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# COMPARISON OF SOME PURITIES, FLATNESSES AND INJECTIVITIES

WALID AL-KAWARIT AND FRANÇOIS COUCHOT

ABSTRACT. In this paper, we compare  $(n, m)$ -purities for different pairs of positive integers  $(n, m)$ . When  $R$  is a commutative ring, these purities are not equivalent if  $R$  doesn't satisfy the following property: there exists a positive integer  $p$  such that, for each maximal ideal  $P$ , every finitely generated ideal of  $R_P$  is  $p$ -generated. When this property holds, then the  $(n, m)$ -purity and the  $(n, m')$ -purity are equivalent if  $m$  and  $m'$  are integers  $\geq np$ . These results are obtained by a generalization of Warfield's methods. There are also some interesting results when  $R$  is a semiperfect strongly  $\pi$ -regular ring. We also compare  $(n, m)$ -flatnesses and  $(n, m)$ -injectivities for different pairs of positive integers  $(n, m)$ . In particular, if  $R$  is right perfect and right self  $(\aleph_0, 1)$ -injective, then each  $(1, 1)$ -flat right  $R$ -module is projective. In several cases, for each positive integer  $p$ , all  $(n, p)$ -flatnesses are equivalent. But there are some examples where the  $(1, p)$ -flatness is not equivalent to the  $(1, p + 1)$ -flatness.

All rings in this paper are associative with unity, and all modules are unital. Let  $n$  and  $m$  be two positive integers. A right  $R$ -module  $M$  is said to be  $(n, m)$ -presented if it is the factor module of a free right module of rank  $n$  modulo a  $m$ -generated submodule. A short exact sequence  $(\Sigma)$  of left  $R$ -modules is called  $(n, m)$ -pure if it remains exact when tensoring it with any  $(n, m)$ -presented right module. We say that  $(\Sigma)$  is  $(\aleph_0, m)$ -pure exact (respectively  $(n, \aleph_0)$ -pure exact if, for each positive integer  $n$  (respectively  $m$ )  $(\Sigma)$  is  $(n, m)$ -pure exact. Let us observe that the  $(1, 1)$ -pure exact sequences are exactly the RD-exact sequences (see (Warfield, 1969a)) and the  $(\aleph_0, \aleph_0)$ -exact sequences are the pure-exact sequences in the Cohn's sense. Similar results as in the classical theories of purity hold, with similar proofs. In particular, a left  $R$ -module is  $(n, m)$ -pure-projective if and only if it is a summand of a direct sum of  $(m, n)$ -presented left modules, and each left  $R$ -module has a  $(n, m)$ -pure-injective hull which is unique up to an isomorphism.

In this paper, we compare  $(n, m)$ -purities for different pairs of positive integers  $(n, m)$ . When  $R$  is commutative, we shall see that these purities are not equivalent if  $R$  doesn't satisfy the following property: there exists a positive integer  $p$  such that, for each maximal ideal  $P$ , every finitely generated ideal of  $R_P$  is  $p$ -generated. When this property holds, then the  $(n, m)$ -purity and the  $(n, m')$ -purity are equivalent if  $m$  and  $m'$  are integers  $\geq np$ . These results are obtained by using the following: if  $R$  is a local commutative ring for which there exists a  $(p + 1)$ -generated ideal, where  $p$  is a positive integer, then, for each positive integer  $n$ , for each integer  $m$ ,  $n(p - 1) + 1 \leq m \leq np + 1$ , there exists a  $(n, m)$ -presented  $R$ -module whose

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endomorphism ring is local. It is a generalization of the Warfield's construction of indecomposable finitely presented modules when  $R$  is not a valuation ring.

When  $R$  is semiperfect and strongly  $\pi$ -regular, we show that there exists an integer  $m > 0$  such that, for any integer  $n > 0$ , each  $(n, m)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact if and only if there exists an integer  $p > 0$  such that every finitely generated left ideal is  $p$ -generated.

As in (Zhang et al., 2005) we define  $(n, m)$ -flat modules and  $(n, m)$ -injective modules. We also compare  $(n, m)$ -flatnesses and  $(n, m)$ -injectivities for different pairs of positive integers  $(n, m)$ . In particular, if  $R$  is a right perfect ring which is right self  $(\aleph_0, 1)$ -injective, then each  $(1, 1)$ -flat right  $R$ -module is projective. For many classes of rings, for each positive integer  $p$ , we show that the  $(1, p)$ -flatness implies the  $(\aleph_0, p)$ -flatness, but we have no general result. If  $R$  is a local commutative ring with a non finitely generated maximal ideal  $P$  satisfying  $P^2 = 0$ , then for each positive integer  $p$ , there exists an  $R$ -module which is  $(\aleph_0, p)$ -flat (resp,  $(\aleph_0, p)$ -injective) and which is not  $(1, p + 1)$ -flat (resp,  $(1, p + 1)$ -injective).

As in (Zhang et al., 2005) we define left  $(n, m)$ -coherent rings. When  $R$  is a commutative locally perfect ring which is  $(1, 1)$ -coherent and self  $(1, 1)$ -injective, we show that  $R$  is an IF-ring, each  $(1, 1)$ -flat  $R$ -module is flat and each  $(1, 1)$ -injective  $R$ -module is FP-injective. For other classes of rings, for each positive integer  $p$ , we show that the left  $(1, p)$ -coherence implies the left  $(\aleph_0, p)$ -coherence, but we have no general result. If  $R = V[[X]]$ , the power series ring over a valuation domain  $V$  whose order group is not isomorphic to  $\mathbb{R}$ , then  $R$  is a  $(\aleph_0, 1)$ -coherent ring which is not  $(1, 2)$ -coherent.

### 1. $(n, m)$ -PURE EXACT SEQUENCES

By using a standard technique, (see for instance (Fuchs and Salce, 2001, Chapter I, Section 8)), we can prove the following theorem, and similar results hold if we replace  $n$  or  $m$  with  $\aleph_0$ .

**Theorem 1.1.** *Assume that  $R$  is an algebra over a commutative ring  $S$  and  $E$  is an injective  $S$ -cogenerator. Then, for each exact sequence  $(\Sigma)$  of left  $R$ -modules  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , the following conditions are equivalent:*

- (1)  $(\Sigma)$  is  $(n, m)$ -pure;
- (2) for each  $(m, n)$ -presented left module  $G$  the sequence  $\text{Hom}_R(G, (\Sigma))$  is exact;
- (3) every system of  $n$  equations over  $A$

$$\sum_{j=1}^m r_{i,j} x_j = a_i \in A \quad (i = 1, \dots, n)$$

with coefficients  $r_{i,j} \in R$  and unknowns  $x_1, \dots, x_m$  has a solution in  $A$  whenever it is solvable in  $B$ ;

- (4) the exact sequence of right  $R$ -modules  $\text{Hom}_S((\Sigma), E)$  is  $(m, n)$ -pure.

Propositions 1.2 and 1.3 can be deduced from (Warfield, 1969b, Theorem 1).

A left  $R$ -module  $G$  is called  $(n, m)$ -pure-projective if for each  $(n, m)$ -pure exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  the sequence

$$0 \rightarrow \text{Hom}_R(G, A) \rightarrow \text{Hom}_R(G, B) \rightarrow \text{Hom}_R(G, C) \rightarrow 0$$

is exact. Similar definitions can be given by replacing  $n$  or  $m$  by  $\aleph_0$ . From Theorem 1.1 and by using standard technique we get the following proposition in which  $n$  or  $m$  can be replaced by  $\aleph_0$ :

**Proposition 1.2.** *Let  $G$  be a left  $R$ -module. Then the following assertions hold:*

- (1) *there exists a  $(n, m)$ -pure exact sequence of left modules*

$$0 \rightarrow K \rightarrow F \rightarrow G \rightarrow 0$$

*where  $F$  is a direct sum of  $(m, n)$ -presented left modules;*

- (2)  *$G$  is  $(n, m)$ -pure projective if and only if it is a summand of a direct sum of  $(m, n)$ -presented left modules.*

A left  $R$ -module  $G$  is called  $(n, m)$ -pure-injective if for each  $(n, m)$ -pure exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  the sequence

$$0 \rightarrow \text{Hom}_R(C, G) \rightarrow \text{Hom}_R(B, G) \rightarrow \text{Hom}_R(A, G) \rightarrow 0$$

is exact.

If  $M$  is a left module we put  $M^\sharp = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ . Thus  $M^\sharp$  is a right module. It is the *character module* of  $M$ .

