation domain V whose order group is not isomorphic to \mathbb{R} , then R is a $(\aleph_0, 1)$ -coherent ring which is not $(1, 2)$ -coherent.

1. (n, m) -PURE EXACT SEQUENCES

By using a standard technique, (see for instance [4, Chapter I, Section 8]), we can prove the following theorem, and similar results hold if we replace n or m with \aleph_0 .

Theorem 1.1. *Assume that R is an algebra over a commutative ring S and E is an injective S -cogenerator. Then, for each exact sequence (Σ) of left R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the following conditions are equivalent:*

- (1) (Σ) is (n, m) -pure;
- (2) for each (m, n) -presented left module G the sequence $\text{Hom}_R(G, (\Sigma))$ is exact;
- (3) every system of n equations over A

$$\sum_{j=1}^m r_{i,j} x_j = a_i \in A \quad (i = 1, \dots, n)$$

with coefficients $r_{i,j} \in R$ and unknowns x_1, \dots, x_m has a solution in A whenever it is solvable in B ;

- (4) the exact sequence of right R -modules $\text{Hom}_S((\Sigma), E)$ is (m, n) -pure.

Propositions 1.2 and 1.3 can be deduced from [13, Theorem 1].

A left R -module G is called (n, m) -pure-projective if for each (n, m) -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the sequence

$$0 \rightarrow \text{Hom}_R(G, A) \rightarrow \text{Hom}_R(G, B) \rightarrow \text{Hom}_R(G, C) \rightarrow 0$$

is exact. Similar definitions can be given by replacing n or m by \aleph_0 . From Theorem 1.1 and by using standard technique (see for instance [4, Chapter VI, Section 12]) we get the following proposition in which n or m can be replaced by \aleph_0 :

Proposition 1.2. *Let G be a left R -module. Then the following assertions hold:*

- (1) *there exists a (n, m) -pure exact sequence of left modules*

$$0 \rightarrow K \rightarrow F \rightarrow G \rightarrow 0$$

where F is a direct sum of (m, n) -presented left modules;

- (2) *G is (n, m) -pure projective if and only if it is a summand of a direct sum of (m, n) -presented left modules.*

A left R -module G is called (n, m) -pure-injective if for each (n, m) -pure exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the sequence

$$0 \rightarrow \text{Hom}_R(C, G) \rightarrow \text{Hom}_R(B, G) \rightarrow \text{Hom}_R(A, G) \rightarrow 0$$

is exact.

If M is a left module we put $M^\sharp = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. Thus M^\sharp is a right module. It is the *character module* of M .

If A is a submodule of a left R -module B , we say that B is a (n, m) -pure essential extension of A if A is a (n, m) -pure submodule of B and for each nonzero submodule K of B such that $A \cap K = 0$, $(A + K)/K$ is not a (n, m) -pure submodule of B/K . If, in addition, B is (n, m) -pure injective, we say that B is a (n, m) -pure injective hull of A . In these above definitions and in the following proposition n or m can be replaced by \aleph_0 .

Proposition 1.3. *The following assertions hold:*

- (1) *each left R -module is a (n, m) -pure submodule of a (n, m) -pure injective left module;*
(2) *each left R -module has a (n, m) -pure injective hull which is unique up to an isomorphism.*

Proof. (1). Let M be a left R -module. By Proposition 1.2 there exists a (m, n) -pure exact sequence of right R -modules $0 \rightarrow K \rightarrow F \rightarrow M^\sharp \rightarrow 0$ where F is a direct sum of (n, m) -presented right modules. From Theorem 1.1 it follows that $(M^\sharp)^\sharp$ is a (n, m) -pure submodule of F^\sharp . By [3, Corollary 1.30] M is isomorphic to a pure submodule of $(M^\sharp)^\sharp$. So, M is isomorphic to a (n, m) -pure submodule of F^\sharp . By using the canonical isomorphism $(F \otimes_R -)^\sharp \cong \text{Hom}_R(-, F^\sharp)$ we get that F^\sharp is (n, m) -pure injective since F is a direct sum of (n, m) -presented modules.

(2). Since (1) holds and every direct limit of (n, m) -pure exact sequences is (n, m) -pure exact too, we can adapt the method of Warfield's proof of existence of pure-injective hull to show (2)(see [11, Proposition 6]). We can also use [10, Proposition 4.5]. \square

Proposition 1.4. *Let R be a commutative ring and let (Σ) be a short exact sequence of R -modules. Then (Σ) is (n, m) -pure if and only if, for each maximal ideal P $(\Sigma)_P$ is (n, m) -pure.*

Proof. Assume that (Σ) is (n, m) -pure and let M be a (n, m) -presented R_P -module where P is a maximal ideal. There exists a (n, m) -presented R -module M' such that $M \cong M'_P$ and $M \otimes_{R_P} (\Sigma)_P \cong (M' \otimes_R (\Sigma))_P$. We deduce that $(\Sigma)_P$ is (n, m) -pure.

Conversely, suppose that (Σ) is the sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$. Let M be a (n, m) -presented R -module. Then, for each maximal ideal P , $(\Sigma)_P$ is (n, m) -pure over R since $M \otimes_R (\Sigma)_P \cong M_P \otimes_{R_P} (\Sigma)_P$. On the other hand, since $M \otimes_R (\prod_{P \in \text{Max } R} (\Sigma)_P) \cong (\prod_{P \in \text{Max } R} M \otimes_R (\Sigma)_P)$, $\prod_{P \in \text{Max } R} A_P$ is a (n, m) -pure

submodule of $\prod_{P \in \text{Max } R} B_P$. By [2, Lemme 1.3] A is isomorphic to a pure submodule of $\prod_{P \in \text{Max } R} A_P$. We successively deduce that A is a (n, m) -pure submodule of $\prod_{P \in \text{Max } R} B_P$ and B . \square

2. COMPARISON OF PURITIES OVER A SEMIPERFECT RING

In this section we shall compare (n, m) -purities for different pairs of integers (n, m) . In [7] some various purities are also compared. In particular some necessary and sufficient conditions on a ring R are given for the $(1, 1)$ -purity to be equivalent to the (\aleph_0, \aleph_0) -purity.

The following lemma is due to Lawrence Levy, see [15, Lemma 1.3]. If M be a finitely generated left (or right) R -module, we denote by $\text{gen } M$ its minimal number of generators.

