Bulking II: Classifications of Cellular Automata

Marianne Delorme, Jacques Mazoyer, Nicolas Ollinger, Guillaume Theyssier

Bulking II: Classifications of Cellular Automata[☆]

M. Delorme[a], J. Mazoyer[a], N. Ollinger[b], G. Theyssier[c,*]

[a] *LIP, ENS Lyon, CNRS, 46 allée d'Italie, 69 007 Lyon, France*
[b] *LIF, Aix-Marseille Université, CNRS, 39 rue Joliot-Curie, 13 013 Marseille, France*
[c] *LAMA, Université de Savoie, CNRS, 73 376 Le Bourget-du-Lac Cedex, France*

Abstract

This paper is the second part of a serie of two papers dealing with bulking: a quasi-order on cellular automata comparing space-time diagrams up to some rescaling. Bulking is a generalization of grouping taking into account universality phenomena, giving rise to a maximal equivalence class. In the present paper, we introduce 3 notions of simulation between cellular automata and study the quasi-order structures induced by these simulation relations on the whole set of cellular automata. Various aspects of these quasi-orders are considered (induced equivalence relations, maximum elements, induced orders, etc) providing several formal tools to classify cellular automata.

Key words: cellular automata, bulking, grouping, classification

1. Introduction

In the first paper [8], we have developped a general theory of bulking aimed at defining quasi-orders on cellular automata based on the idea of space-time rescaling. The present paper focuses on three instances of such quasi-orders and uses them as classification tools over the set of one-dimensional cellular automata.

To classify has no sense without additional assumptions (some criteria of classification). If in Wolframs papers [40] these criteria were implicit, several classifications with explicit criteria have been proposed since [10, 4, 19]. Usually, the criteria are those of dynamical systems and consist in a finite list of qualitative behaviors. Our approach here is different: the criterion is the quasi-order and various aspects of the order structure are considered which provide several classification tools. The bulking quasi-orders we are going to define are based on two basic notions of comparison of two systems:

- the *injection* of a small system (B) *into* a larger one (A),
- the *projection* of a large system (A) *onto* a smaller one (B).

In the context of comparison of local rules of cellular automata, the first notion translates into the *subautomaton* relation (B obtained from A by forgetting some

[☆] The results presented here first appeared to a great extent in French in the PhD theses of Ollinger [30] and Theyssier [37]

[*] Corresponding author (Guillaume.Theyssier@univ-savoie.fr)

states) and the second one into the *quotient* relation (B obtained from A by identifying some sates). The subautomaton relation was already introduced in [8] and its importance in cellular automata is illustrated by theorems 4 and 5 of that paper. The quotient relation can be seen as a particular case of the notion of factor in dynamical systems theory and symbolic dynamics (homomorphism between shift-commuting continuous global maps, see [21]). More generally, the quotient relation is a means to extract coarse-grained information (B) from a complex system (A) (see [17]). For instance, the use of the metaphor of particles moving in a stable background in the literature [3] follow this idea: some information is hidden by identification of states. In the context of algorithmic constructions in cellular automata, the notion of signal [25] is fundamental and it also implicitly uses state identification: the pertinent information carried by the signal is not at the level of individual states, but group of states.

Following the bulking formalism of [8], three quasi-orders can be derived from these relations as follows: $\preccurlyeq_i$ is the subautomaton relation up to rescalings, $\preccurlyeq_s$ is the quotient relation up to rescalings and $\preccurlyeq_m$ is the relation "quotient of a subautomaton" up to rescalings. We study the structure of these orders. At finite level (classes with a finite non zero number of classes below them), there is infinitely many classes. Orders are not upper or lower semi-lattices. For example, infinite increasing sequences may be due to the following fact: some integer may be coded by states or (at the limit) by extracting this integer from the data of the initial configuration. Orders $\preccurlyeq_i$ and $\preccurlyeq_m$ have a maximal class. Classes having Turing-universality are obtained by simulating (in a way closed to Smith III [34]) an universal Turing machine. Such a class is not necessarily at the top (intrinsically universal). Moreover there exists a cellular automaton which, in some sense, handles potential infinite. Precisely, for $\preccurlyeq_m$-simulation, there exists an infinite increasing sequence of cellular automata (simulating n copies of a Turing machine but not $n+1$ copies) and the limit cellular automata $\preccurlyeq_m$-simulates any finite number of copies but not an infinite number. Every class contains a cellular automaton with radius 1 but, more surprising, there exist classes which do not contain cellular automata with two states of any neighborhood. This fact implies that some global properties needs more than two states to appear whatever is the neighborhood. Finally, many global properties of cellular automata as dynamical systems (reversibility, sensitivity, expansivity, etc) or cellular automata as computational devices (ability to simulate a turing head, or to propagate some signal) form an ideal or a filter in our orders.

Section 1.1 introduces 3 different comparison relations which are 3 different instances of the bulking theory developed in the companion paper [8]. Section 2 sets the definitions of these 3 notions of simulations and establish some of their basic properties. Section 3 studies the 'bottom' of each of the 3 quasi-orders induced on CA, *i.e.* CA or classes of CA of least complexity. Section 4 focuses on the order structure with respect to various classical properties of CA, and from a computability point of view. Then section 5 explores the set of CA at the 'top' of these quasi-orders: universal CA. Once again, the point of view is both structural and computational. Finally, section 6 is devoted to the construction of noticeable induced orders (like infinite chains), and the study of how simple families of CA spread over those quasi-orders.

Figures 1, 3 and 2 give an overview of our results on the 3 quasi-orders studied in the paper.

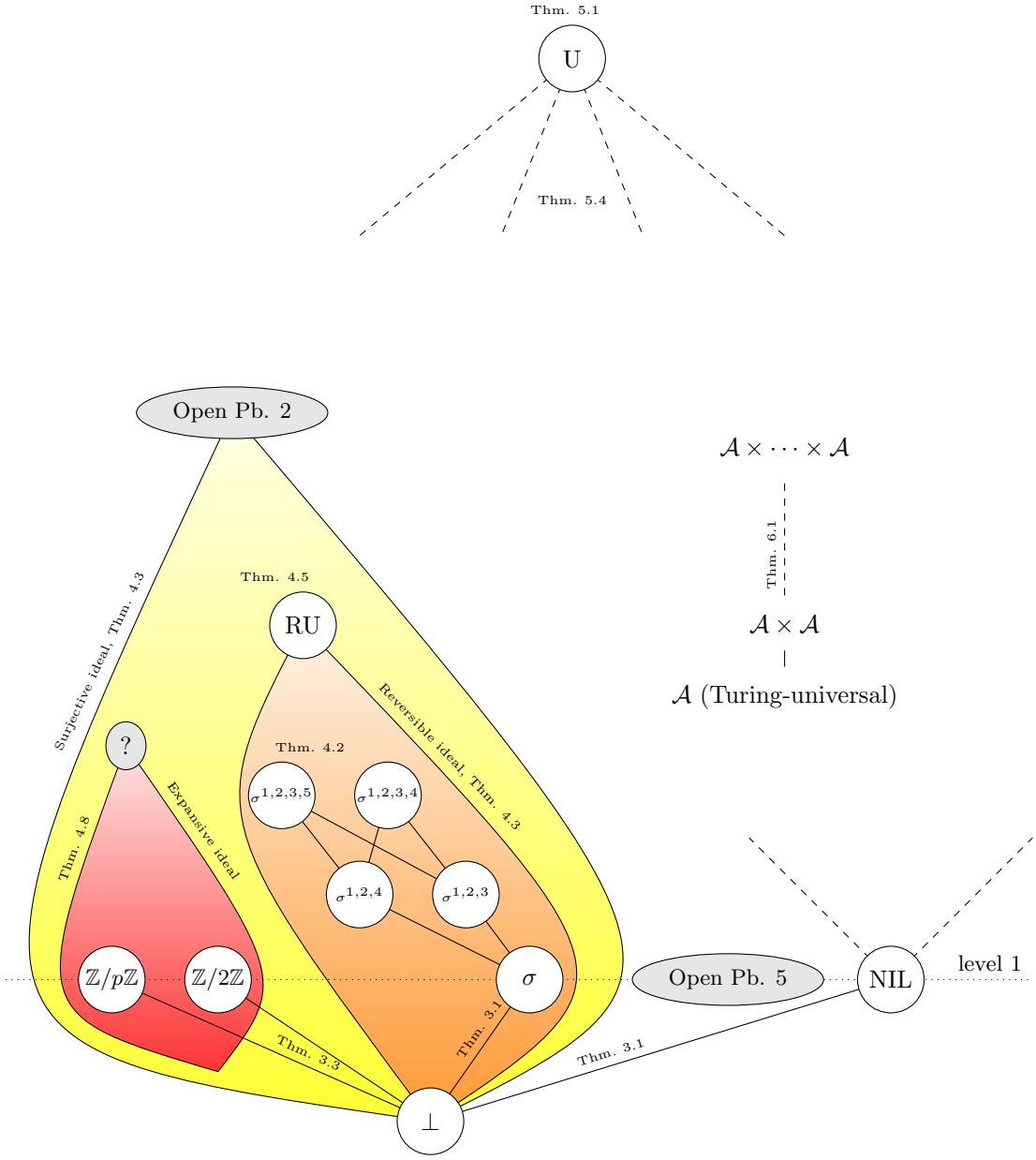


Figure 1: Injective bulking (quasi-order $\preccurlyeq_i$)

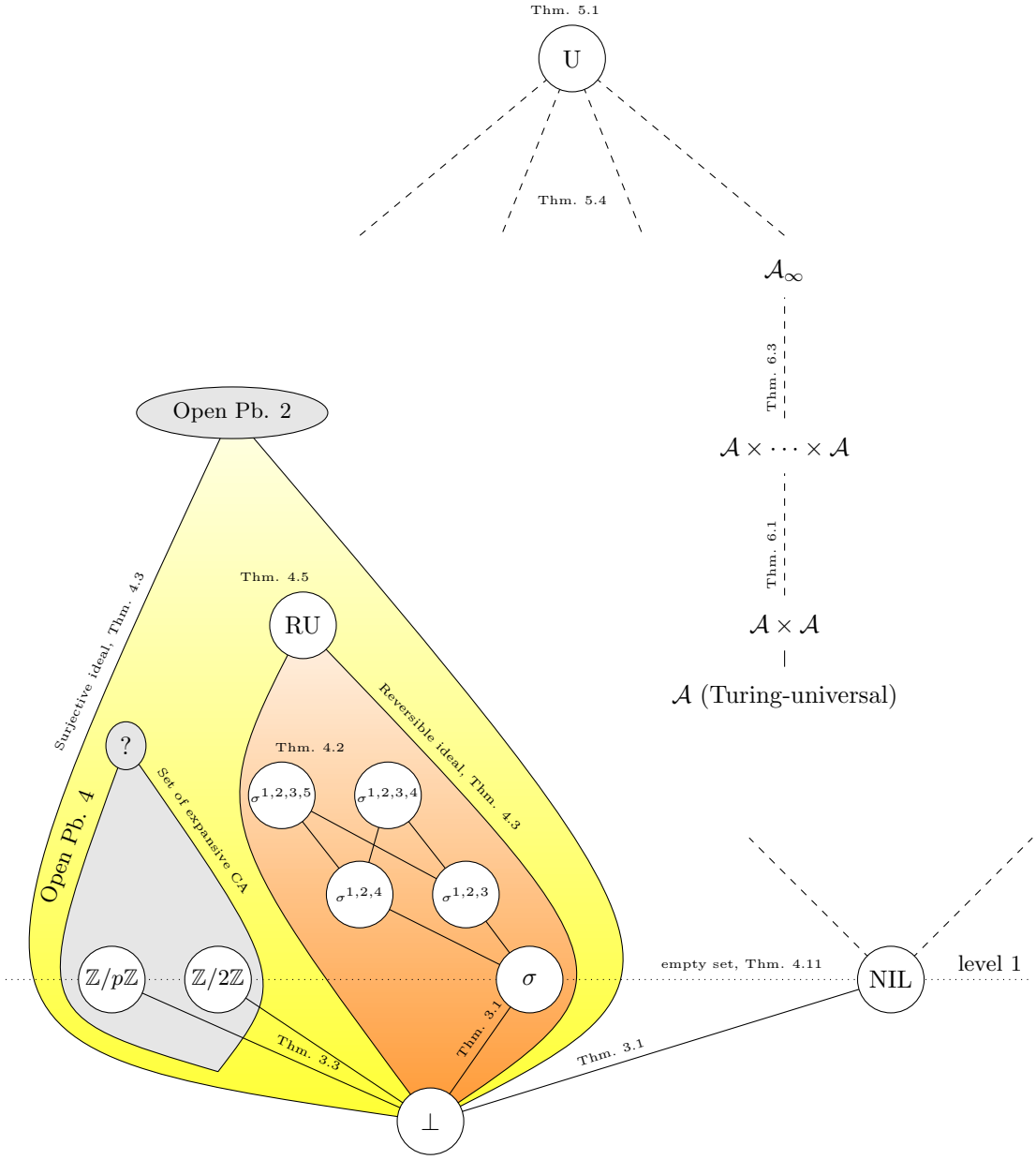


Figure 2: Mixed bulking (quasi-order $\preccurlyeq_m$)