If  $A$  is a submodule of a left  $R$ -module  $B$ , we say that  $B$  is a  $(n, m)$ -pure essential extension of  $A$  if  $A$  is a  $(n, m)$ -pure submodule of  $B$  and for each submodule  $K$  of  $B$  such that  $A \cap K = 0$ ,  $(A + K)/K$  is not a  $(n, m)$ -pure submodule of  $B/K$ . If, in addition,  $B$  is  $(n, m)$ -pure injective, we say that  $B$  is a  $(n, m)$ -pure injective hull of  $A$ . In these above definitions and in the following proposition  $n$  or  $m$  can be replaced by  $\aleph_0$ .

**Proposition 1.3.** *The following assertions hold:*

- (1) *each left  $R$ -module is a  $(n, m)$ -pure submodule of a  $(n, m)$ -pure injective left module;*  
(2) *each left  $R$ -module has a  $(n, m)$ -pure injective hull which is unique up to an isomorphism.*

*Proof.* (1). Let  $M$  be a left  $R$ -module. By Proposition 1.2 there exists a  $(m, n)$ -pure exact sequence of right  $R$ -modules  $0 \rightarrow K \rightarrow F \rightarrow M^\sharp \rightarrow 0$  where  $F$  is a direct sum of  $(n, m)$ -presented right modules. From Theorem 1.1 it follows that  $(M^\sharp)^\sharp$  is a  $(n, m)$ -pure submodule of  $F^\sharp$ . By (Facchini, 1998, Corollary 1.30)  $M$  is isomorphic to a pure submodule of  $(M^\sharp)^\sharp$ . So,  $M$  is isomorphic to a  $(n, m)$ -pure submodule of  $F^\sharp$ . By using the canonical isomorphism  $(F \otimes_R -)^\sharp \cong \text{Hom}_R(-, F^\sharp)$  we get that  $F^\sharp$  is  $(n, m)$ -pure injective since  $F$  is a direct sum of  $(n, m)$ -presented modules.

(2). Since (1) holds and every direct limit of  $(n, m)$ -pure exact sequences is  $(n, m)$ -pure exact too, we can adapt the method of Warfield's proof of existence of pure-injective hull to show (2)(see (Warfield, 1969a, Proposition 6)). We can also use (Stenström, 1967, Proposition 4.5).  $\square$

**Proposition 1.4.** *Let  $R$  be a commutative ring and let  $(\Sigma)$  be a short exact sequence of  $R$ -modules. Then  $(\Sigma)$  is  $(n, m)$ -pure if and only if, for each maximal ideal  $P$   $(\Sigma)_P$  is  $(n, m)$ -pure.*

*Proof.* Assume that  $(\Sigma)$  is  $(n, m)$ -pure and let  $M$  be a  $(n, m)$ -presented  $R_P$ -module where  $P$  is a maximal ideal. There exists a  $(n, m)$ -presented  $R$ -module  $M'$  such that  $M \cong M'_P$  and  $M \otimes_{R_P} (\Sigma)_P \cong (M' \otimes_R (\Sigma))_P$ . We deduce that  $(\Sigma)_P$  is  $(n, m)$ -pure.

Conversely, suppose that  $(\Sigma)$  is the sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ . Let  $M$  be a  $(n, m)$ -presented  $R$ -module. Then, for each maximal ideal  $P$ ,  $(\Sigma)_P$  is  $(n, m)$ -pure over  $R$  since  $M \otimes_R (\Sigma)_P \cong M_P \otimes_{R_P} (\Sigma)_P$ . On the other hand, since  $M \otimes_R (\prod_{P \in \text{Max } R} (\Sigma)_P) \cong (\prod_{P \in \text{Max } R} M \otimes_R (\Sigma)_P)$ ,  $\prod_{P \in \text{Max } R} A_P$  is a  $(n, m)$ -pure submodule of  $\prod_{P \in \text{Max } R} B_P$ . By (Couchot, 1982, Lemme 1.3)  $A$  is isomorphic to a pure submodule of  $\prod_{P \in \text{Max } R} A_P$ . We successively deduce that  $A$  is a  $(n, m)$ -pure submodule of  $\prod_{P \in \text{Max } R} B_P$  and  $B$ .  $\square$

## 2. COMPARISON OF PURITIES OVER A SEMIPERFECT RING

In this section we shall compare  $(n, m)$ -purities for different pairs of integers  $(n, m)$ . In (Puninski et al., 1999) some various purities are also compared. In particular some necessary and sufficient conditions on a ring  $R$  are given for the  $(1, 1)$ -purity to be equivalent to the  $(\aleph_0, \aleph_0)$ -purity.

The following lemma is due to Lawrence Levy, see (Wiegand and Wiegand, 1975, Lemma 1.3). If  $M$  be a finitely generated left (or right)  $R$ -module, we denote by  $\text{gen } M$  its minimal number of generators.

**Lemma 2.1.** *Let  $R$  be a ring. Assume there exists a positive integer  $p$  such that  $\text{gen } A \leq p$  for each finitely generated left ideal  $A$  of  $R$ . Then  $\text{gen } N \leq p \times \text{gen } M$ , if  $N$  is a finitely generated submodule of a finitely generated left  $R$ -module  $M$ .*

From this lemma and Theorem 1.1 we deduce the following:

**Proposition 2.2.** *Let  $R$  be a ring. Assume there exists a positive integer  $p$  such that  $\text{gen } A \leq p$  for each finitely generated left ideal  $A$  of  $R$ . Then, for each positive integer  $n$ :*

- (1) *each  $(n, np)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;*
- (2) *each  $(np, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact.*

**Corollary 2.3.** *Let  $R$  be a left Artinian ring. Then there exists a positive integer  $p$  such that, for each positive integer  $n$ :*

- (1) *each  $(n, np)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;*
- (2) *each  $(np, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact.*

*Proof.* Each finitely generated left  $R$ -module  $M$  has a finite length denoted by  $\text{length } M$ , and  $\text{gen } M \leq \text{length } M$ . So, for each left ideal  $A$  we have  $\text{gen } A \leq \text{length } R$ . We choose  $p = \sup\{\text{gen } A \mid A \text{ left ideal of } R\}$  and we apply the previous proposition.  $\square$

Let  $R$  be a ring and  $J$  its Jacobson radical. Recall that  $R$  is *semiperfect* if  $R/J$  is semisimple and idempotents lift modulo  $J$ .

**Theorem 2.4.** *Let  $R$  be semiperfect ring. Assume that each indecomposable finitely presented cyclic left  $R$ -module has a local endomorphism ring. The following conditions are equivalent:*

- (1) *there exists an integer  $p > 0$  such that, for each integer  $n > 0$ , each  $(n, np)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;*
- (2) *there exists an integer  $p > 0$  such that, for each integer  $n > 0$ , each  $(np, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact;*
- (3) *there exists an integer  $q > 0$  such that each  $(1, q)$ -pure exact sequence of right modules is  $(1, \aleph_0)$ -pure exact;*

- (4) there exists an integer  $q > 0$  such that each  $(q, 1)$ -pure exact sequence of left modules is  $(\aleph_0, 1)$ -pure exact;
- (5) there exists an integer  $q > 0$  such that each indecomposable finitely presented cyclic left module is  $q$ -related;
- (6) there exists an integer  $p > 0$  such that  $\text{gen } A \leq p$  for each finitely generated left ideal  $A$  of  $R$ .

Moreover, if each indecomposable finitely presented left  $R$ -module has a local endomorphism ring, these conditions are equivalent to the following:

- (7) there exist two positive integers  $n, m$  such that each  $(n, m)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;
- (8) there exist two positive integers  $n, m$  such that each  $(m, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact;

*Proof.* By Proposition 2.2 (6)  $\Rightarrow$  (1). By Theorem 1.1 (1)  $\Leftrightarrow$  (2), (3)  $\Leftrightarrow$  (4) and (7)  $\Leftrightarrow$  (8). It is obvious that (2)  $\Rightarrow$  (4) and (2)  $\Rightarrow$  (7).

(4)  $\Rightarrow$  (5). Let  $C$  be an indecomposable finitely presented cyclic left module. Then  $C$  is  $(q, 1)$ -pure-projective. So,  $C$  is a direct summand of a finite direct sum of  $(1, q)$ -presented left modules. Since  $R$  is semiperfect, we may assume that these  $(1, q)$ -presented left modules are indecomposable. So, by Krull-Schmidt theorem  $C$  is  $(1, q)$ -presented.