Lemma 2.1. *Let R be a ring. Assume there exists a positive integer p such that $\text{gen } A \leq p$ for each finitely generated left ideal A of R . Then $\text{gen } N \leq p \times \text{gen } M$, if N is a finitely generated submodule of a finitely generated left R -module M .*

From this lemma and Theorem 1.1 we deduce the following:

Proposition 2.2. *Let R be a ring. Assume there exists a positive integer p such that $\text{gen } A \leq p$ for each finitely generated left ideal A of R . Then, for each positive integer n :*

- (1) *each (n, np) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (2) *each (np, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact.*

Corollary 2.3. *Let R be a left Artinian ring. Then there exists a positive integer p such that, for each positive integer n :*

- (1) *each (n, np) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (2) *each (np, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact.*

Proof. Each finitely generated left R -module M has a finite length denoted by $\text{length } M$, and $\text{gen } M \leq \text{length } M$. So, for each left ideal A we have $\text{gen } A \leq \text{length } R$. We choose $p = \sup\{\text{gen } A \mid A \text{ left ideal of } R\}$ and we apply the previous proposition. \square

Let R be a ring and J its Jacobson radical. Recall that R is *semiperfect* if R/J is semisimple and idempotents lift modulo J .

Theorem 2.4. *Let R be semiperfect ring. Assume that each indecomposable finitely presented cyclic left R -module has a local endomorphism ring. The following conditions are equivalent:*

- (1) *there exists an integer $p > 0$ such that, for each integer $n > 0$, each (n, np) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (2) *there exists an integer $p > 0$ such that, for each integer $n > 0$, each (np, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact;*
- (3) *there exists an integer $q > 0$ such that each $(1, q)$ -pure exact sequence of right modules is $(1, \aleph_0)$ -pure exact;*
- (4) *there exists an integer $q > 0$ such that each $(q, 1)$ -pure exact sequence of left modules is $(\aleph_0, 1)$ -pure exact;*
- (5) *there exists an integer $q > 0$ such that each indecomposable finitely presented cyclic left module is q -related;*

- (6) *there exists an integer $p > 0$ such that $\text{gen } A \leq p$ for each finitely generated left ideal A of R .*

Moreover, if each indecomposable finitely presented left R -module has a local endomorphism ring, these conditions are equivalent to the following:

- (7) *there exist two positive integers n, m such that each (n, m) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
 (8) *there exist two positive integers n, m such that each (m, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact;*

Proof. By Proposition 2.2 (6) \Rightarrow (1). By Theorem 1.1 (1) \Leftrightarrow (2), (3) \Leftrightarrow (4) and (7) \Leftrightarrow (8). It is obvious that (2) \Rightarrow (4) and (2) \Rightarrow (7).

(4) \Rightarrow (5). Let C be an indecomposable finitely presented cyclic left module. Then C is $(q, 1)$ -pure-projective. So, C is a direct summand of a finite direct sum of $(1, q)$ -presented left modules. Since R is semiperfect, we may assume that these $(1, q)$ -presented left modules are indecomposable. So, by Krull-Schmidt theorem C is $(1, q)$ -presented.

(5) \Rightarrow (6). Let A be a finitely generated left ideal. Then $R/A = \bigoplus_{i=1}^t R/A_i$ where, for each $i = 1, \dots, t$, A_i is a left ideal and R/A_i is indecomposable. We have the following commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \bigoplus_{i=1}^t A_i & \rightarrow & R^t & \rightarrow & \bigoplus_{i=1}^t R/A_i & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A & \rightarrow & R & \rightarrow & R/A & \rightarrow & 0 \end{array}$$

Since the right vertical map is an isomorphism, we deduce from the snake lemma that the other two vertical homomorphisms have isomorphic cokernels. It follows that $\text{gen } A \leq tq + 1$ because $\text{gen } A_i \leq q$ by (5). On the other hand, let P be a projective cover of R/A . Then P is isomorphic to a direct summand of R . We know that the left module R is a finite direct sum of indecomposable projective modules. Let s the number of these indecomposable summands. It is easy to show that $t \leq s$. So, if $p = sq + 1$, then $\text{gen } A \leq p$.

(8) \Rightarrow (5). Let C be an indecomposable finitely presented cyclic left module. Then C is (m, n) -pure-projective. So, C is a direct summand of a finite direct sum of (n, m) -presented left modules. Since R is semiperfect, we may assume that these (n, m) -presented left modules are indecomposable. So, by the Krull-Schmidt theorem C is $(1, m)$ -presented. \square

A ring R is said to be *strongly π -regular* if, for each $r \in R$, there exist $s \in R$ and an integer $q \geq 1$ such that $r^q = r^{q+1}s$. By [3, Theorem 3.16] each strongly π -regular R satisfies the following condition: for each $r \in R$, there exist $s \in R$ and an integer $q \geq 1$ such that $r^q = sr^{q+1}$. Recall that a left R -module M is said to be *Fitting* if for each endomorphism f of M there exists a positive integer t such that $M = \ker f^t \oplus f^t(M)$.

Lemma 2.5. *Let R be a strongly π -regular semiperfect ring. Then:*

- (1) *each finitely presented cyclic left (or right) R -module is Fitting;*
 (2) *each indecomposable finitely presented cyclic left (or right) R -module has a local endomorphism ring.*

Proof. In [3, Lemma 3.21] it is proven that every finitely presented R -module is a Fitting module if R is a semiperfect ring with $M_n(R)$ strongly π -regular for all n . We do a similar proof to show (1).

(2). By [3, Lemma 2.21] each indecomposable Fitting module has a local endomorphism ring. \square

Corollary 2.6. *Let R be a strongly π -regular semiperfect ring. The following conditions are equivalent:*

- (1) *there exists an integer $p > 0$ such that, for each integer $n > 0$, each (n, np) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (2) *there exists an integer $p > 0$ such that, for each integer $n > 0$, each (np, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact;*
- (3) *there exists an integer $q > 0$ such that each $(1, q)$ -pure exact sequence of right modules is $(1, \aleph_0)$ -pure exact;*
- (4) *there exists an integer $q > 0$ such that each $(q, 1)$ -pure exact sequence of left modules is $(\aleph_0, 1)$ -pure exact;*
- (5) *there exists an integer $q > 0$ such that each indecomposable finitely presented cyclic left module is q -related;*
- (6) *there exists an integer $p > 0$ such that $\text{gen } A \leq p$ for each finitely generated left ideal A of R .*

Moreover, if $M_n(R)$ is strongly π -regular for all $n > 0$, these conditions are equivalent to the following:

- (7) *there exist two positive integers n, m such that each (n, m) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (8) *there exist two positive integers n, m such that each (m, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact;*

Proof. By Lemma 2.5 each indecomposable finitely presented cyclic left R -module has a local endomorphism ring. If $M_n(R)$ is strongly π -regular for all n , then by [3, Lemmas 3.21 and 2.21] each indecomposable finitely presented left R -module has a local endomorphism ring. So, we apply Theorem 2.4. \square

Recall that a ring R is *right perfect* if each flat right R -module is projective.