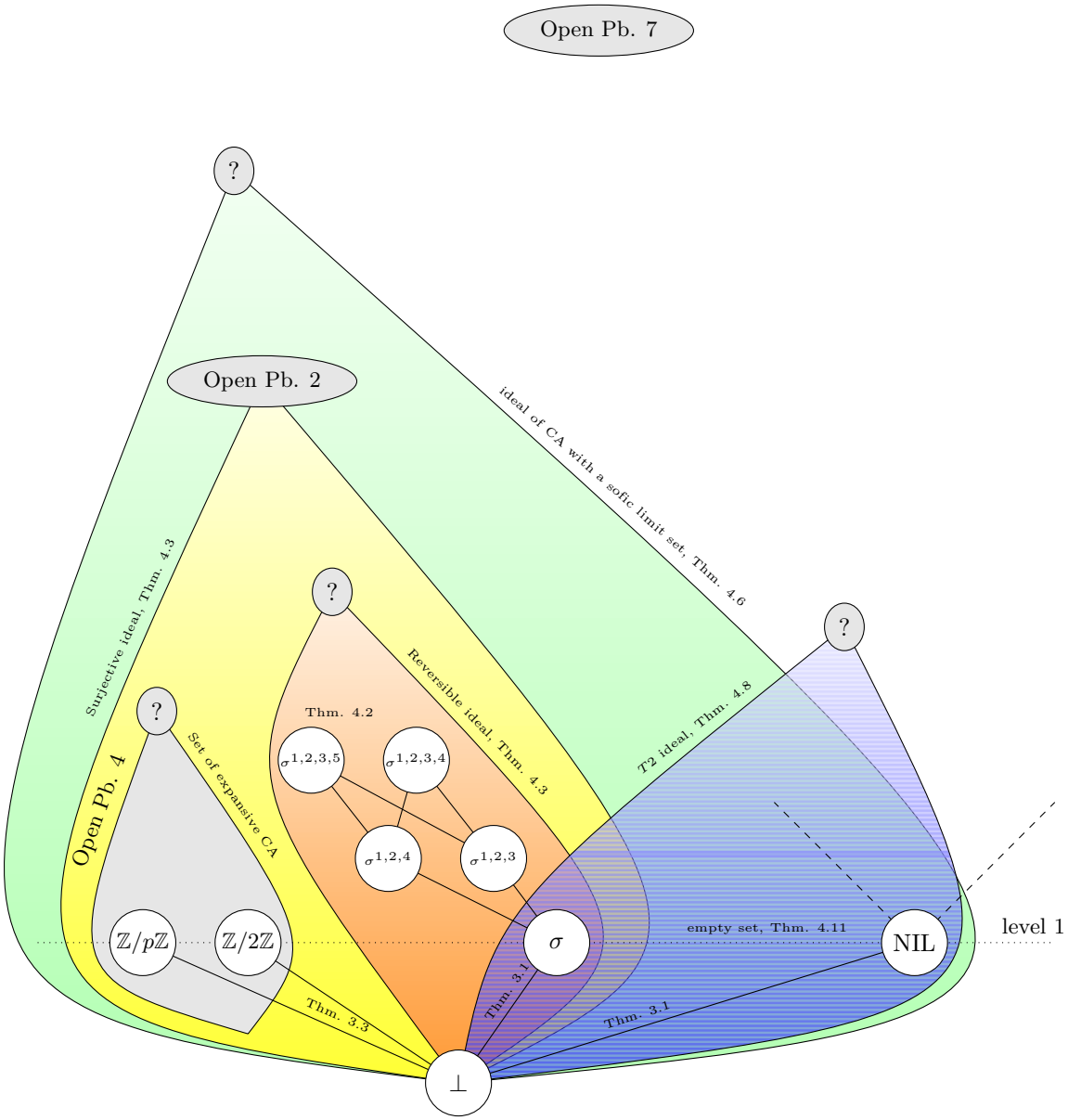


Figure 3: Surjective bulking (quasi-order $\preccurlyeq_s$)

1.1. Definitions

In this paper, we adopt the setting of one-dimensional cellular automata with a canonical neighborhood (connected and centered). A cellular automaton (CA) is a triple $\mathcal{A} = (S, r, f)$ where:

- S is a (finite) *state set*,
- r is the neighborhood *radius*,
- $f : S^{2r+1} \to S$ is the *local transition function*.

A coloring of the lattice $\mathbb{Z}$ with states from S (*i.e.* an element of $S^{\mathbb{Z}}$) is called a *configuration*. To $\mathcal{A}$ we associate a global function G acting on configurations by synchronous and uniform application of the local transition function. Formally, $G : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ is defined by:

$$G(x)_z = f(x_{z-r}, \ldots, x_{z+r})$$

for all $z \in \mathbb{Z}$. Several CA can share the same global function although there are syntactically different (different radii and local functions). However we are mainly interested in global functions and will sometimes define CA through their global function without specifying a particular syntactical representation. In addition, the Curtis-Heldund-Lyndon theorem [13] allows us to freely compose global CA functions to construct new CA without manipulating explicitly the underlying syntactical representation.

When dealing with several CA simultaneously, we use index notation to denote their respective state sets, radii and local functions. For instance, to $\mathcal{A}$ we associate $S_{\mathcal{A}}$, $r_{\mathcal{A}}$ and $f_{\mathcal{A}}$.

This paper will make an intensive use of $\tilde{\mathbf{P}}\mathbf{CS}$ transforms defined in section 4.2 of [8], but restricted to dimension 1. With this restriction, a $\tilde{\mathbf{P}}\mathbf{CS}$ transform α has the form $\alpha = \langle m, \tau, T, s\rangle$ where m and T are positive integers, s is a (possibly negative) integer and τ is either 1 or -1.

For any CA $\mathcal{A}$, we denote by $\mathcal{A}^{\langle\alpha\rangle}$ or more explicitly $\mathcal{A}^{\langle m,\tau,T,s\rangle}$ the application of α to $\mathcal{A}$, which is, according to notations of [8], a CA of state set $S_{\mathcal{A}}^m$ and global rule:

$$\langle \boxplus_m, V_\tau \odot \square_m\rangle \circ \sigma_s \circ G_{\mathcal{A}}^T \circ \langle \boxplus_m, V_\tau \odot \square_m\rangle^{-1}.$$

To simplify notation we will use a shortcut for purely temporal transforms: for any CA $\mathcal{A}$ we denote by $\mathcal{A}^t$ the CA $\mathcal{A}^{\langle 1,1,t,0\rangle}$. Finally, as another special case, we denote by $\mathcal{A}^{[n]}$ the grouped instance of $\mathcal{A}$ of parameter n: it corresponds to the transform $\langle n, 1, n, 0\rangle$ (see [8] for a detailed exposition of grouping).

2. Canonical orders

In this section we introduce the 3 bulking quasi-orders that are studied all along the paper. They are obtained by applying the bulking axiomatics developed in the companion paper [8] to 3 'canonical' relations between local rules of CA.

Those 3 'canonical' relations are in turn based on 2 classical notions of morphism between local transition rules of CA: sub-automaton and quotient-automaton. As it is shown below, the 3 relations we consider are exactly the reflexive and transitive relations that can be defined by composition of one or more such morphisms.

2.1. From 3 Local Relations to 3 Bulking Quasi-Orders

A *sub-automaton* is a restriction of a CA to a stable sub-alphabet. A *quotient* is a projection of a CA onto a smaller alphabet and compatible with the local transition rule[1]. Both define a kind of morphism between cellular automata:

- $\mathcal{A}$ is a *sub-automaton* of $\mathcal{B}$, denoted $\mathcal{A} \sqsubseteq \mathcal{B}$, if there is an injective map $\iota : S_{\mathcal{A}} \to S_{\mathcal{B}}$ such that $\overline{\iota} \circ G_{\mathcal{A}} = G_{\mathcal{B}} \circ \overline{\iota}$, where $\overline{\iota} : S_{\mathcal{A}}^{\mathbb{Z}} \to S_{\mathcal{B}}^{\mathbb{Z}}$ denotes the uniform extension of ι. We often write $\mathcal{A} \sqsubseteq_{\iota} \mathcal{B}$ to make the map ι explicit.
- $\mathcal{A}$ is a *quotient* of $\mathcal{B}$, denoted $\mathcal{A} \trianglelefteq \mathcal{B}$, if there is a surjective (onto) map s from $S_{\mathcal{B}}$ to $S_{\mathcal{A}}$ such that $\overline{s} \circ G_{\mathcal{B}} = G_{\mathcal{A}} \circ \overline{s}$, where $\overline{s} : S_{\mathcal{B}}^{\mathbb{Z}} \to S_{\mathcal{A}}^{\mathbb{Z}}$ denotes the uniform extension of s. We also write $\mathcal{A} \trianglelefteq_{s} \mathcal{B}$ to make the map s explicit.

Relations $\sqsubseteq$ and $\trianglelefteq$ are quasi-orders (reflexive and transitive) and it is straightforward to check that their induced equivalence relation is the relation of isomorphism between cellular automata (equality up to state renaming) denoted by $\equiv$.

It is also straightforward to check that $\sqsubseteq$ and $\trianglelefteq$ are incomparable (none of them is implied by the other). It is thus interesting to consider compositions of them. The composition of two relations R_1 and R_2 is the relation $R_1 \cdot R_2$ defined by

$$R_1 \cdot R_2 = \{(x, y) : \exists z, (x, z) \in R_1 \text{ and } (z, y) \in R_2\}.$$

We denote by $\mathcal{R}$ the set of relations obtained by (finite) composition of $\trianglelefteq$ and $\sqsubseteq$. Any relation of $\mathcal{R}$ is a priori interesting, but the following theorem justifies that we restrict to $\trianglelefteq$, $\sqsubseteq$ and the composition $\trianglelefteq \cdot \sqsubseteq$ only. In the sequel $\trianglelefteq \cdot \sqsubseteq$ is denoted by $\trianglelefteq\!\!\sqsubseteq$ and, as for $\sqsubseteq$ and $\trianglelefteq$, we use the infix notation ($\mathcal{A} \trianglelefteq\!\!\sqsubseteq \mathcal{B}$).

Theorem 2.1.

