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# Bulking I: an Abstract Theory of Bulking<sup>★</sup>

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## Abstract

This paper is the first part of a serie of two papers dealing with bulking: a quasi-order on cellular automata comparing space-time diagrams up to some rescaling. Bulking is a generalization of grouping taking into account universality phenomena, giving rise to a maximal equivalence class. In the present paper, we discuss the proper components of grouping and study the most general extensions. We identify the most general space-time transforms and give an axiomatization of bulking quasi-order. Finally, we study some properties of intrinsically universal cellular automata obtained by comparing grouping to bulking.

*Key words:* cellular automata, bulking, grouping, classification

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Bulking is introduced as a tool to structure cellular automata, considered as the sets of their orbits. To achieve this goal, sets of orbits are considered up to spatio-temporal transforms. Such quotients are then compared according to algebraic relations to obtain quasi-orders on the set of cellular automata, in a way similar to reductions in the case of recursive functions. Curiously, the obtained equivalence classes tend to capture relevant properties: in particular, the greatest element, when it exists, corresponds to a notion of intrinsic universality. The first and present paper is concerned with the choice of the main ingredients to define an interesting bulking. The second paper, *Bulking II: Classifications of Cellular Automata* [4], studies the structure of the main three varieties of bulking.

A cellular automaton is a discrete dynamical system consisting of a network of cells fulfilling the following properties: each cell acts as a finite state machine;

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<sup>★</sup> The results presented here first appeared to a great extent in French in the PhD theses of Ollinger [13] and Theyssier [14]

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the network is regular; interactions are local, uniform and synchronous. A space-time diagram is the geometrical representation of an orbit obtained by piling up the successive configurations. The present paper aims at structuring the sets of space-time diagrams generated by cellular automata.

To deal with the richness of these objects, some families of space-time diagrams are identified through spatio-temporal transforms preserving the notion of cellular automata. In particular, one type of transform, commonly used in algorithmic constructions in the cellular automata literature, is to be taken into account: cells grouping.

A very common grouping transform appears early in algorithmic constructions on cellular automata as a tool to simplify the description of the algorithm. A typical use of this tool appear in the work of Fischer [6]. To recognize prime numbers in real-time, a first construction is given to recognize the primality of  $n$  at time  $3n$ , the cells are then packed in  $3 \times 3$  blocks defining a new cellular automaton achieving real-time recognition, as depicted on Fig. 1.

			$q_{c,6,1}$	$q_{c,6,2}$	$q_{c,6,3}$			
		$q_{l,5,3}$	$q_{c,5,1}$	$q_{c,5,2}$	$q_{c,5,3}$	$q_{r,5,1}$		
	$q_{l,4,2}$	$q_{l,4,3}$	$q_{c,4,1}$	$q_{c,4,2}$	$q_{c,4,3}$	$q_{r,4,1}$	$q_{r,4,2}$	
$q_{l,3,1}$	$q_{l,3,2}$	$q_{l,3,3}$	$q_{c,3,1}$	$q_{c,3,2}$	$q_{c,3,3}$	$q_{r,3,1}$	$q_{r,3,2}$	$q_{r,3,3}$
$q_{l,2,1}$	$q_{l,2,2}$	$q_{l,2,3}$	$q_{c,2,1}$	$q_{c,2,2}$	$q_{c,2,3}$	$q_{r,2,1}$	$q_{r,2,2}$	$q_{r,2,3}$
$q_{l,1,1}$	$q_{l,1,2}$	$q_{l,1,3}$	$q_{c,1,1}$	$q_{c,1,2}$	$q_{c,1,3}$	$q_{r,1,1}$	$q_{r,1,2}$	$q_{r,1,3}$

Fig. 1. Grouping cells *à la* Fischer

Rescaling also appears when comparing neighborhoods, in particular the relation between first neighbors and the one-way neighborhood. To simulate, up to a translation, a first neighbors cellular automaton by a one-way cellular automaton, Choffrut and Čulik II [3] proposes to add to the set of states every pair of original states and compute a transition in two time steps, as depicted on Fig. 2. Ibarra *et al.* [8] proposes to achieve the simulation in real-time by grouping pairs of consecutive cells in a single cell, as depicted on Fig. 3.



Fig. 2. Grouping cells *à la* Choffrut et Čulik II



Fig. 3. Grouping cells *à la* Ibarra *et al.*

Rescaling can be used as a tool to describe a dynamical property like nilpotency, where every configuration evolves in uniform time  $T$  into a single fix-point configuration. Every nilpotent cellular automaton with  $n$  states and a transition time  $T$  is simulated by a cellular automaton with 2 states and

transition time 2 by grouping every  $T$  transitions in the first automaton and grouping cells by at least  $\lceil \log_2 n \rceil$  in the second automaton, as depicted on Fig. 4.

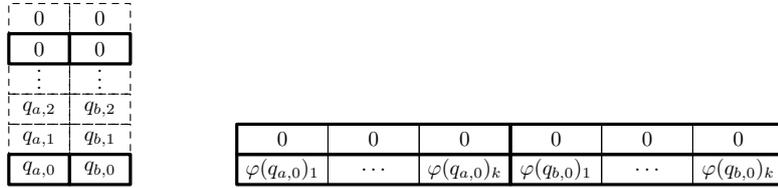


Fig. 4. Grouping cells for nilpotency

A last example of the classical use of rescaling and simulation is the notion of intrinsic universality. A cellular automaton is intrinsically universal if it can simulate, up to rescaling, every cellular automaton. The existence of such cellular automata appears in the work of Banks [2] and more and more precise definitions were successively proposed by Albert and Čulik [1], Martin [10], Durand and Róka [5]. The notion of simulation used in these articles is the following. To simulate a given cellular automaton, each cell of the initial configuration is encoded as a segment of cells of the universal automaton and each transition is simulated by a fixed number of transitions, as depicted on Fig. 5.

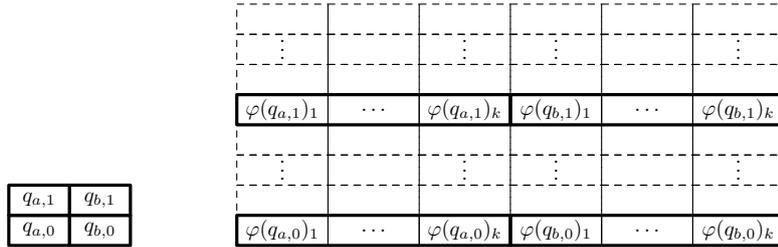


Fig. 5. Grouping cells for intrinsic universality

Bulking generalizes the use of rescaling as a tool to compare cellular automata. Figures 6, 7 and 8 show examples of simulations of some cellular automata by others. For each of them, both geometric transformation of space-time and local transformation on cellular automata rules are shown. These examples are intentionally simple, but the simulation relations are studied more in depth in the second paper. All cellular automata chosen in these examples have the same neighborhood (cell itself, left and right immediate neighbors). Their local rules are the following.

**Just gliders.** Two states interpreted as particules moving left ( $\boxtimes$ ) and right ( $\boxleftarrow$ ) evolve in a quiescent background state ( $\square$ ). When two opposite particles meet they annihilate, leaving a background state ( $\square$ ).

**ECA 184.** The line of cells is interpreted as a highway where  $\square$  states represent cars and  $\blacksquare$  represent free portions of highways. Cars move to the right

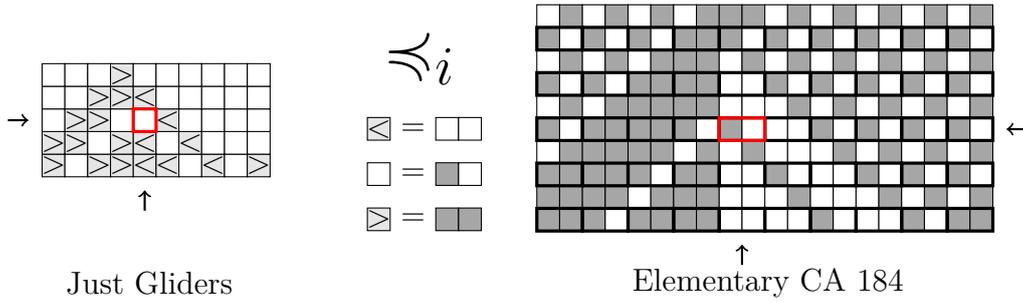


Fig. 6. Injective simulation of 'Just gliders' by ECA 184.

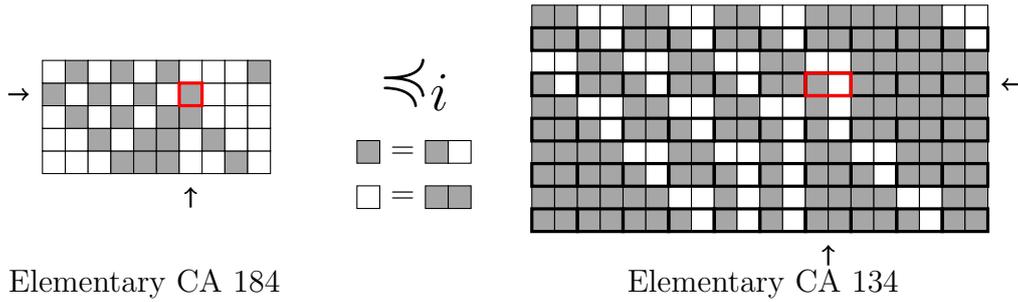


Fig. 7. Injective simulation of ECA 184 by ECA 134.