Corollary 2.7. *Let R be a right perfect ring. The following conditions are equivalent:*

- (1) *there exists an integer $p > 0$ such that, for each integer $n > 0$, each (n, np) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (2) *there exists an integer $p > 0$ such that, for each integer $n > 0$, each (np, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact;*
- (3) *there exists an integer $q > 0$ such that each $(1, q)$ -pure exact sequence of right modules is $(1, \aleph_0)$ -pure exact;*
- (4) *there exist two positive integers n, m such that each (n, m) -pure exact sequence of right modules is (n, \aleph_0) -pure exact;*
- (5) *there exist two positive integers n, m such that each (m, n) -pure exact sequence of left modules is (\aleph_0, n) -pure exact;*
- (6) *there exists an integer $p > 0$ such that $\text{gen } A \leq p$ for each finitely generated left ideal A of R .*

Proof. For all $n > 0$, $M_n(R)$ is right perfect. Since each right perfect ring satisfies the descending chain condition on finitely generated left ideals, then $M_n(R)$ is strongly π -regular for all $n > 0$. We apply Corollary 2.6. \square

3. COMPARISON OF PURITIES OVER A COMMUTATIVE RING

In the sequel of this section R is a commutative local ring, except in Theorem 3.5. We denote respectively by P and k its maximal ideal and its residue field

Let M be a finitely presented R -module. Recall that $\text{gen } M = \dim_k M/PM$. Let F_0 be a free R -module whose rank is $\text{gen } M$ and let $\phi : F_0 \rightarrow M$ be an epimorphism. Then $\ker \phi \subseteq PF_0$. We put $\text{rel } M = \text{gen } \ker \phi$. Let F_1 be a free R -module whose rank is $\text{rel } M$ and let $f : F_1 \rightarrow F_0$ be a homomorphism such that $\text{im } f = \ker \phi$. Then $\ker f \subseteq PF_1$. For any R -module N , we put $N^* = \text{Hom}_R(N, R)$. Let $f^* : F_0^* \rightarrow F_1^*$ be the homomorphism deduced from f . We set $D(M) = \text{coker } f^*$ the Auslander and Bridger's dual of M . The following proposition is the version in commutative case of [14, Theorem 2.4]:

Proposition 3.1. *Assume that M has no projective summand. Then:*

- (1) $\ker f^* \subseteq PF_0^*$ and $\text{im } f^* \subseteq PF_1^*$;
- (2) $M \cong D(D(M))$ and $D(M)$ has no projective summand;
- (3) $\text{gen } D(M) = \text{rel } M$ and $\text{rel } D(M) = \text{gen } M$;
- (4) if $M = M_1 \oplus M_2$ then
 $\text{gen } M = \text{gen } M_1 + \text{gen } M_2$ and $\text{rel } M = \text{rel } M_1 + \text{rel } M_2$.
- (5) $\text{End}_R(D(M))$ is local if and only if so is $\text{End}_R(M)$.

Lemma 3.2. *Let M be a finitely generated R -module, s an endomorphism of M and \bar{s} the endomorphism of M/PM induced by s . Then s is an isomorphism if and only if so is \bar{s} .*

Proof. If s is an isomorphism it is obvious that so is \bar{s} . Conversely, $\text{coker } s = 0$ by Nakayama lemma. So, s is surjective. By using a Vasconcelos's result (see [4, Theorem V.2.3]) s is bijective. \square

Proposition 3.3. *Assume that there exists an ideal A with $\text{gen } A = p + 1$ where p is a positive integer. Then, for each positive integers n and m with $(n - 1)p + 1 \leq m \leq np + 1$, there exists a finitely presented R -module $W_{p,n,m}$ whose endomorphism ring is local and such that $\text{gen } W_{p,n,m} = n$ and $\text{rel } W_{p,n,m} = m$.*

Proof. Suppose that A is generated by a_1, \dots, a_p, a_{p+1} . Let F be a free module of rank n with basis e_1, \dots, e_n and let K be the submodule of F generated by x_1, \dots, x_m where these elements are defined in the following way: if $j = pq + r$ where $1 \leq r \leq p$, $x_j = a_r e_{q+1}$ if $r \neq 1$ or $q = 0$, and $x_j = a_{p+1} e_q + a_1 e_{q+1}$ else; when $m = pn + 1$, $x_m = a_{p+1} e_n$. We put $W_{p,n,m} = F/K$. We can say that $W_{p,n,m}$ is named by the following $n \times m$ matrix, where $r = m - p(n - 1)$:

$$\begin{pmatrix} a_1 & \dots & a_p & a_{p+1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & a_1 & a_2 & \dots & a_{p+1} & 0 & \dots & \dots & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & & 0 & a_1 & \dots & a_p & a_{p+1} & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & a_2 & \dots & a_r \end{pmatrix}$$

Since $K \subseteq PF$, $\text{gen } W_{p,n,m} = n$. Now we consider the following relation: $\sum_{j=0}^m c_j x_j = 0$. From the definition of the x_j we get the following equality:

$$\sum_{q=0}^{n-2} \left(\sum_{i=1}^{p+1} c_{pq+i} a_i \right) e_{q+1} + \left(\sum_{i=1}^r c_{p(n-1)+i} a_i \right) e_n = 0.$$

Since $\{e_1, \dots, e_n\}$ is a basis and $\text{gen } A = p+1$ we deduce that $c_j \in P$, $\forall j$, $1 \leq j \leq m$. So, $\text{rel } W_{p,n,m} = m$.

