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# New global stability estimates for the Gel'fand-Calderon inverse problem

R.G. Novikov

CNRS (UMR 7641), Centre de Mathématiques Appliquées, Ecole Polytechnique,  
91128 Palaiseau, France  
e-mail: novikov@cmap.polytechnique.fr

**Abstract.** We prove new global stability estimates for the Gel'fand-Calderon inverse problem in 3D. For sufficiently regular potentials this result of the present work is a principal improvement of the result of [G.Alessandrini, Stable determination of conductivity by boundary measurements, Appl. Anal. **27** (1988), 153-172].

## 1. Introduction

We consider the equation

$$-\Delta\psi + v(x)\psi = 0, \quad x \in D, \quad (1.1)$$

where

$$D \text{ is an open bounded domain in } \mathbb{R}^d, \quad d \geq 2, \quad \partial D \in C^2, \quad v \in L^\infty(D). \quad (1.2)$$

Equation (1.1) arises, in particular, in quantum mechanics, acoustics, electrodynamics. Formally, (1.1) looks as the Schrödinger equation with potential  $v$  at zero energy.

We consider the map  $\Phi$  such that

$$\frac{\partial\psi}{\partial\nu}\bigg|_{\partial D} = \Phi(\psi|_{\partial D}) \quad (1.3)$$

for all sufficiently regular solutions  $\psi$  of (1.1) in  $\bar{D} = D \cup \partial D$ , where  $\nu$  is the outward normal to  $\partial D$ . Here we assume also that

$$0 \text{ is not a Dirichlet eigenvalue for the operator } -\Delta + v \text{ in } D. \quad (1.4)$$

The map  $\Phi$  is called the Dirichlet-to-Neumann map for equation (1.1) and is considered as boundary measurements for (physical model described by) (1.1).

We consider the following inverse boundary value problem for equation (1.1):

**Problem 1.1.** Given  $\Phi$ , find  $v$ .

This problem can be considered as the Gel'fand inverse boundary value problem for the Schrödinger equation at zero energy (see [9], [16]). This problem can be also considered as a generalization of the Calderon problem of the electrical impedance tomography (see [5], [23], [16]).

We remind that the simplest interpretation of  $D$ ,  $v$  and  $\Phi$  in the framework of the electrical impedance tomography consists in the following (see, for example, [16], [14], [13]):  $D$  is a body with isotropic conductivity  $\sigma(x)$  (where  $\sigma \geq \sigma_{min} > 0$ ),

$$v(x) = (\sigma(x))^{-1/2} \Delta(\sigma(x))^{1/2}, \quad x \in D, \quad (1.5)$$

$$\Phi = \sigma^{-1/2} \left( \Lambda \sigma^{-1/2} + \frac{\partial \sigma^{1/2}}{\partial \nu} \right), \quad (1.6)$$

where  $\Delta$  is the Laplacian,  $\Lambda$  is the voltage-to-current map on  $\partial D$ , and  $\sigma^{-1/2}$ ,  $\partial\sigma^{1/2}/\partial\nu$  in (1.6) denote the multiplication operators by the functions  $\sigma^{-1/2}|_{\partial D}$ ,  $(\partial\sigma^{1/2}/\partial\nu)|_{\partial D}$ , respectively. In addition, (1.4) is always fulfilled if  $v$  is given by (1.5) (where  $\sigma \geq \sigma_{\min} > 0$  and  $\sigma$  is twice differential in  $L^\infty(D)$ ).

Problem 1.1 includes, in particular, the following questions: (a) uniqueness, (b) reconstruction, (c) stability.

Global uniqueness theorems for Problem 1.1 (in its Calderon or Gel'fand form) in dimension  $d \geq 3$  were obtained for the first time in [23] and [16]. In particular, according to [16], under assumptions (1.2), (1.4), the map  $\Phi$  uniquely determines  $v$ .

A global reconstruction method for Problem 1.1 in dimension  $d \geq 3$  was proposed for the first time in [16]. In its simplest form, this method of [16] consists in reducing Problem 1.1 to Problem 4.1 via formulas and equations (5.1), (5.2) for  $v_2 = v$ ,  $\psi_2 = \psi$ ,  $h_1 = h$ ,  $v_1 \equiv 0$ ,  $\psi_1(x, k) = e^{ikx}$ ,  $R_1(x, y, k) = G(x - y, k)$ ,  $h_1 \equiv 0$  and in solving Problem 4.1 via formula (4.10), see Sections 3, 4 and 5 of the present paper.

Global stability estimates for Problem 1.1 in dimension  $d \geq 3$  were given for the first time in [1]. A variation of this result is presented as Theorem 2.1 of Section 2 of the present paper.

We remind that in global results one does not assume that  $\sigma$  or  $v$  is small in some sense or that  $\sigma$  or  $v$  is piecewise constant or piecewise real-analytic. For piecewise constant or piecewise real-analytic  $\sigma$  the first uniqueness results for the Calderon version of Problem 1.1 in dimension  $d \geq 2$  were given in [6], [11].

As regards global results given in the literature on Problem 1.1 in dimension  $d = 2$ , the reader is referred to Corollary 2 of [16] and to [15], [13], [4] and to references given in [4].

In the present work we continue studies of [16], [17], [19], [21], [20]. One of the main purposes of these studies was developing an efficient reconstruction algorithm for Problem 1.1 in dimension  $d = 3$ . Note that the aforementioned global reconstruction method of [16] is fine in the sense that it consists in solving Fredholm linear integral equations of the second type and using explicit formulas but this reconstruction is not optimal with respect to its stability properties. An effectivization of this reconstruction of [16] with respect to its stability properties was developed in [17], [19], [20]. In [21] and in the present work we illustrate our progress in stability by proving new stability estimates for Problem 1.1 in 3D and, in particular, by proving Theorem 2.2 of Section 2 in 3D. For sufficiently regular potentials this theorem is a principle improvement of the aforementioned Alessandrini stability result of [1].

Note that algorithms developed in [17], [19], [20] for finding  $v$  from  $\Phi$  in 3D can be used even if  $v$  has discontinuities. In this case these algorithms can be considered as methods for finding the smooth part of  $v$  from  $\Phi$ . In particular, the global algorithms of [17], [20] can be considered as methods for approximate finding the Fourier transform  $\hat{v}(p)$ ,  $|p| \leq 2\rho$ , from  $\Phi$  for sufficiently great  $\rho$ , where  $\hat{v}(p)$  is defined by (4.12),  $d = 3$ . However, for the case when  $v$  is not smooth enough we did not manage yet to formalize our principle progress of [20] in global stability as a rigorous mathematical theorem.

Note that in [21] our new stability estimates of Theorem 2.2 of the next section were proved in the Born approximation (that is in the linear approximation near zero potential)

only. Besides, a scheme of proof of these estimates was also mentioned in [21] for potentials with sufficiently small norm in dimension  $d = 3$ . In the present work we give a complete proof of Theorem 2.2 in the general (or by other words global) case in dimension  $d = 3$ .

Our new stability estimates for Problem 1.1 (that is the estimates of Theorem 2.2) are presented and discussed in Section 2. In addition, relations between Theorems 2.1, 2.2 and global reconstructions of [16], [17], [20] are explained in Section 3. The proof of Theorem 2.2 for the global case in 3D is given in Section 8.

## 2. Stability estimates

As in [21] we assume for simplicity that

$$\begin{aligned} D \text{ is an open bounded domain in } \mathbb{R}^d, \partial D \in C^2, \\ v \in W^{m,1}(\mathbb{R}^d) \text{ for some } m > d, \text{ supp } v \subset D, d \geq 2, \end{aligned} \quad (2.1)$$

where

$$W^{m,1}(\mathbb{R}^d) = \{v : \partial^J v \in L^1(\mathbb{R}^d), |J| \leq m\}, \quad m \in \mathbb{N} \cup 0, \quad (2.2)$$

where

$$J \in (\mathbb{N} \cup 0)^d, \quad |J| = \sum_{i=1}^d J_i, \quad \partial^J v(x) = \frac{\partial^{|J|} v(x)}{\partial x_1^{J_1} \dots \partial x_d^{J_d}}.$$

Let

$$\|v\|_{m,1} = \max_{|J| \leq m} \|\partial^J v\|_{L^1(\mathbb{R}^d)}. \quad (2.3)$$

Let

$$\begin{aligned} \|A\| \text{ denote the norm of an operator} \\ A : L^\infty(\partial D) \rightarrow L^\infty(\partial D). \end{aligned} \quad (2.4)$$

We recall that if  $v_1, v_2$  are potentials satisfying (1.2), (1.3), where  $D$  is fixed, then

$$\Phi_1 - \Phi_2 \text{ is a compact operator in } L^\infty(\partial D), \quad (2.5)$$

where  $\Phi_1, \Phi_2$  are the DtN maps for  $v_1, v_2$  respectively, see [16], [17]. Note also that (2.1)  $\Rightarrow$  (1.2).

**Theorem 2.1** (variation of the result of [1]). *Let conditions (1.4), (2.1) hold for potentials  $v_1$  and  $v_2$ , where  $D$  is fixed,  $d \geq 3$ . Let  $\|v_j\|_{m,1} \leq N$ ,  $j = 1, 2$ , for some  $N > 0$ . Let  $\Phi_1, \Phi_2$  denote the DtN maps for  $v_1, v_2$ , respectively. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_1 (\ln(3 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_1}, \quad (2.6)$$

where  $c_1 = c_1(N, D, m)$ ,  $\alpha_1 = (m - d)/m$ ,  $\|\Phi_1 - \Phi_2\|$  is defined according to (2.4).

As it was mentioned in [21], Theorem 2.1 follows from formulas (4.9)-(4.11), (5.1) (of Sections 4 and 5).

A disadvantage of estimate (2.6) is that

$$\alpha_1 < 1 \text{ for any } m > d \text{ even if } m \text{ is very great.} \quad (2.7)$$

**Theorem 2.2.** *Let the assumptions of Theorem 2.1 hold. Then*

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_2(\ln(3 + \|\Phi_1 - \Phi_2\|^{-1}))^{-\alpha_2}, \quad (2.8)$$

where  $c_2 = c_2(N, D, m)$ ,  $\alpha_2 = m - d$ ,  $\|\Phi_1 - \Phi_2\|$  is defined according to (2.4).

A principal advantage of estimate (2.8) in comparison with (2.6) is that

$$\alpha_2 \rightarrow +\infty \text{ as } m \rightarrow +\infty, \quad (2.9)$$

in contrast with (2.7).

In the Born approximation, that is in the linear approximation near zero potential, Theorem 2.2 was proved in [21].

For sufficiently small  $N$  in dimension  $d = 3$ , a scheme of proof of Theorem 2.2 was also mentioned in [21]. This scheme involves, in particular, results of [17], [19].