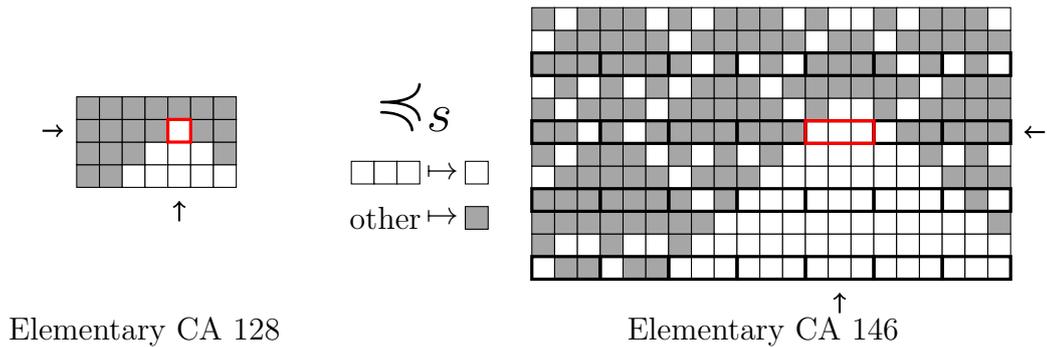


Fig. 8. Surjective simulation of ECA 128 by ECA 146.

by one cell if they can (no car present on the next cell), otherwise they don't move.

**ECA 134.** If state  $\square$  is interpreted as 1 and  $\blacksquare$  as 0 then the rule is simply the addition modulo 2 of neighbouring states, except when the left cell is 1 and others 0.

**ECA 128.** The state  $\blacksquare$  is spreading over the quiescent state  $\square$ .

**ECA 146.** If state  $\square$  is interpreted as 1 and  $\blacksquare$  as 0 then the rule is simply the addition modulo 2 of neighbouring states, except when the central cell is 1 and others 0.

This first paper, discussing the proper components of bulking is organized as follows. In section 1, definitions are given with a geometrical bias. In section 2, grouping is presented. In section 3, possible extensions are investigated, searching for good candidates of geometrical transforms and elementary simu-

lation relation. In section 4, bulking is defined as a formal family of simulation quasi-orders and an extension of grouping is chosen. In section 5, this extension is compared to grouping and a first result concerning intrinsically universal cellular automata is obtained.

## 1 Patterns, Colorings and Cellular Automata

### 1.1 Patterns

A *pattern*  $\mathcal{P}$  is a subset of  $\mathbb{Z}^d$ . The *m-rectangular pattern*  $\boxplus_m$  is the pattern  $\{0, \dots, m_1 - 1\} \times \dots \times \{0, \dots, m_d - 1\}$ . Using the natural extension of  $+$  on sets, the *translation*  $\mathcal{P} + u$  of  $\mathcal{P}$  by a vector  $u \in \mathbb{Z}^d$  is the pattern  $\{z + u | z \in \mathcal{P}\}$  and the *sum* of two patterns  $\mathcal{P}$  and  $\mathcal{P}'$  is the pattern  $\{z + z' | z \in \mathcal{P}, z' \in \mathcal{P}'\}$ . An *elementary translation* is a translation by a vector  $\varsigma_k \in \mathbb{Z}^d$  with all coordinates equal to 0 but the  $k$ th which is equal to 1 or  $-1$ . Every translation is obtained by composition of elementary translations.

A *basis*  $V \in (\mathbb{Z}^d)^d$  is a tuple of  $d$  non-zero linearly independent vectors  $(v_1, \dots, v_d)$ . The *m-rectangular basis*  $\square_m$  is the basis  $(m_1\delta_1, \dots, m_d\delta_d)$  where  $\delta_k$  has all coordinates equal to 0 but the  $k$ th which is equal to 1. The *image*  $V \odot z$  by  $V$  of a point  $z \in \mathbb{Z}^d$  is the point  $\sum_{i=1}^d z_i v_i$ . The *image*  $V \odot \mathcal{P}$  by  $V$  of a pattern  $\mathcal{P}$  is the pattern  $\{V \odot z | z \in \mathcal{P}\}$ . The *image* by a basis  $V'$  of a basis  $V$  is the basis  $V' \odot V = (V' \odot v_1, \dots, V' \odot v_d)$ . Notice that  $\square_m \odot \square_{m'} = \square_{mm'}$  where for all  $k$ ,  $(mm')_k = m_k m'_k$ .

A *tiling of space* is a pair  $(\mathcal{P}, V)$  where  $\mathcal{P}$  is a pattern that tiles the plane with the basis  $V$ , *i.e.* such that  $\{\mathcal{P} + V \odot z | z \in \mathbb{Z}^d\}$  is a partition of  $\mathbb{Z}^d$ . The *m-rectangular tiling* is the tiling  $(\boxplus_m, \square_m)$ . The *composition*  $(\mathcal{P}', V') \circ (\mathcal{P}, V)$  of two tilings of space  $(\mathcal{P}', V')$  and  $(\mathcal{P}, V)$  is the tiling of space  $(\mathcal{P} + V \odot \mathcal{P}', V' \odot V)$ . Notice that  $(\boxplus_m, \square_m) \circ (\boxplus_{m'}, \square_{m'}) = (\boxplus_{mm'}, \square_{mm'})$ . Notice that the size of  $\mathcal{P}$  has to be  $|\det V|$ . Given a basis  $V$ , the equivalence relation  $\equiv_V$  on  $\mathbb{Z}^d$  is defined by  $z \equiv_V z'$  if  $z' - z \in V \odot \mathbb{Z}^d$ . It defines precisely  $|\det V|$  equivalence classes. Valid patterns are precisely patterns consisting of one point in each equivalence class of  $\equiv_V$ .

### 1.2 Colorings

A *coloring*  $\mathcal{C} \in \Sigma^{\mathcal{P}}$  is a covering of its support  $\mathcal{P} \subseteq \mathbb{Z}^d$ , denoted as  $\text{Sup}(\mathcal{C})$ , by letters of a finite alphabet  $\Sigma$ . A *singleton coloring* is a coloring with a singleton support. A *finite coloring* is a coloring with finite support. A *full*

*coloring* is a coloring with support  $\mathbb{Z}^d$ . A coloring  $\mathcal{C}'$  is a *subcoloring* of a coloring  $\mathcal{C}$ , denoted as  $\mathcal{C}' \ll \mathcal{C}$ , if  $\mathcal{C}'$  is a restriction of  $\mathcal{C}$ , i.e.  $\mathcal{C}' = \mathcal{C}|_{\text{Sup}(\mathcal{C}'})$ . The *translation*  $u \cdot \mathcal{C}$  of  $\mathcal{C}$  by a vector  $u \in \mathbb{Z}^d$  is the coloring with support  $\text{Sup}(\mathcal{C}) + u$  satisfying, for all  $z \in \text{Sup}(\mathcal{C})$ ,  $u \cdot \mathcal{C}(z + u) = \mathcal{C}(z)$ . The *u-shift* is the translation map over full colorings  $\sigma_u : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$  defined for all coloring  $\mathcal{C}$  by  $\sigma_u(\mathcal{C}) = u \cdot \mathcal{C}$ . An *elementary shift* is a shift by an elementary translation. A coloring  $\mathcal{C}$  occurs in a coloring  $\mathcal{C}'$ , denoted as  $\mathcal{C} \Subset \mathcal{C}'$  if a translation of  $\mathcal{C}$  is a subcoloring of  $\mathcal{C}'$ . A coloring  $\mathcal{C}$  is *periodic*, with periodicity vector  $u \in \mathbb{Z}^d$ , if for all  $z \in \text{Sup}(\mathcal{C}) \cap (\text{Sup}(\mathcal{C}) - u)$ ,  $\mathcal{C}(z) = \mathcal{C}(z + u)$ . A coloring  $\mathcal{C}$  is *s-finite* if  $\mathcal{C}$  is equal to  $s$  everywhere but on a finite support.

The *cylinder* generated by a coloring  $\mathcal{C}$  over an alphabet  $\Sigma$  is the set of full colorings  $[\mathcal{C}] = \{\mathcal{C}' \in \Sigma^{\mathbb{Z}^d} \mid \mathcal{C} \ll \mathcal{C}'\}$ . The *Cantor topology* on  $\Sigma^{\mathbb{Z}^d}$  is the product topology of the discrete topology on  $\Sigma$ . Its open sets are generated by the cylinders of finite colorings. This topology is compact, metric and perfect.

The *packing map* with tiling  $(\mathcal{P}, V)$  over the alphabet  $\Sigma$  is the map  $\langle \mathcal{P}, V \rangle : \Sigma^{\mathbb{Z}^d} \rightarrow (\Sigma^{\mathcal{P}})^{\mathbb{Z}^d}$  defined for all full coloring  $\mathcal{C} \in \Sigma^{\mathbb{Z}^d}$  and all point  $z \in \mathbb{Z}^d$  by  $\langle \mathcal{P}, V \rangle(\mathcal{C})(z) = ((-V \odot z) \cdot \mathcal{C})|_{\mathcal{P}}$ . The *rectangular packing map*  $\square^m$  is the packing map  $\langle \boxplus_m, \square_m \rangle$ , its inverse is denoted as  $\square^{-m}$ .

### 1.3 Cellular Automata

A *d-dimensional cellular automaton* (*d-CA*)  $\mathcal{A}$  is a triple  $(S, N, f)$  where  $S$  is a finite set of states,  $N$  is the neighborhood, a finite pattern of  $\mathbb{Z}^d$  and  $f : S^N \rightarrow S$  is the local rule of  $\mathcal{A}$ . A *configuration* of  $\mathcal{A}$  is a mapping  $c \in S^{\mathbb{Z}^d}$ . The *global transition function*  $G : S^{\mathbb{Z}^d} \rightarrow S^{\mathbb{Z}^d}$  of  $\mathcal{A}$  maps every configuration  $c \in S^{\mathbb{Z}^d}$  to the configuration  $G(c) \in S^{\mathbb{Z}^d}$  such that, for all  $z \in \mathbb{Z}^d$ ,  $G(c)(z) = f((-z \cdot c)|_N)$ . The *space-time diagram* of  $\mathcal{A}$  starting from a configuration  $c_0$  is a mapping  $\Delta \in S^{\mathbb{N} \times \mathbb{Z}^d}$  encoding an infinite sequence of successive orbits of the dynamical system  $(S^{\mathbb{Z}^d}, G)$  by  $\Delta(0) = c_0$  and, for all  $t \in \mathbb{Z}^+$ ,  $\Delta(t) = G(\Delta(t-1))$ . The set of space-time diagrams of a CA  $\mathcal{A}$  is denoted as  $\text{Diag } \mathcal{A}$ . A *d-CA* is *autarkic* if its neighborhood is  $\{0\}$ . Every *d-CA* with a singleton neighborhood is the composition of a shift by an autarkic CA.