Let $s \in \text{End}_R(W_{p,n,m})$. Then s is induced by an endomorphism \tilde{s} of F which satisfies $\tilde{s}(K) \subseteq K$. For each j , $1 \leq j \leq n$, there exists a family $(\alpha_{i,j})$ of elements of R such that:

$$(1) \quad \tilde{s}(e_j) = \sum_{i=1}^n \alpha_{i,j} e_i$$

Since $\tilde{s}(K) \subseteq K$, $\forall j$, $1 \leq j \leq m$, \exists a family $(\beta_{i,j})$ of elements of R such that:

$$(2) \quad \tilde{s}(x_j) = \sum_{i=1}^m \beta_{i,j} x_i$$

From (1), (2) and the equality $x_1 = a_1 e_1$ it follows that:

$$\sum_{q=1}^n \alpha_{q,1} a_1 e_q = \sum_{q=0}^{n-2} \left(\sum_{i=1}^{p+1} \beta_{p(q-1)+i,1} a_i \right) e_{q+1} + \left(\sum_{i=1}^r \beta_{p(n-1)+i,1} a_i \right) e_n.$$

Then, we get:

$$\forall q, 1 \leq q \leq n-1, \quad \alpha_{q,1} a_1 = \sum_{i=1}^{p+1} \beta_{p(q-1)+i,1} a_i$$

$$\text{and} \quad \alpha_{n,1} a_1 = \sum_{i=1}^r \beta_{p(n-1)+i,1} a_i.$$

We deduce that: $\forall q$, $2 \leq q \leq n$, $\beta_{p(q-2)+p+1,1} \in P$ and $\beta_{p(q-1)+1,1} \equiv \alpha_{q,1} [P]$. So,

$$(3) \quad \forall q, 2 \leq q \leq n, \quad \alpha_{q,1} \in P.$$

Now, let $j = p\ell + 1$ where $1 \leq \ell \leq (n-1)$. In this case, $x_j = a_{p+1} e_\ell + a_1 e_{\ell+1}$. From (1) and (2) it follows that:

$$\sum_{q=1}^n (\alpha_{q,\ell} a_{p+1} + \alpha_{q,\ell+1} a_1) e_q = \sum_{q=0}^{n-2} \left(\sum_{i=1}^{p+1} \beta_{p(q-1)+i,j} a_i \right) e_{q+1} + \left(\sum_{i=1}^r \beta_{p(n-1)+i,j} a_i \right) e_n.$$

Then, we get:

$$\forall q, 1 \leq q \leq n-1, \quad \alpha_{q,\ell} a_{p+1} + \alpha_{q,\ell+1} a_1 = \sum_{i=1}^{p+1} \beta_{p(q-1)+i,j} a_i$$

$$\text{and} \quad \alpha_{n,\ell} a_{p+1} + \alpha_{n,\ell+1} a_1 = \sum_{i=1}^r \beta_{p(n-1)+i,j} a_i.$$

We deduce that

$$\forall q, \ell, 1 \leq q, \ell \leq (n-1), \quad \alpha_{q,\ell} \equiv \beta_{p(q-1)+p+1,j} [P] \text{ and } \alpha_{q+1,\ell+1} \equiv \beta_{p(q+1),j} [P],$$

whence $\alpha_{q,\ell} \equiv \alpha_{q+1,\ell+1} [P]$. Consequently, $\forall q$, $1 \leq q \leq n$, $\alpha_{q,q} \equiv \alpha_{1,1} [P]$ and $\forall t$, $1 \leq t \leq (n-1)$, $\forall q$, $1 \leq q \leq (n-t)$, $\alpha_{q+t,q} \equiv \alpha_{1+t,1} \equiv 0 [P]$ by (3). Let \bar{s} be the endomorphism of $W_{p,n,m}/PW_{p,n,m}$ induced by s . If $\alpha_{1,1}$ is a unit then \bar{s} is an isomorphism, else $\mathbf{1}_{W_{p,n,m}} - \bar{s}$ is an isomorphism. By Lemma 3.2 we conclude that either s or $(\mathbf{1}_{W_{p,n,m}} - s)$ is an isomorphism. Hence, $\text{End}_R(W_{p,n,m})$ is local. \square

Remark 3.4. Observe that $D(W_{1,n-1,n})$ is isomorphic to the indecomposable module built in the proof of [12, Theorem 2].

Theorem 3.5. Let R be a commutative ring. The following assertions hold:

- (1) Assume that, for any integer $p > 0$, there exists a maximal ideal P and a finitely generated ideal A of R_P such that $\text{gen}_{R_P} A \geq p + 1$. Then, if (n, m) and (r, s) are two different pairs of integers, the (n, m) -purity and the (r, s) -purity are not equivalent.
- (2) Assume that, there exists an integer $p > 0$ such that, for each maximal ideal P , for any finitely generated ideal A of R_P , $\text{gen}_{R_P} A \leq p$. Then:
 - (a) for each integer $n > 0$ the (\aleph_0, n) -purity (respectively (n, \aleph_0) -purity) is equivalent to the (np, n) -purity (respectively (n, np) -purity);
 - (b) if $p > 1$, then, for each integer $n > 0$, for each integer m , $1 \leq m \leq n(p-1)$, the (n, m) -purity (respectively (m, n) -purity) is not equivalent to the $(n, m+1)$ -purity (respectively $(m+1, n)$ -purity).

Proof. By Proposition 1.4 we may assume that R is local with maximal P . By Theorem 1.1 the (n, m) -purity and the (r, s) -purity are equivalent if and only if so are the (m, n) -purity and the (s, r) -purity.

(1). Suppose that $r > n$ and let $t = \min(m, s)$. Let q be the greatest divisor of $(r-1)$ which is $\leq t$ and $p = (r-1)/q$. Let A be a finitely generated ideal such that $\text{gen} A > p$. By way of contradiction, suppose that $W_{p,q,r}$ is (n, m) -pure-projective. By Proposition 1.2 $W_{p,q,r}$ is a summand of $\bigoplus_{i \in I} F_i$ where I is a finite set and $\forall i \in I$, F_i is a (m, n) -presented R -module. Since its endomorphism ring is local, $W_{p,q,r}$ is an exchange module (see [3, Theorem 2.8]). So, we have $W_{p,q,r} \oplus (\bigoplus_{i \in I} G_i) \cong (\bigoplus_{i \in I} H_i) \oplus (\bigoplus_{i \in I} G_i)$ where $\forall i \in I$, G_i and H_i are submodules of F_i and $F_i = G_i \oplus H_i$. Let $G = \bigoplus_{i \in I} G_i$. Then G is finitely generated. By [4, Proposition V.7.1] $\text{End}_R(G)$ is semilocal. By using Evans's theorem ([3, Corollary 4.6]) we deduce that $W_{p,q,r} \cong (\bigoplus_{i \in I} H_i)$. Since $W_{p,q,r}$ is indecomposable, we get that it is (m, n) -presented. This contradicts that $\text{rel} W_{p,q,r} = r > n$.