1. *any relation $R \in \mathcal{R}$ is included in $\trianglelefteq\!\!\sqsubseteq$ (i.e., $(\mathcal{A}, \mathcal{B}) \in R$ implies $\mathcal{A} \trianglelefteq\!\!\sqsubseteq \mathcal{B}$) ;*
2. *the transitive relations of $\mathcal{R}$ are exactly: $\trianglelefteq$, $\sqsubseteq$ and $\trianglelefteq\!\!\sqsubseteq$.*

Proof. We first prove that if $\mathcal{A} \sqsubseteq \cdot \trianglelefteq \mathcal{B}$ then $\mathcal{A} \trianglelefteq\!\!\sqsubseteq \mathcal{B}$, which is sufficient to prove (1) by transitivity of $\sqsubseteq$ and of $\trianglelefteq$. So consider $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ such that $\mathcal{A} \sqsubseteq_{\iota} \mathcal{C}$ and $\mathcal{C} \trianglelefteq_{s} \mathcal{B}$. Then consider $Q = s^{-1} \circ \iota(S_{\mathcal{A}})$. We have $G_{\mathcal{B}}(Q^{\mathbb{Z}}) \subseteq Q^{\mathbb{Z}}$ because

$$\overline{s} \circ G_{\mathcal{B}}(Q^{\mathbb{Z}}) = G_{\mathcal{C}} \circ \overline{s}(Q^{\mathbb{Z}}) = G_{\mathcal{C}} \circ \overline{\iota}(S_{\mathcal{A}}^{\mathbb{Z}}) = \overline{\iota} \circ G_{\mathcal{A}}(S_{\mathcal{A}}^{\mathbb{Z}}) \subseteq \overline{\iota}(S_{\mathcal{A}}^{\mathbb{Z}}).$$

The CA $\mathcal{X} = (Q, r_{\mathcal{B}}, f_{\mathcal{B}})$ is thus well-defined and by definition we have $\mathcal{X} \sqsubseteq \mathcal{B}$. Moreover, we have $\mathcal{A} \trianglelefteq_{\iota^{-1} \circ s} \mathcal{X}$ because $\iota^{-1} \circ s : Q \to A$ is well-defined and surjective and because

$$\overline{\iota^{-1} \circ s} \circ G_{\mathcal{X}} = G_{\mathcal{A}} \circ \overline{\iota^{-1} \circ s}$$

since $\overline{s} \circ G_{\mathcal{B}} = G_{\mathcal{C}} \circ \overline{s}$ and $\overline{\iota^{-1}} \circ G_{\mathcal{C}} = G_{\mathcal{A}} \circ \overline{\iota^{-1}}$ over $\big(\iota(A)\big)^{\mathbb{Z}} = \overline{s}\big(Q^{\mathbb{Z}}\big)$. Hence $\mathcal{A} \trianglelefteq\!\!\sqsubseteq \mathcal{B}$ and (1) is proven.

[1] A quotient is a particular kind of *factor*, a classical notion in dynamical systems theory and symbolic dynamics [20]

Given (1) we have $\mathcal{R} = \{\trianglelefteq, \sqsubseteq, \sqsubseteq \cdot \trianglelefteq, \trianglelefteq\!\!\sqsubseteq\}$. To prove (2), it is thus sufficient to prove that $\sqsubseteq \cdot \trianglelefteq$ is not transitive. To do this, consider $S_{\mathcal{A}} = \{0, \ldots, p-1\}$ with p prime, $p \geq 5$, and let $\alpha, a_0, a_1, b_0, b_1$ be 5 distinct elements of $S_{\mathcal{A}}$. Then consider $\mathcal{A}$, the CA of states set $S_{\mathcal{A}}$, radius 1 and local rule $f_{\mathcal{A}}$ defined by:

$$f_{\mathcal{A}}(*, x, y) = \begin{cases} a_{1-i} & \text{if } x \neq \alpha \text{ and } y = a_i, \\ b_{1-i} & \text{if } x \neq \alpha \text{ and } y = b_i, \\ y + 1 \bmod p & \text{else.} \end{cases}$$

$f_{\mathcal{A}}$ depends only on two variables. Suppose now that there is some AC $\mathcal{B}$ with at least 2 states such that $\mathcal{B} \trianglelefteq_\pi \mathcal{A}$. We will show that π must be one-to-one. Suppose indeed by contradiction that there are distinct elements e and f in $S_{\mathcal{A}}$ such that $\pi(e) = \pi(f)$. Then $\pi(e + 1 \bmod p) = \pi(f + 1 \bmod p)$ and more generally $\pi(e + i \bmod p) = \pi(f + i \bmod p)$ for all $i \in \mathbb{N}$ because $f_{\mathcal{A}}(\alpha, e) = e + 1 \bmod p$ and $f_{\mathcal{A}}(\alpha, f) = f + 1 \bmod p$. So, supposing without loss of generality $f > e$, let $k = f - e$. We deduce from above that $\pi(f) = \pi(f + jk \bmod p)$ for all $k \in \mathbb{N}$ and, by elementary group theory, that π is constant equal to $\pi(f)$ (because p is prime and $k \neq 0$). This is in contradiction with the fact that π is surjective onto $S_{\mathcal{B}}$ which has at least 2 elements. So π is one-to-one and $\mathcal{B}$ is isomorphic to $\mathcal{A}$. Now consider $\mathcal{C}$, the identity CA over state set $S_{\mathcal{C}} = \{0, 1\}$. Since $\mathcal{C}$ possesses 2 quiescent states and $\mathcal{A}$ has no quiescent state (straightforward from the definition of $f_{\mathcal{A}}$ above), we have $\mathcal{C} \not\trianglelefteq \mathcal{A}$. With the discussion above, we can conclude that $\mathcal{C} \sqsubseteq \cdot \not\trianglelefteq \mathcal{A}$.

However, we have $\mathcal{B} \trianglelefteq\!\!\sqsubseteq \mathcal{A}$ because the states $\{a_0, a_1, b_0, b_1\}$ induce a sub-automaton $\mathcal{C}$ of $\mathcal{A}$ which verifies $\mathcal{B} \trianglelefteq_s \mathcal{C}$ where $s : \{a_0, a_1, b_0, b_1\} \to \{0, 1\}$ is defined by $s(a_i) = 0$ et $s(b_i) = 1$. (2) follows since the relation $\trianglelefteq\!\!\sqsubseteq$ is included in the composition of the relation $\sqsubseteq \cdot \trianglelefteq$ with itself. ■

Like $\sqsubseteq$ (already considered in [8]), $\trianglelefteq$ and $\trianglelefteq\!\!\sqsubseteq$ are quasi-orders on CA and therefore constitute natural candidates for the **divide** relation of bulking axiomatics (definition 8 of [8]).

Inspired by definition 14 of [8], we now define 3 bulking quasi-orders using $\tilde{\mathbf{P}}\mathbf{C}\mathbf{S}$ transforms.

Definition 2.1. *$\mathcal{B}$ simulates $\mathcal{A}$* injectively*, denoted $\mathcal{A} \preccurlyeq_i \mathcal{B}$, if there exists two $\tilde{\mathbf{P}}\mathbf{C}\mathbf{S}$ transforms α and β such that $\mathcal{A}^{\langle\alpha\rangle} \sqsubseteq \mathcal{B}^{\langle\beta\rangle}$.*

We will occasionally use the notion of simulation by grouping introduced in [24] and discussed in [8]: we denote by $\mathcal{A} \leqslant_\square \mathcal{B}$ the fact that there is n and m such that $\mathcal{A}^{[n]} \sqsubseteq \mathcal{B}^{[m]}$. This is a special case of the injective simulation above.

Definition 2.2. *$\mathcal{B}$ simulates $\mathcal{A}$* surjectively*, denoted $\mathcal{A} \preccurlyeq_s \mathcal{B}$, if there exists two $\tilde{\mathbf{P}}\mathbf{C}\mathbf{S}$ transforms α and β such that $\mathcal{A}^{\langle\alpha\rangle} \trianglelefteq \mathcal{B}^{\langle\beta\rangle}$.*

Definition 2.3. *$\mathcal{B}$ simulates $\mathcal{A}$ in a mixed way, denoted $\mathcal{A} \preccurlyeq_m \mathcal{B}$, if there exists two $\tilde{\mathbf{P}}\mathbf{C}\mathbf{S}$ transforms α and β such that $\mathcal{A}^{\langle\alpha\rangle} \trianglelefteq\!\!\sqsubseteq \mathcal{B}^{\langle\beta\rangle}$*

For each notion of simulation above, we say that the simulation is *strong* if the transformation α applied to the simulated CA is trivial: $\alpha = \langle 1, 1, 1, 0\rangle$ so that $\mathcal{A}^{\langle\alpha\rangle} = \mathcal{A}$.

Theorem 2.2. *$\big(CA, \preccurlyeq_i\big)$, $\big(CA, \preccurlyeq_s\big)$ and $\big(CA, \preccurlyeq_m\big)$ are quasi-orders.*

PROOF. We show that $\preccurlyeq_i$ and $\preccurlyeq_m$ correspond exactly to models of bulking developped in [8]: the proof of theorem 15 of [8] contains the case of injective simulation. The case of $\preccurlyeq_m$ follows immediately (axiom (B_4) is straightforward and axiom (B_5) is true because $\trianglelefteq\!\!\sqsubset$ contains $\sqsubseteq$). For $\preccurlyeq_s$, the proof of each axiom is similar except for axiom (B_5).

With or without axiom (B_5), theorem 10 of [8] can be applied in each case and show the theorem. ■

Lemma 2.1. *Let $\triangleleft$ be any relation among $\sqsubseteq$, $\trianglelefteq$ and $\trianglelefteq\!\!\sqsubset$. Then the following propositions are equivalent:*

- *there exists two* $\tilde{\mathbf{P}}\mathbf{CS}$ *transforms α and β such that $\mathcal{A}^{\langle\alpha\rangle} \triangleleft \mathcal{B}^{\langle\beta\rangle}$,*
- *there exist a* $\tilde{\mathbf{P}}\mathbf{CS}$ *transform α and and integer t such that $\mathcal{A}^{\langle\alpha\rangle} \triangleleft \mathcal{B}^{[t]}$,*
- *there exist a* $\tilde{\mathbf{P}}\mathbf{CS}$ *transform β and and integer t such that $\mathcal{A}^{[t]} \triangleleft \mathcal{B}^{\langle\beta\rangle}$,*

PROOF. We use the property of compatibility of relation $\triangleleft$ with respect to geometrical transforms (axiom B_4 of [8]). The lemma follows from the following property: for any transform α, there exist a transform β and an integer t such that

$$\forall F : \left(F^{\langle\alpha\rangle}\right)^{\langle\beta\rangle} = F^{[t]}.$$

If $\alpha = \langle m, t, z, 0\rangle$, β can be chosen as the composition of $\langle 1, m, 0, 0\rangle$, $\langle 1, 1, -z, 0\rangle$ and $\langle t, 1, 0, 0\rangle$. ■

In the sequel, if $\preccurlyeq$ denotes a simulation quasi-order we denote by $\sim$ the induced equivalence relation and by $[\mathcal{A}]$ the equivalence class of $\mathcal{A}$ with respect to $\sim$. For instance, to $\preccurlyeq_i$ we associate the notations $\sim_i$ and $[\mathcal{A}]_i$. We use similar notations for $\preccurlyeq_s$ and $\preccurlyeq_m$.

Before entering into details concerning various aspects of the 3 simulation relations defined above, we can already make a clear (yet informal) distinction between $\preccurlyeq_i$ and $\preccurlyeq_m$ on one hand, and $\preccurlyeq_s$ on the other hand. For the two former, the simulation take place on a subset of configurations and nothing can be said *a priori* about the behavior of the simulator outside this subset of configurations. For $\preccurlyeq_s$, however, the simulation occur on any configuration and the simulator's behavior on any configuration is in some way affected by the simulation. Section 4.2 give several illustrations of this difference.

2.2. First Properties

We now establish a set of basic general facts about $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$ while next sections of the paper focus on particular aspects.

Theorem 2.3. *Let $\mathcal{A}$ be any CA and $\preccurlyeq$ be any relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. Then we have:*

1. *there is some $\mathcal{B} \in [\mathcal{A}]$ having a quiescent state,*
2. *there is some $\mathcal{B} \in [\mathcal{A}]$ with radius 1,*
3. *$\perp \preccurlyeq \mathcal{A}$ where $\perp$ is the CA with a single state,*
4. *$\mathcal{A} \preccurlyeq \mathcal{A} \times \mathcal{B}$ and $\mathcal{A} \preccurlyeq \mathcal{B} \times \mathcal{A}$ for any $\mathcal{B}$.*

PROOF.