In the general (or by other words global) case Theorem 2.2 in dimension  $d = 3$  is proved in Section 8. This proof involves, in particular, results of [17], [20]. We see no principal difficulties (except restrictions in time) to give a similar proof in dimension  $d > 3$ .

We would like to mention that, under the assumptions of Theorems 2.1 and 2.2, according to the Mandache results of [13], the estimate (2.8) can not hold with  $\alpha_2 > m(2d - 1)/d$ . However, we think that this instability result of [13] can be sharpened considerably and that in this sense our Theorem 2.2 is almost optimal already.

For additional information concerning stability and instability results given in the literature for Problem 1.1 (in its Calderon or Gel'fand form) the reader is referred to [1], [13], [21], [22] and references therein.

### 3. Theorems 2.1 and 2.2 and global reconstructions of [16], [17], [20]

In this section we explain relations between Theorems 2.1, 2.2 and global reconstructions of [16], [17], [20] for Problem 1.1, where we assume that dimension  $d = 3$ . (The case  $d > 3$  is similar to the case  $d = 3$ .) The scheme of the aforementioned global reconstructions consists in the following:

$$\Phi \rightarrow h|_{\Xi} \rightarrow \hat{v}_{\rho, \tau} \rightarrow v, \quad (3.1)$$

where  $h$  is the Faddeev generalized scattering amplitude (at zero energy) defined in the complex domain  $\Theta$  (see formulas (4.4), (4.5), (4.1)-(4.3) of Section 4 for  $d = 3$ ),  $\Xi$  is an appropriate subset of  $\Theta$ , for example  $\Xi$  is the subset of  $\Theta$  with the imaginary part length equal to  $\rho$ ,  $\hat{v}_{\rho, \tau}$  is an approximation to  $\hat{v}$  on  $\mathcal{B}_{2\tau\rho}$ ,

$$\hat{v}_{\rho, \tau} \rightarrow \hat{v} \text{ on } \mathcal{B}_{2\tau\rho} \text{ as } \rho \rightarrow +\infty, \quad (3.2)$$

where  $\hat{v}$  is the Fourier transform of  $v$  (see formula (4.12),  $d = 3$ ),  $\mathcal{B}_r$  is the ball of the radius  $r$ ,  $\tau$  is a fixed parameter,  $0 < \tau \leq 1$ .

*3.1. Theorem 2.1 and global reconstructions of [16], [17].* For the case when

$$\Xi = b\Theta_\rho \text{ (see formulas (6.2)),} \quad (3.3a)$$

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$$h|_{b\Theta_\rho} \text{ is determined from } \Phi \text{ via} \quad (3.3b)$$

$$(5.1), (5.2) \text{ considered for } v_2 = v, \psi_2 = \psi, h_2 = h,$$

$$v_1 \equiv 0, \psi_1(x, k) = e^{ikx}, R_1(x, y, k) = G(x - y, k), h_1 \equiv 0,$$

$$\begin{aligned} \hat{v}_{\rho, \tau}(p) &= h(k(p), l(p)), \quad p \in \mathcal{B}_{2\tau\rho}, \quad \tau = 1, \\ \text{for some vector - functions } &k(p), l(p) \\ \text{such that } &k(p), l(p) \in b\Theta_\rho, \quad k(p) - l(p) = p, \end{aligned} \quad (3.3c)$$

reconstruction (3.1) was, actually, given in [16]. In this case (3.2) holds according to (4.10), (4.11). (In the framework of [16] less precise version of (4.11) was known.)

General formulas and equations (5.1)-(5.3) for finding  $h|_{b\Theta_\rho}$  from  $\Phi$  with known background potential  $v_1 \not\equiv 0$  were given for the first time in [17]. If unknown potential  $v$  is sufficiently close to some known background potential  $v_1 \not\equiv 0$ , then formulas and equations (5.1)-(5.3) considered for  $v_2 = v, \psi_2 = \psi, h_2 = h$  give more stable determination of  $h|_{b\Theta_\rho}$  from  $\Phi$  than these formulas and equations for  $v_1 \equiv 0$  (see [17] for more detailed discussion of this issue).

Note that Theorem 2.1 follows from formulas (4.10), (4.11), (5.1) mentioned above in connection with the versions of (3.1), (3.2) of [16], [17].

*3.2. Theorem 2.2 and global reconstructions of [17], [20].* The main disadvantage of the global reconstruction (3.1), (3.2) in the framework of [16], [17] is related with the following two facts:

(1) The determination of  $h|_{\Xi}$  for  $\Xi = b\Theta_\rho$  or, more generally, for  $\Xi \subseteq \bar{\Theta}_\rho, \Xi \cap b\Theta_\rho \neq \emptyset$  (see formulas (6.2)) from  $\Phi$  via formulas and equations (5.1)-(5.3) for  $v_2 = v, \psi_2 = \psi, h_2 = h$  with some known background potential  $v_1$  is stable for relatively small  $\rho$ , but is very unstable for  $\rho \rightarrow +\infty$  in the points of  $\Xi$  with sufficiently great imaginary part (see [17], [19], [20] for more detailed discussion of this issue).

(2) For  $\hat{v}_{\rho, \tau}$  defined as in (3.3c) the decay of the error  $\hat{v} - \hat{v}_{\rho, \tau}$  on  $\mathcal{B}_{2\rho\tau}$  is very slow for  $\rho \rightarrow +\infty$  (not faster than  $O(\rho^{-1})$  even for infinitely smooth compactly supported  $v$ ) (see [20] for more detailed discussion of this issue).

As a corollary, the global reconstruction (3.1), (3.2) in the framework of [16] and even in the framework of [17] is not optimal with respect to its stability properties. A principle effectivization of the global reconstruction (3.1), (3.2) with respect to its stability properties was given in [20].

The main new result of [20] consists in stable construction of some  $\hat{v}_{\rho, \tau}$  on  $\mathcal{B}_{2\tau\rho}$  from  $h$  on  $b\Theta_{\rho, \tau} \subseteq b\Theta_\rho$  (see definitions (6.2)) in such a way that

$$\begin{aligned} |\hat{v}(p) - \hat{v}_{\rho, \tau}(p)| &\leq C(m, \mu_0, \tau) N^2 (1 + |p|)^{-\mu_0} \rho^{-(m - \mu_0)} \\ \text{for } &p \in \mathcal{B}_{2\tau\rho}, \quad \rho \geq \rho_1 \text{ and fixed } \tau \in ]0, \tau_1[, \end{aligned} \quad (3.4)$$

under the assumptions that

$$\begin{aligned} v &\in W^{m, 1}(\mathbb{R}^2), \quad \|v\|_{m, 1} < N \text{ (see definitions (2.2), (2.3)),} \\ m &> 2, \quad 2 \leq \mu_0 < m, \quad \tau_1 = \tau_1(m, \mu_0, N), \quad \rho_1 = \rho_1(m, \mu_0, N), \end{aligned} \quad (3.5)$$

where  $C(m, \mu_0, \tau)$ ,  $\tau_1(m, \mu_0, N)$ ,  $\rho_1(m, \mu_0, N)$  are special constants and, in particular,  $0 < \tau_1 < 1$ , see Theorem 2.1 of [20]. The scheme of this construction of [20] consists in the following:

$$\begin{aligned} h|_{b\Theta_{\rho,\tau}} &\rightarrow H|_{b\Omega_{\rho,\tau}} \rightarrow H|_{b\Lambda_{\rho,\tau,\nu}} \\ &\rightarrow H^0|_{\Lambda_{\rho,\tau,\nu}} \rightarrow \tilde{H}_{\rho,\tau} \quad \text{on } \Lambda_{\rho,\tau,\nu} \\ &\rightarrow \hat{v}_{\rho,\tau} \quad \text{on } \mathcal{B}_{2\tau\rho}. \end{aligned} \quad (3.6)$$

Here  $h|_{b\Theta_{\rho,\tau}}$  and  $H|_{b\Omega_{\rho,\tau}}$  are related by (4.13) (see also the definitions of  $b\Theta_{\rho,\tau}$  and  $b\Omega_{\rho,\tau}$  given in (6.2), (6.3)),  $H|_{b\Lambda_{\rho,\tau,\nu}}$  is defined in terms of  $H|_{b\Omega_{\rho,\tau}}$  by means of (7.4) (where  $b\Lambda_{\rho,\tau,\nu}$  is defined in (6.13)),  $H^0|_{\Lambda_{\rho,\tau,\nu}}$  is defined in terms of  $H|_{b\Lambda_{\rho,\tau,\nu}}$  by (7.8) (see also formulas (6.13), (6.16)),  $\tilde{H}_{\rho,\tau}$  is defined as the solution of (7.7) for  $H$  on  $\Lambda_{\rho,\tau,\nu}$ , where  $Q_{\rho,\tau}$  is replaced by zero in (7.7), finally,  $\hat{v}_{\rho,\tau}$  on  $\mathcal{B}_{2\tau\rho}$  is defined as  $\hat{v}_{\rho,\tau} = \hat{v}_{\rho,\tau}^+$  or as  $\hat{v}_{\rho,\tau} = \hat{v}_{\rho,\tau}^-$ , where

$$\begin{aligned} \hat{v}_{\rho,\tau}^+(p) &= \lim_{\lambda \rightarrow 0} \tilde{H}_{\rho,\tau}(\lambda, p), \\ \hat{v}_{\rho,\tau}^-(p) &= \lim_{\lambda \rightarrow \infty} \tilde{H}_{\rho,\tau}(\lambda, p), \end{aligned} \quad (3.7)$$

where definitions (3.7) are similar to formulas (7.16). For more detailed discussion of (3.6) see Section 5 of [20].

Note that in [20] we deal with approximate finding  $v$  from  $h|_{b\Theta_{\rho,\tau}}$  in a similar way with approximate finding  $v$  from the scattering amplitude  $f$  at fixed energy  $E$  of [18]. The parameter  $\rho$  of [20] plays the role of the parameter  $\sqrt{E}$  of [18].

One can see that for sufficiently regular potentials  $v$  that is for  $v \in W^{m,1}(\mathbb{R}^3)$ , where  $m$  is sufficiently great, and for  $\hat{v}_{\rho,\tau}$  constructed from  $h|_{b\Theta_{\rho,\tau}}$  via (3.6) as in [20] the decay rate of the error  $\hat{v} - \hat{v}_{\rho,\tau}$  on  $\mathcal{B}_{2\tau\rho}$  is  $O(\rho^{-(m-2)})$  as  $\rho \rightarrow +\infty$  (see (3.4) for  $\mu_0 = 2$ ) and is much faster than for  $\hat{v}_{\rho,\tau}$  given by (3.3c) as in [16], [17]. Therefore, for Problem 1.1 in dimension  $d = 3$  the global reconstruction of [20] (using also (5.1)-(5.3) for finding  $h|_{b\Theta_{\rho,\tau}}$  from  $\Phi$  as in [17]) has much more optimal stability properties than the global reconstructions of [16], [17].