(5)  $\Rightarrow$  (6). Let  $A$  be a finitely generated left ideal. Then  $R/A = \bigoplus_{i=1}^t R/A_i$  where, for each  $i = 1, \dots, t$ ,  $A_i$  is a left ideal and  $R/A_i$  is indecomposable. We have the following commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \bigoplus_{i=1}^t A_i & \rightarrow & R^t & \rightarrow & \bigoplus_{i=1}^t R/A_i & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & A & \rightarrow & R & \rightarrow & R/A & \rightarrow & 0 \end{array}$$

Since the right vertical map is an isomorphism, we deduce from snake lemma that the other two vertical homomorphisms have isomorphic cokernels. It follows that  $\text{gen } A \leq tq + 1$  because  $\text{gen } A_i \leq q$  by (5). On the other hand, let  $P$  be a projective cover of  $R/A$ . Then  $P$  is isomorphic to a direct summand of  $R$ . We know that the left module  $R$  is a finite direct sum of indecomposable projective modules. Let  $s$  be the number of these indecomposable summands. It is easy to show that  $t \leq s$ . So, if  $p = sq + 1$ , then  $\text{gen } A \leq p$ .

(8)  $\Rightarrow$  (5). Let  $C$  be an indecomposable finitely presented cyclic left module. Then  $C$  is  $(m, n)$ -pure-projective. So,  $C$  is a direct summand of a finite direct sum of  $(n, m)$ -presented left modules. Since  $R$  is semiperfect, we may assume that these  $(n, m)$ -presented left modules are indecomposable. So, by Krull-Schmidt theorem  $C$  is  $(1, m)$ -presented.  $\square$

A ring  $R$  is said to be *strongly  $\pi$ -regular* if, for each  $r \in R$ , there exist  $s \in R$  and an integer  $q \geq 1$  such that  $r^q = r^{q+1}s$ . By (Facchini, 1998, Théorème 3.16) each strongly  $\pi$ -regular  $R$  satisfies the following condition: for each  $r \in R$ , there exist  $s \in R$  and an integer  $q \geq 1$  such that  $r^q = sr^{q+1}$ . Recall that a left  $R$ -module  $M$  is said to be *Fitting* if for each endomorphism  $f$  of  $M$  there exists a positive integer  $t$  such that  $M = \ker f^t \oplus f^t(M)$ .

**Lemma 2.5.** *Let  $R$  be a strongly  $\pi$ -regular semiperfect ring. Then:*

- (1) each finitely presented cyclic left (or right)  $R$ -module is Fitting;

- (2) each indecomposable finitely presented cyclic left (or right)  $R$ -module has a local endomorphism ring.

*Proof.* In (Facchini, 1998, Lemma 3.21) it is proven that every finitely presented  $R$ -module is a Fitting module if  $R$  is a semiperfect ring with  $M_n(R)$  strongly  $\pi$ -regular for all  $n$ . We do a similar proof to show (1).

(2). By (Facchini, 1998, Lemma 2.21) each indecomposable Fitting module has a local endomorphism ring.  $\square$

**Corollary 2.6.** *Let  $R$  be a strongly  $\pi$ -regular semiperfect ring. The following conditions are equivalent:*

- (1) there exists an integer  $p > 0$  such that, for each integer  $n > 0$ , each  $(n, np)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;
- (2) there exists an integer  $p > 0$  such that, for each integer  $n > 0$ , each  $(np, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact;
- (3) there exists an integer  $q > 0$  such that each  $(1, q)$ -pure exact sequence of right modules is  $(1, \aleph_0)$ -pure exact;
- (4) there exists an integer  $q > 0$  such that each  $(q, 1)$ -pure exact sequence of left modules is  $(\aleph_0, 1)$ -pure exact;
- (5) there exists an integer  $q > 0$  such that each indecomposable finitely presented cyclic left module is  $q$ -related;
- (6) there exists an integer  $p > 0$  such that  $\text{gen } A \leq p$  for each finitely generated left ideal  $A$  of  $R$ .

Moreover, if  $M_n(R)$  is strongly  $\pi$ -regular for all  $n > 0$ , these conditions are equivalent to the following:

- (7) there exist two positive integers  $n, m$  such that each  $(n, m)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;
- (8) there exist two positive integers  $n, m$  such that each  $(m, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact;

*Proof.* By Lemma 2.5 each indecomposable finitely presented cyclic left  $R$ -module has a local endomorphism ring. If  $M_n(R)$  is strongly  $\pi$ -regular for all  $n$ , then by (Facchini, 1998, Lemmas 3.21 and 2.21) each indecomposable finitely presented left  $R$ -module has a local endomorphism ring. So, we apply Theorem 2.4.  $\square$

Recall that a ring  $R$  is *right perfect* if each flat right  $R$ -module is projective.

**Corollary 2.7.** *Let  $R$  be a right perfect ring. The following conditions are equivalent:*

- (1) there exists an integer  $p > 0$  such that, for each integer  $n > 0$ , each  $(n, np)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;
- (2) there exists an integer  $p > 0$  such that, for each integer  $n > 0$ , each  $(np, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact;
- (3) there exists an integer  $q > 0$  such that each  $(1, q)$ -pure exact sequence of right modules is  $(1, \aleph_0)$ -pure exact;
- (4) there exist two positive integers  $n, m$  such that each  $(n, m)$ -pure exact sequence of right modules is  $(n, \aleph_0)$ -pure exact;
- (5) there exist two positive integers  $n, m$  such that each  $(m, n)$ -pure exact sequence of left modules is  $(\aleph_0, n)$ -pure exact;
- (6) there exists an integer  $p > 0$  such that  $\text{gen } A \leq p$  for each finitely generated left ideal  $A$  of  $R$ .

*Proof.* For all  $n > 0$ ,  $M_n(R)$  is right perfect. Since each right perfect ring satisfies the descending chain condition on finitely generated left ideals, then  $M_n(R)$  is strongly  $\pi$ -regular for all  $n > 0$ . We apply Corollary 2.6.  $\square$

3. COMPARISON OF PURITIES OVER A COMMUTATIVE RING

In the sequel of this section  $R$  is a commutative local ring, except in Theorem 3.5. We denote respectively by  $P$  and  $k$  its maximal ideal and its residue field

Let  $M$  be a finitely presented  $R$ -module. Recall that  $\text{gen } M = \dim_k M/PM$ . Let  $F_0$  be a free  $R$ -module whose rank is  $\text{gen } M$  and let  $\phi : F_0 \rightarrow M$  be an epimorphism. Then  $\ker \phi \subseteq PF_0$ . We put  $\text{rel } M = \text{gen } \ker \phi$ . Let  $F_1$  be a free  $R$ -module whose rank is  $\text{rel } M$  and let  $f : F_1 \rightarrow F_0$  be a homomorphism such that  $\text{im } f = \ker \phi$ . Then  $\ker f \subseteq PF_1$ . For any  $R$ -module  $N$ , we put  $N^* = \text{Hom}_R(N, R)$ . Let  $f^* : F_0^* \rightarrow F_1^*$  be the homomorphism deduced from  $f$ . We set  $D(M) = \text{coker } f^*$  the Auslander and Bridger's dual of  $M$ . The following proposition holds:

**Proposition 3.1.** *Assume that  $M$  has no projective summand. Then:*

- (1)  $\ker f^* \subseteq PF_0^*$  and  $\text{im } f^* \subseteq PF_1^*$ ;
- (2)  $M \cong D(D(M))$  and  $D(M)$  has no projective summand;
- (3)  $\text{gen } D(M) = \text{rel } M$  and  $\text{rel } D(M) = \text{gen } M$ ;
- (4) if  $M = M_1 \oplus M_2$  then  
 $\text{gen } M = \text{gen } M_1 + \text{gen } M_2$  and  $\text{rel } M = \text{rel } M_1 + \text{rel } M_2$ ;
- (5)  $\text{End}_R(D(M))$  is isomorphic to the opposite ring of  $\text{End}_R(M)$ .