1. there exists some uniform configuration x and some $t \geq 1$ such that $G_{\mathcal{A}}^t(x) = x$. So $\mathcal{A}^t$ has a quiescent state and it clearly belongs to $[\mathcal{A}]$.
2. $\mathcal{A}^{\langle r_{\mathcal{A}},1,1,0\rangle}$ admits a syntactical representation with radius 1 and clearly belongs to $[\mathcal{A}]$.
3. First, one always has $\perp \trianglelefteq_\pi \mathcal{A}$ where π is the trivial surjection sending each state of $\mathcal{A}$ to the single state of $\perp$. So assertion 3 is proven for $\preccurlyeq_s$ and $\preccurlyeq_m$. Second, one has $\perp \sqsubseteq_i \mathcal{B}$ if $\mathcal{B}$ has a quiescent state where i is the trivial injection sending the single state of $\perp$ to the quiescent state of $\mathcal{B}$. Assertion 3 follows for $\preccurlyeq_i$ by assertion 1.
4. We show only the first relation, the second being rigorously symmetric. First, one has always $\mathcal{A} \trianglelefteq_{\pi_1} \mathcal{A} \times \mathcal{B}$ where $\pi_1 : S_{\mathcal{A}} \times S_{\mathcal{B}} \to S_{\mathcal{A}}$ is the projection over the first component. Second, if $\mathcal{B}$ has a quiescent state q, one has $\mathcal{A} \sqsubseteq_\iota \mathcal{A} \times \mathcal{B}$ where ι is the injection defined by $\iota(x) = (x, q)$ for all $x \in S_{\mathcal{A}}$ (the equality $\overline{\tau} \circ G_{\mathcal{A}} = (G_{\mathcal{A}} \times G_{\mathcal{B}}) \circ \overline{\tau}$ is indeed true over $S_{\mathcal{A}}^{\mathbb{Z}}$). If $\mathcal{B}$ has no quiescent state, just consider $\mathcal{B}^t$ and apply the previous reasoning to obtain:
$$\mathcal{A}^t \sqsubseteq \mathcal{A}^t \times \mathcal{B}^t = \left(\mathcal{A} \times \mathcal{B}\right)^t$$
and thus $\mathcal{A} \preccurlyeq_i \mathcal{A} \times \mathcal{B}$.

■

The 3 simulation quasi-orders are derived through bulking axiomatics from 3 different relations on local rules (see 2.1). There is *a priori* no reason why the differences between local relations should extend to differences between the 3 simulation quasi-orders. The following theorem shows that $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$ are nevertheless different.

Theorem 2.4. *The relations $\preccurlyeq_i$ and $\preccurlyeq_s$ are incomparable (no inclusion in either direction) and both strictly included in $\preccurlyeq_m$.*

Proof. We first show that there are CA $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \sqsubseteq \mathcal{B}$ but $\mathcal{A} \not\preccurlyeq_s \mathcal{B}$. Let $\mathcal{A} = \sigma \times \sigma^{-1}$ defined over states set $S_{\mathcal{A}} = \{0,1\} \times \{0,1\}$ and let $\mathcal{B}$ be the CA of radius 1 defined over $S_{\mathcal{B}} = S_{\mathcal{A}} \cup \{\#\}$ by:

$$f_{\mathcal{B}}(x,y,z) = \begin{cases} f_{\mathcal{A}}(x,y,z) & \text{if } x,y,z \in S_{\mathcal{A}}, \\ y & \text{else.} \end{cases}$$

One clearly has $\mathcal{A} \sqsubseteq_{Id} \mathcal{B}$. Now suppose $\mathcal{A} \preccurlyeq_s \mathcal{B}$. Without loss of generality we can assume that there are geometrical transforms $\alpha =< m, \tau, T, s >$ and $\beta =< m', 1, T', 0 >$ such that $\mathcal{A}^{\langle\alpha\rangle} \trianglelefteq_\pi \mathcal{B}^{\langle\beta\rangle}$. But, by definition of $\mathcal{B}$, there exist some state q_0 of $\mathcal{B}^{\langle\beta\rangle}$ (one can choose $\#^{m'}$) which is left invariant by iteration of $\mathcal{B}^{\langle\beta\rangle}$ whatever the context. Then $\pi(q_0)$ must be a state of $\mathcal{A}^{\langle\alpha\rangle}$ with the same property. This is impossible since either $s \neq T$ or $s \neq -T$ and thus some component of the future state of a cell of $\mathcal{A}^{\langle\alpha\rangle}$ is dependent of the state of a neighboring cell.

Now we show that there are $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \trianglelefteq \mathcal{B}$ but $\mathcal{A} \not\preccurlyeq_i \mathcal{B}$ and the theorem follows. Intuitively, $\mathcal{A}$ is a CA with two states, 0 and 1, whose behavior is to reduce ranges of 1's progressively until they reach size 1: at each time step the cells at each ends of a range of size 3 or more are turned into state 0 (only the right cell of range of size 2 is turned into 0). $\mathcal{B}$ has 3 states (0, 1 and 2) and has the following behavior: ranges of size 3 or more of non-zero states are reduced in

a similar way by the two ends (states inside ranges are left unchanged), ranges of size 2 become an isolated 2 (left cell becomes 2 and right cell 0), and ranges of size 1 become an isolated 1. In a word, $\mathcal{B}$ reduce size of non-zero until size 1 but keeps the parity information at the end: an even range becomes eventually an isolated 2 and an odd range becomes an isolated 1 (see figure 4).

Formally, let $\pi : \{0,1,2\} \to \{0,1\}$ be the surjective function defined by $\pi(0) = 0$ and $\pi(x) = 1$ if $x \neq 0$. Now let $\mathcal{A}$ be the CA of radius 2 and states set $S_{\mathcal{A}} = \{0,1,2\}$ with local rule:

$$f_{\mathcal{A}}(x,y,z,t,u) = \begin{cases} 1 & \text{if } \overline{\pi}(xyztu) = 01110, \\ 2 & \text{if } \overline{\pi}(yztu) = 0110, \\ z & \text{if } \overline{\pi}(yzt) = 111 \text{ and if } \overline{\pi}(xyztu) \neq 01110, \\ z & \text{if } \overline{\pi}(yzt) = 010, \\ 0 & \text{in any other case.} \end{cases}$$

Finally, let $\mathcal{B}$ be the CA with states set $B = \{0,1\}$, radius 2 and local transition function

$$f_{\mathcal{B}}(x,y,z,t,u) = \begin{cases} 1 & \text{if } yzt = 111 \text{ or } yzt = 010 \text{ or } yztu = 0110, \\ 0 & \text{else.} \end{cases}$$

By construction, we have $\mathcal{A} \trianglelefteq_{\pi} \mathcal{B}$. Now suppose by contradiction that $\mathcal{A} \preccurlyeq_i \mathcal{B}$

Figure 4: Behavior of $\mathcal{A}$ (left) and $\mathcal{B}$ (right). Time goes from bottom to top.

and more precisely:

$$\mathcal{A}^{\langle m',\tau',t',s' \rangle} \sqsubseteq_{\phi} \mathcal{B}^{\langle m,1,t,0 \rangle},$$

where $\alpha =< m', \tau', t', s' >$ and $\beta =< m, 1, t, 0 >$ are suitable geometrical transforms. Let $u = 1^{m'}$ and $v = 0^{m'}$ (u and v are particular states of $\mathcal{A}^{\langle \alpha \rangle}$) and consider $U = \phi(u)$ and $V = \phi(v)$ (U and V belongs to $S_{\mathcal{B}}^{m'}$). Since configurations $\overline{u}$ and $\overline{v}$ are fixed points of $\mathcal{A}^{\langle \alpha \rangle}$, so are $\overline{U}$ and $\overline{V}$ for $\mathcal{B}^{\langle \beta \rangle}$. Moreover, one can check from the definition above that the state 0 is a 'blocking state' for $\mathcal{B}$: the half-configuration on the left of an occurrence of 0 evolve independently of the half-configuration on its right. So, if U contains one or more zero's, then any configuration of $\mathcal{B}^{\langle \beta \rangle}$ containing U^3 will contain at least one occurrence of U for ever (because it is the case for the configuration $\overline{U}$): this is in contradiction with the fact that the orbit of a configuration of the form ${}^{\omega}vuv^{\omega}$ do not contain any occurrence of u after one iteration of $\mathcal{A}^{\langle \alpha \rangle}$. Hence we have $\overline{\pi}(U) = 1^m$.

From this we deduce that $V = 0^m$ because configurations of the form ${}^{\omega}VU^nV^{\omega}$ are transformed into configurations where a single cell is not in state V (just consider the orbit of ${}^{\omega}vuv^{\omega}$ under $\mathcal{A}^{\langle \alpha \rangle}$) and large range of non-zero states are always turned into large range of zero's under $\mathcal{B}$.

Finally, we have $\overline{\pi}\big(\phi(0^{m'-1}1)\big) = 0^{m-1}1$ by considering the orbit of a configuration of the form ${}^{\omega}vu^2v^{\omega}$ under $\mathcal{A}^{\langle \alpha \rangle}$ and its counterpart of the form ${}^{\omega}VU^2V^{\omega}$

under $\mathcal{B}^{\langle\beta\rangle}$ (by the way, we also show that the shift parameter of transform α is 0). Now, letting $u' = 0^{m'-1}1$, we have on one hand the orbits of 2 configurations of the form ${}^{\omega}vu'u^{2n}v^{\omega}$ and ${}^{\omega}vu^{2n}v^{\omega}$ both leading to the same configuration of the form ${}^{\omega}vu'v^{\omega}$ under $\mathcal{A}^{\langle\alpha\rangle}$, and on the other hand, the orbits of ${}^{\omega}V\phi(u')U^{2n}V^{\omega}$ and ${}^{\omega}VU^{2n}V^{\omega}$ leading to different fixed points under $\mathcal{B}^{\langle\beta\rangle}$ due to different parity of non-zero ranges: this is a contradiction since $\overline{\phi} \circ \mathcal{A}^{\langle\alpha\rangle} = \mathcal{B}^{\langle\beta\rangle} \circ \overline{\phi}$. ■

3. Bottoms of the Orders

We have already seen (theorem 2.3) that $\perp$ is a global minimum for the 3 quasi-orders considered here. In this section, we study CA that are at the bottom of the quasi-orders. Formally, the only CA at level 0 is $\perp$ and a CA $\mathcal{A}$ is at level $n+1$ for a quasi-order $\preccurlyeq$ if:

1. $\mathcal{A}$ is not at level n and,
2. $\forall \mathcal{B} : \mathcal{B} \preccurlyeq \mathcal{A} \Rightarrow \mathcal{B} \in [\mathcal{A}]$ or $\mathcal{B}$ is at level i with $i \leq n$.

A cellular automaton is nilpotent if all initial configurations lead to the same configuration after a finite time. The property of nilpotency corresponds to a class at level 1 as shown by the following theorem.

Theorem 3.1. *Let $\preccurlyeq$ be a simulation relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. Then the following CA are at level 1:*

1. *the set of nilpotent CA with 2 or more states, which is an equivalence class for $\sim$,*
2. *the set of CA which are periodic up to translation ($\mathcal{A}^t \circ \sigma_z = Id$) which is exactly the equivalence class for $\sim$ of the identity CA.*

Proof.