A *d-CA*  $\mathcal{A}$  is a *subautomaton* of a *d-CA*  $\mathcal{B}$ , with respect to the injective map  $\varphi : S_{\mathcal{A}} \rightarrow S_{\mathcal{B}}$ , denoted as  $\mathcal{A} \sqsubseteq_{\varphi} \mathcal{B}$ , if  $G_{\mathcal{B}} \circ \bar{\varphi} = \bar{\varphi} \circ G_{\mathcal{A}}$  where  $\bar{\varphi} : S_{\mathcal{A}}^{\mathbb{Z}^d} \rightarrow S_{\mathcal{B}}^{\mathbb{Z}^d}$  is the canonical extension of  $\varphi$  defined for all  $c \in S_{\mathcal{A}}^{\mathbb{Z}^d}$  by  $\bar{\varphi}(c) = \varphi \circ c$ . Equivalently stated, a *d-CA*  $\mathcal{A}$  is a subautomaton of a *d-CA*  $\mathcal{B}$  with respect to  $\varphi$  if and only if  $\bar{\varphi}(\text{Diag } \mathcal{A}) \subseteq \text{Diag } \mathcal{B}$ . A *d-CA*  $\mathcal{A}$  is *isomorphic* to a *d-CA*  $\mathcal{B}$ , denoted as  $\mathcal{A} \equiv \mathcal{B}$ , if both  $\mathcal{A} \sqsubseteq \mathcal{B}$  and  $\mathcal{B} \sqsubseteq \mathcal{A}$ .

In this paper, we focus on CA seen as discrete dynamical systems, that is the

pair  $(S^{\mathbb{Z}^d}, G)$  up to isomorphism and more precisely the orbits of such systems (represented by space-time diagrams). Thanks to the following theorem, we can freely manipulate global rules of CA to generate new CA.

**Theorem 1 (Hedlund [7])** *A map  $G : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$  is the global transition function of a cellular automaton if and only if  $G$  is continuous and commutes with elementary translations.*

**PROOF.** The preimage by a continuous map  $G : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$  of a cylinder generated by a singleton coloring is an open set: a finite union of cylinders generated by finite colorings. If  $G$  commutes with elementary translations, thus with translations, it is a  $d$ -CA. Moreover, the global transition function of a CA is, by construction, continuous and commutes with translations. ■

A CA is injective (*resp.* surjective, bijective) if its global rule is injective (*resp.* surjective, bijective). By previous theorem, the composition of two CAs, the cartesian product of two CAs or the inverse of a bijective CA is a CA. The *cartesian product* of two  $d$ -CA  $\mathcal{A}$  and  $\mathcal{B}$  is the  $d$ -CA  $\mathcal{A} \times \mathcal{B}$  whose global transition function verifies for all  $(c, c') \in S_{\mathcal{A}}^{\mathbb{Z}^d} \times S_{\mathcal{B}}^{\mathbb{Z}^d}$ ,  $G_{\mathcal{A} \times \mathcal{B}}((c, c')) = (G_{\mathcal{A}}(c), G_{\mathcal{B}}(c'))$ .

The *phase space* of a CA  $(S^{\mathbb{Z}^d}, G)$  is the graph with vertices  $S^{\mathbb{Z}^d}$  and two kinds of directed edges: global rule edges are pairs  $(c, G(c))$  labelled by  $G$ , translation edges are pairs  $(c, \varsigma_i \cdot c)$  labelled by  $\varsigma_i$ , for all  $c \in S^{\mathbb{Z}^d}$  and elementary translation  $\varsigma_i$ . Orbits correspond to infinite  $G$ -paths in the phase space. A *periodic point*, with period  $p \in \mathbb{Z}^+$ , is a configuration  $c$  such that  $G^p(c) = c$ . A *fixpoint* is a periodic point with period 1. A *Garden-of-Eden* is a configuration  $c$  with no ancestor, *i.e.* such that  $G^{-1}(c) = \emptyset$ . An *ultimately periodic point*, with transient  $\tau \in \mathbb{N}$  and period  $p \in \mathbb{Z}^+$ , is a configuration  $c$  such that  $G^{p+\tau}(c) = G^\tau(c)$ .

The *limit set*  $\Lambda_G$  of a CA  $(S^{\mathbb{Z}^d}, G)$  is the non-empty translation invariant compact set  $\Lambda_G = \bigcap_{i \in \mathbb{N}} \Lambda_G^{(i)}$  where  $\Lambda_G^{(0)} = S^{\mathbb{Z}^d}$  and for all  $i \in \mathbb{N}$ ,  $\Lambda_G^{(i+1)} = G(\Lambda_G^{(i)})$ . The limit set consists exactly of all configurations that appear in biinfinite space-time diagrams  $\Delta \in S^{\mathbb{Z} \times \mathbb{Z}^d}$  such that, for all  $t \in \mathbb{Z}$ ,  $\Delta(t+1) = G(\Delta(t))$ . A CA is *nilpotent* if its limit set is a singleton. By compactity, a CA  $(S^{\mathbb{Z}^d}, G)$  is nilpotent if and only if there exists a uniform bound  $\tau \in \mathbb{Z}^+$  such that  $G^\tau(S^{\mathbb{Z}^d})$  is a singleton.

A  *$d$ -dimensional partitioned cellular automaton* ( $d$ -PCA)  $\mathcal{A}$  is a triple  $(S, N, \psi)$  where  $S$  is a finite set of states,  $N$  is the neighborhood, a finite pattern of  $\mathbb{Z}^d$  and  $\psi : S^N \rightarrow S^N$  is the local rule of  $\mathcal{A}$ . The  $N$ -mixing rule  $\mu_N : (S^N)^{\mathbb{Z}^d} \rightarrow (S^N)^{\mathbb{Z}^d}$  is defined, for all  $c \in (S^N)^{\mathbb{Z}^d}$ , for all  $z \in \mathbb{Z}^d$  and for all  $u \in N$ , by  $\mu_N(c)(z)(u) = c(z+u)(u)$ . The global transition function of  $\mathcal{A}$  is  $\bar{\psi} \circ \mu_N$ .

Every PCA is a CA. Moreover, (the global transition function of) a PCA is bijective if and only if its local rule is bijective.

## 2 Grouping Cellular Automata

The grouping quasi-order was introduced by Mazoyer and Rapaport [12] as a successful tool to classify CA according to algebraic properties [11]. However, grouping fails to capture several geometrical properties of CA that one would like to see classified by such a geometric classification. In this section, we recall the grouping quasi-order.

Grouping deals with cellular automata of dimension 1 and neighborhood  $\{-1, 0, 1\}$ . Such a cellular automaton will be denoted in this section as a pair  $(S, f)$ . In space-time diagrams of such cellular automata, a  $k$ -uple of state of a segment of  $k$  cells at time  $t$  only depends on states of  $2t + k$  states at time 0, as shown on Fig. 9.

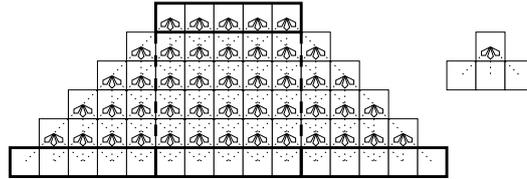


Fig. 9. dependencies in space-time diagrams for grouping

The  $n$ th iteration of the local rule is recursively defined by  $f^1 = f$  and, for all  $n \in \mathbb{N}$ ,

$$f^{n+1}(x_{-n-1}, \dots, x_{n+1}) = f\left(\begin{array}{c} f^n(x_{-n-1}, \dots, x_{n-1}), \\ f^n(x_{-n}, \dots, x_n), \\ f^n(x_{-n+1}, \dots, x_{n+1}) \end{array}\right) .$$

Inspired by dependencies in space-time diagrams and geometrical considerations, one defines the  $n$ th grouped instance  $(S, f)^{[n]}$  of a cellular automaton  $(S, f)$  by  $(S, f)^n = (S^n, f_{\square}^n)$  where  $f_{\square}^n$  is defined for all triple of  $n$ -uple of states by

$$f_{\square}^n((x_1, \dots, x_n), (x_{n+1}, \dots, x_{2n}), (x_{2n+1}, \dots, x_{3n})) = \left(\begin{array}{c} f^n(x_1, \dots, x_{2n+1}), \\ f^n(x_2, \dots, x_{2n+2}), \\ \vdots \\ f^n(x_n, \dots, x_{3n}) \end{array}\right) .$$

For all  $n > 0$ , the space-time diagrams of a cellular automaton  $(S, f)$  are in one-to-one correspondance with the space-time diagrams of its  $n$ th grouped instance  $(S, f)^{[n]}$ .

A cellular automaton  $(S, f)$  is *simulated* by a cellular automaton  $(S', f')$ , denoted by  $(S, f) \leq_{\square} (S', f')$ , if there exists two powers  $m$  and  $n$  such that  $(S, f)^m \sqsubseteq (S', f')^n$ .