Therefore, Theorem 1.2 illustrating the progress in global stability of [17], [20] is much more optimal than Theorem 1.1 related with global reconstruction results of [16] and [17] only.

#### 4. Faddeev functions

We consider the Faddeev functions  $G$ ,  $\psi$  and  $h$  (see [7], [8], [10], [16]):

$$\psi(x, k) = e^{ikx} + \int_{\mathbb{R}^d} G(x - y, k) v(y) \psi(y, k) dy, \quad (4.1)$$

$$G(x, k) = e^{ikx} g(x, k), \quad g(x, k) = -(2\pi)^{-d} \int_{\mathbb{R}^d} \frac{e^{i\xi x} d\xi}{\xi^2 + 2k\xi}, \quad (4.2)$$

where  $x \in \mathbb{R}^d$ ,  $k \in \Sigma$ ,

$$\Sigma = \{k \in \mathbb{C}^d : k^2 = k_1^2 + \dots + k_d^2 = 0\}; \quad (4.3)$$

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$$h(k, l) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ilx} v(x) \psi(x, k) dx, \quad (4.4)$$

where  $(k, l) \in \Theta$ ,

$$\Theta = \{k \in \Sigma, \quad l \in \Sigma : \operatorname{Im} k = \operatorname{Im} l\}. \quad (4.5)$$

One can consider (4.1), (4.4) assuming that  $v$  is a sufficiently regular function on  $\mathbb{R}^d$  with sufficient decay at infinity. For example, one can consider (4.1), (4.4) assuming that (1.2) holds.

We recall that:

$$\Delta G(x, k) = \delta(x), \quad x \in \mathbb{R}^d, \quad k \in \Sigma; \quad (4.6)$$

formula (4.1) at fixed  $k$  is considered as an equation for

$$\psi = e^{ikx} \mu(x, k), \quad (4.7)$$

where  $\mu$  is sought in  $L^\infty(\mathbb{R}^d)$ ; as a corollary of (4.1), (4.2), (4.6),  $\psi$  satisfies (1.1);  $h$  of (4.4) is a generalized "scattering" amplitude in the complex domain at zero energy.

Note that, actually,  $G$ ,  $\psi$ ,  $h$  of (4.1)-(4.5) are zero energy restrictions of functions introduced by Faddeev as extensions to the complex domain of some functions of the classical scattering theory for the Schrödinger equation at positive energies. In addition,  $G$ ,  $\psi$ ,  $h$  in their zero energy restriction were considered for the first time in [2]. The Faddeev functions  $G$ ,  $\psi$ ,  $h$  were, actually, rediscovered in [2].

We recall also that, under the assumptions of Theorem 2.1,

$$\mu(x, k) \rightarrow 1 \quad \text{as} \quad |\operatorname{Im} k| \rightarrow \infty \quad (\text{uniformly in } x) \quad (4.8)$$

and, for any  $\sigma > 1$ ,

$$|\mu(x, k)| < \sigma \quad \text{for} \quad |\operatorname{Im} k| \geq r_1(N, D, m, \sigma), \quad (4.9)$$

where  $x \in \mathbb{R}^d$ ,  $k \in \Sigma$ ;

$$\hat{v}(p) = \lim_{\substack{(k, l) \in \Theta, \quad k-l=p \\ |\operatorname{Im} k| = |\operatorname{Im} l| \rightarrow \infty}} h(k, l) \quad \text{for any } p \in \mathbb{R}^d, \quad (4.10)$$

$$\begin{aligned} |\hat{v}(p) - h(k, l)| &\leq \frac{c_3(D, m)N^2}{\rho} \quad \text{for } (k, l) \in \Theta, \quad p = k - l, \\ |\operatorname{Im} k| = |\operatorname{Im} l| = \rho &\geq r_2(N, D, m), \end{aligned} \quad (4.11)$$

where

$$\hat{v}(p) = \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} e^{ipx} v(x) dx, \quad p \in \mathbb{R}^d. \quad (4.12)$$

Results of the type (4.8), (4.9) go back to [2]. Results of the type (4.10), (4.11) (with less precise right-hand side in (4.11)) go back to [10]. Estimates (4.8), (4.11) are related also with some important  $L_2$ -estimate going back to [23] on the Green function  $g$  of (4.1).



Note also that in some considerations it is convenient to consider  $h$  on  $\Theta$  as  $H$  on  $\Omega$ , where

$$\begin{aligned} h(k, l) &= H(k, k - l), \quad (k, l) \in \Theta, \\ H(k, p) &= h(k, k - p), \quad (k, p) \in \Omega, \end{aligned} \quad (4.13)$$

$$\Omega = \{k \in \mathbb{C}^d, p \in \mathbb{R}^d : k^2 = 0, p^2 = 2kp\}. \quad (4.14)$$

For more information on properties of the Faddeev functions  $G, \psi, h$ , see [10], [17], [20] and references therein.

In the next section we recall that Problem 1.1 (of Introduction) admits a reduction to the following inverse "scattering" problem:

**Problem 4.1.** Given  $h$  on  $\Theta$ , find  $v$  on  $\mathbb{R}^d$ .

### 5. Reduction of [16], [17] of Problem 1.1 to Problem 4.1

Let conditions (1.2), (1.4) hold for potentials  $v_1$  and  $v_2$ , where  $D$  is fixed. Let  $\Phi_i, \psi_i, h_i$  denote the DtN map  $\Phi$  and the Faddeev functions  $\psi, h$  for  $v = v_i, i = 1, 2$ . Let also  $\Phi_i(x, y)$  denote the Schwartz kernel  $\Phi(x, y)$  of the integral operator  $\Phi$  for  $v = v_i, i = 1, 2$ . Then (see [17] for details):

$$h_2(k, l) - h_1(k, l) = \left(\frac{1}{2\pi}\right)^d \int_{\partial D} \int_{\partial D} \psi_1(x, -l) (\Phi_2 - \Phi_1)(x, y) \psi_2(y, k) dy dx, \quad (5.1)$$

where  $(k, l) \in \Theta$ ;

$$\psi_2(x, k) = \psi_1(x, k) + \int_{\partial D} A(x, y, k) \psi_2(y, k) dy, \quad x \in \partial D, \quad (5.2a)$$

$$A(x, y, k) = \int_{\partial D} R_1(x, z, k) (\Phi_2 - \Phi_1)(z, y) dz, \quad x, y \in \partial D, \quad (5.2b)$$

$$R_1(x, y, k) = G(x - y, k) + \int_{\mathbb{R}^d} G(x - z, k) v_1(z) R_1(z, y, k) dz, \quad x, y \in \mathbb{R}^d, \quad (5.3)$$

where  $k \in \Sigma$ . Note that: (5.1) is an explicit formula, (5.2a) is considered as an equation for finding  $\psi_2$  on  $\partial D$  from  $\psi_1$  on  $\partial D$  and  $A$  on  $\partial D \times \partial D$  for each fixed  $k$ , (5.2b) is an explicit formula, (5.3) is an equation for finding  $R_1$  from  $G$  and  $v_1$ , where  $G$  is the function of (4.2).

Note that formulas and equations (5.1)-(5.3) for  $v_1 \equiv 0$  were given in [16] (see also [10] (Note added in proof), [14], [15]). In this case  $h_1 \equiv 0, \psi_1 = e^{ikx}, R_1 = G(x - y, k)$ . Formulas and equations (5.1)-(5.3) for the general case were given in [17].

Formulas and equations (5.1)-(5.3) with fixed background potential  $v_1$  reduce Problem 1.1 (of Introduction) to Problem 4.1 (of Section 4). In this connection we consider (5.1)-(5.3) for  $v_2 = v, \psi_2 = \psi, h_2 = h$  and this reduction consists of the following steps:

- (1)  $v_1 \rightarrow \Phi_1, \psi_1, R_1, h_1$  via formulas and equations of the direct problem for  $v_1$ ;
- (2)  $\Phi, \Phi_1, R_1|_{\partial D \times \partial D \times \Sigma} \rightarrow A$  via (5.2b);
- (3)  $A, \psi_1|_{\partial D \times \Sigma} \rightarrow \psi|_{\partial D \times \Sigma}$  via (5.2a);

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(4)  $h_1, \psi_1|_{\partial D \times \Sigma}, \psi|_{\partial D \times \Sigma}, \Phi, \Phi_1 \rightarrow h$  via (5.1).

Here, on the step (4) we find  $h$  on  $\Theta$  for the unknown potential  $v$  of Problem 1.1.

## 6. Some considerations related with $\Theta$ and $\Omega$

6.1 *Some subsets of  $\Theta$  and  $\Omega$ .* Let

$$\begin{aligned} \mathcal{B}_r &= \{p \in \mathbb{R}^d : |p| < r\}, \quad \partial \mathcal{B}_r = \{p \in \mathbb{R}^d : |p| = r\}, \\ \bar{\mathcal{B}}_r &= \mathcal{B}_r \cup \partial \mathcal{B}_r, \quad \text{where } r > 0. \end{aligned} \quad (6.1)$$

In addition to  $\Theta$  of (4.5), we consider, in particular, the following its subsets:

$$\begin{aligned} \Theta_\rho &= \{(k, l) \in \Theta : |\operatorname{Im} k| = |\operatorname{Im} l| < \rho\}, \\ b\Theta_\rho &= \{(k, l) \in \Theta : |\operatorname{Im} k| = |\operatorname{Im} l| = \rho\}, \\ \bar{\Theta}_\rho &= \Theta_\rho \cup b\Theta_\rho, \\ \Theta_{\rho, \tau}^\infty &= \{(k, l) \in \Theta \setminus \bar{\Theta}_\rho : k - l \in \mathcal{B}_{2\rho\tau}\}, \\ b\Theta_{\rho, \tau} &= \{(k, l) \in b\Theta_\rho : k - l \in \mathcal{B}_{2\rho\tau}\}, \end{aligned} \quad (6.2)$$

where  $\rho > 0$ ,  $0 < \tau < 1$ , and  $\mathcal{B}_r$  is defined in (6.1).

In addition to  $\Omega$  of (4.14), we consider, in particular, the following its subsets:

$$\begin{aligned} \Omega_\rho &= \{(k, p) \in \Omega : |\operatorname{Im} k| < \rho\}, \\ b\Omega_\rho &= \{(k, p) \in \Omega : |\operatorname{Im} k| = \rho\}, \\ \bar{\Omega}_\rho &= \Omega_\rho \cup b\Omega_\rho, \\ \Omega_{\rho, \tau}^\infty &= \{(k, p) \in \Omega \setminus \bar{\Omega}_\rho : p \in \mathcal{B}_{2\rho\tau}\}, \\ b\Omega_{\rho, \tau} &= \{(k, p) \in b\Omega_\rho : p \in \mathcal{B}_{2\rho\tau}\}, \end{aligned} \quad (6.3)$$

where  $\rho > 0$ ,  $0 < \tau < 1$ , and  $\mathcal{B}_r$  is defined in (6.1).