1. Nilpotency is equivalent to the existence of a uniform configuration reached in a fixed finite time from any configuration. This property of phase space is clearly invariant by geometrical transforms and preserved by taking sub-automata or quotient automata. So any nilpotent CA is at level at most 1. Moreover, for any nilpotent $\mathcal{A}$, there is t such that $\mathcal{A}^t$ is a constant function equal to some $\overline{q_a}$. If we consider any nilpotent $\mathcal{B}$ with at least 2 states, there is m such that $|S^m_{\mathcal{B}}| \geq |S_{\mathcal{A}}|$ and t' such that $\mathcal{B}^{t'}$ is a constant function equal to some $\overline{q_b}$. If we consider the geometrical transforms $\alpha =< 1, 1, t, 0 >$ and $\beta =< m, 1, t', 0 >$, then we have both $\mathcal{A}^{\langle\alpha\rangle} \sqsubseteq_i \mathcal{B}^{\langle\beta\rangle}$ and $\mathcal{A}^{\langle\alpha\rangle} \trianglelefteq_\pi \mathcal{B}^{\langle\beta\rangle}$ if i is such that $i(q_a) = q_b$ and π is such that $\pi(x) = q_a \iff x = q_b$.
2. Any CA which is periodic up to a translation is by definition equivalent to some identity CA and two identity CA with different state set are also clearly equivalent. The proof follow since the identity CA is at level 1. ■

The bottom of the quasi-order $\leqslant_\square$ was studied in [24]. The main result is the existence of an infinite familly of mutually incomparable CA at level 1: the familly of CA $\mathcal{Z}_p$ with p a prime number and where $\mathcal{Z}_p$ is a CA of radius 1 and state set $\{0, \ldots, p-1\}$ defined by the following local rule:

$$\delta_{\mathcal{Z}_p}(x, y, z) = x + y + z \bmod p.$$

There are strong connections between $\leqslant_\square$ and $\preccurlyeq_i$ and in fact the set of CA at level 1 are the same for these two quasi-orders.

Lemma 3.1. *If $\mathcal{A}$ is at level 1 for $\leqslant_\square$ then $\mathcal{A}$ is at level 1 for $\preccurlyeq_i$.*

Proof. If $\mathcal{B} \preccurlyeq_i \mathcal{A}$ then by lemma 2.1 there is some integer t and some transform β such that $\mathcal{B}^{\langle\beta\rangle} \sqsubseteq \mathcal{A}^{[t]}$. By theorem 2.3 we can suppose that $\mathcal{A}$ has radius 1 so $\mathcal{B}^{\langle\beta\rangle}$ has radius 1. Since $\mathcal{A}$ is at level 1 for $\leqslant_\square$, then either $\mathcal{B}^{\langle\beta\rangle} \in [\mathcal{A}]_\square$ or $\mathcal{B}^{\langle\beta\rangle} \in [\bot]_\square$. We deduce that either $\mathcal{B} \in [\mathcal{A}]_i$ or $\mathcal{B} \in [\bot]_i$. Hence $\mathcal{A}$ is at level at most 1 for $\preccurlyeq_i$ and it cannot be at level O since it is not in $[\bot]_i = [\bot]_\square$. ■

The previous lemma is not enough to show that the CA $\left(\mathcal{Z}_p\right)_p$ with p prime are mutually $\preccurlyeq_i$-incomparable because several equivalence classes for $\leqslant_\square$ can be included in a single class for $\preccurlyeq_i$. However we are going to show that this familly is a set of mutually incomparable CA for all quasi-orders considered in this paper[2]. Moreover, for $\leqslant_\square$ and $\preccurlyeq_i$, they are all at level 1. The proof relies on the following result already used for the case of $\leqslant_\square$.

A CA (S, r, f) is *LR-permutative* if the two following functions are bijections for all $a_1, \ldots, a_{2r}$:

- $x \mapsto f(a_1, \ldots, a_{2r}, x)$ and
- $x \mapsto f(x, a_1, \ldots, a_{2r})$.

Theorem 3.2 ([22]). *Let p be a prime number and $t \geq 1$. Then we have:*

1. *${\mathcal{Z}_p}^{[t]}$ is LR-permutative;*
2. *if $\mathcal{A} \sqsubseteq {\mathcal{Z}_p}^{[t]}$ then p divides $|S_\mathcal{A}|$.*

To take into account use of $\trianglelefteq$ in simulation we will use the following lemma.

Lemma 3.2. *If $\mathcal{B}$ is LR-permutative and $\mathcal{A} \trianglelefteq \mathcal{B}$ then $|S_\mathcal{A}|$ divides $|S_\mathcal{B}|$.*

Proof. To simplify notations, we suppose that $\mathcal{B}$ is of radius 1 (the proof works the same way for higher radii). Suppose $\mathcal{A} \trianglelefteq_\pi \mathcal{B}$. By surjectivity of π, it is sufficient to show that π is balanced, *i.e.* such that for all $x, y \in S_\mathcal{A}$:

$$|\{e : \pi(e) = x\}| = |\{e : \pi(e) = y\}|.$$

Consider any $x, y \in S_\mathcal{A}$. Let $a, b \in S_\mathcal{B}$ be such that $\pi(a) = x$ and $\pi(b) = y$ and consider any $c \in S_\mathcal{B}$. By R-permutativity there is $d \in S_\mathcal{B}$ such that $f_\mathcal{B}(a, c, d) = b$. Now for any $a' \in S_\mathcal{B}$ such that $\pi(a') = \pi(a)$, we must have $\pi\big(f_\mathcal{B}(a', c, d)\big) = \pi(b)$ because $\mathcal{A} \trianglelefteq_\pi \mathcal{B}$. Moreover, by L-permutativity, all the images $f_\mathcal{B}(a', c, d)$ are different which proves:

$$|\{a : \pi(a) = x\}| \leq |\{b : \pi(b) = y\}|.$$

The balance of π follows by symmetry. ■

The results above are the key ingredient of the following theorem.

Theorem 3.3. *Let $\preccurlyeq$ be any relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. Let p and q be two distinct prime numbers. Then we have:*

[2]The proof we give here was suggested by E. Jeandel.

1. $\mathcal{Z}_p \not\preccurlyeq \mathcal{Z}_q$
2. $\mathcal{Z}_p$ *is at level* 1 *for* $\preccurlyeq_i$*;*

PROOF. 2 follows immedialty from lemma 3.1 and the fact that $\mathcal{Z}_p$ is at level 1 for $\leqslant_\square$ (corollary 2 of [22]). To prove 1 it is enough to prove $\mathcal{Z}_p \not\preccurlyeq_m \mathcal{Z}_q$. Suppose by contradiction that $\mathcal{Z}_p \preccurlyeq_m \mathcal{Z}_q$, or equivalently by lemma 2.1, that there are a CA $\mathcal{A}$, a transform α and an integer t such that ${\mathcal{Z}_p}^{\langle\alpha\rangle} \trianglelefteq \mathcal{A} \sqsubseteq {\mathcal{Z}_q}^{[t]}$. Then, combining lemma 3.2 and theorem 3.2, we deduce that the number of states of ${\mathcal{Z}_p}^{\langle\alpha\rangle}$ is a power of q which contradicts the fact that p and q are two distinct primes. ■

We will now study a familly of CA which shows that there are infinitely many incomparable CA at any finite level greater than 3 for any of the 3 simulation quasi-orders of the paper.

We denote by $\sigma_{n,z}$ the translation CA with n states $\{1,\dots,n\}$ and translation vector z defined by:

$$\sigma_{n,z}(c)_{z'} = c_{z'-z}.$$

We then consider cartesian products of such CA. Since $\sigma_{n,z} \times \sigma_{p,z} \equiv \sigma_{np,z}$, we can focus on consider cartesian products where all vectors are distinct.

The next lemma shows that the structure of product of translation is preserved when taking sub-automata and quotient-automata.

Lemma 3.3. *Let* $\mathcal{B} = \prod_{i=1}^{p} \sigma_{n_i,z_i}$ *(with* z_i *all distinct) and suppose* $\mathcal{A}$ *is such that* $\mathcal{A} \trianglelefteq\!\!\sqsubseteq \mathcal{B}$ *then* $\mathcal{A}$ *is isomorphic to* $\prod_{j=1}^{k} \sigma_{n'_{i_j},z_{i_j}}$ *where* $1 \leq i_j \leq n$ *and* $2 \leq n'_{i_j} \leq n_{i_j}$*.*

PROOF. The lemma is straightforward if we replace $\trianglelefteq\!\!\sqsubseteq$ by $\sqsubseteq$. So it is enough to show that it is also true when replacing $\trianglelefteq\!\!\sqsubseteq$ by $\trianglelefteq$. Suppose that $\mathcal{A} \trianglelefteq_h \mathcal{B}$. Let i be fixed between 1 and p. Consider any $x, x' \in \{1,\dots,n_i\}$ and any $q, q' \in S_\mathcal{B}$ such that $\pi_i(q) = \pi_i(q') = x$ (where π_i denotes the projection on the ith coordinate). Denote by q_+ and q'_+ the states obtained from q and q' by changing the ith coordinate to x'. Then we have:

$$h(q) = h(q') \Rightarrow h(q_+) = h(q'_+)$$

because one can build two configurations c and c' of $\mathcal{B}$ such that $c' = \mathcal{B}(c)$ and $c(0) = q$ and $c'(-z_i) = q'$, and obtain c'_+ and c' by changing q to q_+ at $c(0)$ and q' to q'_+ at $c'(-z_i)$. Hence, if there exist two states q and q' with the same image by f and which agree on all components except component i where q equals x and q' equals x', then x and x' in ith component can always be exchanged without afecting the image by h whatever the content of other components. Therefore, taking the apropriate quotient automaton $\mathcal{C}$, we have $\mathcal{A} \trianglelefteq \mathcal{C} \trianglelefteq \mathcal{B}$ where $\mathcal{C}$ equals $\mathcal{B}$ but with 1 state less in ith component. Applying this reasoning inductively, we finally have $\mathcal{A} \trianglelefteq_g \mathcal{C}_0 \trianglelefteq \mathcal{B}$ where $\mathcal{C}_0$ is of the form $\prod_{j=1}^{k} \sigma_{n'_{i_j},z_{i_j}}$ where $1 \leq i_j \leq n$ and $2 \leq n'_{i_j} \leq n_{i_j}$ (component reduced to 1 state during the induction can be eliminated) and g is such that changing the value of any component will change the image by g. Now suppose by contradiction that g is not injective. Then there are states q and q' of $\mathcal{C}_0$ such that $g(q) = g(q')$. Let $c = \overline{q}$ and c' be equal to c except on position 0 where it is in state q'. By hypothesis, at any position z, $G_{\mathcal{C}_0}(c)$ and $G_{\mathcal{C}_0}(c')$ must be in states having the

same image by g. But since $\mathcal{C}_0$ is a product of translations with distinct vectors, there must be some position z where $G_{\mathcal{C}_0}(c)$ and $G_{\mathcal{C}_0}(c')$ are in states which differ on one component only: this is in contradiction with the hypothesis on g. Hence g is injective and therefore $\mathcal{A} \equiv \mathcal{C}_0$. ■

If $\mathcal{A}$ is a proudct of translations with vectors $z_1 < \ldots < z_a$, we denote by $\chi(\mathcal{A})$ the following characteristic sequence (provided $a \geq 3$):

$$\chi(\mathcal{A}) = \left(\frac{z_3 - z_1}{z_2 - z_1}, \ldots, \frac{z_a - z_1}{z_2 - z_1} \right).$$

The usefulness of this notion of characterisitic sequence is revealed by the following theorem.