**Theorem 2 ([12])** *The relation  $\leq_{\square}$  is a quasi-order relation.*

**Theorem 3 ([12])**  *$\leq_{\square}$  admits no maximal element.*

### 3 Towards a Generalization of Grouping

Several extensions of grouping are possible: change the elementary simulation order (subautomaton), several extensions of the allowed geometrical transformations. Subsection 3.1 points out the interest of the subautomaton relation to obtain all CA as an algebraic closure, subsection 3.2 shows connections between grouping and stability of a certain kind of subshifts and the need of new geometrical transformations to that extent, subsection 3.3 characterizes the most general family of space-time transformations preserving CA uniformity.

#### 3.1 An Algebraic Characterization of Cellular Automata

Every CA is the subautomaton of a PCA. The mixing part of a PCA rule is a cartesian product of shifts, that is composition of elementary shifts; the local rule of a PCA acts as an autarkic CA. When restricting PCA to RPCA, all RCA are generated. From there, one derives the following algebraic characterizations of CA and RCA, pointing out the use of the subautomaton relation to hide blueprint marks.

**Theorem 4** *The set of  $d$ -CA is the algebraic closure of autarkic CA and elementary shifts by composition, cartesian product and subautomaton.*

**PROOF.** Autarkic CA and shifts being CA, the closure generates only CA.

Let  $(S, N, f)$  be a  $d$ -CA  $\mathcal{A}$ . Let  $\varphi : S \rightarrow S^N$  map  $s$  to  $(s, \dots, s)$ . By construction,  $\mathcal{A}$  is a sub-automaton of the PCA  $(S, N, \varphi \circ f)$  with respect to  $\varphi$ . The global transition function of the PCA is the composition of  $\overline{\varphi \circ f}$ , which is an autarkic CA, by  $\mu_N$ . The  $N$ -mixing map  $\mu_N$  is the product of  $|N|$  shifts, each of which can be obtained as a composition of elementary shifts. ■

The restriction to RCA uses the following fact: every RCA is a subautomaton of a RPCA with neighborhood a valid neighborhood for both the RCA and its reverse. As reversibility is undecidable starting from dimension 2, there exists RCA with arbitrarily larger RPCA representation than PCA representation.

**Theorem 5** *The set of  $d$ -RCA is the algebraic closure of bijective autarkic CA and elementary shifts by composition, cartesian product and subautomaton.*

**PROOF.** Injectivity being preserved by composition, cartesian product and subautomaton, the closure generates only RCA.

Let  $(S, N, f)$  be a  $d$ -RCA  $\mathcal{A}$  with its reverse  $(S, N, g)$  a  $d$ -RCA  $\mathcal{B}$ . Let  $S_\bullet = S \cup \{\bullet\}$ . To conclude, we introduce three RPCA  $\mathcal{A}^\bullet = (S_\bullet^2, N, f_\bullet)$ ,  $\mathcal{B}^\bullet = (S_\bullet^2, -N, g_\bullet)$  and  $\mathcal{S} = (S_\bullet^2, N, h)$  such that  $\mathcal{A} \sqsubseteq_\varphi \mathcal{S} \circ \mathcal{B}^\bullet \circ \mathcal{A}^\bullet$  where  $\varphi$  maps  $s$  to  $((s, \bullet), \dots, (s, \bullet))$ .

The bijective map  $f_\bullet : S_\bullet^2 \rightarrow S_\bullet^2$  is given by the following partial injective definition. For all  $(s_1, \dots, s_k) \in S^N$ , let  $f_\bullet((s_1, \bullet), \dots, (s_k, \bullet)) = ((s_1, s'), \dots, (s_k, s'))$  where  $s' = f(s_1, \dots, s_k)$ .

The bijective map  $g_\bullet : S_\bullet^2 \rightarrow S_\bullet^2$  is given by the following partial injective definition. For all  $(s_1, \dots, s_k) \in S^N$ , let  $g_\bullet((s', s_1), \dots, (s', s_k)) = ((\bullet, s_1), \dots, (\bullet, s_k))$  where  $s' = g(s_1, \dots, s_k)$ .

The bijective map  $h : S_\bullet^2 \rightarrow S_\bullet^2$  is given by the following partial injective definition. For all  $(s_1, \dots, s_k) \in S^N$ , let  $h((\bullet, s_1), \dots, (\bullet, s_k)) = ((s_1, \bullet), \dots, (s_k, \bullet))$ .

Using the arguments of the proof of Thm 4, the global transition function of every RPCA is expressible in the closure. ■

### 3.2 Grouping and stability of bloc subshifts

*Note: for clarity and within this subsection only, we restrict to dimension 1.*

A subautomaton of a given CA  $\mathcal{A}$  is always induced by a subset of states which is stable under iterations, *i.e.* a set  $T \subseteq S_{\mathcal{A}}$  such that  $G_{\mathcal{A}}(T^{\mathbb{Z}}) \subseteq T^{\mathbb{Z}}$ . Sets of the form  $T^{\mathbb{Z}}$  are called *full-shifts* which are particular *subshifts* (closed translation-invariant set of configurations).

We can establish a similar connection between the grouping relation  $\leq_{\square}$  and a particular kind of subshifts that we call *bloc subshifts*. A bloc subshift is the set of configuration obtained by (infinite) catenation of finite words of same length from a given set. Formally, given an integer  $m$  and a set  $X$  of words

of length  $m$ , the bloc subshift  $\Sigma_X$  associated to  $X$  is the set of configurations whose language is  $X^*$  closed by subword (a subshift is characterised by the language of its configurations, see [9]). If  $G_{\mathcal{A}} \sqsubseteq_{\phi} G_{\mathcal{B}}^{[i]}$ , then  $X = \phi(S_{\mathcal{A}})$  is a set of words of length  $i$  over the alphabet  $S_{\mathcal{B}}$ . It is straightforward to check that the bloc subshift  $\Sigma_X$  is (weakly) stable under the action of  $G_{\mathcal{B}}$ , more precisely:

$$G_{\mathcal{B}}^i(\Sigma_X) \subseteq \Sigma_X.$$

Therefore any subautomaton with  $q$  states of a grouped instance of  $\mathcal{B}$  is induced by a bloc subshift made from  $q$  words which is (weakly) stable under the action  $\mathcal{B}$ . The converse is false as shown by the following example: a CA can have a weakly stable bloc subshift made from  $q$  words without any subautomaton with  $q$  states in the corresponding grouped instance.

**Example 6** Consider  $\mathcal{A}$  over state set  $S_{\mathcal{A}} = \{0, 1\} \times \{0, 1\}$  with neighbourhood  $\{0, 1\}$  and local rule  $f$  defined by

$$f((a_1, b_1), (a_2, b_2)) = (b_1, a_2).$$

$G_{\mathcal{A}}^2$  is the elementary right-shift CA over state set  $S_{\mathcal{A}}$ . So for any set  $X$  of words of length 2 over alphabet  $S_{\mathcal{A}}$ , the bloc subshift  $\Sigma_X$  is stable under  $G_{\mathcal{A}}^2$ . Now consider  $G_{\mathcal{A}}^{[2]}$ . The only stable subset of states it admits are of the form  $Q \times Q \subseteq S_{\mathcal{A}} \times S_{\mathcal{A}}$  since  $G_{\mathcal{A}}^2$  is an elementary shift. Therefore, a subautomaton of  $G_{\mathcal{A}}^{[2]}$  must have a square number of states. ■

However, as shown by the following theorem, a larger set of geometrical transformations allows to capture all weakly stable bloc subshifts. This constitutes an additional motivation for the generalisation of grouping presented in the sequel.

**Theorem 7** Let  $i, m, q$  be positive integers, and  $\mathcal{B}$  be a CA. The two following propositions are equivalent:

- (1) there exists a set  $X$  of  $q$  words of length  $m$  such that  $G_{\mathcal{B}}^i(\Sigma_X) \subseteq \Sigma_X$  ;
- (2) there exists a translation  $s$  and a CA  $\mathcal{A}$  with  $q$  states which is a subautomaton of  $\square^m \circ s \circ G_{\mathcal{B}}^i \circ \square^{-m}$

**PROOF.** First, for (2)  $\Rightarrow$  (1), we suppose  $G_{\mathcal{A}} \sqsubseteq_{\phi} \square^m \circ s \circ G_{\mathcal{B}}^i \circ \square^{-m}$  and it suffices to check that  $X = \phi(S_{\mathcal{A}})$  is a set of  $q$  words of length  $m$  and that  $G_{\mathcal{B}}^i(\Sigma_X) \subseteq \Sigma_X$  (by definition of  $\sqsubseteq$  and by commutation of  $G_{\mathcal{B}}$  with translations).

For (1)  $\Rightarrow$  (2), we suppose 1 and consider the set  $E_p$  of configurations from  $\Sigma_X$  for which the catenation of words of  $X$  is aligned with position  $p$  of the

lattice, formally:

$$E_p = \left\{ c \in \Sigma_X : \forall k \in \mathbb{Z}, c(km + p) \cdots c(km + p + k - 1) \in X \right\}.$$

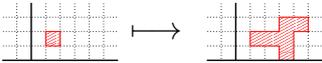
Clearly,  $E_p$  are closed sets and  $\Sigma_X = \cup_p E_p$ . Moreover,  $\square^m(E_0)$  is a full-shift of alphabet  $X$ . Now consider a configuration  $c \in \square^m(E_0)$  whose language is  $X^*$  (a “universal” configuration as called sometimes in the literature) and let  $c' = \square^{-m}(c)$ . By hypothesis,  $G_{\mathcal{B}}^i(c') \in \Sigma_X$  so it belongs to some  $E_p$ . By choice of  $c'$ , any  $c'' \in E_0$  is obtained as the limit of some sequence  $(t_n(c'))_n$  where each  $t_n$  is a translation of a vector multiple of  $m$ . By continuity and commutation with translations of  $G_{\mathcal{B}}$  we deduce that  $G_{\mathcal{B}}(c'')$  is the limit of elements of  $E_p$  so it belongs to  $E_p$  because this set is closed. Hence, we have  $G_{\mathcal{B}}^i(E_0) \subseteq E_p$  and  $s \circ G_{\mathcal{B}}^i(E_0) \subseteq E_0$  if  $s$  is the suitable translation. From this we deduce that  $\square^m(E_0)$  is a non-trivial stable full-shift of  $\square^m \circ s \circ G_{\mathcal{B}}^i \circ \square^{-m}$  and the theorem follows by the discussion at the beginning of this section. ■

### 3.3 A Characterization of the Most General Geometrical Space-Time Transforms

#### 3.3.1 Geometrical Space-Time Transforms

Grouping and classical transforms described in previous section consists of purely geometrical transforms: transforms that do not depend on the state set of the transformed CA, and thus can be applied to all CA. Such a transform maps a space-time diagram to a new space-time diagram, each space-time cell of which consists of a tuple of space-time cells of the initial diagram.