Note that

$$\begin{aligned} \Omega &\approx \Theta, \quad \Omega_\rho \approx \Theta_\rho, \quad b\Omega_\rho \approx b\Theta_\rho, \\ \Omega_{\rho, \tau}^\infty &\approx \Theta_{\rho, \tau}^\infty, \quad b\Omega_{\rho, \tau} \approx b\Theta_{\rho, \tau}, \end{aligned} \quad (6.4)$$

or, more precisely,

$$\begin{aligned} (k, p) \in \Omega &\Rightarrow (k, k - p) \in \Theta, \quad (k, l) \in \Theta \Rightarrow (k, k - l) \in \Omega \\ \text{and the same for } \Omega_\rho, b\Omega_\rho, \Omega_{\rho, \tau}^\infty, b\Omega_{\rho, \tau} & \\ \text{and } \Theta_\rho, b\Theta_\rho, \Theta_{\rho, \tau}^\infty, b\Theta_{\rho, \tau}, & \text{ respectively, in place of } \Omega \text{ and } \Theta. \end{aligned} \quad (6.5)$$

We consider also, in particular,

$$\begin{aligned} \Omega_\nu &= \{(k, p) \in \Omega : p \notin \mathcal{L}_\nu\}, \\ \Omega_{\rho, \tau, \nu}^\infty &= \Omega_{\rho, \tau}^\infty \cap \Omega_\nu, \quad b\Omega_{\rho, \tau, \nu} = b\Omega_{\rho, \tau} \cap \Omega_\nu, \end{aligned} \quad (6.6)$$

where

$$\mathcal{L}_\nu = \{p \in \mathbb{R}^d : p = t\nu, t \in \mathbb{R}\}, \quad (6.7)$$

$\nu \in \mathbb{S}^{d-1}$ ,  $\rho > 0$ ,  $0 < \tau < 1$ .

*6.2. Coordinates on  $\Omega$  for  $d = 3$ .* In this subsection we assume that  $d = 3$  in formulas (4.5), (4.14), (6.1)-(6.7).

For  $p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu$  we consider  $\theta(p)$  and  $\omega(p)$  such that

$$\begin{aligned} \theta(p), \omega(p) & \text{ smoothly depend on } p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu, \\ & \text{take values in } \mathbb{S}^2, \text{ and} \\ \theta(p)p = 0, \omega(p)p = 0, \theta(p)\omega(p) = 0, \end{aligned} \quad (6.8)$$

where  $\mathcal{L}_\nu$  is defined by (6.7) (for  $d = 3$ ).

Assumptions (6.8) imply that

$$\omega(p) = \frac{p \times \theta(p)}{|p|} \quad \text{for } p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu \quad (6.9a)$$

or

$$\omega(p) = -\frac{p \times \theta(p)}{|p|} \quad \text{for } p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu, \quad (6.9b)$$

where  $\times$  denotes vector product.

To satisfy (6.8), (6.9a) we can take

$$\theta(p) = \frac{\nu \times p}{|\nu \times p|}, \quad \omega(p) = \frac{p \times \theta(p)}{|p|}, \quad p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu. \quad (6.10)$$

Let  $\theta, \omega$  satisfy (6.8). Then (according to [19]) the following formulas give a diffeomorphism between  $\Omega_\nu$  and  $(\mathbb{C} \setminus 0) \times (\mathbb{R}^3 \setminus \mathcal{L}_\nu)$ :

$$(k, p) \rightarrow (\lambda, p), \quad \text{where } \lambda = \lambda(k, p) = \frac{2k(\theta(p) + i\omega(p))}{i|p|}, \quad (6.11a)$$

$$\begin{aligned} (\lambda, p) & \rightarrow (k, p), \quad \text{where } k = k(\lambda, p) = \kappa_1(\lambda, p)\theta(p) + \kappa_2(\lambda, p)\omega(p) + \frac{p}{2}, \\ \kappa_1(\lambda, p) &= \frac{i|p|}{4}\left(\lambda + \frac{1}{\lambda}\right), \quad \kappa_2(\lambda, p) = \frac{|p|}{4}\left(\lambda - \frac{1}{\lambda}\right), \end{aligned} \quad (6.11b)$$

where  $(k, p) \in \Omega_\nu$ ,  $(\lambda, p) \in (\mathbb{C} \setminus 0) \times (\mathbb{R}^3 \setminus \mathcal{L}_\nu)$ . In addition, formulas (6.11a), (6.11b) for  $\lambda(k)$  and  $k(\lambda)$  at fixed  $p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu$  give a diffeomorphism between  $Z_p = \{k \in \mathbb{C}^3 : (k, p) \in \Omega\}$  for fixed  $p$  and  $\mathbb{C} \setminus 0$ .

In addition, for  $k$  and  $\lambda$  of (6.11) we have that

$$|Im k| = \frac{|p|}{4}\left(|\lambda| + \frac{1}{|\lambda|}\right), \quad |Re k| = \frac{|p|}{4}\left(|\lambda| + \frac{1}{|\lambda|}\right), \quad (6.12)$$

where  $(k, p) \in \Omega_\nu$ ,  $(\lambda, p) \in (\mathbb{C} \setminus 0) \times (\mathbb{R}^3 \setminus \mathcal{L}_\nu)$ .

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Let

$$\Lambda_{\rho,\nu} = \{(\lambda, p) : \lambda \in \mathcal{D}_{\rho/|p|}, \quad p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu\}, \quad (6.13)$$

$$\Lambda_{\rho,\tau,\nu} = \{(\lambda, p) : \lambda \in \mathcal{D}_{\rho/|p|}, \quad p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu, \quad |p| < 2\tau\rho\},$$

$$b\Lambda_{\rho,\tau,\nu} = \{(\lambda, p) : \lambda \in \mathcal{T}_{\rho/|p|}, \quad p \in \mathbb{R}^3 \setminus \mathcal{L}_\nu, \quad |p| < 2\tau\rho\},$$

where  $\rho > 0$ ,  $0 < \tau < 1$ ,  $\nu \in \mathbb{S}^2$ ,

$$\mathcal{D}_r = \{\lambda \in \mathbb{C} \setminus 0 : \frac{1}{4}(|\lambda| + |\lambda|^{-1}) > r\}, \quad r > 0, \quad (6.14)$$

$$\mathcal{T}_r = \{\lambda \in \mathbb{C} : \frac{1}{4}(|\lambda| + |\lambda|^{-1}) = r\}, \quad r \geq 1/2, \quad (6.15)$$

$\mathcal{L}_\nu$  is defined by (6.7) (for  $d = 3$ ).

Note that

$$\begin{aligned} \Lambda_{\rho,\tau,\nu} &= \Lambda_{\rho,\tau,\nu}^+ \cup \Lambda_{\rho,\tau,\nu}^-, \quad \Lambda_{\rho,\tau,\nu}^+ \cap \Lambda_{\rho,\tau,\nu}^- = \emptyset, \\ b\Lambda_{\rho,\tau,\nu} &= b\Lambda_{\rho,\tau,\nu}^+ \cup b\Lambda_{\rho,\tau,\nu}^-, \end{aligned} \quad (6.16)$$

where

$$\begin{aligned} \Lambda_{\rho,\tau,\nu}^\pm &= \{(\lambda, p) : \lambda \in \mathcal{D}_{\rho/|p|}^\pm, \quad p \in \mathcal{B}_{2\tau\rho} \setminus \mathcal{L}_\nu\}, \\ b\Lambda_{\rho,\tau,\nu}^\pm &= \{(\lambda, p) : \lambda \in \mathcal{T}_{\rho/|p|}^\pm, \quad p \in \mathcal{B}_{2\tau\rho} \setminus \mathcal{L}_\nu\}, \end{aligned} \quad (6.17)$$

$$\mathcal{D}_r^\pm = \{\lambda \in \mathbb{C} \setminus 0 : \frac{1}{4}(|\lambda| + |\lambda|^{-1}) > r, \quad |\lambda|^{\pm 1} < 1\}, \quad (6.18)$$

$$\mathcal{T}_r^\pm = \{\lambda \in \mathbb{C} : \frac{1}{4}(|\lambda| + |\lambda|^{-1}) = r, \quad |\lambda|^{\pm 1} \leq 1\}, \quad r > 1/2,$$

where  $\rho > 0$ ,  $\tau \in ]0, 1[$ ,  $\nu \in \mathbb{S}^2$ .

Using (6.12) one can see that formulas (6.11) give also the following diffeomorphisms

$$\begin{aligned} \Omega_\nu \setminus \bar{\Omega}_\rho &\approx \Lambda_{\rho,\nu}, \quad \Omega_{\rho,\tau,\nu}^\infty \approx \Lambda_{\rho,\tau,\nu}, \\ b\Omega_{\rho,\tau,\nu} &\approx b\Lambda_{\rho,\tau,\nu}, \\ Z_{p,\rho}^\infty &= \{k \in \mathbb{C}^3 : (k, p) \in \Omega_\nu \setminus \bar{\Omega}_\rho\} \approx \mathcal{D}_{\rho/|p|} \quad \text{for fixed } p, \end{aligned} \quad (6.19)$$

where  $\rho > 0$ ,  $0 < \tau < 1$ ,  $\nu \in \mathbb{S}^2$  (and where we use the definitions (6.3), (6.6), (6.13)).

In [19]  $\lambda, p$  of (6.11) were used as coordinates on  $\Omega$ . In the present work we use them also as coordinates on  $\Omega \setminus \Omega_\rho$  (or more precisely on  $\Omega_\nu \setminus \Omega_\rho$ ).

## 7. An integral equation of [20] and some related formulas

In the main considerations of [20] it is assumed that  $d = 3$  and the basic assumption on  $v$  consists in the following condition on its Fourier transform:

$$\hat{v} \in L_\mu^\infty(\mathbb{R}^3) \cap \mathcal{C}(\mathbb{R}^3) \quad \text{for some real } \mu \geq 2, \quad (7.1)$$

where  $\hat{v}$  is defined by (4.12) (for  $d = 3$ ),

$$\begin{aligned} L_\mu^\infty(\mathbb{R}^d) &= \{u \in L^\infty(\mathbb{R}^d) : \|u\|_\mu < +\infty\}, \\ \|u\|_\mu &= \text{ess sup}_{p \in \mathbb{R}^d} (1 + |p|)^\mu |u(p)|, \quad \mu > 0, \end{aligned} \quad (7.2)$$

and  $\mathcal{C}$  denotes the space of continuous functions.

Note that

$$\begin{aligned} v \in W^{m,1}(\mathbb{R}^d) &\implies \hat{v} \in L_\mu^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d), \\ \|\hat{v}\|_\mu &\leq c_4(m, d)\|v\|_{m,1} \quad \text{for } \mu = m, \end{aligned} \quad (7.3)$$

where  $W^{m,1}$ ,  $L_\mu^\infty$  are the spaces of (2.2), (7.2).