**Lemma 3.2.** *Let  $M$  be a finitely generated  $R$ -module,  $s$  an endomorphism of  $M$  and  $\bar{s}$  the endomorphism of  $M/PM$  induced by  $s$ . Then  $s$  is an isomorphism if and only if so is  $\bar{s}$ .*

*Proof.* If  $s$  is an isomorphism it is obvious that so is  $\bar{s}$ . Conversely,  $\text{coker } s = 0$  by Nakayama lemma. So,  $s$  is surjective. By using a Vasconcelos's result (see (Fuchs and Salce, 2001, Theorem V.2.3))  $s$  is bijective.  $\square$

**Proposition 3.3.** *Assume that there exists an ideal  $A$  with  $\text{gen } A = p + 1$  where  $p$  is a positive integer. Then, for each positive integers  $n$  and  $m$  with  $(n - 1)p + 1 \leq m \leq np + 1$ , there exists a finitely presented  $R$ -module  $W_{p,n,m}$  whose endomorphism ring is local and such that  $\text{gen } W_{p,n,m} = n$  and  $\text{rel } W_{p,n,m} = m$ .*

*Proof.* Suppose that  $A$  is generated by  $a_1, \dots, a_p, a_{p+1}$ . Let  $F$  be a free module of rank  $n$  with basis  $e_1, \dots, e_n$  and let  $K$  be the submodule of  $F$  generated by  $x_1, \dots, x_m$  where these elements are defined in the following way: if  $j = pq + r$  where  $1 \leq r \leq p$ ,  $x_j = a_r e_{q+1}$  if  $r \neq 1$  or  $q = 0$  and  $x_j = a_{p+1} e_q + a_1 e_{q+1}$  else; when  $m = pn + 1$ ,  $x_m = a_{p+1} e_n$ . We put  $W_{p,n,m} = F/K$ . We can say that  $W_{p,n,m}$  is named by the following  $n \times m$  matrix, where  $r = m - p(n - 1)$ :

$$\begin{pmatrix} a_1 & \dots & a_p & a_{p+1} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & a_1 & a_2 & \dots & a_{p+1} & 0 & \dots & \dots & 0 \\ \vdots & & & & & & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & \dots & a_p & a_{p+1} & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 & a_1 & a_2 & \dots & a_r \end{pmatrix}$$



Since  $K \subseteq PF$ ,  $\text{gen } W_{p,n,m} = n$ . Now we consider the following relation:  $\sum_{j=0}^m c_j x_j = 0$ . From the definition of the  $x_j$  we get the following equality:

$$\sum_{q=0}^{n-2} \left( \sum_{i=1}^{p+1} c_{pq+i} a_i \right) e_{q+1} + \left( \sum_{i=1}^r c_{p(n-1)+i} a_i \right) e_n = 0.$$

Since  $\{e_1, \dots, e_n\}$  is a basis and  $\text{gen } A = p+1$  we deduce that  $c_j \in P$ ,  $\forall j$ ,  $1 \leq j \leq m$ . So,  $\text{rel } W_{p,n,m} = m$ .

Let  $s \in \text{End}_R(W_{p,n,m})$ . Then  $s$  is induced by an endomorphism  $\tilde{s}$  of  $F$  which satisfies  $\tilde{s}(K) \subseteq K$ . For each  $j$ ,  $1 \leq j \leq n$ , there exists a family  $(\alpha_{i,j})$  of elements of  $R$  such that:

$$(1) \quad \tilde{s}(e_j) = \sum_{i=1}^n \alpha_{i,j} e_i$$

Since  $\tilde{s}(K) \subseteq K$ ,  $\forall j$ ,  $1 \leq j \leq m$ ,  $\exists$  a family  $(\beta_{i,j})$  of elements of  $R$  such that:

$$(2) \quad \tilde{s}(x_j) = \sum_{i=1}^m \beta_{i,j} x_i$$

From (1), (2) and the equality  $x_1 = a_1 e_1$  it follows that:

$$\sum_{q=1}^n \alpha_{q,1} a_1 e_q = \sum_{q=0}^{n-2} \left( \sum_{i=1}^{p+1} \beta_{pq+i,1} a_i \right) e_{q+1} + \left( \sum_{i=1}^r \beta_{p(n-1)+i,1} a_i \right) e_n.$$

Then, we get:

$$\forall q, 1 \leq q \leq n-1, \quad \alpha_{q,1} a_1 = \sum_{i=1}^{p+1} \beta_{p(q-1)+i,1} a_i$$

$$\text{and} \quad \alpha_{n,1} a_1 = \sum_{i=1}^r \beta_{p(n-1)+i,1} a_i.$$

We deduce that:  $\forall q$ ,  $2 \leq q \leq n$ ,  $\beta_{p(q-2)+p+1,1} \in P$  and  $\beta_{p(q-1)+1,1} \equiv \alpha_{q,1} [P]$ . So,

$$(3) \quad \forall q, 2 \leq q \leq n, \quad \alpha_{q,1} \in P.$$

Now, let  $j = p\ell + 1$  where  $1 \leq \ell \leq (n-1)$ . In this case,  $x_j = a_{p+1} e_\ell + a_1 e_{\ell+1}$ . From (1) and (2) it follows that:

$$\sum_{q=1}^n (\alpha_{q,\ell} a_{p+1} + \alpha_{q,\ell+1} a_1) e_q = \sum_{q=0}^{n-2} \left( \sum_{i=1}^{p+1} \beta_{pq+i,j} a_i \right) e_{q+1} + \left( \sum_{i=1}^r \beta_{p(n-1)+i,j} a_i \right) e_n.$$

Then, we get:

$$\forall q, 1 \leq q \leq n-1, \quad \alpha_{q,\ell} a_{p+1} + \alpha_{q,\ell+1} a_1 = \sum_{i=1}^{p+1} \beta_{p(q-1)+i,j} a_i$$

$$\text{and} \quad \alpha_{n,\ell} a_{p+1} + \alpha_{n,\ell+1} a_1 = \sum_{i=1}^r \beta_{p(n-1)+i,j} a_i.$$

We deduce that

$$\forall q, \ell, 1 \leq q, \ell \leq (n-1), \quad \alpha_{q,\ell} \equiv \beta_{p(q-1)+p+1,j} [P] \text{ and } \alpha_{q+1,\ell+1} \equiv \beta_{pq+1,j} [P],$$

whence  $\alpha_{q,\ell} \equiv \alpha_{q+1,\ell+1} [P]$ . Consequently,  $\forall q, 1 \leq q \leq n, \alpha_{q,q} \equiv \alpha_{1,1} [P]$  and  $\forall t, 1 \leq t \leq (n-1), \forall q, 1 \leq q \leq (n-t), \alpha_{q+t,q} \equiv \alpha_{1+t,1} \equiv 0 [P]$  by (3). Let  $\bar{s}$  be the endomorphism of  $\overline{W_{p,n,m}/PW_{p,n,m}}$  induced by  $s$ . If  $\alpha_{1,1}$  is a unit then  $\bar{s}$  is an isomorphism, else  $\overline{\mathbf{1}_{W_{p,n,m}} - s}$  is an isomorphism. By Lemma 3.2 we conclude that either  $s$  or  $(\mathbf{1}_{W_{p,n,m}} - s)$  is an isomorphism. Hence,  $\text{End}_R(W_{p,n,m})$  is local.  $\square$

**Remark 3.4.** *Observe that  $D(W_{1,n-1,n})$  is isomorphic to the indecomposable module built in the proof of (Warfield, 1970, Theorem 2).*

**Theorem 3.5.** *Let  $R$  be a commutative ring. The following assertions hold:*

- (1) *Assume that, for any integer  $p > 0$ , there exists a maximal ideal  $P$  and a finitely generated ideal  $A$  of  $R_P$  such that  $\text{gen}_{R_P} A \geq p + 1$ . Then, if  $(n, m)$  and  $(r, s)$  are two different pairs of integers, the  $(n, m)$ -purity and the  $(r, s)$ -purity are not equivalent.*
- (2) *Assume that, there exists an integer  $p > 0$  such that, for each maximal ideal  $P$ , for any finitely generated ideal  $A$  of  $R_P$ ,  $\text{gen}_{R_P} A \leq p$ . Then:*
  - (a) *for each integer  $n > 0$  the  $(\aleph_0, n)$ -purity (respectively  $(n, \aleph_0)$ -purity) is equivalent to the  $(np, n)$ -purity (respectively  $(n, np)$ -purity);*
  - (b) *if  $p > 1$ , then, for each integer  $n > 0$ , for each integer  $m, 1 \leq m \leq n(p-1)$ , the  $(n, m)$ -purity (respectively  $(m, n)$ -purity) is not equivalent to the  $(n, m+1)$ -purity (respectively  $(m+1, n)$ -purity).*

*Proof.* By Proposition 1.4 we may assume that  $R$  is local of maximal ideal  $P$ . By Theorem 1.1 the  $(n, m)$ -purity and the  $(r, s)$ -purity are equivalent if and only if so are the  $(m, n)$ -purity and the  $(s, r)$ -purity.