(2)(a) is an immediate consequence of Proposition 2.2.

(2)(b). There exist two integers q, t such that $m+1 = (q-1)(p-1) + t$ with $n \geq q \geq 1$ and $1 \leq t \leq p$. As in (1) we prove that $W_{p-1,q,m+1}$ is not (m, n) -pure-projective. \square

Remark 3.6. In the previous theorem, when there exists an integer $p > 1$ such that, for any finitely generated ideal A $\text{gen} A \leq p$, we don't know if the (n, m) -purity and the $(n, m+1)$ -purity are equivalent when $n(p-1) + 1 \leq m \leq np-1$. If R is a local ring with maximal P with residue field k such that $P^2 = 0$ and $\dim_k P = p$ it is easy to show that each finitely presented R -module F with $\text{gen} F = n$ and $\text{rel} F = np$ is semisimple. So, the (np, n) -purity is equivalent to the $(np-1, n)$ -purity.

4. (n, m) -FLAT MODULES AND (n, m) -INJECTIVE MODULES

Let M be a right R -module. We say that M is (n, m) -flat if for any m -generated submodule K of a n -generated free left R -module F , the natural map: $M \otimes_R K \rightarrow M \otimes_R F$ is a monomorphism. We say that M is (\aleph_0, m) -flat (respectively (n, \aleph_0) -flat) if M is (n, m) -flat for each integer $n > 0$ (respectively $m > 0$). We say that M is (n, m) -injective if for any m -generated submodule K of a n -generated free right R -module F , the natural map: $\text{Hom}_R(F, M) \rightarrow \text{Hom}_R(K, M)$ is an epimorphism.

We say that M is (\aleph_0, m) -injective (respectively (n, \aleph_0) -injective) if M is (n, m) -injective for each integer $n > 0$ (respectively $m > 0$). A ring R is called *left self (n, m) -injective* if R is (n, m) -injective as left R -module.

If R is a commutative domain, then an R -module is $(1, 1)$ -flat (respectively $(1, 1)$ -injective) if and only if it is torsion-free (respectively divisible).

The following propositions can be proved with standard technique: see [16, Theorem 4.3 and Proposition 2.3]. In these propositions the integers n or m can be replaced with \aleph_0 .

Proposition 4.1. *Assume that R is an algebra over a commutative ring S and let E be an injective S -cogenerator. Let M be a right R -module. The following conditions are equivalent:*

- (1) M is (n, m) -flat;
- (2) each exact sequence $0 \rightarrow L \rightarrow N \rightarrow M \rightarrow 0$ is (n, m) -pure, where L and N are right R -modules;
- (3) for each (m, n) -presented right module F , every homomorphism $f : F \rightarrow M$ factors through a free right R -module;
- (4) $\text{Hom}_S(M, E)$ is a (n, m) -injective left R -module.

Proposition 4.2. *Let M be a right module. The following conditions are equivalent:*

- (1) M is (n, m) -injective;
- (2) each exact sequence $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$ is (m, n) -pure, where L and N are right R -modules;
- (3) M is a (m, n) -pure submodule of its injective hull.

Proposition 4.3. *Let R be a commutative ring. Then an R -module M is (n, m) -flat if and only if, for each maximal ideal P , M_P is (n, m) -flat over R_P .*

Lemma 4.4. *Let M be p -generated right R -module where p is a positive integer. Then M is flat if and only if it is $(1, p)$ -flat.*

Proof. Only “if” requires a proof. Let A be a left ideal. Assume that M is generated by x_1, \dots, x_p . So, if $z \in M \otimes_R A$, $z = \sum_{i=1}^p x_i \otimes a_i$ where $a_1, \dots, a_p \in A$. Suppose that the image of z in $M \otimes_R R$ is 0. If A' is the left ideal generated by a_1, \dots, a_p , if z' is the element of $M \otimes_R A'$ defined by $z' = \sum_{i=1}^p x_i \otimes a_i$, then z (respectively 0) is the image of z' in $M \otimes_R A$ (respectively $M \otimes_R R$). Since M is $(1, p)$ -flat we successively deduce that $z' = 0$ and $z = 0$. \square

It is well known that each $(1, \aleph_0)$ -flat right module is (\aleph_0, \aleph_0) -flat. **For each positive integer p , is each $(1, p)$ -flat right module (\aleph_0, p) -flat?**

The following theorem and Theorem 4.11 give a partial answer to this question.

Theorem 4.5. *Let p be a positive integer and let R be a ring. For each positive integer n , assume that, for each p -generated submodule G of the left R -module $R^n \oplus R$, $(G \cap R^n)$ is the direct limit of its p -generated submodules. Then a right R -module M is $(1, p)$ -flat if and only if it is (\aleph_0, p) -flat.*

Proof. We shall prove that M is (n, p) -flat by induction on n . Let G be a p -generated submodule of the left R -module $R^{n+1} = R^n \oplus R$. Let π be the projection of R^{n+1} onto R and $G' = \pi(G)$. Then G' is a p -generated left module. We put

$H = G \cap R^n$. We have the following commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccc} M \otimes_R H & \rightarrow & M \otimes_R G & \xrightarrow{1_M \otimes \pi} & M \otimes_R G' & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M \otimes_R R^n & \rightarrow & M \otimes_R R^{n+1} & \xrightarrow{1_M \otimes \pi} & M \otimes_R R \rightarrow 0 \end{array}$$

Let $u : G \rightarrow R^{n+1}$, $u' : G' \rightarrow R$, $w : R^n \rightarrow R^{n+1}$ be the inclusion maps and let $v = u|_H$. Then $(1_M \otimes u')$ is injective. Let H' be a p -generated submodule of H . By the induction hypothesis M is (n, p) -flat. So, $(1_M \otimes (v|_{H'}))$ is injective. It follows that $(1_M \otimes v)$ is injective too. We conclude that $(1_M \otimes u)$ is injective and M is (\aleph_0, p) -flat. \square

Corollary 4.6. *Let p be a positive integer and let R be a ring such that each left ideal is $(1, p)$ -flat. Then, for each positive integer $q \leq p$, a right R -module M is $(1, q)$ -flat if and only if it is (\aleph_0, q) -flat.*

Proof. Let the notations be as in the previous theorem. Since G' is a flat left R -module by Lemma 4.4, H is a pure submodule of G . Let $\{g_1, \dots, g_q\}$ be a spanning set of G and let $h_1, \dots, h_t \in H$. For each k , $1 \leq k \leq t$ there exist $a_{k,1}, \dots, a_{k,q} \in R$ such that $h_k = \sum_{i=1}^q a_{k,i} g_i$. It follows that there exist $g'_1, \dots, g'_q \in H$ such that $\forall k$, $1 \leq k \leq t$, $h_k = \sum_{i=1}^q a_{k,i} g'_i$. So, each finitely generated submodule of H is contained in a q -generated submodule. We conclude by applying Theorem 4.5. \square

Corollary 4.7. *Let R be a commutative local ring with maximal P . Assume that $P^2 = 0$. Let q a positive integer. Then:*

- (1) *each $(1, q)$ -flat module is (\aleph_0, q) -flat;*
- (2) *each $(1, q)$ -injective module is (\aleph_0, q) -injective.*

Proof. Let the notations be as in the previous theorem. We may assume that $G \subseteq PR^{n+1}$. Then G is a semisimple module and H is a direct summand of G . So, (1) is a consequence of Theorem 4.5.