Formally, a *geometrical transform* is a pair  $(k, \mathbf{\Lambda})$  where  $k \in \mathbb{Z}^+$  and  $\mathbf{\Lambda}$  maps  $\mathbb{N} \times \mathbb{Z}^d$  to  $(\mathbb{N} \times \mathbb{Z}^d)^k$ . To help the reader visualize the transform, we will depict geometrical transforms as follows:

$$\mathbf{\Lambda} : \mathbb{N} \times \mathbb{Z}^d \longrightarrow (\mathbb{N} \times \mathbb{Z}^d)^k$$


The *space-time diagram transform* over a state set  $S$  by a geometrical transform  $(k, \mathbf{\Lambda})$ , is the map  $\overline{\mathbf{\Lambda}}_S$  from  $S^{\mathbb{N} \times \mathbb{Z}^d}$  to  $S^{(\mathbb{N} \times \mathbb{Z}^d)^k}$  defined, for all space-time diagram  $\Delta \in S^{\mathbb{N} \times \mathbb{Z}^d}$  and for all space-time position  $\xi \in \mathbb{N} \times \mathbb{Z}^d$ , by  $\overline{\mathbf{\Lambda}}_S(\xi) = (\Delta(\lambda_1), \dots, \Delta(\lambda_k))$  where  $\mathbf{\Lambda}(\xi) = (\lambda_1, \dots, \lambda_k)$ .

**Example 8** In 1D, the geometrical transform  $(3, \mathbf{\Lambda}^{(3,4,1)})$  defined, for all  $(t, p) \in \mathbb{N} \times \mathbb{Z}$  by  $\mathbf{\Lambda}^{(3,4,1)}(t, p) = ((4t, 3p + t), (4t, 3p + t + 1), (4t, 3p + t + 2))$  transforms the set of space-time diagrams  $\text{Diag } \mathcal{A}$  of every CA  $\mathcal{A}$  into the set of

space-time diagrams  $\text{Diag } \mathcal{A}^{(3,4,1)}$  of a new CA  $\mathcal{A}^{(3,4,1)}$ . Fig. 10 depicts the transform.  $\diamond$

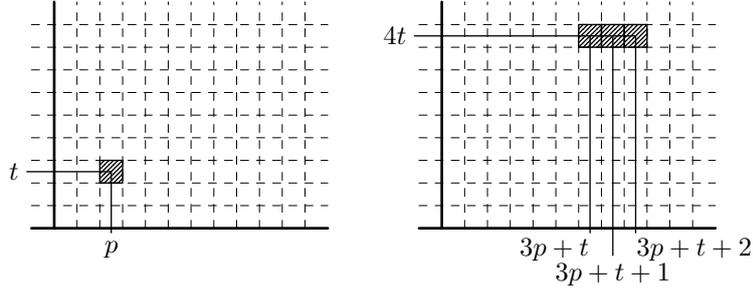


Fig. 10. Sample geometrical transform:  $(3, \mathbf{\Lambda}^{(3,4,1)})$

The composition  $(k', \mathbf{\Lambda}') \circ (k, \mathbf{\Lambda})$  of two geometrical transforms  $(k, \mathbf{\Lambda})$  and  $(k', \mathbf{\Lambda}')$  is the geometrical transform  $(kk', \mathbf{\Lambda}' \circ \mathbf{\Lambda})$  where, for all  $\xi \in \mathbb{N} \times \mathbb{Z}^d$ :

$$(\mathbf{\Lambda}' \circ \mathbf{\Lambda})(\xi) = \left( \mathbf{\Lambda}(\mathbf{\Lambda}'(\xi)_1)_1, \dots, \mathbf{\Lambda}(\mathbf{\Lambda}'(\xi)_{k'})_k \right) .$$

Let  $\tilde{\mathbf{\Lambda}}$  map each space-time cell to the set of associated space-time cells by  $\mathbf{\Lambda}$ :

$$\begin{aligned} \tilde{\mathbf{\Lambda}} : 2^{\mathbb{N} \times \mathbb{Z}^d} &\longrightarrow 2^{\mathbb{N} \times \mathbb{Z}^d} \\ X &\longmapsto \bigcup_{\xi \in X} \{ \mathbf{\Lambda}(\xi)_1, \dots, \mathbf{\Lambda}(\xi)_k \} \end{aligned}$$

A *nice geometrical transform* is a geometrical transform which plays nicely with space-time diagrams and can be used to extend grouping. It should transform sets of space-time diagrams into sets of space-time diagrams and be non-trivial (to avoid some cheating).

Formally, a geometrical transform  $(k, \mathbf{\Lambda})$  is *nice* if it satisfies the following conditions:

- (i) for all CA  $\mathcal{A}$ , there exists a CA  $\mathcal{B}$  such that  $\overline{\mathbf{\Lambda}}_{S_{\mathcal{A}}}(\text{Diag } \mathcal{A}) = \text{Diag } (\mathcal{B})$
- (ii) for all time  $t \in \mathbb{N}$ ,  $\tilde{\mathbf{\Lambda}}(\{t+1\} \times \mathbb{Z}^d) \not\subseteq \tilde{\mathbf{\Lambda}}(\{t\} \times \mathbb{Z}^d)$ .

Nice geometrical transforms are close by composition.

### 3.3.2 Packing, Cutting and Shifting

The classical transforms of previous section can be expressed as composition of three kinds of nice transforms, the action of which can be expressed easily in an algebraic way as global rules compositions.

**Packing.** A purely spatial geometrical transform can be defined using a tiling of space to cut space regularly. The *packing transform*  $\mathbf{P}_{\mathcal{P},V}$ , with tiling of space  $(\mathcal{P}, V)$ , is defined for all  $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$  by:

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \\ \mathbf{P}_{\mathcal{P},V}(t, p) = \{t\} \times (\mathcal{P} + V \odot p) \quad . \end{array}$$

Let  $\mathcal{A}$  be a CA and  $(\mathcal{P}, V)$  a tiling of space, the image of  $\text{Diag } \mathcal{A}$  by  $\mathbf{P}_{\mathcal{P},V}$  is the set of space-time diagrams of the CA with global rule  $\langle \mathcal{P}, V \rangle \circ G_{\mathcal{A}} \circ \langle \mathcal{P}, V \rangle^{-1}$ .

**Cutting.** A purely temporal geometrical transform can be defined by cutting unwanted time steps. The *cutting transform*  $\mathbf{C}_T$ , with rate  $T \in \mathbb{Z}^+$ , is defined for all  $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$  by:

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \\ \mathbf{C}_T(t, p) = (tT, p) \end{array}$$

Let  $\mathcal{A}$  be a CA and  $T \in \mathbb{Z}^+$  a rate, the image of  $\text{Diag } \mathcal{A}$  by  $\mathbf{C}_T$  is the set of space-time diagrams of the CA with global rule  $G_{\mathcal{A}}^T$ .

**Shifting.** A pure translation geometrical transform can be defined by shifting space-time. The *shifting transform*  $\mathbf{S}_s$ , with translation vector  $s \in \mathbb{Z}^d$ , is defined for all  $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$  by:

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \\ \hline \end{array} \\ \mathbf{S}_s(t, p) = (t, p + ts) \end{array}$$

Let  $\mathcal{A}$  be a CA and  $s \in \mathbb{Z}^d$  a translation vector, the image of  $\text{Diag } \mathcal{A}$  by  $\mathbf{S}_s$  is the set of space-time diagrams of the CA with global rule  $\sigma_s \circ G_{\mathcal{A}}$ .

**Composition.** Let  $(\mathcal{P}, V)$  be a tiling of space,  $s$  be a translation vector and  $T$  be a rate, the nice geometrical transform  $\mathbf{PCS}_{\mathcal{P},V,T,s}$  is the composition  $\mathbf{P}_{\mathcal{P},V} \circ \mathbf{S}_s \circ \mathbf{C}_T$ . For all  $(t, p) \in \mathbb{N} \times \mathbb{Z}^d$  it satisfies:

$$\mathbf{PCS}_{\mathcal{P},V,T,s}(t, p) = \{tT\} \times (\mathcal{P} + V \odot v + ts) \quad .$$

Let  $\mathcal{A}$  be a CA, the image of  $\text{Diag } \mathcal{A}$  by  $\mathbf{PCS}_{\mathcal{P},V,T,s}$  is the set of space-time diagrams of the CA with global rule

$$\langle \mathcal{P}, V \rangle \circ \sigma_s \circ G_{\mathcal{A}}^T \circ \langle \mathcal{P}, V \rangle^{-1} \quad .$$

The set of **PCS** transforms generated by pure **P**, **C** and **S** transforms is closed

by composition:

$$\mathbf{PCS}_{\mathcal{P}_2, V_2, T_2, s_2} \circ \mathbf{PCS}_{\mathcal{P}_1, V_1, T_1, s_1} = \mathbf{PCS}_{\mathcal{P}_1 + (\mathcal{P}_2 \circ V_1), V_2 \circ V_1, T_1 T_2, (s_2 \circ V_1) + T_2 s_1} \quad .$$

### 3.3.3 Characterizing the Most General Transforms

**Theorem 9** *Every nice geometrical transform is a PCS transform.*

**PROOF.** Let  $(k, \Lambda)$  be a nice geometrical transform. By definition:

- (i) for all CA  $\mathcal{A}$ , there exists a CA  $\mathcal{B}$  such that  $\overline{\Lambda}_{S_{\mathcal{A}}}(\text{Diag } \mathcal{A}) = \text{Diag } (\mathcal{B})$
- (ii) for all time  $t \in \mathbb{N}$ ,  $\tilde{\Lambda}(\{t+1\} \times \mathbb{Z}^d) \not\subseteq \tilde{\Lambda}(\{t\} \times \mathbb{Z}^d)$ .