(1). Suppose that  $r > n$  and let  $t = \min(m, s)$ . Let  $q$  be the greatest divisor of  $(r-1)$  which is  $\leq t$  and  $p = (r-1)/q$ . Let  $A$  be a finitely generated ideal such that  $\text{gen } A > p$ . By way of contradiction, suppose that  $W_{p,q,r}$  is  $(n, m)$ -pure-projective. By Proposition 1.2  $W_{p,q,r}$  is a summand of  $\bigoplus_{i \in I} F_i$  where  $I$  is a finite set and  $\forall i \in I, F_i$  is a  $(m, n)$ -presented  $R$ -module. Since its endomorphism ring is local,  $W_{p,q,r}$  is an exchange module (see (Facchini, 1998, Theorem 2.8)). So, we have  $W_{p,q,r} \oplus (\bigoplus_{i \in I} G_i) \cong (\bigoplus_{i \in I} H_i) \oplus (\bigoplus_{i \in I} G_i)$  where  $\forall i \in I, G_i$  and  $H_i$  are submodules of  $F_i$  and  $F_i = G_i \oplus H_i$ . Let  $G = \bigoplus_{i \in I} G_i$ . Then  $G$  is finitely generated. By (Fuchs and Salce, 2001, Proposition V.7.1)  $\text{End}_R(G)$  is semilocal. By using Evans's theorem ((Facchini, 1998, Corollary 4.6)) we deduce that  $W_{p,q,r} \cong (\bigoplus_{i \in I} H_i)$ . Since  $W_{p,q,r}$  is indecomposable, we get that it is  $(m, n)$ -presented. This contradicts that  $\text{rel } W_{p,q,r} = r > n$ .

(2)(a) is an immediate consequence of Proposition 2.2.

(2)(b). There exist two integers  $q, t$  such that  $m+1 = (q-1)(p-1) + t$  with  $n \geq q \geq 1$  and  $1 \leq t \leq p$ . As in (1) we prove that  $W_{p-1,q,m+1}$  is not  $(m, n)$ -pure-projective.  $\square$

**Remark 3.6.** *In the previous theorem, when there exists an integer  $p > 1$  such that, for any finitely generated ideal  $A$   $\text{gen } A \leq p$ , we don't know if the  $(n, m)$ -purity and the  $(n, m+1)$ -purity are equivalent when  $n(p-1) + 1 \leq m \leq np-1$ . If  $R$  is a local ring of maximal ideal  $P$  with residue field  $k$  such that  $P^2 = 0$  and  $\dim_k P = p$  it is easy to show that each finitely presented  $R$ -module  $F$  with  $\text{gen } F = n$  and  $\text{rel } F = np$  is semisimple. So, the  $(np, n)$ -purity is equivalent to the  $(np-1, n)$ -purity.*

4.  $(n, m)$ -FLAT MODULES AND  $(n, m)$ -INJECTIVE MODULES

Let  $M$  be a right  $R$ -module. We say that  $M$  is  $(n, m)$ -flat if for any  $m$ -generated submodule  $K$  of a  $n$ -generated free left  $R$ -module  $F$ , the natural map:  $M \otimes_R K \rightarrow M \otimes_R F$  is a monomorphism. We say that  $M$  is  $(\aleph_0, m)$ -flat (respectively  $(n, \aleph_0)$ -flat) if  $M$  is  $(n, m)$ -flat for each integer  $n > 0$  (respectively  $m > 0$ ). We say that  $M$  is  $(n, m)$ -injective if for any  $m$ -generated submodule  $K$  of a  $n$ -generated free right  $R$ -module  $F$ , the natural map:  $\text{Hom}_R(F, M) \rightarrow \text{Hom}_R(K, M)$  is an epimorphism. We say that  $M$  is  $(\aleph_0, m)$ -injective (respectively  $(n, \aleph_0)$ -injective) if  $M$  is  $(n, m)$ -injective for each integer  $n > 0$  (respectively  $m > 0$ ). A ring  $R$  is called *left self  $(n, m)$ -injective* if  $R$  is  $(n, m)$ -injective as left  $R$ -module.

If  $R$  is a commutative domain, then an  $R$ -module is  $(1, 1)$ -flat (respectively  $(1, 1)$ -injective) if and only if it is torsion-free (respectively divisible).

The following propositions can be proved with standard technique: see (Zhang et al., 2005, Theorem 4.3 and Proposition 2.3). In these propositions the integers  $n$  or  $m$  can be replaced with  $\aleph_0$ .

**Proposition 4.1.** *Assume that  $R$  is an algebra over a commutative ring  $S$  and let  $E$  be an injective  $S$ -cogenerator. Let  $M$  be a right  $R$ -module. The following conditions are equivalent:*

- (1)  $M$  is  $(n, m)$ -flat;
- (2) each exact sequence  $0 \rightarrow L \rightarrow N \rightarrow M \rightarrow 0$  is  $(n, m)$ -pure, where  $L$  and  $N$  are right  $R$ -modules;
- (3) for each  $(m, n)$ -presented right module  $F$ , every homomorphism  $f : F \rightarrow M$  factors through a free right  $R$ -module;
- (4)  $\text{Hom}_S(M, E)$  is a  $(n, m)$ -injective left  $R$ -module.

**Proposition 4.2.** *Let  $M$  be a right module. The following conditions are equivalent:*

- (1)  $M$  is  $(n, m)$ -injective;
- (2) each exact sequence  $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$  is  $(m, n)$ -pure, where  $L$  and  $N$  are right  $R$ -modules;
- (3)  $M$  is a  $(m, n)$ -pure submodule of its injective hull.

**Proposition 4.3.** *Let  $R$  be a commutative ring. Then an  $R$ -module  $M$  is  $(n, m)$ -flat if and only if, for each maximal ideal  $P$ ,  $M_P$  is  $(n, m)$ -flat over  $R_P$ .*

**Lemma 4.4.** *Let  $M$  be  $p$ -generated right  $R$ -module where  $p$  is a positive integer. Then  $M$  is flat if and only if it is  $(1, p)$ -flat.*

*Proof.* Only “if” requires a proof. Let  $A$  be a left ideal. Assume that  $M$  is generated by  $x_1, \dots, x_p$ . So, if  $z \in M \otimes_R A$ ,  $z = \sum_{i=1}^p x_i \otimes a_i$  where  $a_1, \dots, a_p \in A$ . Suppose that the image of  $z$  in  $M \otimes_R R$  is 0. If  $A'$  is the left ideal generated by  $a_1, \dots, a_p$ , if  $z'$  is the element of  $M \otimes_R A'$  defined by  $z' = \sum_{i=1}^p x_i \otimes a_i$ , then  $z$  (respectively 0) is the image of  $z'$  in  $M \otimes_R A$  (respectively  $M \otimes_R R$ ). Since  $M$  is  $(1, p)$ -flat we successively deduce that  $z' = 0$  and  $z = 0$ .  $\square$

It is well known that each  $(1, \aleph_0)$ -flat right module is  $(\aleph_0, \aleph_0)$ -flat. **For each positive integer  $p$ , is each  $(1, p)$ -flat right module  $(\aleph_0, p)$ -flat?**

The following theorem and Theorem 4.11 give a partial answer to this question.

**Theorem 4.5.** *Let  $p$  be a positive integer and let  $R$  be a ring. For each positive integer  $n$ , assume that, for each  $p$ -generated submodule  $G$  of the left  $R$ -module  $R^n \oplus R$ ,  $(G \cap R^n)$  is the direct limit of its  $p$ -generated submodules. Then a right  $R$ -module  $M$  is  $(1, p)$ -flat if and only if it is  $(\aleph_0, p)$ -flat.*

*Proof.* We shall prove that  $M$  is  $(n, p)$ -flat by induction on  $n$ . Let  $G$  be a  $p$ -generated submodule of the left  $R$ -module  $R^{n+1} = R^n \oplus R$ . Let  $\pi$  be the projection of  $R^{n+1}$  onto  $R$  and  $G' = \pi(G)$ . Then  $G'$  is a  $p$ -generated left module. We put  $H = G \cap R^n$ . We have the following commutative diagram with exact horizontal sequences:

$$\begin{array}{ccccccc} M \otimes_R H & \rightarrow & M \otimes_R G & \xrightarrow{1_M \otimes \pi} & M \otimes_R G' & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M \otimes_R R^n & \rightarrow & M \otimes_R R^{n+1} & \xrightarrow{1_M \otimes \pi} & M \otimes_R R \rightarrow 0 \end{array}$$

Let  $u : G \rightarrow R^{n+1}$ ,  $u' : G' \rightarrow R$ ,  $w : R^n \rightarrow R^{n+1}$  be the inclusion maps and let  $v = u|_H$ . Then  $(1_M \otimes u')$  is injective. Let  $H'$  be a  $p$ -generated submodule of  $H$ . By the induction hypothesis  $M$  is  $(n, p)$ -flat. So,  $(1_M \otimes (v|_{H'}))$  is injective. It follows that  $(1_M \otimes v)$  is injective too. We conclude that  $(1_M \otimes u)$  is injective and  $M$  is  $(\aleph_0, p)$ -flat.  $\square$