(2). Let M be a $(1, q)$ -injective module. We shall prove by induction on n that M is (n, q) -injective. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_R(R, M) & \rightarrow & \text{Hom}_R(R^{n+1}, M) & \rightarrow & \text{Hom}_R(R^n, M) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Hom}_R(G', M) & \rightarrow & \text{Hom}_R(G, M) & \rightarrow & \text{Hom}_R(H, M) \rightarrow 0 \end{array}$$

where the horizontal sequences are exact. By the induction hypothesis the left and the right vertical maps are surjective. It follows that the middle vertical map is surjective too. \square

By [9, Example 5.2] or [6, Theorem 2.3], for each integer $n > 0$, there exists a ring R for which each finitely generated left ideal is $(1, n)$ -flat (hence (\aleph_0, n) -flat by Corollary 4.6) but there is a finitely generated left ideal which is not $(1, n+1)$ -flat. The following proposition gives other examples in the commutative case.

Proposition 4.8. *Let R be a commutative local ring with maximal ideal P and residue field k . Assume that $P^2 = 0$ and $\dim_k P > 1$. Then, for each positive integer $p < \dim_k P$, there exists:*

- (1) *a $(p+1, 1)$ -presented R -module which is (\aleph_0, p) -flat but not $(1, p+1)$ -flat;*
- (2) *a (\aleph_0, p) -injective R -module which is not $(1, p+1)$ -injective.*

Proof. (1). Let F be a free R -module of rank $(p+1)$ with basis $\{e_1, \dots, e_p, e_{p+1}\}$, let $(a_1, \dots, a_p, a_{p+1})$ be a family of linearly independent elements of P , let K be the submodule of F generated by $\sum_{i=1}^{p+1} a_i e_i$ and let $M = F/K$. Then $M \cong D(W_{p,1,p+1})$ (see the proof of Proposition 3.3). First, we show that K is a $(1, p)$ -pure submodule of F . We consider the following equation:

$$(4) \quad \sum_{j=1}^p r_j x_j = s \left(\sum_{i=1}^{p+1} a_i e_i \right)$$

where $r_1, \dots, r_p, s \in R$ and with unknowns x_1, \dots, x_p . Assume that this equation has a solution in F . Suppose there exists ℓ , $1 \leq \ell \leq p$, such that r_ℓ is a unit. For each j , $1 \leq j \leq p$, we put $x'_j = \delta_{j,\ell} r_\ell^{-1} s (\sum_{i=1}^{p+1} a_i e_i)$. It is easy to check that (x'_1, \dots, x'_p) is a solution of (4) in K . Now we assume that $r_j \in P$, $\forall j$, $1 \leq j \leq p$. Suppose that (x_1, \dots, x_p) is a solution of (4) in F . For each j , $1 \leq j \leq p$, $x_j = \sum_{i=1}^{p+1} c_{j,i} e_i$, where $c_{j,i} \in R$. We get the following equality:

$$(5) \quad \sum_{i=1}^{p+1} \left(\sum_{j=1}^p r_j c_{j,i} \right) e_i = \sum_{i=1}^{p+1} s a_i e_i$$

We deduce that:

$$(6) \quad \forall i, \quad 1 \leq i \leq p+1, \quad \sum_{j=1}^p r_j c_{j,i} = s a_i$$

So, if s is a unit, $\forall i$, $1 \leq i \leq p+1$, $a_i \in \sum_{j=1}^p R r_j$. It follows that

$$\dim_k \left(\sum_{i=1}^{p+1} R a_i \right) \leq p$$

that is false. So, $s \in P$. In this case (4) has the nil solution. Hence M is (\aleph_0, p) -flat by Proposition 4.1(2) and Corollary 4.7.

By way of contradiction suppose that M is $(1, p+1)$ -flat. It follows that K is a $(1, p+1)$ -pure submodule of F by Proposition 4.1. Since M is $(1, p+1)$ -pure-projective we deduce that M is free. This is false.

(2). Let E be an injective R -cogenerator. Then $\text{Hom}_R(M, E)$ is (\aleph_0, p) -injective but not $(1, p+1)$ -injective by Proposition 4.1(4). \square

In a similar way we show the following proposition.

Proposition 4.9. *Let R be a commutative local ring with maximal ideal P . Assume that $P^2 = 0$. Let M be a $(m, 1)$ -presented R -module with $m > 1$, let $\{x_1, \dots, x_m\}$ be a spanning set of M and let $\sum_{j=1}^m a_j x_j = 0$ be the relation of M , where $a_1, \dots, a_m \in P$. If $p = \text{gen}(\sum_{j=1}^m R a_j) - 1 > 0$, then:*

- (1) M is (\aleph_0, p) -flat but not $(1, m)$ -flat;
- (2) $\text{Hom}_R(M, E)$ is (\aleph_0, p) -injective but not $(1, m)$ -injective, where E is an injective R -cogenerator.

When R is an arithmetical commutative ring, i.e. its lattice of ideals is distributive, each $(1, 1)$ -flat module is flat and by [4, Theorem VI.9.10] the converse holds if R is a commutative domain (it is also true if each principal ideal is flat). However we shall see that there exist non-arithmetical commutative rings for which each

(1, 1)-flat module is flat. Recall that a left (or right) R -module M is *torsionless* if the natural map $M \rightarrow (M^*)^*$ is injective.

Proposition 4.10. *For each ring R the following conditions are equivalent:*

- (1) R is right self $(\aleph_0, 1)$ -injective;
- (2) each finitely presented cyclic left R -module is torsionless;
- (3) each finitely generated left ideal A satisfies $A = 1 - \text{ann}(r - \text{ann}(A))$.