Let

$$H(\lambda, p) = H(k(\lambda, p), p), \quad (\lambda, p) \in (\mathbb{C} \setminus 0) \times (\mathbb{R}^3 \setminus \mathcal{L}_\mu), \quad (7.4)$$

where  $H$  is the function of (4.13),  $\lambda, p$  are the coordinates of Subsection 6.2 under assumption (6.9a).

Let

$$\begin{aligned} L_\mu^\infty(\Lambda_{\rho,\tau,\nu}) &= \{U \in L^\infty(\Lambda_{\rho,\tau,\nu}) : |||U|||_{\rho,\tau,\mu} < \infty\}, \\ |||U|||_{\rho,\tau,\mu} &= \text{ess sup}_{(\lambda,p) \in \Lambda_{\rho,\tau,\nu}} (1 + |p|)^\mu |U(\lambda, p)|, \quad \mu > 0, \end{aligned} \quad (7.5)$$

where  $\Lambda_{\rho,\tau,\nu}$  is defined in (6.13),  $\rho > 0$ ,  $\tau \in ]0, 1[$ ,  $\nu \in \mathbb{S}^2$ ,  $\mu > 0$ .

Let  $v$  satisfy (7.1) and  $\|\hat{v}\|_\mu \leq C$ . Let

$$\eta(C, \rho, \mu) \stackrel{\text{def}}{=} a(\mu)C(\ln \rho)^2 \rho^{-1} < 1, \quad \ln \rho \geq 2, \quad (7.6)$$

where  $a(\mu)$  is the constant  $c_2(\mu)$  of [20]. Let  $H(\lambda, p)$  be defined by (7.4) and be considered as a function on  $\Lambda_{\rho,\tau,\nu}$  of (6.13). Then (see Section 5 of [20]):

$$H = H^0 + M_{\rho,\tau}(H) + Q_{\rho,\tau}, \quad \tau \in ]0, 1[, \quad (7.7)$$

where

$$H^0(\lambda, p) = \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^+} H(\zeta, p) \frac{d\zeta}{\zeta - \lambda}, \quad (\lambda, p) \in \Lambda_{\rho,\tau,\nu}^+, \quad (7.8a)$$

$$H^0(\lambda, p) = -\frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^-} H(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda)}, \quad (\lambda, p) \in \Lambda_{\rho,\tau,\nu}^-, \quad (7.8b)$$

where  $\Lambda_{\rho,\tau,\nu}^\pm$ ,  $\mathcal{T}_r^\pm$  are defined in (6.17), (6.18) (and where the integrals along  $\mathcal{T}_r^\pm$  are taken in the counter-clock wise direction);

$$\begin{aligned} M_{\rho,\tau}(U)(\lambda, p) &= M_{\rho,\tau}^+(U)(\lambda, p) = \\ &= -\frac{1}{\pi} \int \int_{\mathcal{D}_{\rho/|p|}^+} (U, U)_{\rho,\tau}(\zeta, p) \frac{d\text{Re } \zeta d\text{Im } \zeta}{\zeta - \lambda}, \quad (\lambda, p) \in \Lambda_{\rho,\tau,\nu}^+, \end{aligned} \quad (7.9a)$$

$$\begin{aligned} M_{\rho,\tau}(U)(\lambda, p) &= M_{\rho,\tau}^-(U)(\lambda, p) = \\ &= -\frac{1}{\pi} \int \int_{\mathcal{D}_{\rho/|p|}^-} (U, U)_{\rho,\tau}(\zeta, p) \frac{\lambda d\text{Re } \zeta d\text{Im } \zeta}{\zeta(\zeta - \lambda)}, \quad (\lambda, p) \in \Lambda_{\rho,\tau,\nu}^-, \end{aligned} \quad (7.9b)$$

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$$\begin{aligned} (U_1, U_2)_{\rho, \tau}(\zeta, p) &= \{\chi_{2\tau\rho} U_1', \chi_{2\tau\rho} U_2'\}(\zeta, p), \quad (\zeta, p) \in \Lambda_{\rho, \tau, \nu}, \\ \chi_{2\tau\rho} U_j'(k, p) &= U_j(\lambda(k, p), p), \quad (k, p) \in \Omega_{\rho, \tau, \nu}^\infty, \\ \chi_{2\tau\rho} U_j'(k, p) &= 0, \quad |p| \geq 2\tau\rho, \quad j = 1, 2, \end{aligned} \quad (7.10)$$

where  $U, U_1, U_2$  are test functions on  $\Lambda_{\rho, \tau, \nu}$ ,  $\Omega_{\rho, \tau, \nu}^\infty$  is defined in (6.6),  $\lambda(k, p)$  is defined in (6.11a),  $\{\cdot, \cdot\}$  is defined by the formula

$$\begin{aligned} \{F_1, F_2\}(\lambda, p) &= -\frac{\pi}{4} \int_{-\pi}^{\pi} \left( \frac{|p|}{2} \frac{|\lambda|^2 - 1}{\lambda|\lambda|} (\cos \varphi - 1) - \frac{|p|}{\lambda} \sin \varphi \right) \times \\ &F_1(k(\lambda, p), -\xi(\lambda, p, \varphi)) F_2(k(\lambda, p) + \xi(\lambda, p, \varphi), p + \xi(\lambda, p, \varphi)) d\varphi, \end{aligned} \quad (7.11)$$

for  $(\lambda, p) \in \Lambda_{\rho, \nu}$ , where  $F_1, F_2$  are test functions on  $\Omega \setminus \bar{\Omega}_\rho$ ,  $k(\lambda, p)$  is defined in (6.11b),  $\Lambda_{\rho, \nu}$  is defined in (6.13),

$$\xi(\lambda, p, \varphi) = \operatorname{Re} k(\lambda, p) (\cos \varphi - 1) + k^\perp(\lambda, p) \sin \varphi, \quad (7.12)$$

$$k^\perp(\lambda, p) = \frac{\operatorname{Im} k(\lambda, p) \times \operatorname{Re} k(\lambda, p)}{|\operatorname{Im} k(\lambda, p)|}, \quad (7.13)$$

where  $\times$  in (7.13) denotes vector product;

$$H, H^0, Q_{\rho, \tau} \in L_\mu^\infty(\Lambda_{\rho, \tau, \nu}), \quad (7.14)$$

$$|||H|||_{\rho, \tau, \mu_0} \leq \frac{C}{1 - \eta(C, \rho, \mu)}, \quad (7.15a)$$

$$|||H^0|||_{\rho, \tau, \mu_0} \leq \frac{C}{1 - \eta(C, \rho, \mu)} \left( 1 + \frac{c_5(\mu_0)C}{1 - \eta(C, \rho, \mu)} \right), \quad (7.15b)$$

$$|||Q_{\rho, \tau}|||_{\rho, \tau, \mu_0} \leq \frac{3c_5(\mu_0)C^2}{(1 - \eta(C, \rho, \mu))^2(1 + 2\tau\rho)^{\mu - \mu_0}}, \quad (7.15c)$$

where  $2 \leq \mu_0 \leq \mu$ ,  $c_5$  is the constant  $b_4$  of [20],  $\eta(C, \rho, \mu)$  is defined by (7.6).

Following [20] we consider (7.7) as an approximate integral equation for finding  $H$  on  $\Lambda_{\rho, \tau, \nu}$  from  $H^0$  on  $\Lambda_{\rho, \tau, \nu}$  with unknown remainder  $Q_{\rho, \tau}$ .

Note also that if  $\hat{v}$  satisfies (7.1), then (see [19], [20])

$$\begin{aligned} H(\lambda, p) &\rightarrow \hat{v}(p) \quad \text{as } \lambda \rightarrow 0, \\ H(\lambda, p) &\rightarrow \hat{v}(p) \quad \text{as } \lambda \rightarrow \infty, \end{aligned} \quad (7.16)$$

where  $p \in \mathcal{B}_{2\tau\rho} \setminus \mathcal{L}_\nu$ ,  $H$  is defined by (7.4) and is considered as a function on  $\Lambda_{\rho, \tau, \nu}$ ,  $\rho > 0$ ,  $0 < \tau < 1$ ,  $\nu \in \mathbb{S}^2$ .

## 8. Proof of Theorem 2.2 for $d = 3$

Our proof of Theorem 2.2 for  $d = 3$  is based on formulas (7.16) of Section 7 and on Lemma 8.4 given below in the present section. In turn, Lemma 8.4 follows from formula (7.3) of Section 7 and from Lemmas 8.1, 8.2, 8.3 given below in the present section. In turn:

- (1) Lemma 8.1 follows from some estimates and Lemmas of [20] related with studies of the non-linear integral equation given as equation (7.7) of the present paper;
- (2) Lemma 8.2 does not follow immediately from results of preceding works and its proof is given in Section 9;
- (3) Lemma 8.3 follows from formulas (4.9), (5.1) and from definitions, see Section 9 for proof of this lemma.

**Lemma 8.1.** *Let  $\hat{v}_i$  satisfy (7.1) and  $\|\hat{v}_i\|_\mu < C$ , where  $i = 1, 2$ . Let*

$$0 < \tau \leq \tau_1(\mu, \mu_0, C, \delta), \quad \rho \geq \rho_1(\mu, \mu_0, C, \delta), \quad (8.1)$$

where  $\tau_1, \rho_1$  are the constants of Section 4 of [20] and where  $\delta = 1/2$ ,  $2 \leq \mu_0 < \mu$ . Then

$$|||H_2 - H_1|||_{\rho, \tau, \mu_0} \leq 2(|||H_2^0 - H_1^0|||_{\rho, \tau, \mu_0} + |||Q_{\rho, \tau}^2 - Q_{\rho, \tau}^1|||_{\rho, \tau, \mu_0}), \quad (8.2)$$

where  $H_i$ ,  $H_i^0$ ,  $Q_{\rho, \tau}^i$  are the functions of (7.4), (7.7), (7.8), (7.14), (7.15) for  $v = v_i$ ,  $i = 1, 2$ ,  $|||\cdot|||_{\rho, \tau, \mu_0}$  is defined as in (7.5). In addition,

$$|||Q_{\rho, \tau}^2 - Q_{\rho, \tau}^1|||_{\rho, \tau, \mu_0} \leq \frac{24c_5(\mu_0)C^2}{(1 + 2\tau\rho)^{\mu - \mu_0}}. \quad (8.3)$$

In connection with (8.1) we remind that  $\tau_1 \in ]0, 1[$  is sufficiently small and  $\rho_1$  is sufficiently great, see [20].

Lemma 8.1 follows from estimates mentioned as estimates (7.14), (7.15) of the present paper (see estimates (4.3), (5.20), (5.22) of [20]) and from Lemmas A.1, A.2, A.3 and estimate (A.5) (see Appendix to the present paper).