**Corollary 4.6.** *Let  $p$  be a positive integer and let  $R$  be a ring such that each left ideal is  $(1, p)$ -flat. Then, for each positive integer  $q \leq p$ , a right  $R$ -module  $M$  is  $(1, q)$ -flat if and only if it is  $(\aleph_0, q)$ -flat.*

*Proof.* Let the notations be as in the previous theorem. Since  $G'$  is a flat left  $R$ -module by Lemma 4.4,  $H$  is a pure submodule of  $G$ . Let  $\{g_1, \dots, g_q\}$  be a spanning set of  $G$  and let  $h_1, \dots, h_t \in H$ . For each  $k$ ,  $1 \leq k \leq t$  there exist  $a_{k,1}, \dots, a_{k,q} \in R$  such that  $h_k = \sum_{i=1}^q a_{k,i} g_i$ . It follows that there exist  $g'_1, \dots, g'_q \in H$  such that  $\forall k$ ,  $1 \leq k \leq t$ ,  $h_k = \sum_{i=1}^q a_{k,i} g'_i$ . So, each finitely generated submodule of  $H$  is contained in a  $q$ -generated submodule. We conclude by applying Theorem 4.5.  $\square$

**Corollary 4.7.** *Let  $R$  be a commutative local ring of maximal ideal  $P$ . Assume that  $P^2 = 0$ . Let  $q$  a positive integer. Then:*

- (1) *each  $(1, q)$ -flat module is  $(\aleph_0, q)$ -flat;*
- (2) *each  $(1, q)$ -injective module is  $(\aleph_0, q)$ -injective.*

*Proof.* Let the notations be as in the previous theorem. We may assume that  $G \subseteq PR^{n+1}$ . Then  $G$  is a semisimple module and  $H$  is a direct summand of  $G$ . So, (1) is a consequence of Theorem 4.5.

(2). Let  $M$  be a  $(1, q)$ -injective module. We shall prove by induction on  $n$  that  $M$  is  $(n, q)$ -injective. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}_R(R, M) & \rightarrow & \text{Hom}_R(R^{n+1}, M) & \rightarrow & \text{Hom}_R(R^n, M) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \text{Hom}_R(G', M) & \rightarrow & \text{Hom}_R(G, M) & \rightarrow & \text{Hom}_R(H, M) \rightarrow 0 \end{array}$$

where the horizontal sequences are exact. By the induction hypothesis the left and the right vertical maps are surjective. It follows that the middle vertical map is surjective too.  $\square$

By (Shamsuddin, 2001, Example 5.2) or (Jøndrup, 1971, Theorem 2.3), for each integer  $n > 0$ , there exists a ring  $R$  for which each finitely generated left ideal is

$(1, n)$ -flat (hence  $(\aleph_0, n)$ -flat by Corollary 4.6) but there is a finitely generated left ideal which is not  $(1, n + 1)$ -flat. The following proposition gives other examples in the commutative case.

**Proposition 4.8.** *Let  $R$  be a commutative local ring of maximal ideal  $P$  with residue field  $k$ . Assume that  $P^2 = 0$  and  $\dim_k P > 1$ . Then, for each positive integer  $p < \dim_k P$ , there exists:*

- (1) a  $(p + 1, 1)$ -presented  $R$ -module which is  $(\aleph_0, p)$ -flat but not  $(1, p + 1)$ -flat;
- (2) a  $(\aleph_0, p)$ -injective  $R$ -module which is not  $(1, p + 1)$ -injective.

*Proof.* (1). Let  $F$  be a free  $R$ -module of rank  $(p + 1)$  with basis  $\{e_1, \dots, e_p, e_{p+1}\}$ , let  $(a_1, \dots, a_p, a_{p+1})$  be a family of linearly independent elements of  $P$ , let  $K$  be the submodule of  $F$  generated by  $\sum_{i=1}^{p+1} a_i e_i$  and let  $M = F/K$ . Then  $M \cong D(W_{p,1,p+1})$ . First, we show that  $K$  is a  $(1, p)$ -pure submodule of  $F$ . We consider the following equation:

$$(4) \quad \sum_{j=1}^p r_j x_j = s \left( \sum_{i=1}^{p+1} a_i e_i \right)$$

where  $r_1, \dots, r_p, s \in R$  and with unknowns  $x_1, \dots, x_p$ . Assume that this equation has a solution in  $F$ . Suppose there exists  $\ell$ ,  $1 \leq \ell \leq p$ , such that  $r_\ell$  is a unit. For each  $j$ ,  $1 \leq j \leq p$ , we put  $x'_j = \delta_{j,\ell} r_\ell^{-1} s \left( \sum_{i=1}^{p+1} a_i e_i \right)$ . It is easy to check that  $(x'_1, \dots, x'_p)$  is a solution of (4) in  $K$ . Now we assume that  $r_j \in P$ ,  $\forall j$ ,  $1 \leq j \leq p$ . Suppose that  $(x_1, \dots, x_p)$  is a solution of (4) in  $F$ . For each  $j$ ,  $1 \leq j \leq p$ ,  $x_j = \sum_{i=1}^{p+1} c_{j,i} e_i$ , where  $c_{j,i} \in R$ . We get the following equality:

$$(5) \quad \sum_{i=1}^{p+1} \left( \sum_{j=1}^p r_j c_{j,i} \right) e_i = \sum_{i=1}^{p+1} s a_i e_i$$

We deduce that:

$$(6) \quad \forall i, \quad 1 \leq i \leq p + 1, \quad \sum_{j=1}^p r_j c_{j,i} = s a_i$$

So, if  $s$  is a unit,  $\forall i$ ,  $1 \leq i \leq p + 1$ ,  $a_i \in \sum_{j=1}^p R r_j$ . It follows that

$$\dim_k \left( \sum_{i=1}^{p+1} R a_i \right) \leq p$$

that is false. So,  $s \in P$ . In this case (4) has the nil solution. Hence  $M$  is  $(\aleph_0, p)$ -flat by Propositions 4.1(2) and 4.7.

By way of contradiction suppose that  $M$  is  $(1, p + 1)$ -flat. It follows that  $K$  is a  $(1, p + 1)$ -pure submodule of  $F$  by Proposition 4.1. Since  $M$  is  $(1, p + 1)$ -pure-projective we deduce that  $M$  is free. This is false.

(2). Let  $E$  be an injective  $R$ -cogenerator. Then  $\text{Hom}_R(M, E)$  is  $(\aleph_0, p)$ -injective but not  $(1, p + 1)$ -injective by Proposition 4.1(4).  $\square$

In a similar way we show the following proposition.

**Proposition 4.9.** *Let  $R$  be a commutative local ring of maximal ideal  $P$ . Assume that  $P^2 = 0$ . Let  $M$  be a  $(m, 1)$ -presented  $R$ -module with  $m > 1$ , let  $\{x_1, \dots, x_m\}$  be*

a spanning set of  $M$  and let  $\sum_{j=1}^m a_j x_j = 0$  be the relation of  $M$ , where  $a_1, \dots, a_m \in P$ . If  $p = \text{gen}(\sum_{j=1}^m Ra_j) - 1 > 0$ , then:

- (1)  $M$  is  $(\aleph_0, p)$ -flat but not  $(1, m)$ -flat;
- (2)  $\text{Hom}_R(M, E)$  is  $(\aleph_0, p)$ -injective but not  $(1, m)$ -injective, where  $E$  is an injective  $R$ -cogenerator.

When  $R$  is an arithmetical commutative ring, i.e. its lattice of ideals is distributive, each  $(1, 1)$ -flat module is flat and by (Fuchs and Salce, 2001, Theorem VI.9.10) the converse holds if  $R$  is a commutative domain (it is also true if each principal ideal is flat). However we shall see that there exist non-arithmetical commutative rings for which each  $(1, 1)$ -flat module is flat. Recall that a left (or right)  $R$ -module  $M$  is *torsionless* if the natural map  $M \rightarrow (M^*)^*$  is injective.

**Proposition 4.10.** *For each ring  $R$  the following conditions are equivalent:*

- (1)  $R$  is right self  $(\aleph_0, 1)$ -injective;
- (2) each finitely presented cyclic left  $R$ -module is torsionless;
- (3) each finitely generated left ideal  $A$  satisfies  $A = 1 - \text{ann}(r - \text{ann}(A))$ .