Proof. We prove (1) \Leftrightarrow (2) as [5, Theorem 2.3] and (2) \Leftrightarrow (3) is easy. \square

Theorem 4.11. *Let R be a right perfect ring which is right self $(\aleph_0, 1)$ -injective. Then each (1, 1)-flat right module is projective.*

Proof. Let M be a (1, 1)-flat right R -module. It is enough to show that M is flat. Let A be a finitely generated left ideal of R . Assume that $\{a_1, \dots, a_n\}$ is a minimal system of generators of A with $n > 1$. Let $z \in M \otimes_R A$ such that its image in M is 0. We have $z = \sum_{i=1}^n y_i \otimes a_i$, where $y_1, \dots, y_n \in M$, and $\sum_{i=1}^n y_i a_i = 0$. For each i , $1 \leq i \leq n$, we set $A_i = \sum_{\substack{j=1 \\ j \neq i}}^n R a_j$. Then, $\forall i$, $1 \leq i \leq n$, $A_i \subset A$. For each finitely generated left ideal B we have $B = 1 - \text{ann}(r - \text{ann}(B))$. It follows that, $\forall i$, $1 \leq i \leq n$, $r - \text{ann}(A) \subset r - \text{ann}(A_i)$. Let $b_i \in r - \text{ann}(A_i) \setminus r - \text{ann}(A)$. Then $y_i a_i b_i = 0$. From the (1, 1)-flatness of M we deduce that $y_i = \sum_{k=1}^{m_i} y'_{i,k} c_{i,k}$, where $y'_{i,1}, \dots, y'_{i,m_i} \in M$ and $c_{i,1}, \dots, c_{i,m_i} \in R$ with $c_{i,k} a_i b_i = 0$, $\forall k$, $1 \leq k \leq m_i$. It follows that $z = \sum_{i=1}^n (\sum_{k=1}^{m_i} y'_{i,k} \otimes c_{i,k} a_i)$. Let $A^{(1)}$ be the left ideal generated by $\{c_{i,k} a_i \mid 1 \leq i \leq n, 1 \leq k \leq m_i\}$. Then $A^{(1)} \subset A$; else, $\forall i$, $1 \leq i \leq n$, $a_i = \sum_{j=1}^n (\sum_{k=1}^{m_j} d_{i,j,k} c_{j,k} a_j)$ with $d_{i,j,k} \in R$; we get that $a_i b_i = \sum_{j=1}^n (\sum_{k=1}^{m_j} d_{i,j,k} c_{j,k} a_j b_i)$; but $a_j b_i = 0$ if $j \neq i$ and $c_{i,k} a_i b_i = 0$; so, there is a contradiction because the second member of the previous equality is 0 while $a_i b_i \neq 0$. Let $\{a_1^{(1)}, \dots, a_{n_1}^{(1)}\}$ be a minimal system of generators of $A^{(1)}$. So, $z = \sum_{i=1}^{n_1} y_i^{(1)} \otimes a_i^{(1)}$ where $y_1^{(1)}, \dots, y_{n_1}^{(1)} \in M$, and z is the image of $z^{(1)} \in M \otimes_R A^{(1)}$ defined by $z^{(1)} = \sum_{i=1}^{n_1} y_i^{(1)} \otimes a_i^{(1)}$. If $n_1 \leq 1$ we conclude that $z^{(1)} = 0$ since M is (1, 1)-flat, and $z = 0$. If $n_1 > 1$, in the same way we get that $z^{(1)}$ is the image of an element $z^{(2)} \in M \otimes_R A^{(2)}$ where $A^{(2)}$ is a left ideal such that $A^{(2)} \subset A^{(1)}$. If $\text{gen } A^{(2)} > 1$ we repeat this process, possibly several times, until we get a left ideal $A^{(l)}$ with $\text{gen } A^{(l)} \leq 1$; this is possible because R satisfies the descending chain condition on finitely generated left ideals since it is right perfect (see [8, Théorème 5 p.130]). The (1, 1)-flatness of M implies that $z^{(l)} = 0$ and $z = 0$. So, M is projective. \square

Let \mathcal{P} be a ring property. We say that a commutative ring R is *locally \mathcal{P}* if R_P satisfies \mathcal{P} for each maximal ideal P .

The following corollary is a consequence of Theorem 4.11 and Proposition 4.3.

Corollary 4.12. *Let R be a commutative ring which is locally perfect and locally self $(\aleph_0, 1)$ -injective. Then each (1, 1)-flat R -module is flat.*

5. (n, m) -COHERENT RINGS

We say that a ring R is left (n, m) -coherent if each m -generated submodule of a n -generated free left R -module is finitely presented. We say that R is left (\aleph_0, m) -coherent (respectively (n, \aleph_0) -coherent) if for each integer $n > 0$ (respectively $m > 0$) R is left (n, m) -coherent. The following theorem can be proven with standard

technique: see [16, Theorems 5.1 and 5.7]. In this theorem the integers n or m can be replaced with \aleph_0 .

Theorem 5.1. *Let R be a ring and n, m two fixed positive integers. Assume that R is an algebra over a commutative ring S . Let E be an injective S -cogenerator. Then the following conditions are equivalent:*

- (1) R is left (n, m) -coherent;
- (2) any direct product of right (n, m) -flat R -modules is (n, m) -flat;
- (3) for any set Λ , R^Λ is a (n, m) -flat right R -module;
- (4) any direct limit of a direct system of (n, m) -injective left R -modules is (n, m) -injective;
- (5) for any exact sequence of left modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, C is (n, m) -injective if so is B and if A is a (\aleph_0, m) -pure submodule of B ;
- (6) for each (n, m) -injective left R -module M , $\text{Hom}_S(M, E)$ is (n, m) -flat.

It is well known that each left $(1, \aleph_0)$ -coherent ring is left (\aleph_0, \aleph_0) -coherent. **For each positive integer p , is each left $(1, p)$ -coherent ring left (\aleph_0, p) -coherent?**

Propositions 5.2 and 5.3 and Theorem 5.4 give a partial answer to this question.

Proposition 5.2. *Let p be a positive integer and let R be a ring. For each positive integer n , assume that, for each p -generated submodule G of the left R -module $R^n \oplus R$, $(G \cap R^n)$ is the direct limit of its p -generated submodules. Then the following conditions are equivalent:*

- (1) R is left $(1, p)$ -coherent;
- (2) R is left (\aleph_0, p) -coherent.