The proof proceeds, in 7 steps, by enforcing using (i) successive constraints on  $\Lambda$  until the **PCS** nature becomes clear.

**Step 1.** Let us first prove the following property:

$$\begin{array}{c} \begin{array}{|c|c|c|c|} \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{green}{\square} & \color{green}{\square} & \color{green}{\square} & \color{green}{\square} \\ \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{green}{\square} & \color{green}{\square} & \color{green}{\square} & \color{green}{\square} \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|} \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{green}{\square} & \color{green}{\square} & \color{green}{\square} & \color{green}{\square} \\ \hline \color{red}{\square} & \color{red}{\square} & \color{red}{\square} & \color{red}{\square} \\ \hline \color{green}{\square} & \color{green}{\square} & \color{green}{\square} & \color{green}{\square} \\ \hline \end{array} \\ \forall t > 0, \forall (t', p_{t'}) \in \tilde{\Lambda}(\{t\} \times \mathbb{Z}^d) \setminus \tilde{\Lambda}(\{t-1\} \times \mathbb{Z}^d), \\ \exists t'' < t', \quad \{t''\} \times \mathbb{Z}^d \subseteq \tilde{\Lambda}(\{t-1\} \times \mathbb{Z}^d) \end{array}$$

This property states that nice transforms preserves some temporal dependencies: in a transformed space-time diagram, states of cells at time  $t$  is completely determined by states of cells at time  $t-1$ .

Assume that the property is not satisfied at time  $t$ . Thus, there exists a time  $t'$  and a sequence of spatial positions  $(p_0, \dots, p_{t'})$  such that  $(t', p_{t'})$  participates to a transformed cell at time  $t$  and for all time  $i$  the cell  $(i, p_i)$  does not participate to a transformed cell at time  $t-1$ .

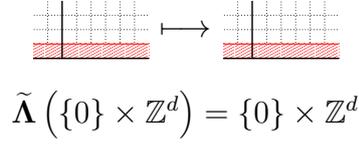
$$\begin{cases} \forall i \leq t', & (i, p_i) \notin \tilde{\Lambda}(\{t-1\} \times \mathbb{Z}^d) \\ & (t', p_{t'}) \in \tilde{\Lambda}(\{t\} \times \mathbb{Z}^d) \end{cases} \quad \begin{array}{|c|c|c|c|} \hline \color{red}{\square} & & & \\ \hline \color{green}{\square} & & & \\ \hline \color{red}{\square} & & & \\ \hline \color{green}{\square} & & & \\ \hline \end{array}$$

Let  $\mathcal{A}$  be a CA with state set  $\{\perp, 0, \dots, t'\}$ , neighborhood radius at least  $\max_{i,j} |p_j - p_i|$  and such that, starting from an initial configuration uniformly equal to  $\perp$  but in position  $p_0$  where it is equal to 0, it generates a space-time diagram  $\Delta$  whose  $t'+1$  first configurations are uniformly equal to  $\perp$  but, for each time  $i$ , the position  $p_i$  is equal to  $i$ .



The transformed coloring  $\overline{\Lambda}_{S_{\mathcal{A}}}(\Delta)$  is not the space-time diagram of a CA as the configuration at time  $t - 1$  is uniform and the configuration at time  $t$  is not: a CA cannot break such a symmetry.

**Step 2.** We now show a property on initial configurations:



$$\tilde{\Lambda}(\{0\} \times \mathbb{Z}^d) = \{0\} \times \mathbb{Z}^d$$

This property states that the initial configuration of a transformed space-time diagram is obtained by a purely spatial transformation of the original initial configuration.

It follows from the fact that the image of  $\text{Diag } \mathcal{A}$  has to be the whole set of space-time diagram of a CA, that is all initial configuration  $(S_{\mathcal{A}}^k)^{\mathbb{Z}^d}$  should be obtained.

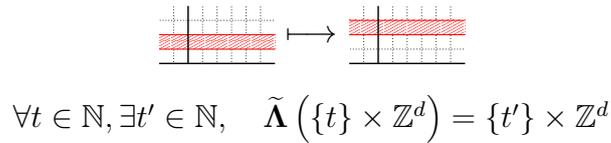


Assume that the property is not satisfied. Let  $p$  be a position such that there exists  $(t, p') \in \tilde{\Lambda}(0, p)$  with  $t' > 0$ . Let  $\mathcal{A}$  be the CA with two states  $\{0, 1\}$  and local rule  $f_{\mathcal{A}}$  constantly equal to 0. For all space-time diagrams of  $\mathcal{A}$ , the cell  $(t, p')$  has state 0, contradicting (i).

With a similar argument, one show that the images of cells at time 0 are composed of disjointed cells:

$$\forall p, q, \quad \text{card}(\tilde{\Lambda}(0, p)) = k \quad \wedge \quad p \neq q \Rightarrow \tilde{\Lambda}(0, p) \cap \tilde{\Lambda}(0, q) = \emptyset$$

**Step 3.** Previous property is extended to every time step:



$$\forall t \in \mathbb{N}, \exists t' \in \mathbb{N}, \quad \tilde{\Lambda}(\{t\} \times \mathbb{Z}^d) = \{t'\} \times \mathbb{Z}^d$$

Let  $t \in \mathbb{N}$  be a time step and, by **(step 1)**, let  $t'$  satisfy

$$\tilde{\Lambda}(\{t\} \times \mathbb{Z}^d) \supseteq \{t'\} \times \mathbb{Z}^d.$$

We show first that each cell at time  $t$  contributes a same number  $l$  of cells to image cells at time  $t'$ . Formally:

$$\begin{aligned} & \exists l, \forall p, \exists i_1 < i_2 \cdots < i_l, (\mathbf{\Lambda}(t, p)_{i_1}, \dots, \mathbf{\Lambda}(t, p)_{i_l}) \in (\{t'\} \times \mathbb{Z}^d)^l \\ \wedge & \quad \forall i \notin \{i_1, \dots, i_l\}, \mathbf{\Lambda}(t, p)_i \notin \{t'\} \times \mathbb{Z}^d \end{aligned}$$



diagram  $\Delta'_n$  has a periodic initial configuration with periodic coloring the  $d$ -dimensional ball with radius  $n$  filled with states  $(1, \dots, k), (k+1, \dots, 2k), \dots, (n^d - 1)k + 1, \dots, n^d k$ . By symmetry considerations, the configuration of  $\Delta'_n$  at time  $t$  is periodic with a smaller period. By **(step 1)**, the periodic coloring contains all possible states  $1, 2, \dots, n^d k$ . Thus, all cells at distance less than or equal to  $n$  corresponds to disjointed cells.

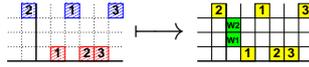
**Step 5.** Next step consists in proving a property of uniformity in time:



$$\forall p, \exists s_0, T_0, \forall t, \quad \mathbf{\Lambda}(t, p) = \{tT_0\} \times (\pi_2(\mathbf{\Lambda}(0, p)) + ts_0)$$

where  $\pi_2$  projects sets of space-time cells to their space components. This property states that the successive images of a given cell are regularly aligned in space-time.

Let  $p$  be a position in space. Let  $(p_1, \dots, p_k), (p'_1, \dots, p'_k)$  and  $t$  be such that  $\mathbf{\Lambda}(0, p) = ((0, p_1), \dots, (0, p_k))$  and  $\mathbf{\Lambda}(1, p) = ((t, p'_1), \dots, (t, p'_k))$ .

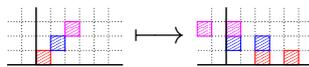


Let  $\mathcal{A}$  be the CA with state set  $\{\perp, 1, \dots, k, W_1, \dots, W_{t-1}\}$ , neighborhood radius  $2 \max \{\|p_1\|_\infty, \dots, \|p_k\|_\infty, \|p'_1\|_\infty, \dots, \|p'_k\|_\infty\}$  and such that one of its space-time diagrams  $\Delta$  is filled with state  $\perp$  for all times between 0 and  $t$  but:

$$\begin{aligned} \Delta(0, p_1) = 1, \dots, \Delta(0, p_k) = k, \\ \Delta(1, 0) = W_1, \dots, \Delta(t-1, 0) = W_{t-1}, \\ \Delta(t, p'_1) = 1, \dots, \Delta(t, p'_k) = k. \end{aligned}$$

All undefined transitions are mapped to  $\perp$ . Let  $\Delta'$  be the transformed diagram of  $\Delta$ . By construction, at time steps 0 and 1, the configurations of  $\Delta'$  are filled with  $(\perp, \dots, \perp)$  but at position  $p$  where it is equal to  $(1, \dots, k)$ . As  $\Delta'$  is a space-time diagram of a CA and as its configurations at time 0 and 1 are equal, all the configurations of  $\Delta'$  are equal. By construction, it implies that all the cells are uniformly shifted:  $\exists s, p_1 = p'_1 + s \wedge \dots \wedge p_k = p'_k + s$ . The construction straightforwardly extends to all time steps.

**Step 6.** Previous property stands up to an elementary shift:



$$\forall i, p, \exists s_i, T_i, \forall t, \quad \mathbf{\Lambda}(t, p + ts_i) = \{tT_i\} \times (\pi_2(\mathbf{\Lambda}(0, p)) + ts_i)$$

Replay the same arguments as for previous property but considering translated cells at time 1.