*Proof.* We prove (1)  $\Leftrightarrow$  (2) as (Jain, 1973, Theorem 2.3) and (2)  $\Leftrightarrow$  (3) is easy.  $\square$

**Theorem 4.11.** *Let  $R$  be a right perfect ring which is right self  $(\aleph_0, 1)$ -injective. Then each  $(1, 1)$ -flat right module is projective.*

*Proof.* Let  $M$  be a  $(1, 1)$ -flat right  $R$ -module. It is enough to show that  $M$  is flat. Let  $A$  be a finitely generated left ideal of  $R$ . Assume that  $\{a_1, \dots, a_n\}$  is a minimal system of generators of  $A$  with  $n > 1$ . Let  $z \in M \otimes_R A$  such that its image in  $M$  is 0. We have  $z = \sum_{i=1}^n y_i \otimes a_i$ , where  $y_1, \dots, y_n \in M$ , and  $\sum_{i=1}^n y_i a_i = 0$ . For each  $i$ ,  $1 \leq i \leq n$ , we set  $A_i = \sum_{(j \neq i)}^n Ra_j$ . Then,  $\forall i$ ,  $1 \leq i \leq n$ ,  $A_i \subset A$ . For each finitely generated left ideal  $B$  we have  $B = 1 - \text{ann}(r - \text{ann}(B))$ . It follows that,  $\forall i$ ,  $1 \leq i \leq n$ ,  $r - \text{ann}(A) \subset r - \text{ann}(A_i)$ . Let  $b_i \in r - \text{ann}(A_i) \setminus r - \text{ann}(A)$ . Then  $y_i a_i b_i = 0$ . From the  $(1, 1)$ -flatness of  $M$  we deduce that  $y_i = \sum_{k=1}^{m_i} y'_{i,k} c_{i,k}$ , where  $y'_{i,1}, \dots, y'_{i,m_i} \in M$  and  $c_{i,1}, \dots, c_{i,m_i} \in R$  with  $c_{i,k} a_i b_i = 0$ ,  $\forall k$ ,  $1 \leq k \leq m_i$ . It follows that  $z = \sum_{i=1}^n (\sum_{k=1}^{m_i} y'_{i,k} \otimes c_{i,k} a_i)$ . Let  $A^{(1)}$  be the left ideal generated by  $\{c_{k,i} a_i \mid 1 \leq i \leq n, 1 \leq k \leq m_i\}$ . Then  $A^{(1)} \subset A$ ; else,  $\forall i$ ,  $1 \leq i \leq n$ ,  $a_i = \sum_{j=1}^n (\sum_{k=1}^{m_j} d_{i,j,k} c_{j,k} a_j)$  with  $d_{i,j,k} \in R$ ; we get that  $a_i b_i = \sum_{j=1}^n (\sum_{k=1}^{m_j} d_{i,j,k} c_{j,k} a_j b_i)$ ; but  $a_j b_i = 0$  if  $j \neq i$  and  $c_{i,k} a_i b_i = 0$ ; so, there is a contradiction because the second member of the previous equality is 0 while  $a_i b_i \neq 0$ . Let  $\{a_1^{(1)}, \dots, a_{n_1}^{(1)}\}$  be a minimal system of generators of  $A^{(1)}$ . So,  $z = \sum_{i=1}^{n_1} y_i^{(1)} \otimes a_i^{(1)}$  where  $y_1^{(1)}, \dots, y_{n_1}^{(1)} \in M$ , and  $z$  is the image of  $z^{(1)} \in M \otimes_R A^{(1)}$  defined by  $z^{(1)} = \sum_{i=1}^{n_1} y_i^{(1)} \otimes a_i^{(1)}$ . If  $n_1 \leq 1$  we conclude that  $z^{(1)} = 0$  since  $M$  is  $(1, 1)$ -flat, and  $z = 0$ . If  $n_1 > 1$ , in the same way we get that  $z^{(1)}$  is the image of an element  $z^{(2)} \in M \otimes_R A^{(2)}$  where  $A^{(2)}$  is a left ideal such that  $A^{(2)} \subset A^{(1)}$ . If  $\text{gen } A^{(2)} > 1$  we repeat this process, possibly several times, until we get a left ideal  $A^{(l)}$  with  $\text{gen } A^{(l)} \leq 1$ ; this is possible because  $R$  satisfies the descending chain condition on finitely generated left ideals since it is right perfect (see (Renault, 1975, Théorème 5 p.130)). The  $(1, 1)$ -flatness of  $M$  implies that  $z^{(l)} = 0$  and  $z = 0$ . So,  $M$  is projective.  $\square$

Let  $\mathcal{P}$  be a ring property. We say that a commutative ring  $R$  is *locally  $\mathcal{P}$*  if  $R_P$  satisfies  $\mathcal{P}$  for each maximal ideal  $P$ .

The following corollary is a consequence of Theorem 4.11 and Proposition 4.3.

**Corollary 4.12.** *Let  $R$  be a commutative ring which is locally perfect and locally self  $(\aleph_0, 1)$ -injective. Then each  $(1, 1)$ -flat  $R$ -module is flat.*

## 5. $(n, m)$ -COHERENT RINGS

We say that a ring  $R$  is left  $(n, m)$ -coherent if each  $m$ -generated submodule of a  $n$ -generated free left  $R$ -module is finitely presented. We say that  $R$  is left  $(\aleph_0, m)$ -coherent (respectively  $(n, \aleph_0)$ -coherent) if for each integer  $n > 0$  (respectively  $m > 0$ )  $R$  is left  $(n, m)$ -coherent. The following theorem can be proven with standard technique: see (Zhang et al., 2005, Theorems 5.1 and 5.7). In this theorem the integers  $n$  or  $m$  can be replaced with  $\aleph_0$ .

**Theorem 5.1.** *Let  $R$  be a ring and  $n, m$  two fixed positive integers. Assume that  $R$  is an algebra over a commutative ring  $S$ . Let  $E$  be an injective  $S$ -cogenerator. Then the following conditions are equivalent:*

- (1)  $R$  is left  $(n, m)$ -coherent;
- (2) any direct product of right  $(n, m)$ -flat  $R$ -modules is  $(n, m)$ -flat;
- (3) for any set  $\Lambda$ ,  $R^\Lambda$  is a  $(n, m)$ -flat right  $R$ -module;
- (4) any direct limit of a direct system of  $(n, m)$ -injective left  $R$ -modules is  $(n, m)$ -injective;
- (5) for any exact sequence of left modules  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ ,  $C$  is  $(n, m)$ -injective if so is  $B$  and if  $A$  is a  $(\aleph_0, m)$ -pure submodule of  $B$ ;
- (6) for each  $(n, m)$ -injective left  $R$ -module  $M$ ,  $\text{Hom}_S(M, E)$  is  $(n, m)$ -flat.

It is well known that each left  $(1, \aleph_0)$ -coherent ring is left  $(\aleph_0, \aleph_0)$ -coherent. **For each positive integer  $p$ , is each left  $(1, p)$ -coherent ring left  $(\aleph_0, p)$ -coherent?**

Propositions 5.2 and 5.3 and Theorem 5.4 give a partial answer to this question.

**Proposition 5.2.** *Let  $p$  be a positive integer and let  $R$  be a ring. For each positive integer  $n$ , assume that, for each  $p$ -generated submodule  $G$  of the left  $R$ -module  $R^n \oplus R$ ,  $(G \cap R^n)$  is the direct limit of its  $p$ -generated submodules. Then the following conditions are equivalent:*

- (1)  $R$  is left  $(1, p)$ -coherent;
- (2)  $R$  is left  $(\aleph_0, p)$ -coherent.

Moreover, when these conditions hold each  $(1, p)$ -injective left module is  $(\aleph_0, p)$ -injective.

*Proof.* It is obvious that (2)  $\Rightarrow$  (1).

(1)  $\Rightarrow$  (2). Let  $\Lambda$  be a set. By Theorem 5.1  $R^\Lambda$  is a  $(1, p)$ -flat right module. From Theorem 4.5 we deduce that  $R^\Lambda$  is a  $(\aleph_0, p)$ -flat right module. By using again Theorem 5.1 we get (2).