**Lemma 8.2.** *Let  $\hat{v}_i$  satisfy (7.1) and  $\|\hat{v}_i\|_\mu < C$ , where  $i = 1, 2$ . Let*

$$\eta(C, \rho_0, \mu) \leq 1/2, \quad \ln \rho_0 \geq 2, \quad (8.4)$$

where  $\eta$  is defined in (7.6). Let

$$0 < \tau_0 < 1, \quad 2 \leq \mu_0 < \mu, \quad \rho = 2\rho_0, \quad \tau = \tau_0/2.$$

Then

$$|||H_2^0 - H_1^0|||_{\rho, \tau, \mu_0} \leq (c_6 + 4c_7(\mu_0, \tau_0, \rho_0)C)\Delta_{\rho_0, \rho, \mu_0}, \quad (8.5)$$

$$\Delta_{\rho_0, \rho, \mu_0} \stackrel{\text{def}}{=} |||\chi_{\rho_0, \rho}(H_2 - H_1)|||_{\rho_0, \mu_0}, \quad (8.6)$$

where  $H_i$ ,  $H_i^0$  are the functions of (4.13), (7.8) for  $v = v_i$ ,  $i = 1, 2$ ,  $\chi_{\rho_0, \rho}$  is the characteristic function of  $\bar{\Omega}_\rho \setminus \bar{\Omega}_{\rho_0}$ ,  $|||\cdot|||_{\rho, \tau, \mu_0}$  in (8.5) is defined as in (7.5),  $|||\cdot|||_{\rho_0, \mu_0}$  in (8.6) is defined as in (A.12) of Appendix,  $c_6$  is defined by (9.9),  $c_7$  is the constant  $c_8$  of [20] (that is  $c_7(\mu, \tau, \rho) = 3b_1(\mu)\tau^2 + 4b_2(\mu)\rho^{-1} + 4b_3(\mu)\tau$ , where  $b_1, b_2, b_3$  are the constants of [20]).

Lemma 8.2 is proved in Section 9.

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**Lemma 8.3.** *Let the assumptions of Theorem 2.1 hold (for  $d = 3$ ). Let*

$$\rho_0 \geq r_1(N, D, m, \sigma) \text{ for some } \sigma > 1, \quad (8.7)$$

where  $r_1$  is the number of (4.9). Let  $0 < \tau_0 < 1$ ,  $0 < \mu_0$ ,  $\rho = 2\rho_0$ ,  $\tau = \tau_0/2$ . Then

$$\Delta_{\rho_0, \rho, \mu_0} \leq c_8 \sigma^2 e^{2\rho L} \|\Phi_2 - \Phi_1\| (1 + 2\rho)^{\mu_0}, \quad (8.8)$$

where  $\Delta_{\rho_0, \rho, \mu_0}$  is defined by (8.6),

$$c_8 = (2\pi)^{-d} \int_{\partial D} dx, \quad L = \max_{x \in \partial D} |x|, \quad (8.9)$$

$\|\Phi_2 - \Phi_1\|$  is defined according to (2.4).

Lemma 8.3 is proved in Section 9.

**Lemma 8.4.** *Let the assumptions of Theorem 2.1 hold (for  $d = 3$ ). Let*

$$0 < \tau \leq \tau_2(m, \mu_0, N), \quad \rho \geq \rho_2(m, \mu_0, N, D, \sigma), \quad (8.10)$$

where  $2 \leq \mu_0 < m$ ,  $\sigma > 1$ , and  $\tau_2$ ,  $\rho_2$  are constants such that (8.10) implies that

$$\tau \leq \tau_1(m, \mu_0, c_4(m, 3)N, 1/2), \quad \rho \geq \rho_1(m, \mu_0, c_4(m, 3)N, 1/2), \quad (8.11a)$$

$$\tau < 1/2, \quad \eta(c_4(m, 3)N, \rho/2, m) \leq 1/2, \quad \ln(\rho/2) \geq 2, \quad (8.11b)$$

$$\rho/2 \geq r_1(N, D, m, \sigma), \quad (8.11c)$$

where  $\tau_1$ ,  $\rho_1$ ,  $\eta$ ,  $r_1$  are the same that in (8.1), (8.4), (8.7),  $c_4$  is the constant of (7.3). Then

$$\|H_2 - H_1\|_{\rho, \tau, \mu_0} \leq c_9(N, D, m, \mu_0, \sigma, \tau) e^{2L\rho} \rho^{\mu_0} \|\Phi_2 - \Phi_1\| + c_{10}(N, m, \mu_0, \tau) \rho^{-(m-\mu_0)}, \quad (8.12)$$

where  $c_9$ ,  $c_{10}$  are some constants which can be given explicitly.

Lemma 8.4 follows from formula (7.3) and Lemmas 8.1, 8.2, 8.3.

The final part of the proof of Theorem 2.2 for  $d = 3$  consists of the following. Using the inverse Fourier transform formula

$$v(x) = \int_{\mathbb{R}^3} e^{-ipx} \hat{v}(p) dp, \quad x \in \mathbb{R}^3, \quad (8.13)$$

we have that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq \sup_{x \in \bar{D}} \left| \int_{\mathbb{R}^3} e^{-ipx} (\hat{v}_2(p) - \hat{v}_1(p)) dp \right| \leq \quad (8.14)$$

$$I_1(r) + I_2(r) \text{ for any } r > 0,$$



where

$$\begin{aligned} I_1(r) &= \int_{|p| \leq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp, \\ I_2(r) &= \int_{|p| \geq r} |\hat{v}_2(p) - \hat{v}_1(p)| dp. \end{aligned} \quad (8.15)$$

Using (7.3), (7.5), (7.14) for  $H$ , (7.16) we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq \|H_2 - H_1\|_{\rho, \tau, 2} (1 + |p|)^{-2} \quad (8.16)$$

for  $|p| < 2\tau\rho$ , at least, under the assumptions of Lemma 8.4 for  $\mu_0 = 2$ .

Using (7.3) we obtain that

$$|\hat{v}_2(p) - \hat{v}_1(p)| \leq 2c_4(m, 3)N(1 + |p|)^{-m}, \quad p \in \mathbb{R}^3. \quad (8.17)$$

Using (8.14)-(8.17) we have that

$$\begin{aligned} \|v_1 - v_2\|_{L^\infty(D)} &\leq I_1(2\tau\rho) + I_2(2\tau\rho) \leq \\ &\|H_2 - H_1\|_{\rho, \tau, 2} \int_{|p| \leq 2\tau\rho} \frac{dp}{(1 + |p|)^2} + 2c_4(m, 3)N \int_{|p| \geq 2\tau\rho} \frac{dp}{(1 + |p|)^m} \leq \\ &8\pi\tau\rho \|H_2 - H_1\|_{\rho, \tau, 2} + \frac{8\pi c_4(m, 3)N}{(m-3)(2\tau)^{m-3}} \frac{1}{\rho^{m-3}}, \end{aligned} \quad (8.18)$$

at least, under the assumptions of Lemma 8.4 for  $\mu_0 = 2$ .

Under the assumptions of Lemma 8.4 for  $\mu_0 = 2$ , using (8.18) and (8.12) we obtain that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_{11}(N, D, m, \sigma, \tau) e^{2L\rho} \rho^3 \|\Phi_2 - \Phi_1\| + c_{12}(N, m, \tau) \rho^{-(m-3)}, \quad (8.19)$$

where  $c_{11}$ ,  $c_{12}$  are related in a simple way with  $c_9$ ,  $c_{10}$  for  $\mu_0 = 2$ .

Let now

$$\alpha \in ]0, 1[, \quad \beta = \frac{1 - \alpha}{2L}, \quad \delta = \|\Phi_1 - \Phi_2\|, \quad \rho = \beta \ln(3 + \delta^{-1}), \quad (8.20)$$

where  $\delta$  is so small that  $\rho \geq \rho_2(m, 2, N, D, \sigma)$ , where  $\rho_2$  is the constant of (8.10). Then, due to (8.19),

$$\begin{aligned} \|v_1 - v_2\|_{L^\infty(D)} &\leq c_{11}(N, D, m, \sigma, \tau) (3 + \delta^{-1})^{2L\beta} (\beta \ln(3 + \delta^{-1}))^3 \delta + \\ &c_{12}(N, D, m, \tau) (\beta \ln(3 + \delta^{-1}))^{-(m-3)} = \\ &c_{11}(N, D, m, \sigma, \tau) \beta^3 (1 + 3\delta)^{1-\alpha} \delta^\alpha (\ln(3 + \delta^{-1}))^3 + \\ &c_{12}(N, D, m, \tau) \beta^{-(m-3)} (\ln(3 + \delta^{-1}))^{-(m-3)}, \end{aligned} \quad (8.21)$$

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where  $\sigma, \tau$  are the same that in (8.10) for  $\mu_0 = 2$  and where  $\alpha, \beta$  and  $\delta$  are the same that in (8.20).

Using (8.21) we obtain that

$$\|v_1 - v_2\|_{L^\infty(D)} \leq c_{13}(N, D, m)(\ln(3 + \|\Phi_1 - \Phi_2\|^{-1}))^{-(m-3)} \quad (8.22)$$

for  $\delta = \|\Phi_1 - \Phi_2\| \leq \delta_0(N, D, m)$ , where  $\delta_0$  is sufficiently small positive constant. Estimate (8.22) in the general case (with modified  $c_{13}$ ) follows from (8.22) for

$\delta = \|\Phi_1 - \Phi_2\| \leq \delta_0(N, D, m)$  and the property that  $\|v_j\|_{L^\infty(D)} \leq c_{14}(m)N$  (for  $d = 3$ ).

Thus, Theorem 2.2 for  $d = 3$  is proved.

### 9. Proofs of Lemmas 8.2 and 8.3

*Proof of Lemma 8.2.* Using the maximum principle for holomorphic functions it is sufficient to prove that

$$\begin{aligned} \sup_{(\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^\pm} (1 + |p|)^{\mu_0} |H_2^0(\lambda(1 \mp 0), p) - H_1^0(\lambda(1 \mp 0), p)| \leq \\ (c_6 + 4c_7(\mu_0, \tau_0, \rho_0)C)\Delta_{\rho_0, \rho, \mu_0}, \end{aligned} \quad (9.1)$$

where  $b\Lambda_{\rho, \tau, \nu}^+$ ,  $b\Lambda_{\rho, \tau, \nu}^-$  are defined in (6.17) (and where  $H_i^0(\lambda(1 - 0), p)$ ,  $i = 1, 2$ , are considered for  $(\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^+$ ,  $H_i^0(\lambda(1 + 0), p)$ ,  $i = 1, 2$ , are considered for  $(\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^-$ ).