**Step 7.** We can now conclude that  $\Lambda = \text{PCS}_{\pi_2(\Lambda(0,0)),(s_1-s_0,\dots,s_d-s_0),T_0,s_0}$  as elementary translations form a base for translations in  $\mathbb{Z}^d$ . ■

## 4 Axiomatics of Bulking Quasi-Orders

The basic ingredients of a bulking simulation are now clear: a set of objects with an elementary comparison relation and an algebra of (geometrical) transforms to apply on objects. Simulation is then defined by comparison up to transformation on both sides. Strong simulation is defined by comparison up to transformation on the simulator only.

In this section, the basic ingredients of bulking are formalized and properties of quasi-ordering and strong universality are established. Then, a model of bulking based on most general space-time transforms is discussed. Finally, a first model of bulking based on rectangular transforms and the subautomaton relation is introduced.

### 4.1 Theory of Bulking

The properties being sufficiently elementary, we choose to present bulking in a very formal way, as a first-order theory, rather than using a more classical algebraic definition. The two points of view are equivalent. Grouping being a model of bulking, it will be used to illustrate formal stuff.

**Definition 10** *The bulking is the theory  $\Phi_{\mathbf{b}}$  on the two-sorted signature*

$$\left( \begin{array}{l} \text{Obj, Trans; } \mathbf{apply} : \text{Obj} \times \text{Trans} \rightarrow \text{Obj}, \\ \mathbf{divide} \subseteq \text{Obj} \times \text{Obj}, \\ \mathbf{combine} : \text{Trans} \times \text{Trans} \rightarrow \text{Trans} \end{array} \right)$$

*defined by the following axioms, where latin letters  $(x, y, \dots)$  denotes elements of the sort **Obj**, greek letters  $(\alpha, \beta, \dots)$  and number 1 denotes elements of the sort **Trans**,  $x^\alpha$  denotes  $\mathbf{apply}(x, \alpha)$ ,  $x \mid y$  denotes  $\mathbf{divide}(x, y)$  and  $\alpha \cdot \beta$  denotes*

**combine**( $\alpha, \beta$ ):

- (B<sub>1</sub>)  $\exists 1 \forall \alpha (\alpha \cdot 1 = \alpha \wedge 1 \cdot \alpha = \alpha) \wedge \forall \alpha \forall \beta \forall \gamma ((\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma))$
- (B<sub>2</sub>)  $\forall x (x^1 = x) \wedge \forall x \forall \alpha \forall \beta ((x^\alpha)^\beta = x^{\alpha \cdot \beta})$
- (B<sub>3</sub>)  $\forall x (x | x) \wedge \forall x \forall y \forall z ((x | y \wedge y | z) \rightarrow x | z)$
- (B<sub>4</sub>)  $\forall x \forall y \forall \alpha (x | y \rightarrow x^\alpha | y^\alpha)$
- (B<sub>5</sub>)  $\forall \alpha \forall x \exists y (x | y^\alpha)$
- (B<sub>6</sub>)  $\forall \beta \exists \gamma \forall \alpha \exists \delta (\alpha \cdot \gamma = \beta \cdot \delta)$

**Definition 11** The simulation relation  $x \preceq y$  is syntactically defined for all  $x, y \in \mathbf{Obj}$  by the formula  $\exists \alpha \exists \beta (x^\alpha | y^\beta)$ .

Objects, transforms and relations between them can be visualized graphically by considering elements of  $\mathbf{Obj}$  as vertices and two kind of edges: a wiggly edge labelled by an element of  $\mathbf{Trans}$  represents **apply**, a regular edge represents the relation **divide**. The simulation relation is depicted on Fig. 11.

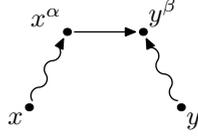


Fig. 11. Visual representation of the simulation relation

Grouping is a model of bulking where  $\mathbf{Obj}$  is the set of global rules of 1D CA with neighborhood  $\{-1, 0, 1\}$ ,  $\mathbf{Trans}$  is the set of square transforms  $\mathbb{N}$ , **apply**( $G, n$ ) is the global rule  $\square^n \circ G^n \circ \square^{-n}$ , **divide** is the subautomaton relation  $\sqsubseteq$  and **combine** is the classical arithmetical product on  $\mathbb{N}$ .

Axioms of  $\Phi_{\mathbf{b}}$  formalize necessary algebraic properties:

- (B<sub>1</sub>) The structure  $(\mathbf{Trans}, \cdot)$  is a monoid.
- (B<sub>2</sub>) The operator **apply** is an action of the monoid  $(\mathbf{Trans}, \cdot)$  on the set  $\mathbf{Obj}$ .
- (B<sub>3</sub>) The relation **divide** is a quasi-order on  $\mathbf{Obj}$ .
- (B<sub>4</sub>) The operator **apply** is compatible with **divide**.
- (B<sub>5</sub>) The operator **apply** preserves the diversity of objects.
- (B<sub>6</sub>) The monoid  $(\mathbf{Trans}, \cdot)$  admits a strong diamond property.

The (B<sub>5</sub>) axiom ensures transforms do not pathologically weakens the elementary simulation relation.

The (B<sub>6</sub>) axiom might seem less natural. In fact, in the case of grouping, the square transforms commute and (B<sub>6</sub>) could be replaced by a commutation axiom  $\forall \alpha \forall \beta (\alpha \cdot \beta = \beta \cdot \alpha)$ . Due to the shift transform, even rectangular shifted transforms do not commute but satisfy the strong diamond property of (B<sub>6</sub>).

The simulation relation is a quasi-order given only  $(B_{1-5})$  if and only if it satisfies  $\forall x \forall \alpha \forall \beta \exists \gamma \exists \delta (x^{\alpha \cdot \gamma} \mid x^{\beta \cdot \delta})$ . By letting  $\gamma$  depend only on  $x$  and  $\beta$  one can derive Thm. 13 on strongly universal objects.

**Theorem 12** *The simulation relation is a quasi-order, formally:*

$$\Phi_{\mathbf{b}} \vdash \forall x (x \preceq x) \wedge \forall x \forall y \forall z ((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z)$$

**PROOF.** The simulation relation is reflexive:  $\Phi_{\mathbf{b}} \vdash \forall x (x \preceq x)$ . By combining  $(B_2)$  and  $(B_3)$ , it holds  $\Phi_{\mathbf{b}} \vdash \forall x (x^1 \mid x^1)$ .

The simulation relation is transitive:  $\Phi_{\mathbf{b}} \vdash \forall x \forall y \forall z ((x \preceq y \wedge y \preceq z) \rightarrow x \preceq z)$ . Let  $x, y, z$  and  $\alpha, \beta, \gamma, \delta$  be such that  $x^\alpha \mid y^\beta$  and  $y^\gamma \mid z^\delta$ . By  $(B_6)$  there exists  $\eta$  and  $\nu$  such that  $\beta \cdot \eta = \gamma \cdot \nu$ . By  $(B_4)$  and  $(B_2)$ ,  $x^{\alpha \cdot \eta} \mid y^{\beta \cdot \eta}$  and  $y^{\gamma \cdot \nu} \mid z^{\delta \cdot \nu}$ . Thus, by  $(B_3)$ ,  $x^{\alpha \cdot \eta} \mid z^{\delta \cdot \nu}$ , implying  $x \preceq z$ . ■

**Theorem 13** *If a strongly universal object exists then all universal objects are strongly universal, formally:*

$$\Phi_{\mathbf{b}} \vdash \exists u \forall x \exists \alpha (x \mid u^\alpha) \rightarrow \forall x (\forall y (y \preceq x) \rightarrow \forall y \exists \beta (y \mid x^\beta))$$

**PROOF.** Let  $u$  be a strongly universal object:  $\forall x \exists \alpha (x \mid u^\alpha)$ . Let  $x$  be a universal object:  $\forall y (y \preceq x)$ . By universality of  $x$ , there exists  $\alpha$  and  $\beta$  such that  $u^\alpha \mid x^\beta$ . By  $(B_6)$ , there exists  $\gamma$  such that  $\forall \eta \exists \delta (\eta \cdot \gamma = \alpha \cdot \delta)$ . Let  $y$  be an object. By  $(B_5)$ , there exists  $z$  such that  $y \mid z^\gamma$ . By strong universality of  $u$ , there exists  $\eta$  such that  $z \mid u^\eta$ . By  $(B_4)$  and  $(B_2)$ ,  $z^\gamma \mid u^{\eta \cdot \gamma}$ . Let  $\delta$  be such that  $\eta \cdot \gamma = \alpha \cdot \delta$ . By  $(B_4)$  and  $(B_2)$ ,  $u^{\alpha \cdot \delta} \mid x^{\beta \cdot \delta}$ . Thus, by  $(B_3)$ ,  $y \mid x^{\beta \cdot \delta}$ . ■

#### 4.2 CA Bulking: A First Attempt

Thanks to previous discussion, one might try to build a model of bulking using  $d$ -CA as sort **Obj**, **PCS** transforms as sort **Trans** (following Thm. 9) and the subautomaton relation  $\sqsubseteq$  as elementary simulation relation (following Thm. 4). However, axiom  $(B_6)$  is not satisfied by **PCS** transforms. More precisely, the problem roots in the structure of the monoid of composition of tilings of space which does not admit the diamond property.