Let  $M$  be a  $(1, p)$ -injective left module. By Theorem 5.1  $M^\sharp$  is a  $(1, p)$ -flat right  $R$ -module. Then it is also  $(\aleph_0, p)$ -flat. We deduce that  $(M^\sharp)^\sharp$  is a  $(\aleph_0, p)$ -injective left module. Since  $M$  is a pure submodule of  $(M^\sharp)^\sharp$ , it follows that  $M$  is  $(\aleph_0, p)$ -injective too.  $\square$

**Proposition 5.3.** *Let  $R$  be a commutative perfect ring. Then  $R$  is Artinian if and only if it is  $(1, 1)$ -coherent.*

*Proof.* Suppose that  $R$  is  $(1, 1)$ -coherent. Since  $R$  is perfect,  $R$  is a finite product of local rings. So, we may assume that  $R$  is local of maximal ideal  $P$ . Let  $S$  be a minimal non-zero ideal of  $R$  generated by  $s$ . Then  $P$  is the annihilator of  $s$ . So,  $P$  is finitely generated and it is the sole prime ideal of  $R$ . Since all prime ideals of  $R$  are finitely generated,  $R$  is Noetherian. On the other hand  $R$  satisfies the descending chain condition on finitely generated ideals. We conclude that  $R$  is Artinian.  $\square$

Except in some particular cases, we don't know if each  $(1, p)$ -injective module is  $(\aleph_0, p)$ -injective, even if we replace  $p$  by  $\aleph_0$ .

**Theorem 5.4.** *Let  $R$  be a ring which is right perfect, left  $(1, 1)$ -coherent and right self  $(\aleph_0, 1)$ -injective. Then each  $(1, 1)$ -injective left module is  $(\aleph_0, \aleph_0)$ -injective and  $R$  is left coherent.*

*Proof.* Let  $M$  be a left  $(1, 1)$ -injective module. By Theorem 5.1  $M^\#$  is  $(1, 1)$ -flat. Whence  $M^\#$  is projective by Theorem 4.11. We do as in the proof of the previous proposition to conclude that  $M$  is  $(\aleph_0, \aleph_0)$ -injective.

For each set  $\Lambda$ ,  $R^\Lambda$  is a  $(1, 1)$ -flat right module by Theorem 5.1. It follows that  $R^\Lambda$  is a projective right module by Theorem 4.11.  $\square$

Recall that a ring is *quasi-Frobenius* if it is Artinian and self-injective.

**Corollary 5.5.** *Let  $R$  be a quasi-Frobenius ring. Then, for each right (or left)  $R$ -module  $M$ , the following conditions are equivalent:*

- (1)  $M$  is  $(1, 1)$ -flat;
- (2)  $M$  is projective;
- (3)  $M$  is injective;
- (4)  $M$  is  $(1, 1)$ -injective.

*Proof.* It is well known that (2)  $\Leftrightarrow$  (3). By Theorem 4.11 (1)  $\Leftrightarrow$  (2) because  $R$  satisfies the conditions of this theorem, and it is obvious that (3)  $\Rightarrow$  (4) and the converse holds by Theorem 5.4.  $\square$

We prove the following theorem as (Couchot, 1982, Théorème 1.4).

**Theorem 5.6.** *Let  $R$  be a commutative ring and  $n, m$  two fixed positive integers. The following conditions are equivalent:*

- (1)  $R$  is  $(n, m)$ -coherent;
- (2) for each multiplicative subset  $S$  of  $R$ ,  $S^{-1}R$  is  $(n, m)$ -coherent, and for each  $(n, m)$ -injective  $R$ -module  $M$ ,  $S^{-1}M$  is  $(n, m)$ -injective over  $S^{-1}R$ ;
- (3) For each maximal ideal  $P$ ,  $R_P$  is  $(n, m)$ -coherent and for each  $(n, m)$ -injective  $R$ -module  $M$ ,  $M_P$  is  $(n, m)$ -injective over  $R_P$ .

Recall that a ring  $R$  is a *right IF-ring* if each right injective  $R$ -module is flat.

**Theorem 5.7.** *Let  $R$  be a commutative ring which is locally perfect,  $(1, 1)$ -coherent and self  $(1, 1)$ -injective. Then:*

- (1)  $R$  is coherent, self  $(\aleph_0, \aleph_0)$ -injective and locally quasi-Frobenius;
- (2) each  $(1, 1)$ -flat module is flat;
- (3) each  $(1, 1)$ -injective module is  $(\aleph_0, \aleph_0)$ -injective.



*Proof.* By Theorem 5.6  $R_P$  is  $(1, 1)$ -coherent and  $(1, 1)$ -injective for each maximal ideal  $P$ . Let  $a$  be a generator of a minimal non-zero ideal of  $R_P$ . Then  $PR_P$  is the annihilator of  $a$  and consequently  $PR_P$  is finitely generated over  $R_P$ . Since all prime ideals of  $R_P$  are finitely generated, we deduce that  $R_P$  is Artinian for each maximal ideal  $P$ . Moreover, the  $(1, 1)$ -injectivity of  $R_P$  implies that the socle of  $R_P$  (the sum of all minimal non-zero ideals) is simple. It follows that  $R_P$  is quasi-Frobenius for each maximal ideal  $P$ .

Let  $M$  be a  $(\aleph_0, \aleph_0)$ -injective  $R$ -module. By Theorem 5.6  $M_P$  is  $(1, 1)$ -injective for each maximal ideal  $P$ . By Corollary 5.5  $M_P$  is injective for each maximal ideal  $P$ . We conclude that  $R$  is self  $(\aleph_0, \aleph_0)$ -injective and it is coherent by Theorem 5.6.

If  $M$  is  $(1, 1)$ -injective, we prove as above that  $M_P$  is injective for each maximal ideal  $P$ . It follows that  $M$  is  $(\aleph_0, \aleph_0)$ -injective.

The second assertion is an immediate consequence of Corollary 4.12.  $\square$

The following proposition is easy to prove:

**Proposition 5.8.** *A ring  $R$  is left  $(\aleph_0, 1)$ -coherent if and only if each finitely generated right ideal has a finitely generated left annihilator.*

**Example 5.9.** *Let  $V$  be a non-Noetherian (commutative) valuation domain whose order group is not the additive group of real numbers and let  $R = V[[X]]$  be the power series ring in one indeterminate over  $V$ . Since  $R$  is a domain,  $R$  is  $(\aleph_0, 1)$ -coherent. But, in (Anderson and Watkins, 1987) it is proven that there exist two elements  $f$  and  $g$  of  $R$  such that  $Rf \cap Rg$  is not finitely generated. By using the exact sequence  $0 \rightarrow Rf \cap Rg \rightarrow Rf \oplus Rg \rightarrow Rf + Rg \rightarrow 0$  we get that  $Rf + Rg$  is not finitely presented. So,  $R$  is not  $(1, 2)$ -coherent.*

#### REFERENCES

- Anderson, M. and Watkins, J. ((1987)). Coherence of power series rings over pseudo-Bezout domains. *J. Algebra*, 107:187–194.
- Couchot, F. ((1982)). Exemples d’anneaux auto fp-injectifs. *Comm. Algebra*, 10(4):339–360.
- Facchini, A. ((1998)). *Module theory. Endomorphism rings and direct sum decompositions in some classes of modules*. Progress in Mathematics (Boston, Mass.). Birkhäuser, Basel.
- Fuchs, L. and Salce, L. ((2001)). *Modules over Non-Noetherian Domains*. Number 84 in Mathematical Surveys and Monographs. American Mathematical Society, Providence.
- Jain, S. ((1973)). Flatness and FP-injectivity. *Proc. Amer. Math. Soc.*, 41(2):437–442.
- Jøndrup, S. ((1971)). p.p.rings and finitely generated flat ideals. *Proc. Amer. Math. Soc.*, 28:431–435.
- Puninski, G., Prest, M., and Rothmaler, P. ((1999)). Rings described by various purities. *Comm. Algebra*, 27(5):2127–2162.
- Renault, G. ((1975)). *Algèbre non commutative*. Collection “Varia Mathematica”. Gauthier-Villars.
- Shamsuddin, A. ((2001)).  $n$ -injective and  $n$ -flat modules. *Comm. Algebra*, 29(5):2039–2050.
- Stenström, B. ((1967)). Pure submodules. *Ark. Mat.*, 7:159–171.

- Warfield, R. ((1969)a). Purity and algebraic compactness for modules. *Pac. J. Math.*, 28(3):689–719.
- Warfield, R. ((1970)). Decomposability of finitely presented modules. *Proc. Amer. Math. Soc.*, 25:167–172.
- Warfield, R. B. ((1969)b). Relatively injective modules. *unpublished paper*.
- Wiegand, R. and Wiegand, S. ((1975)). Finitely generated modules over Bezout rings. *Pac. J. Math.*, 58(2):655–664.
- Zhang, X., Chen, J., and Zhang, J. ((2005)). On  $(m, n)$ -injective modules and  $(m, n)$ -coherent rings. *Algebra Colloq.*, 12(1):149–160.

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