Using (7.8) and the Sohotsky-Plemelj formula, we have that

$$H^0(\lambda(1 - 0), p) = \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^+} H(\zeta, p) \frac{d\zeta}{\zeta - \lambda(1 + 0)} + H(\lambda, p), \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^+, \quad (9.2a)$$

$$H^0(\lambda(1 + 0), p) = -\frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^-} H(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda(1 - 0))} + H(\lambda, p), \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^- \quad (9.2b)$$

In addition, using the Cauchy-Green formula we have that

$$H(\lambda, p) = -\frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^+} H(\zeta, p) \frac{d\zeta}{\zeta - \lambda(1 + 0)} + \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^+} H(\zeta, p) \frac{d\zeta}{\zeta - \lambda} - \quad (9.3a)$$

$$\frac{1}{\pi} \int_{\mathcal{D}_{\rho_0/|p|}^+ \setminus \mathcal{D}_{\rho/|p|}^+} \frac{\partial H(\zeta, p)}{\partial \bar{\zeta}} \frac{dRe\zeta dIm\zeta}{\zeta - \lambda}, \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^+,$$

$$H(\lambda, p) = \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^-} H(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda(1 - 0))} - \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho/|p|}^-} H(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda)} - \quad (9.3b)$$

$$\frac{1}{\pi} \int_{\mathcal{D}_{\rho_0/|p|}^- \setminus \mathcal{D}_{\rho/|p|}^-} \frac{\partial H(\zeta, p)}{\partial \bar{\zeta}} \frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta - \lambda)}, \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^-$$

(where the integrals along  $\mathcal{T}_r^\pm$  are taken in the counter- clockwise direction). In addition (see equation (A.11) of Appendix),

$$\frac{\partial H(\zeta, p)}{\partial \bar{\zeta}} = \{H, H\}(\zeta, p), \quad (\zeta, p) \in \Lambda_{\rho_0, \tau_0, \nu}, \quad (9.4)$$

where  $H(\zeta, p)$  in the left-hand side of (9.4) is defined according to (7.4),  $H$  in the right-hand side of (9.4) is the function of (4.13),  $\{\cdot, \cdot\}$  is defined by (7.11).

Using (9.2), (9.3) we obtain that

$$H^0(\lambda(1-0), p) = \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho_0/|p|}^+} H(\zeta, p) \frac{d\zeta}{\zeta - \lambda} - \quad (9.5a)$$

$$\frac{1}{\pi} \int_{\mathcal{D}_{\rho_0/|p|}^+} \int_{\mathcal{D}_{\rho/|p|}^+ \setminus \mathcal{D}_{\rho_0/|p|}^+} \frac{\partial H(\zeta, p)}{\partial \bar{\zeta}} \frac{dRe\zeta dIm\zeta}{\zeta - \lambda}, \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^+,$$

$$H^0(\lambda(1+0), p) = -\frac{1}{2\pi i} \int_{\mathcal{T}_{\rho_0/|p|}^-} H(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda)} - \quad (9.5b)$$

$$\frac{1}{\pi} \int_{\mathcal{D}_{\rho_0/|p|}^-} \int_{\mathcal{D}_{\rho/|p|}^- \setminus \mathcal{D}_{\rho_0/|p|}^-} \frac{\partial H(\zeta, p)}{\partial \bar{\zeta}} \frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta - \lambda)}, \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^-.$$

Using (9.5), (9.4) for  $H^0 = H_i^0$ ,  $H = H_i$ ,  $i = 1, 2$ , we obtain that:

$$(H_2^0 - H_1^0)(\lambda(1-0), p) = A^+(\lambda, p) + B^+(\lambda, p), \quad (9.6a)$$

$$A^+(\lambda, p) = \frac{1}{2\pi i} \int_{\mathcal{T}_{\rho_0/|p|}^+} (H_2 - H_1)(\zeta, p) \frac{d\zeta}{\zeta - \lambda},$$

$$B^+(\lambda, p) = -\frac{1}{\pi} \int_{\mathcal{D}_{\rho_0/|p|}^+} \int_{\mathcal{D}_{\rho/|p|}^+ \setminus \mathcal{D}_{\rho_0/|p|}^+} (\{H_2 - H_1, H_2\} + \{H_1, H_2 - H_1\})(\zeta, p) \frac{dRe\zeta dIm\zeta}{\zeta - \lambda},$$

where  $(\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^+$ ;

$$(H_2^0 - H_1^0)(\lambda(1+0), p) = A^-(\lambda, p) + B^-(\lambda, p), \quad (9.6b)$$

$$A^-(\lambda, p) = -\frac{1}{2\pi i} \int_{\mathcal{T}_{\rho_0/|p|}^-} (H_2 - H_1)(\zeta, p) \frac{\lambda d\zeta}{\zeta(\zeta - \lambda)},$$

$$B^-(\lambda, p) = -\frac{1}{\pi} \int_{\mathcal{D}_{\rho_0/|p|}^-} \int_{\mathcal{D}_{\rho/|p|}^- \setminus \mathcal{D}_{\rho_0/|p|}^-} (\{H_2 - H_1, H_2\} + \{H_1, H_2 - H_1\})(\zeta, p) \frac{\lambda dRe\zeta dIm\zeta}{\zeta(\zeta - \lambda)},$$

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where  $(\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^-$ .

Estimates (9.1) follow from formulas (9.6) and from the estimates

$$|A^\pm(\lambda, p)| \leq c_6(1 + |p|)^{-\mu_0} \Delta_{\rho_0, \rho, \mu_0}, \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^\pm, \quad (9.7)$$

$$|B^\pm(\lambda, p)| \leq 4c_7(\mu_0, \tau_0, \rho_0)C(1 + |p|)^{-\mu_0} \Delta_{\rho_0, \rho, \mu_0}, \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^\pm, \quad (9.8)$$

where

$$c_6 = \sup_{r \in ]1/2, +\infty[} \frac{q(r)}{q(r) - q(2r)}, \quad (9.9)$$

$$q(r) = 2r \left(1 - \left(1 - \frac{1}{4r^2}\right)^{1/2}\right).$$

Note that

$$0 < c_6 \leq (2\sqrt{3} - 3)^{-1}, \quad (9.10)$$

where  $c_6$  is defined by (9.9). Estimate (9.10) follows from the formulas

$$c_6 = \frac{1}{1 - 2\sigma}, \quad \sigma = \sup_{\tau \in ]0, 1[} \frac{1 - (1 - (1/4)\tau)^{1/2}}{1 - (1 - \tau)^{1/2}}, \quad (9.11)$$

$$(1 - \tau)^{1/2} \leq 1 - (1/2)\tau, \quad 1 - (1/4)\tau \geq a(1 - (1/4)\tau) + 1 - a, \quad (9.12)$$

$$a = 2(2 - \sqrt{3}), \quad \tau \in ]0, 1[.$$

Estimates (9.7) follow from the property that

$$(\zeta, p) \in b\Lambda_{\rho_0, \tau_0, \nu} \quad \text{if} \quad \zeta \in \mathcal{T}_{\rho_0/|p|}^\pm, \quad |p| < 2\tau\rho, \quad \rho = 2\rho_0 > 0, \quad \tau = \tau_0/2, \quad 0 < \tau_0 < 1, \quad (9.13)$$

the properties that

$$H_i \in \mathcal{C}(\Lambda_{\rho_0, \tau_0, \nu} \cup b\Lambda_{\rho_0, \tau_0, \nu}), \quad i = 1, 2, \quad (9.14)$$

$$|(H_2 - H_1)(\lambda, p)| \leq (1 + |p|)^{-\mu_0} \Delta_{\rho_0, \rho, \mu_0}, \quad (\lambda, p) \in b\Lambda_{\rho_0, \tau_0, \nu}$$

(see formulas (A.8), (A.9), (A.12), (6.19), (7.4)) and from the formulas

$$\frac{1}{2\pi} \int_{\mathcal{T}_r^-} \frac{|\lambda| |d\zeta|}{|\zeta| |\zeta - \lambda|} = \frac{1}{2\pi} \int_{\mathcal{T}_r^+} \frac{|d\zeta'|}{|\zeta' - \lambda'|} \leq \frac{q(r)}{q(r) - q(2r)}, \quad (9.15)$$

$\lambda \in \mathcal{T}_{2r}^-$ ,  $\lambda' \in \mathcal{T}_{2r}^+$ ,  $r > 1/2$ . In turn, formulas (9.15) follow from the property that  $z^{-1} \in \mathcal{T}_r^+$  if  $z \in \mathcal{T}_r^-$  and from the formula that  $q(r)$  is the radius of  $\mathcal{T}_r^+$ , where  $r > 1/2$ .

Estimates (9.8) follow from formulas (A.8), (A.9) (for  $\rho = \rho_0$ ), formula (8.4),

Lemma A.5 (for  $\rho = \rho_0$ ,  $\mu = \mu_0$ ), Lemma A.6 (for  $\rho = \rho_0$ ) and the property

$$\lambda \in \mathcal{T}_{\rho/|p|}^\pm \subset \mathcal{D}_{\rho_0/|p|}^\pm \quad \text{if} \quad (\lambda, p) \in b\Lambda_{\rho, \tau, \nu}^\pm, \quad \rho = 2\rho_0 > 0, \quad \tau = \tau_0/2, \quad 0 < \tau_0 < 1. \quad (9.16)$$

Lemma 8.2 is proved.

*Proof of Lemma 8.3.* Using (A.12), the formulas

$$\Omega \approx \Theta, \quad \Omega_\rho \approx \Theta_\rho, \quad \bar{\Omega}_\rho \approx \bar{\Theta}_\rho$$

(see (6.4)), and formulas (6.2), (4.13), we have that

$$\Delta_{\rho_0, \rho, \mu_0} \leq \sup_{(k, l) \in \bar{\Theta}_\rho \setminus \Theta_{\rho_0}} (1 + |k - l|)^{\mu_0} |h_2(k, l) - h_1(k, l)|, \quad (9.17)$$

where  $\Delta_{\rho_0, \rho, \mu_0}$  is defined by (8.6),  $h_1, h_2$  are the functions of Section 5.

Estimate (8.8) follows from formulas (9.17), (5.1), (4.7), (4.9), the property that  $|k - l| \leq 2\rho$  for  $(k, l) \in \bar{\Theta}_\rho$  and the property that  $|e^{ikx}| \leq e^{\rho L}$ ,  $|e^{ilx}| \leq e^{\rho L}$  for  $k, l \in \bar{\Theta}_\rho$ ,  $x \in \partial D$ .

Lemma 8.3 is proved.

### Appendix

In this appendix we remind some results of [20] used in Sections 8 and 9 of the present work.

Consider the equation

$$U = U^0 + M_{\rho, \tau}(U), \quad \rho > 0, \quad \tau \in ]0, 1[, \quad (A.1)$$

for unknown  $U$ , where  $U^0, U$  are functions on  $\Lambda_{\rho, \tau, \nu}$  defined in (6.13),  $M_{\rho, \tau}$  is the map defined by (7.9).