Theorem 3.4. *Let $\preccurlyeq$ be a relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. Let $\mathcal{A}$ be a product of $a \geq 3$ translations with distinct vectors and with characterisitic sequence $\chi(\mathcal{A}) = (\alpha_1, \ldots, \alpha_{a-2})$. If $\mathcal{B} \preccurlyeq \mathcal{A}$ then $\mathcal{B}$ is equivalent to some $\mathcal{C}$ which is a product of a subset of b translations of $\mathcal{A}$. Moreover, we have the following properties:*

1. *if $b = a$ then $\mathcal{C}$ has the same characteristic sequence than $\mathcal{A}$;*
2. *if $b = a - 1$ and $b \geq 3$ then the characteristic sequence of $\mathcal{C}$ has one of the following form:*
   - $(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{a-2})$
   - $\left(\frac{\alpha_2}{\alpha_1}, \ldots, \frac{\alpha_{a-2}}{\alpha_1} \right)$
   - $\left(\frac{\alpha_2 - 1}{\alpha_1 - 1}, \ldots, \frac{\alpha_{a-2} - 1}{\alpha_1 - 1} \right)$
3. *if $\mathcal{C}$ has not $\chi(\mathcal{A})$ for characteristic sequence then $\mathcal{A} \not\preccurlyeq \mathcal{C}$.*

Proof. Let $z_1 < z_2 < \ldots < z_a$ be the ordered list of translation vectors of $\mathcal{A}$. Since $\mathcal{B} \preccurlyeq \mathcal{A}$, there is some $\mathcal{C}$ equivalent to $\mathcal{B}$ and some integer $t \geq 1$ such that $\mathcal{C} \trianglelefteq \mathcal{A}^{[t]}$ (by lemma 2.1). We deuduce from lemma 3.3 that $\mathcal{C}$ is isomorphic to a product of translations whose vectors are a subset of the familly (z_i) since $\mathcal{A}$ and $\mathcal{A}^{[t]}$ have identical translation vectors, $\mathcal{C}$ must have the same characteristic sequence than $\mathcal{A}$ if it has the same number of translation vectors. When $b = a - 1$ and $b \geq 3$, it is straightforward to check that the 3 possible forms of the characteristic sequence of $\mathcal{C}$ correspond to the case where the missing vector is z_i, z_2 and z_1 respectively. To prove the last assertion of the theorem, it is sufficient to check that for any transform α of the form $< m, 1, mt, mz >$ (we can restrict to such transforms by lemma 2.1, $\mathcal{C}$ and $\mathcal{C}^{\langle\alpha\rangle}$ are products of translations with the same characteristic sequence because each vector z_i of $\mathcal{C}$ becomes $tz_i + z$ in $\mathcal{C}^{\langle\alpha\rangle}$. ■

Notice that the previous theorem implies that there are CA $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ which are all equivalent but such that $\mathcal{A} \times \mathcal{A}$ is not equivalent to $\mathcal{B} \times \mathcal{C}$.

The next lemma give canonical members of their equivalence classes.

Lemma 3.4. *Let $\sim$ be the equivalence relation induced by any of the quasi-order $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. Consider any $t \neq 0$, any z and any product of translations of the form $\mathcal{A} = \prod_{i=1}^{p} \sigma_{n_i, tz_i + z}$, . Then we have:*

$$\mathcal{A} \sim \prod_{1 \leq i \leq p} \sigma_{2, z_i}$$

PROOF. Let $\mathcal{B} = \prod_{i=1}^{p} \sigma_{2,z_i}$ and let $m = \max n_i$. It is straightforward to check that one has $\mathcal{A} \trianglelefteq \mathcal{B}^{\langle m,1,mt,mz\rangle}$ and $\mathcal{A} \sqsubseteq \mathcal{B}^{\langle m,1,mt,mz\rangle}$ on one hand, and, on the other hand, $\mathcal{B}^{\langle 1,1,t,z\rangle} \trianglelefteq \mathcal{A}$ and $\mathcal{B}^{\langle 1,1,t,z\rangle} \sqsubseteq \mathcal{A}$. ■

Theorem 3.4 and lemma 3.4 give a complete characterisation of the position of products of shift in the quasi-orders considered in this paper. We will use it later in section 4.1 but we now state the main result of this section concerning levels at the bottom of the quasi-orders.

Corollary 3.1. *Let $\preccurlyeq$ be a relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. For any $n \geq 3$, there are infinitely many incomparable CA at level n for $\preccurlyeq$.*

PROOF. We have shown in theorem 3.1 that translation CA are at level 1. Lemma 3.3 together with lemma 3.4 show that a product of two translations (with distinct vectors) is at level 2. By theorem 3.4 we conclude that any product of n translations with distinct vectors is at level n and two such CA are incomparable if they have different characteristic sequences provided $n \geq 3$. ■

4. Structural properties

In this section we study in various ways the order structure induced by the simulation relations defined above.

4.1. Cartesian Products and Lack of (Semi-)Lattice Structure

The next theorem shows how some simulations by Cartesian products of CA can be transposed to components of the product.

Theorem 4.1. *Let $\mathcal{A}$ be a CA with 2 states and let $\preccurlyeq$ be a simulation relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. For any $\mathcal{B}$ and $\mathcal{C}$, if $\mathcal{B} \times \mathcal{C}$ strongly $\preccurlyeq$-simulates $\mathcal{A}$ then either $\mathcal{A} \preccurlyeq \mathcal{B}$ or $\mathcal{A} \preccurlyeq \mathcal{C}$.*

PROOF. Let $S_{\mathcal{A}} = \{a_1, a_2\}$. First, we show that $\mathcal{A} \sqsubseteq_\iota \mathcal{B} \times \mathcal{C}$ implies either $\mathcal{A} \sqsubseteq \mathcal{B}$ or $\mathcal{A} \sqsubseteq \mathcal{C}$ which is sufficient to prove the theorem for $\preccurlyeq_i$ and $\preccurlyeq_m$. Since $\iota(a_1) \neq \iota(a_2)$ we have either $\pi_1(i(a_1)) \neq \pi_1(i(a_2))$ or $\pi_2(i(a_1)) \neq \pi_2(i(a_2))$ where π_1 and π_2 are projection over first and second component respectively. We suppose the first case (the second is symmetric) and so $\pi_1 \circ i : S_{\mathcal{A}} \to S_{\mathcal{B}}$ is injective. Moreover, since

$$\overline{\pi_1 \circ \iota} \circ G_{\mathcal{A}} = \overline{\pi_1} \circ G_{\mathcal{B}\times\mathcal{C}} \circ \overline{\iota} = G_{\mathcal{B}} \circ \overline{\pi_1 \circ \iota},$$

we conclude that $\mathcal{A} \sqsubseteq_{\pi_1 \circ \iota} \mathcal{B}$.

Second, we show that $\mathcal{A} \trianglelefteq_s \mathcal{B} \times \mathcal{C}$ implies either $\mathcal{A} \preccurlyeq_s \mathcal{B}$ or $\mathcal{A} \preccurlyeq_s \mathcal{C}$ which is sufficient to prove the theorem for $\preccurlyeq_s$. Let $I_{\mathcal{A}}$ be the set of states that can be reached after one step of $\mathcal{A}$ (formally, $I_{\mathcal{A}} = f_{\mathcal{A}}(S_{\mathcal{A}}, \ldots, S_{\mathcal{A}})$) and $I_{\mathcal{B}}$ and $I_{\mathcal{C}}$ be similar sets for $\mathcal{B}$ and $\mathcal{C}$.

- We first suppose that $\mathcal{B}$ and $\mathcal{C}$ are such that each uniform configuration is either a fixed-point or without any uniform antecedent. If $a_1 \notin I_{\mathcal{A}}$ then for any b and c such that $s(b, c) = a_1$ we have either $b \notin I_{\mathcal{B}}$ or $c \notin I_{\mathcal{C}}$. We suppose the first case (the second is analogous) and then we have $\mathcal{A} \trianglelefteq_\zeta \mathcal{B}$ where $\zeta : S_{\mathcal{B}} \to S_{\mathcal{A}}$ is defined by $\zeta(b) = a_1$ and $\zeta(x) = a_2$ for $x \neq b$.

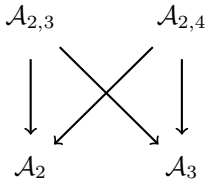
If $a_2 \notin I_{\mathcal{A}}$ we apply the same reasoning and so we are left with the case $I_{\mathcal{A}} = S_{\mathcal{A}}$. Since pairs of $S_{\mathcal{B}} \times S_{\mathcal{C}}$ are 2-colored via s, there must be two pairs of different colors which agree on a component. Suppose it is the first component (the other case is symmetric), we have $b_1, b_2 \in S_{\mathcal{B}}$ and $c \in S_{\mathcal{C}}$ such that $s(b_1, c) = a_1$ and $s(b_2, c) = a_2$. Consider the set $X = \{(b_1, c), (b_2, c)\}$. Since $\overline{s} \circ (G_{\mathcal{B}} \times G_{\mathcal{C}}) = G_{\mathcal{A}} \circ \overline{s}$ and $\overline{s}\big(X^{\mathbb{Z}}\big) = S_{\mathcal{A}}^{\mathbb{Z}}$ and $I_{\mathcal{A}} = S_{\mathcal{A}}$ we necessarily have $s(S_{\mathcal{B}}, d) = S_{\mathcal{A}}$ where d defined by $\overline{d} = G_{\mathcal{C}}(\overline{c})$. d is quiescent by hypothesis on $\mathcal{C}$. So we have $\mathcal{A} \trianglelefteq_{\zeta} \mathcal{B}$ with $\zeta : S_{\mathcal{B}} \to S_{\mathcal{A}}$ defined by $\zeta(x) = s(x, d)$ (ζ is onto by choice of d).

- Now suppose that the hypothesis on $\mathcal{B}$ and $\mathcal{C}$ are not fulfilled. Then, if $t = |S_{\mathcal{B}}|! \times |S_{\mathcal{C}}|!$, both $\mathcal{B}^t$ and $\mathcal{C}^t$ are guarantied to fulfill the required hypothesis (because any uniform configuration is either in a cycle of uniform configurations, or without uniform antecedent arbitrarily far in the past). Since $\mathcal{A}^t \trianglelefteq \big(\mathcal{B} \times \mathcal{C}\big)^t = \mathcal{B}^t \times \mathcal{C}^t$, it suffices to apply the previous reasoning on $\mathcal{A}^t$, $\mathcal{B}^t$ and $\mathcal{C}^t$ to conclude either $\mathcal{A}^t \trianglelefteq \mathcal{B}^t$ or $\mathcal{A}^t \trianglelefteq \mathcal{C}^t$. In either case the theorem follows. ■

Despite the properties of Cartesian product with respect to simulation quasi-orders (theorem 4.1 and assertion 4 of theorem 2.3), this natural operation is not a supremum in any of the quasi-order. In fact these quasi-orders don't admit any supremum operation as shown by the theorem below. The proof relies on the study of products of translation CA.

Theorem 4.2. *Let $\preccurlyeq$ be a relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. Then the ordered structure $\big(AC/\sim, \preccurlyeq\big)$ is neither an upper semi-lattice, nor a lower semi-lattice.*

Proof. Let $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_{2,3}$ and $\mathcal{A}_{2,4}$ be products of translations with characteristic sequences (2), (3), (2, 3) and (2, 4) respectively. Theorem 3.4 and lemma 3.4 show that they induce the following structure in $\preccurlyeq$:



where an arrows from $\mathcal{A}$ to $\mathcal{B}$ means $\mathcal{B} \preccurlyeq \mathcal{A}$ and if $\mathcal{B} \preccurlyeq \mathcal{C} \preccurlyeq \mathcal{A}$ then either $\mathcal{B} \sim \mathcal{C}$ or $\mathcal{C} \sim \mathcal{A}$. This shows that the pair $\mathcal{A}_2$, $\mathcal{A}_3$ has no supremum and that the pair $\mathcal{A}_{2,3}$, $\mathcal{A}_{2,4}$ has no infinimum. ■

4.2. Ideals and Filters

Although the structures $\big(AC, \preccurlyeq\big)$ studied in this paper are not semi-lattices (see above), many classical properties of cellular automata are nicely captured through *ideals* and *filters*. Well-known in lattice theory and algebra, the notions of ideal and filter can also by defined for an arbitrary (quasi-)ordered structure [7]. For the structure $\big(AC, \preccurlyeq\big)$, an ideal I is a set of CA such that:

- if $\mathcal{A} \in I$ and $\mathcal{B} \preccurlyeq \mathcal{A}$ then $\mathcal{B} \in I$;

- for any $\mathcal{A}, \mathcal{B} \in I$ there is some $\mathcal{C} \in I$ such that $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$.

Moreover, I is said *principal* if there is some $\mathcal{A}_I$ such that $\mathcal{A} \in I \iff \mathcal{A} \preccurlyeq \mathcal{A}_I$. The notion of filter and principal filter are dual (replacing all $X \preccurlyeq Y$ by $Y \preccurlyeq X$).

Given a set I of CA, the 3 following conditions are sufficient for I to be an ideal for the simulation $\preccurlyeq_i$ (resp. $\preccurlyeq_s$, or $\preccurlyeq_m$):

1. $\mathcal{A} \in I \iff \mathcal{A}^{\langle\alpha\rangle} \in I$ for any transform α,
2. if $\mathcal{B} \in I$ and $\mathcal{A} \sqsubseteq \mathcal{B}$ (resp. $\mathcal{A} \trianglelefteq \mathcal{B}$, or $\mathcal{A} \trianglelefteq\!\!\sqsubseteq \mathcal{B}$) then $\mathcal{A} \in I$,
3. if $\mathcal{A} \in I$ and $\mathcal{B} \in I$ then $\mathcal{A} \times \mathcal{B} \in I$.