Moreover, when these conditions hold each $(1, p)$ -injective left module is (\aleph_0, p) -injective.

Proof. It is obvious that (2) \Rightarrow (1).

(1) \Rightarrow (2). Let Λ be a set. By Theorem 5.1 R^Λ is a $(1, p)$ -flat right module. From Theorem 4.5 we deduce that R^Λ is a (\aleph_0, p) -flat right module. By using again Theorem 5.1 we get (2).

Let M be a $(1, p)$ -injective left module. By Theorem 5.1 M^\sharp is a $(1, p)$ -flat right R -module. Then it is also (\aleph_0, p) -flat. We deduce that $(M^\sharp)^\sharp$ is a (\aleph_0, p) -injective left module. Since M is a pure submodule of $(M^\sharp)^\sharp$, it follows that M is (\aleph_0, p) -injective too. \square

Proposition 5.3. *Let R be a commutative perfect ring. Then R is Artinian if and only if it is $(1, 1)$ -coherent.*

Proof. Suppose that R is $(1, 1)$ -coherent. Since R is perfect, R is a finite product of local rings. So, we may assume that R is local with maximal P . Let S be a minimal non-zero ideal of R generated by s . Then P is the annihilator of s . So, P is finitely generated and it is the sole prime ideal of R . Since all prime ideals of R are finitely generated, R is Noetherian. On the other hand R satisfies the descending chain condition on finitely generated ideals. We conclude that R is Artinian. \square

Except in some particular cases, we don't know if each $(1, p)$ -injective module is (\aleph_0, p) -injective, even if we replace p by \aleph_0 .

Theorem 5.4. *Let R be a ring which is right perfect, left $(1, 1)$ -coherent and right self $(\aleph_0, 1)$ -injective. Then each $(1, 1)$ -injective left module is (\aleph_0, \aleph_0) -injective and R is left coherent.*

Proof. Let M be a left $(1, 1)$ -injective module. By Theorem 5.1 $M^\#$ is $(1, 1)$ -flat. Whence $M^\#$ is projective by Theorem 4.11. We do as in the proof of Proposition 5.2 to conclude that M is (\aleph_0, \aleph_0) -injective.

For each set Λ , R^Λ is a $(1, 1)$ -flat right module by Theorem 5.1. It follows that R^Λ is a projective right module by Theorem 4.11. \square

Recall that a ring is *quasi-Frobenius* if it is Artinian and self-injective.

Corollary 5.5. *Let R be a quasi-Frobenius ring. Then, for each right (or left) R -module M , the following conditions are equivalent:*

- (1) M is $(1, 1)$ -flat;
- (2) M is projective;
- (3) M is injective;
- (4) M is $(1, 1)$ -injective.

Proof. It is well known that (2) \Leftrightarrow (3). By Theorem 4.11 (1) \Leftrightarrow (2) because R satisfies the conditions of this theorem, and it is obvious that (3) \Rightarrow (4) and the converse holds by Theorem 5.4. \square

We prove the following theorem as [2, Théorème 1.4].

Theorem 5.6. *Let R be a commutative ring and n, m two fixed positive integers. The following conditions are equivalent:*

- (1) R is (n, m) -coherent;
- (2) for each multiplicative subset S of R , $S^{-1}R$ is (n, m) -coherent, and for each (n, m) -injective R -module M , $S^{-1}M$ is (n, m) -injective over $S^{-1}R$;
- (3) For each maximal ideal P , R_P is (n, m) -coherent and for each (n, m) -injective R -module M , M_P is (n, m) -injective over R_P .

Recall that a ring R is a *right IF-ring* if each right injective R -module is flat.

Theorem 5.7. *Let R be a commutative ring which is locally perfect, $(1, 1)$ -coherent and self $(1, 1)$ -injective. Then:*

- (1) R is coherent, self (\aleph_0, \aleph_0) -injective and locally quasi-Frobenius;
- (2) each $(1, 1)$ -flat module is flat;
- (3) each $(1, 1)$ -injective module is (\aleph_0, \aleph_0) -injective.

Proof. By Theorem 5.6 R_P is $(1, 1)$ -coherent and $(1, 1)$ -injective for each maximal ideal P . Let a be a generator of a minimal non-zero ideal of R_P . Then PR_P is the annihilator of a and consequently PR_P is finitely generated over R_P . Since all prime ideals of R_P are finitely generated, we deduce that R_P is Artinian for each maximal ideal P . Moreover, the $(1, 1)$ -injectivity of R_P implies that the socle of R_P (the sum of all minimal non-zero ideals) is simple. It follows that R_P is quasi-Frobenius for each maximal ideal P .

Let M be a (\aleph_0, \aleph_0) -injective R -module. By Theorem 5.6 M_P is $(1, 1)$ -injective for each maximal ideal P . By Corollary 5.5 M_P is injective for each maximal ideal P . We conclude that R is self (\aleph_0, \aleph_0) -injective and it is coherent by Theorem 5.6.

If M is $(1, 1)$ -injective, we prove as above that M_P is injective for each maximal ideal P . It follows that M is (\aleph_0, \aleph_0) -injective.

The second assertion is an immediate consequence of Corollary 4.12. \square

The following proposition is easy to prove:

Proposition 5.8. *A ring R is left $(\mathbb{N}_0, 1)$ -coherent if and only if each finitely generated right ideal has a finitely generated left annihilator.*

Example 5.9. *Let V be a non-Noetherian (commutative) valuation domain whose order group is not the additive group of real numbers and let $R = V[[X]]$ be the power series ring in one indeterminate over V . Since R is a domain, R is $(\mathbb{N}_0, 1)$ -coherent. But, in [1] it is proven that there exist two elements f and g of R such that $Rf \cap Rg$ is not finitely generated. By using the exact sequence $0 \rightarrow Rf \cap Rg \rightarrow Rf \oplus Rg \rightarrow Rf + Rg \rightarrow 0$ we get that $Rf + Rg$ is not finitely presented. So, R is not $(1, 2)$ -coherent.*

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UNIVERSITÉ DE CAEN BASSE-NORMANDIE, CNRS UMR 6139 LMNO, F-14032 CAEN, FRANCE
E-mail address: `walid.al-kawarit@unicaen.fr`

UNIVERSITÉ DE CAEN BASSE-NORMANDIE, CNRS UMR 6139 LMNO, F-14032 CAEN, FRANCE
E-mail address: `francois.couchot@unicaen.fr`