In particular, (7.7) can be written as (A.1) with  $U = H$ ,  $U^0 = H^0 + Q_{\rho, \tau}$ .

**Lemma A.1.** *Let  $\rho > 0$ ,  $\nu \in \mathbb{S}^2$ ,  $\tau \in ]0, 1[$ ,  $\mu \geq 2$  and  $0 < r < (2c_7(\mu, \tau, \rho))^{-1}$ , where  $c_7$  is the constant  $c_8$  of [20] (that is  $c_7(\mu, \tau, \rho) = 3b_1(\mu)\tau^2 + 4b_2(\mu)\rho^{-1} + 4b_3(\mu)\tau$ , where  $b_1, b_2, b_3$  are the constants of [20]). Let  $U^0 \in L^\infty(\Lambda_{\rho, \tau, \nu})$  and  $|||U^0|||_{\rho, \tau, \mu} \leq r/2$  (see definition (7.5)). Then equation (A.1) is uniquely solvable for  $U \in L^\infty(\Lambda_{\rho, \tau, \nu})$ ,  $|||U|||_{\rho, \tau, \mu} \leq r$ , and  $U$  can be found by the method of successive approximations, in addition,*

$$|||U - (M_{\rho, \tau, U^0})^n(0)|||_{\rho, \tau, \mu} \leq \frac{r(2c_7(\mu, \tau, \rho)r)^n}{2(1 - 2c_7(\mu, \tau, \rho)r)}, \quad n \in \mathbb{N}, \quad (A.2)$$

where  $M_{\rho, \tau, U^0}$  denotes the map  $U \rightarrow U^0 + M_{\rho, \tau}(U)$ .

Lemma A.1 corresponds to Lemma 4.4 of [20].

**Lemma A.2.** *Let the assumptions of Lemma A.1 be fulfilled. Let also  $\tilde{U}^0 \in L^\infty(\Lambda_{\rho, \tau, \nu})$ ,  $|||\tilde{U}^0|||_{\rho, \tau, \mu} \leq r/2$ , and  $\tilde{U}$  denote the solution of (A.1) with  $U^0$  replaced by  $\tilde{U}^0$ , where  $\tilde{U} \in L^\infty(\Lambda_{\rho, \tau, \nu})$ ,  $|||\tilde{U}|||_{\rho, \tau, \mu} \leq r$ . Then*

$$|||U - \tilde{U}|||_{\rho, \tau, \mu} \leq (1 - 2c_7(\mu, \tau, \rho)r)^{-1} |||U^0 - \tilde{U}^0|||_{\rho, \tau, \mu}. \quad (A.3)$$

Lemma A.2 corresponds to Lemma 4.5 of [20].

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Let

$$r_{min} \stackrel{\text{def}}{=} \frac{2C}{1 - \eta(C, \rho, \mu)} + \frac{2b_4(\mu)C^2}{(1 - \eta(C, \rho, \mu))^2} \left( 1 + \frac{3}{(1 + 2\tau\rho)^{\mu - \mu_0}} \right), \quad (\text{A.4})$$

where  $C > 0$ ,  $\rho \geq e^2$ ,  $2 \leq \mu_0 < \mu$ ,  $\tau \in ]0, 1[$ ,  $\eta$  is defined as in (7.6),  $b_4$  is the constant of [20] (that is  $b_4(\mu) = \pi^{-1}(b_1(\mu)n_1 + b_2(\mu)n_2 + b_3(\mu)n_3)$ , where  $b_1, b_2, b_3$  are the constants of [20] and  $n_1, n_2, n_3$  are the constants of Lemma 11.1 of [19]). In addition, from estimates (7.14), (7.15) of the present paper (see estimates (3.3), (4.20), (4.22) of [20]) it follows that

$$\begin{aligned} |||H|||_{\rho, \tau, \mu_0} &\leq r_{min} \\ |||H^0|||_{\rho, \tau, \mu_0} + |||Q_{\rho, \tau}|||_{\rho, \tau, \mu_0} &\leq r_{min}/2. \end{aligned} \quad (\text{A.5})$$

**Lemma A.3.** *Let  $C > 0$ ,  $2 \leq \mu_0 < \mu$ ,  $0 < \delta < 1$ ,*

$$0 < \tau \leq \tau_1(\mu, \mu_0, C, \delta), \quad \rho \geq \rho_1(\mu, \mu_0, C, \delta), \quad (\text{A.6})$$

*where  $\tau_1, \rho_1$  are the constants of (4.36) of [20] (in particular,  $\tau_1 \in ]0, 1[$  is sufficiently small and  $\rho_1$  is sufficiently great). Then*

$$\begin{aligned} 0 &\leq \eta(C, \rho, \mu) < \delta, \\ 0 &\leq 2c_7(\mu_0, \tau, \rho)r_{min}(\mu, \mu_0, \tau, \rho, C) < \delta. \end{aligned} \quad (\text{A.7})$$

Lemma A.3 corresponds to estimates (4.36) of [20].

**Lemma A.4.** *Let  $v$  satisfy (7.1),  $\|\hat{v}\|_\mu \leq C$ ,  $\rho$  satisfy (7.6). Let  $H$  be the function of (4.13). Then:*

$$H \in \mathcal{C}(\Omega \setminus \Omega_\rho), \quad (\text{A.8})$$

$$|H(k, p)| \leq \frac{C}{(1 - \eta(C, \rho, \mu))(1 + |p|)^\mu}, \quad (k, p) \in \Omega \setminus \Omega_\rho, \quad (\text{A.9})$$

$$\hat{v}(p) = \lim_{\substack{|k| \rightarrow \infty, \\ (k, p) \in \Omega}} H(k, p), \quad p \in \mathbb{R}^3, \quad (\text{A.10})$$

where  $\mathcal{C}$  denotes the space of continuous functions,  $\Omega, \Omega_\rho$  are defined in (4.14), (6.3),  $\eta$  is defined as in (7.6),  $|k| = ((\text{Re } k)^2 + (\text{Im } k)^2)^{1/2}$ ;

$$\frac{\partial}{\partial \bar{\lambda}} H(k(\lambda, p), p) = \{H, H\}(\lambda, p), \quad (\lambda, p) \in \Lambda_{\rho, \nu}, \quad (\text{A.11})$$

where  $k(\lambda, p)$  is defined in (6.11b),  $\Lambda_{\rho, \nu}$  is defined in (6.13),  $\{\cdot, \cdot\}$  is defined by (7.11).

Lemma A.4 corresponds to formulas (3.2)-(3.4), (3.22) of [20]. Let

$$\begin{aligned} L_\mu^\infty(\Omega \setminus \bar{\Omega}_\rho) &= \{U \in L^\infty(\Omega \setminus \bar{\Omega}_\rho) : |||U|||_{\rho, \mu} < +\infty\}, \\ |||U|||_{\rho, \mu} &= \text{ess sup}_{(k, p) \in \Omega \setminus \bar{\Omega}_\rho} (1 + |p|)^\mu |U(k, p)|, \quad \rho > 0, \quad \mu > 0, \end{aligned} \quad (\text{A.12})$$

where  $\Omega, \bar{\Omega}_\rho$  are defined in (4.14), (6.3).

**Lemma A.5.** *Let  $U_1, U_2 \in L_\mu^\infty(\Omega \setminus \bar{\Omega}_\rho)$ , where  $\rho > 0, \mu \geq 2$ . Let  $\{\cdot, \cdot\}$  be defined by (7.11). Then*

$$\{U_1, U_2\} \in L_{local}^\infty(\Lambda_{\rho, \nu}), \quad (\text{A.13})$$

$$|\{U_1, U_2\}(\lambda, p)| \leq \frac{|||U_1|||_{\rho, \mu} |||U_2|||_{\rho, \mu}}{(1 + |p|)^\mu} b(\mu, |\lambda|, |p|)$$

for almost all  $(\lambda, p) \in \Lambda_{\rho, \nu}$ ,  $b(\mu, |\lambda|, |p|) =$  (A.14)

$$\left( \frac{b_1(\mu)|\lambda|}{(|\lambda|^2 + 1)^2} + \frac{b_2(\mu)|p|(|\lambda|^2 - 1)}{|\lambda|^2(1 + |p|(|\lambda| + |\lambda|^{-1}))^2} + \frac{b_3(\mu)|p|}{|\lambda|(1 + |p|(|\lambda| + |\lambda|^{-1}))} \right),$$

where  $\Lambda_{\rho, \nu}$  is defined in (6.13),  $b_1, b_2, b_3$  are the constants of [20].

Lemma A.5 corresponds to estimates (3.26), (3.27) of [20].

Consider

$$u_1(\zeta, s) = \frac{|\zeta|}{(|\zeta|^2 + 1)^2}, \quad u_2(\zeta, s) = \frac{(|\zeta|^2 + 1)s}{|\zeta|^2(1 + s(|\zeta| + |\zeta|^{-1}))^2},$$

$$u_3(\zeta, s) = \frac{s}{|\zeta|(1 + s(|\zeta| + |\zeta|^{-1}))}, \quad (\text{A.15})$$

where  $\zeta \in \mathbb{C}, s > 0$ .

Note that

$$0 < b(\mu, |\lambda|, |p|) \leq \sum_{j=1}^3 b_j(\mu) u_j(\lambda, |p|), \quad (\text{A.16})$$

where  $b$  is the function of (A.14),  $(\lambda, p) \in \Lambda_{\rho, \nu}, \rho > 0, \mu \geq 2$ .

**Lemma A.6.** *Let  $u_1, u_2, u_3$  be defined by (A.15). Then:*

$$\int_{\mathcal{D}_r^-} u_j(\zeta, s) \frac{|\lambda|}{|\zeta|} \frac{d\text{Re } \zeta d\text{Im } \zeta}{|\zeta - \lambda|} = \int_{\mathcal{D}_r^+} u_j(\zeta, s) \frac{d\text{Re } \zeta d\text{Im } \zeta}{|\zeta - \lambda^{-1}|}, \quad (\text{A.17})$$

$j = 1, 2, 3, r > 1/2, \lambda \in \mathcal{D}_r^-, s > 0$ ,

$$\int_{\mathcal{D}_{\rho/|p|}^+} u_j(\zeta, |p|) \frac{d\text{Re } \zeta d\text{Im } \zeta}{|\zeta - \lambda|} \leq \begin{cases} (3/4)\pi(|p|/\rho)^2 & \text{for } j = 1, \\ 4\pi/\rho & \text{for } j = 2, \\ 2\pi|p|/\rho & \text{for } j = 3, \end{cases} \quad (\text{A.18})$$

$0 < |p| < 2\tau\rho, 0 < \tau < 1, \lambda \in \mathcal{D}_{\rho/|p|}^+$ , where  $\mathcal{D}_r^\pm$  are defined in (6.18).

Lemma A.6 corresponds to formulas (7.2), (7.10) of [20].

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