Most of the proofs below follow this scheme. The following theorem shows that several dynamical properties of global rules of CA correspond to ideals in the quasi-orders. A CA is nilpotent over periodic configurations if there exists a spatially periodic configuration c_0 such that all spatially periodic configurations lead in finite time to c_0.

Theorem 4.3. *Let $\preccurlyeq$ be a simulation relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. For each property $\mathcal{P}$ in the following list, the set of CA having property $\mathcal{P}$ is an ideal of $\big(AC, \preccurlyeq\big)$:*

- *surjectivity,*
- *reversibility,*
- *nilpotency over periodic configurations.*

Proof. First, from the point of view of global maps, a geometric transform consists in iterating or composing with bijective maps. So the properties of being surjective or reversible are left unchanged by geometrical transforms. Besides, geometrical transforms send periodic configurations to periodic configurations, temporal cycles of configuration to temporal cycles (eventually reduced to a single configuration), and attraction basin of such cycles to attraction basin of cycles. Hence, nilpotency over periodic configuration, which is equivalent to the existence of a temporal cycle having all periodic configurations in its attraction basin, is preserved by geometrical transforms. By similar reasoning on the phase space, it is straightforward to check that $\mathcal{A}$ is nilpotent over periodic configurations if $\mathcal{B}$ is and $\mathcal{A} \sqsubseteq \mathcal{B}$ or $\mathcal{A} \trianglelefteq \mathcal{B}$. And $\mathcal{A} \times \mathcal{B}$ is nilpotent over periodic configurations if both $\mathcal{A}$ and $\mathcal{B}$ are. So nilpotency over periodic configurations induces an ideal for $\preccurlyeq$.

It is also clear that surjectivity and reversibility are preserved by cartesian product. Now suppose $\mathcal{A} \trianglelefteq_\pi \mathcal{B}$. If $\mathcal{B}$ is surjective then so is $\mathcal{A}$ since $G_\mathcal{A} \circ \overline{\pi} = \overline{\pi} \circ G_\mathcal{B}$ and π is by definition surjective. If $\mathcal{B}$ is reversible, consider any map ϕ such that $\pi \circ \phi = Id$ and let $\mathcal{A}_{-1}$ be the CA over state set $S_\mathcal{A}$ and defined by the global map $G = \overline{\pi} \circ G_\mathcal{B}^{-1} \circ \overline{\phi}$ (it is a shift-commuting continuous map). Since $G_\mathcal{A} \circ \overline{\pi} = \overline{\pi} \circ G_\mathcal{B}$, one can check that $G_\mathcal{A} \circ G = Id$ so $\mathcal{A}$ is reversible.

Finally, suppose $\mathcal{A} \sqsubseteq_\iota \mathcal{B}$. If $\mathcal{B}$ is reversible then $\mathcal{A}$ is also reversible since $\overline{\iota} \circ G_\mathcal{A} = G_\mathcal{B} \circ \overline{\iota}$ and ι is by definition injective. If $\mathcal{B}$ is surjective, then so is $\mathcal{A}$ because $\mathcal{B}$ being injective over finite configurations (Moore-Myhill theorem[3]) $\mathcal{A}$

[3] In [13], one can find the following theorem: a CA is surjective if and only if there is no pair of finite configurations (*i.e.* uniform except on a finite region) having the same image. The original formulation of the Moore-Myhill theorem [26, 27] supposes the existence of a quiescent state.

is also injective over finite configurations (ι maps finite configurations to finite configurations). ■

Theorem 4.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two reversible CA and $\preccurlyeq$ be a simulation relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. If $\mathcal{A} \preccurlyeq \mathcal{B}$ then $\mathcal{A}^{-1} \preccurlyeq \mathcal{B}^{-1}$.*

Proof. First, it is straightforward to check that the inverse of geometrically transformed instances of $\mathcal{A}$ are transformed instances of the inverse of $\mathcal{A}$. Using what was shown above concerning reversibility, it is thus sufficient to prove the 2 following properties:

- $\mathcal{A} \sqsubseteq_\iota \mathcal{B}$ implies $\mathcal{A}^{-1} \sqsubseteq_\iota \mathcal{A}^{-1}$,
- $\mathcal{A} \trianglelefteq_g \mathcal{B}$ implies $\mathcal{A}^{-1} \trianglelefteq_g \mathcal{A}^{-1}$.

In the first case we have:

$$G_\mathcal{B} \circ \overline{\iota} = \overline{\iota} \circ G_\mathcal{A} \Rightarrow \overline{\iota} = G_\mathcal{B}^{-1} \circ \overline{\iota} \circ G_\mathcal{A} \Rightarrow \overline{\iota} \circ G_\mathcal{A}^{-1} = G_\mathcal{B}^{-1} \circ \overline{\iota}$$

each equality being true on $S_\mathcal{A}^\mathbb{Z}$. In the second case we have:

$$G_\mathcal{A} \circ \overline{g} = \overline{g} \circ G_\mathcal{B} \Rightarrow G_\mathcal{A} \circ \overline{g} \circ G_\mathcal{B}^{-1} = \overline{g} \Rightarrow \overline{g} \circ G_\mathcal{B}^{-1} = G_\mathcal{A}^{-1} \circ \overline{g}$$

each equality being true on $S_\mathcal{B}^\mathbb{Z}$. ■

Open Problem 1. *Consider any simulation relation and $\sim$ the associated equivalence relation. What are the reversible CA F such that $F \sim F^{-1}$?*

Theorem 4.5. *Let $\preccurlyeq$ be $\preccurlyeq_i$ or $\preccurlyeq_m$. Then the ideal of reversible CA is principal: there is a reversible CA $\mathcal{A}$ such that*

$$\mathcal{B} \text{ reversible} \iff \mathcal{B} \preccurlyeq \mathcal{A}.$$

Proof. In [9], a reversible CA $\mathcal{B}$ able to simulate any reversible CA is constructed. The notion of simulation used is included in $\preccurlyeq_i$ and therefore in $\preccurlyeq_m$. The implication $\Rightarrow$ is thus proven and the converse implication is proven by theorem 4.3. ■

For the ideal of surjective CA, the principality is still an open problem in dimension 1.

Open Problem 2. *Is the ideal of surjective CA principal, and for which simulation quasi-order?*

Limit sets of CA has received a lot of attention in the literature [6, 15, 11]. The limit set of $\mathcal{A}$ is the set $\Omega_\mathcal{A}$ of configurations having predecessors arbitrarily far in the past, formally:

$$\Omega_\mathcal{A} = \bigcap_t G_\mathcal{A}^t(S_\mathcal{A}^\mathbb{Z}).$$

The next theorem shows that the class of CA with a sofic limit set is nicely captured by $\preccurlyeq_s$.

Theorem 4.6. *The set of CA with a sofic limit set is an ideal for $\preccurlyeq_s$.*

PROOF. For CA of dimension 1, having a sofic limit set is equivalent to having a regular limit language [39]. It is clear that this latter property is left unchanged by geometrical transforms (the limit language is not affect by iterations and shifts, the regularity of the language is not affected by packing). Hence, it is sufficient to show that if $\mathcal{B}$ has a regular limit language and $\mathcal{A} \trianglelefteq_g \mathcal{B}$ then $\mathcal{A}$ also has a regular limit language. Since regular languages are closed by substitution (a classical result which can be found in [14]), it is sufficient to prove that $\Omega_{\mathcal{A}} = \overline{g}\big(\Omega_{\mathcal{B}}\big)$. This last assertion is a direct consequence of $\mathcal{A} \trianglelefteq_g \mathcal{B}$, since the following equality holds by recurrence on t:

$$\overline{g}\big(G^t_{\mathcal{B}}(S^{\mathbb{Z}}_{\mathcal{B}})\big) = G^t_{\mathcal{A}}(S^{\mathbb{Z}}_{\mathcal{A}}).$$

■

Open Problem 3. *Let $\preccurlyeq$ be $\preccurlyeq_i$ or $\preccurlyeq_m$. Is there a $\preccurlyeq$-universal CA with a sofic limit set?*

The properties considered above are purely dynamic: they can be expressed as structural properties of the phase space with the reachability relation only. We now consider properties from topological dynamics: they are expressed with both the reachability relation and the toppology (Cantor distance) of the space of configurations.

The properties we will consider rely on the equicontinuity classification of P. Kůrka [19]. Let $\mathcal{A}$ be any CA of states set Q and global rule G and denote by d the Cantor distance over $Q^{\mathbb{Z}}$.

- $x \in Q^{\mathbb{Z}}$ is an *equicontinuity point* for $\mathcal{A}$ if

$$\forall \epsilon, \exists \delta, \forall y \in Q^{\mathbb{Z}} : d(x,y) \leq \delta \Rightarrow \forall t, d(G^t(x), G^t(y)) \leq \epsilon.$$

- $\mathcal{A}$ is *sensitive to initial conditions* if

$$\exists \epsilon, \forall \delta, \forall x \in Q^{\mathbb{Z}} \exists y \in Q^{\mathbb{Z}} \exists t : d(x,y) \leq \delta \text{ and } d\big(G^t(x), G^t(y)\big) \geq \epsilon.$$

- $\mathcal{A}$ is *(positively) expansive* if

$$\exists \epsilon, \forall x, y \in Q^{\mathbb{Z}} : x = y \iff \forall t, d\big(G^t(x), G^t(y)\big) \leq \epsilon.$$

The classification of P. Kůrka is the following:

K_1 is the set of CA for which all configurations are equicontinuity points,

K_2 is the set of CA having equicontinuity points,

K_3 is the set of CA sensitive to initial conditions,

K_4 is the set of expansive CA.

The weakness of this classification is its lack of shift-invariance: the identity and the elementary translation belong to different classes (K_1 and K_3 respectively). Several attempts have been made to overcome this problem by changing the topology [5]. More recently, a new approach has been proposed [33]: the Cantor topology is conserved (with all its good properties) but the topological

properties are enriched with a new parameter (a velocity) which is used as the reference direction of information propagation in space-time. The original definitions of P. Kůrka are thus obtained by choosing velocity 0, but now identity and elementary translations are assigned to the same class (with different velocities). This directionnal dynamic approach is more suitable for our study since, by definition, the equivalence classes of any of our quasi-order is shift-invariant. We will define 4 classes based on the existence of some direction for which some dynamical behaviour is observed.

We say that $\mathcal{A}$ is a *rescaling* of $\mathcal{B}$ if there are transforms α and β such that $\mathcal{A}^{\langle\alpha\rangle} \equiv \mathcal{B}^{\langle\beta\rangle}$. We then consider the following 4 classes:

- the set $\mathcal{T}_1$ of CA which are a rescaling of some equicontinuous CA,
- the set $\mathcal{T}_2$ of CA which are a rescaling of some CA having equicontinuity points,
- the set $\mathcal{T}_3$ of CA which are not in $\mathcal{T}_2$, *i.e.* CA which are sensitive in every directions[4],
- the set $\mathcal{T}_4$ of CA which are a rescaling of some (positively) expansive CA.

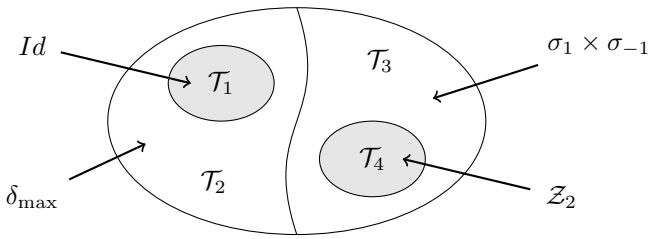


Figure 5: Four kinds of topological dynamics.

Figure 5 is justified by the following theorem.

Theorem 4.7. *We have the following inclusions:*

1. $\mathcal{T}_1 \subseteq \mathcal{T}_2$,
2. $\mathcal{T}_4 \subseteq \mathcal{T}_3$.