**Example 14** *Let  $\mathcal{P}_1 = \{0, 1\}$ ,  $\mathcal{P}_2 = \{0, 3\}$  and  $v = 2$ . Let  $\Theta$  be the set of all  $d$ -tilings of space. The tilings of space  $\langle \mathcal{P}_1, v \rangle$  and  $\langle \mathcal{P}_2, v \rangle$ , depicted on Fig. 12, are incompatible:  $(\Theta \circ \langle \mathcal{P}_1, v \rangle) \cap (\Theta \circ \langle \mathcal{P}_2, v \rangle) = \emptyset$ . ◇*



Fig. 12. Two incompatible tilings of space

Notice that the problem does not come from the composition of bases themselves (product of non-zero determinant matrices with integer coefficients admits a strong diamond property) but really from the geometrical shape of tilings. Intuitively, in Example 14, compositions of  $\mathcal{P}_1$  always contains two consecutive elements on the extreme left whereas compositions of  $\mathcal{P}_2$  contains a single element followed by a gap. Starting from dimension 2, even connected tilings leads to problem, shapes replacing gaps.

**Open Problem 1** *Characterize the submonoids of composition of tilings of space that admits the diamond property.*

As we want to extend grouping, we need to select a submonoid with the strong diamond property that contains square transformations. The set of rectangular tilings constitutes an adequate commutative submonoid. One might enrich a bit this set by allowing permutation and negation of elements of the basis.

**Rectangular Packing.** Let  $m \in (\mathbb{Z}^+)^d$  and  $\tau \in \mathbb{Z}^d$  such that  $\tau$  is a signed permutation of  $(1, \dots, d)$ . The *rectangular packing*  $\tilde{\mathbf{P}}_{m,\tau}$  is the packing  $\mathbf{P}_{\boxplus m, V_\tau \odot \square_m}$  where  $V_\tau$  is the basis where  $V_\tau(k)$  has all its elements equals to 0 but in position  $|\tau_k|$  where it is equal to 1 if  $\tau_k > 0$  and to  $-1$  if  $\tau_k < 0$ .

**Composition.** Let  $(m, \tau)$  be valid rectangular packing parameters,  $s$  be a translation vector and  $T$  be a rate, the nice geometrical transform  $\tilde{\mathbf{PCS}}_{m,\tau,T,s}$  is the composition  $\tilde{\mathbf{P}}_{m,\tau} \circ \mathbf{S}_s \circ \mathbf{C}_T$ . The set of  $\tilde{\mathbf{PCS}}$  transforms generated by pure  $\tilde{\mathbf{P}}$ ,  $\mathbf{C}$  and  $\mathbf{S}$  transforms is closed by composition:

$$\tilde{\mathbf{PCS}}_{(m'_1, \dots, m'_d), \tau', T', s'} \circ \tilde{\mathbf{PCS}}_{(m_1, \dots, m_d), \tau, T, s} = \tilde{\mathbf{PCS}}_{(m''_1, \dots, m''_d), \tau'', T'', s''}$$

with parameters

$$\begin{aligned} m'' &= \varrho(\tau, m, m') \quad \text{where } \varrho(\tau, m, m')_i = m_i m'_{|\tau(i)|} \\ \tau'' &= \tau' \otimes \tau \quad \text{where } (\tau' \otimes \tau)_i = \text{sg}(\tau'_i) \times \tau_{|\tau'_i|} \\ T'' &= TT' \\ s'' &= (s' \odot V_\tau \odot \square_m) + T's \end{aligned}$$

Let  $\tau^{-1}$  denote the inverse of  $\tau$  with respect to  $\otimes$ , that is such that  $\tau^{-1} \otimes \tau = \text{id}$  where  $\text{id} = (1, \dots, d)$ .

To simplify notations, in the rest of the paper,  $\langle m, \tau, T, s \rangle$  denotes a valid  $\tilde{\mathbf{PCS}}_{m,\tau,T,s}$  transform and product on this notation denotes composition. No-

tice that  $\langle 1, \tau^{-1}, 1, 0 \rangle \langle m, \tau, T, s \rangle = \langle m, \text{id}, T, s \rangle$ .

Composition of  $\tilde{\mathbf{PCS}}$  is quite symmetrical but for the shifting component.

**Lemma 15**  $\tilde{\mathbf{PCS}}$  transforms have the strong diamond property, that is  $(B_6)$ .

**PROOF.** Given a  $\tilde{\mathbf{PCS}}$  transform  $\beta = \langle m, \tau, T, s \rangle$ , let

$$\gamma = \langle \text{lcm}(m)(1, \dots, 1), \text{id}, \text{lcm}(m)T, 0 \rangle \quad .$$

For all  $\alpha = \langle m', \tau', T', s' \rangle$ , let  $\delta = \langle \tilde{m}, \tau' \otimes \tau^{-1}, \text{lcm}(m)T', \tilde{s} \rangle$  . By definition,

$$\begin{aligned} \alpha \cdot \gamma &= \langle \text{lcm}(m)m', \tau', \text{lcm}(m)TT', \text{lcm}(m)Ts' \rangle \\ \beta \cdot \delta &= \langle \varrho(\tau, m, \tilde{m}), \tau', \text{lcm}(m)TT', \text{lcm}(m)T's + (\tilde{s} \odot V_\tau \odot \square_m) \rangle \end{aligned}$$

As each component of  $\text{lcm}(m)m'$  is a mutiple of each component of  $m$ , one can choose  $\tilde{m}$  such that  $\varrho(\tau, m, \tilde{m}) = \text{lcm}(m)m'$ . As each component of  $\text{lcm}(m)Ts'$  is a multiple of each component of  $m$ , one can choose  $\tilde{s}$  such that  $\tilde{s} \odot V_\tau \odot \square_m = \text{lcm}(m)(Ts' - T's)$ .  $\blacksquare$

### 4.3 CA Bulking: A Model

**Definition 16** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two  $d$ -CA.  $\mathcal{B}$  simulates  $\mathcal{A}$  injectively, denoted  $\mathcal{A} \preceq_i \mathcal{B}$ , if there exists two  $\tilde{\mathbf{PCS}}$  transforms  $\alpha = \langle m, \tau, T, s \rangle$  and  $\beta = \langle m', \tau', T', s' \rangle$  such that the transform of  $\mathcal{A}$  by  $\alpha$  is a subautomaton of the transform of  $\mathcal{B}$  by  $\beta$ . Formally,

$$\begin{aligned} \langle \boxplus_m, V_\tau \odot \square_m \rangle \circ \sigma_s \circ G_{\mathcal{A}}^T \circ \langle \boxplus_m, V_\tau \odot \square_m \rangle^{-1} \\ \sqsubseteq \langle \boxplus_{m'}, V_{\tau'} \odot \square_{m'} \rangle \circ \sigma_{s'} \circ G_{\mathcal{B}}^{T'} \circ \langle \boxplus_{m'}, V_{\tau'} \odot \square_{m'} \rangle^{-1} \quad . \end{aligned}$$

**Theorem 17** The set of  $d$ -CA equipped with  $\tilde{\mathbf{PCS}}$  transforms and the subautomaton relation  $\sqsubseteq$  is a model of bulking.

**PROOF.** Each axiom has to be checked:

Axiom  $(B_1)$ .  $(\tilde{\mathbf{PCS}}, \circ)$  is a monoid with unit  $\langle (1, \dots, 1), \text{id}, 1, 0 \rangle$ .

Axiom  $(B_2)$ . By definition, applying  $\tilde{\mathbf{PCS}}$  transforms is an action of CA.

Axiom  $(B_3)$ . The subautomaton relation is a straightforward quasi-order:  $\mathcal{A} \subseteq_{\text{id}} \mathcal{A}$  and if  $\mathcal{A} \subseteq_\varphi \mathcal{B}$  and  $\mathcal{B} \subseteq_\psi \mathcal{C}$  then  $\mathcal{A} \subseteq_{\psi \circ \varphi} \mathcal{C}$ .

Axiom ( $B_4$ ). The subautomaton relation is compatible with  $\tilde{\mathbf{PCS}}$  transform by product extension of injective  $\varphi$  functions: if  $\mathcal{A} \sqsubseteq_{\varphi} \mathcal{B}$  then  $\mathcal{A}^{\langle m, \dots \rangle} \sqsubseteq_{\psi} \mathcal{B}^{\langle m, \dots \rangle}$  where  $\psi((s_1, \dots, s_{\prod m_i})) = (\varphi(s_1), \dots, \varphi(s_{\prod m_i}))$ .

Axiom ( $B_5$ ). The application of  $\tilde{\mathbf{PCS}}$  transforms preserves the diversity of objects: let  $\langle m, \tau, T, s \rangle$  be a  $\tilde{\mathbf{PCS}}$  transform and let  $\mathcal{A} = (S, N, f)$  be a CA. Let  $\mathcal{B} = (S, N', f')$  where  $N' = V_{\tau} \odot \square_m \odot N$  and for all  $a \in S^N$ , for all  $f'(V_{\tau} \odot \square_m \odot a)(z) = f(a)$ . By construction,  $\mathcal{A} \sqsubseteq_{\varphi} \mathcal{B}^{\langle m, \tau, T, s \rangle}$  where  $\varphi(s) = (s, \dots, s)$ .

Axiom ( $B_6$ ). This is Lemma 15. ■

**Corollary 18**  $(\text{CA}, \preceq_i)$  is a quasi-ordered set.

**PROOF.** This is Theorem 12. ■

## 5 CA Bulking and Universality

In this section, we investigate elementary properties of the CA bulking introduced in previous section, compare it to grouping and obtain the first results on bulking, in particular with respect to intrinsic universality. This bulking, as well as others, are studied more in depth in *Bulking II: Classifications of Cellular Automata* [4].

**Definition 19** A CA  $\mathcal{U}$  is intrinsically universal if it strongly simulates injectively every CA: for all CA  $\mathcal{A}$ , there exists a  $\tilde{\mathbf{PCS}}$  transform  $\alpha$  such that  $\mathcal{A} \sqsubseteq \mathcal{U}^{\alpha}$ .

**Theorem 20** Every maximal element of  $\preceq_i$  is intrinsically universal.

**PROOF.** This is a consequence of Theorem 13. ■

**Theorem 21** There exists no real-time intrinsically universal CA.

**PROOF.** This is a consequence of Theorem 3. ■

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