Moreover, each of the set $\mathcal{T}_1$, $\mathcal{T}_2 \setminus \mathcal{T}_1$, $\mathcal{T}_3 \setminus \mathcal{T}_4$ and $\mathcal{T}_4$ is non-empty.

Proof. The first inclusion follows from definitions. The second follows from proposition 3.2 of [33] which asserts that the set of direction with equicontinuity points and the set of expansive directions cannot be simultaneously empty.

Non-emptyness of $\mathcal{T}_1$ and $\mathcal{T}_4$ follow from the existence of equicontinuous (*e.g.* the identity) and (positively) expansice CA (*e.g.* $\mathcal{Z}_2$). Moreover, any CA having an equicontinuity point which is not equicontinuous (*e.g.* the CA of local rule $\delta_{\max}(a, b, c) = \max(a, b, c)$) cannot be in $\mathcal{T}_1$ (equicontinuity is preserved by rescaling), so it is in $\mathcal{T}_2 \setminus \mathcal{T}_1$. Finally, by proposition 3.2 of [33],

[4]For one-dimensional CA, the set of sensitive CA is the complement of the set of CA having equicontinuity points (see [19]). In [33], this complementarity is shown for any direction.

$\sigma_1 \times \sigma_{-1} \in \mathcal{T}_3 \setminus \mathcal{T}_4$ because any direction is either a direction of right-expansivity, or a direction of left-expansivity, neither both. ■

Theorem 4.8.

1. *$\mathcal{T}_1$ is an ideal for any simulation $\preccurlyeq$ among $\preccurlyeq_s$, $\preccurlyeq_i$ and $\preccurlyeq_m$;*
2. *$\mathcal{T}_2$ is an ideal for $\preccurlyeq_s$;*
3. *$\mathcal{T}_4$ is an ideal for $\preccurlyeq_i$.*

Proof. First, consider any $\mathcal{A}, \mathcal{B} \in \mathcal{T}_1$. Then there are CA $\mathcal{A}'$ and $\mathcal{B}'$ which are both equicontinuous and $\preccurlyeq$-equivalent to $\mathcal{A}$ and $\mathcal{B}$ respectively. Then, if $\mathcal{C} = \mathcal{A}' \times \mathcal{B}'$ we have $\mathcal{C} \in \mathcal{T}_1$ and by theorem 2.3 we have both $\mathcal{A} \preccurlyeq \mathcal{C}$ and $\mathcal{B} \preccurlyeq \mathcal{C}$. The same reasoning can be applied to $\mathcal{T}_4$ and $\mathcal{T}_2$. Thus we have shown the second condition of the definition of ideals for the 3 properties considered here.

To conclude the theorem, and since the 3 properties considered are by definition invariant by rescaling, it is sufficient to prove:

- if $\mathcal{A} \sqsubseteq \mathcal{B}$ or $\mathcal{A} \trianglelefteq \mathcal{B}$ then $\mathcal{B}$ equicontinuous $\Rightarrow \mathcal{A}$ equicontinuous;
- if $\mathcal{A} \trianglelefteq \mathcal{B}$ then $\mathcal{B}$ has equicontinuous points $\Rightarrow \mathcal{A}$ has equicontinuous points;
- if $\mathcal{A} \sqsubseteq \mathcal{B}$ then $\mathcal{B}$ expansive $\Rightarrow \mathcal{A}$ expansive.

The first assertion follows from the characterisation of equicontinuous CA as ultimately periodic CA [19].

For the second assertion, if $\mathcal{A} \trianglelefteq_\pi \mathcal{B}$ then for all $x, y \in S_\mathcal{B}^\mathbb{Z}$ we have the inequality $d(\pi(x), \pi(y)) \leq d(x, y)$. Moreover, for any $y_1 \in S_\mathcal{A}^\mathbb{Z}$ there is some $y_2 \in S_\mathcal{B}^\mathbb{Z}$ such that $\pi(y_2) = y_1$ and $d(\pi(x), y_1) = d(x, y_2)$ (choose y_2 so that it equals x on the cells around position 0). Hence, if x is an equicontinuous point for $\mathcal{B}$ then $\pi(x)$ is an equicontinuous point for $\mathcal{A}$.

Finally, for the third assertion, it is sufficient to notice that the property of expansivity is defined by a formula using only universal quantifications on configurations so it remains true on a subset of configurations. ■

$\mathcal{T}_2$ is not an ideal for $\preccurlyeq_i$ and neither for $\preccurlyeq_m$ as shown by the following example.

Example 4.1. *Consider $\mathcal{B} \in \mathcal{T}_3$ of radius 1 and let $\mathcal{A}$ be the CA with radius 1, states set $S_\mathcal{A} = S_\mathcal{B} \cup \{M\}$ (with $M \notin S_\mathcal{B}$) with local rule $f_\mathcal{A}$ defined by*

$$f_\mathcal{A}(x, y, z) = \begin{cases} f_\mathcal{B}(x, y, z) & \text{if } \{x, y, z\} \subseteq S_\mathcal{B}, \\ y & \text{else.} \end{cases}$$

$\mathcal{B} \sqsubseteq \mathcal{A}$ so $\mathcal{B} \preccurlyeq_i \mathcal{A}$. However $\mathcal{A} \in \mathcal{T}_2$ since the configuration ${}^\omega M^\omega$ is an equicontinuous point. ■

Notice alos that $\mathcal{T}_3$ cannot be an ideal because $\sigma_1 \times \sigma_{-1} \in \mathcal{T}_3$ simulates $\sigma \in \mathcal{T}_1$.

Open Problem 4. *Are there $\mathcal{A} \notin \mathcal{T}_4$ and $\mathcal{B} \in \mathcal{T}_4$ such that $\mathcal{A} \preccurlyeq_s \mathcal{B}$ (i.e. the simulator CA is expansive up to rescaling but the simulated CA is not expansive, even up to rescaling)?*

4.3. (Un)decidability

The fact that many properties related to the simulation quasi-orders are undecidable comes with no surprise. For instance the nilpotency property, which is an undecidable problem [18], corresponds to an equivalence class in the 3 quasi-orders (theorem 3.1). However, there are non-trivial properties of these quasi-orders which are decidable (see below) and the edge between decidable and undecidable properties is hard to catch.

In this section, we consider two kind of problems in simulation quasi-orders: lower bounds (being above some fixed CA or set of CA) and upper bounds (being simulated by some fixed CA or some CA from a fixed set).

Theorem 4.9 ([23]). *The set of CA of radius* 1 *with a spreading state and nilpotent over periodic configurations is not co-recursively enumerable.*

Theorem 4.10. *Let $\mathcal{A}$ be any CA which is not nilpotent over periodic configurations. Let $\preccurlyeq$ be either $\preccurlyeq_i$ or $\preccurlyeq_m$. Then the set of CA $\mathcal{B}$ such that $\mathcal{A} \preccurlyeq \mathcal{B}$ is not co-recursively enumerable.*

Proof. We describe a computable transformation which, given a CA $\mathcal{C}$ of radius 1 with a spreading state, produce a CA $\mathcal{B}$ with the following properties:

- if $\mathcal{C}$ is not nilpotent over periodic configurations then $\mathcal{A} \preccurlyeq \mathcal{B}$;
- if $\mathcal{C}$ is nilpotent over periodic configurations then so is $\mathcal{B}$.

The theorem follows by theorem 4.3 since we have reduced the problem '$\mathcal{A} \preccurlyeq \mathcal{B}$?' to the problem of nilpotency over periodic configurations (reduced to CA of radius 1 with a spreading state).

We now describe the construction of $\mathcal{B}$ from $\mathcal{C}$. Suppose $\mathcal{C}$ has a spreading state q. $\mathcal{B}$ is the CA of radius 1 and states set $S_\mathcal{B} = (S_\mathcal{C} \setminus \{q\}) \times S_\mathcal{A} \cup \{q\}$ with local rule $f_\mathcal{B}$ defined by:

$$f_\mathcal{B}(a,b,c) = \begin{cases} \big(f_\mathcal{C}(a_1,b_1,c_1), f_\mathcal{A}(a_2,b_2,c_2)\big) & \text{if } \begin{cases} a,b,c \in S_\mathcal{B} \setminus \{q\} \text{ and} \\ f_\mathcal{C}(a_1,b_1,c_1) \neq q, \end{cases} \\ q & \text{in any other case,} \end{cases}$$

where a_i, b_i and c_i represent component i of a, b and c. Any periodic configuration c of $\mathcal{B}$ either leads to the uniform configuration $\overline{q}$, or contain a periodic configuration of $\mathcal{C}$ in its first component. Hence, if $\mathcal{C}$ is nilpotent over periodic configurations, then so is $\mathcal{B}$ (because q is precisely the spreading state of $\mathcal{C}$). If $\mathcal{C}$ is not nilpotent over periodic configurations, then there is a word $u \in (S_\mathcal{C} \setminus \{q\})^m$ and an integer $t \geq 1$ such that the periodic configuration c of period u verifies $G^t_\mathcal{C}(c) = c$. Therefore, by definition of $\mathcal{B}$, we have $\mathcal{A}^{\langle m,1,t,0\rangle} \sqsubseteq_i \mathcal{C}^{\langle m,1,t,0\rangle}$ where $i : S^m_\mathcal{A} \to S_\mathcal{B}$ is defined by:

$$i(a_1,a_2,\ldots,a_m) = \big((a_1,u_1),\ldots,(a_m,u_m)\big).$$

■

This result shows that it is generally undecidable to know whether a CA is lower-bounded by a given (fixed) one. However, there are noticeable exception in dimension 1 for $\preccurlyeq_s$ and $\preccurlyeq_m$.

Theorem 4.11. *Let $\preccurlyeq$ be either $\preccurlyeq_s$ or $\preccurlyeq_m$ and let $\mathcal{A}$ be a nilpotent CA. Then the problem of determining if a given $\mathcal{B}$ is above $\mathcal{A}$ for $\preccurlyeq$ is decidable.*

Proof. We are going to show that $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $\mathcal{B}$ is not surjective and the theorem follows by decidability of surjectivity in one dimension [1]. First, by theorem 4.3, if $\mathcal{B}$ is surjective then $\mathcal{A} \not\preccurlyeq \mathcal{B}$. Suppose now that $\mathcal{B}$ is not surjective, *i.e.* that $\mathcal{B}$ possesses some Eden word $u \in S_{\mathcal{B}}^m$ for some length m. Then, denoting by $\mathcal{C}$ the CA over states set $S_{\mathcal{C}} = \{0, 1\}$ which is constant equal to 0, we have $\mathcal{C} \trianglelefteq_\pi \mathcal{B}^{\langle m,1,1,0\rangle}$ if $\pi : S_{\mathcal{B}}^m \to S_{\mathcal{A}}$ verifies $\pi(w) = 0$ if and only if $w = u$. We deduce by theorem 3.1 that $\mathcal{A} \preccurlyeq \mathcal{B}$. ■

Open Problem 5. *Is there a non-surjective CA $\mathcal{A}$ which cannot injectively simulate any nilpotent CA? Is the problem of being above the class of nilpotent CA for injective simulation a decidable problem?*

Concerning upper-bound problems, the edge between decidability and undecidability is also non-trivial. For instance, theorem 4.5 shows the existence of a CA $\mathcal{A}$ such that the upper-bound decision problem '$\mathcal{B} \preccurlyeq \mathcal{A}$?' is decidable in dimension 1.

5. Tops of the Orders

In this section we study the maximal elements of the quasi-orders. This CA are able to simulate any other CA

Definition 5.1. *Let $\preccurlyeq$ be any relation among $\preccurlyeq_i$, $\preccurlyeq_s$ and $\preccurlyeq_m$. A CA $\mathcal{A}$ is said* $\preccurlyeq$-universal *if for any $\mathcal{B}$ we have $\mathcal{B} \preccurlyeq \mathcal{A}$. It is* strongly $\preccurlyeq$-universal *if it strongly $\preccurlyeq$-simulates any other CA.*

The notion of strong $\preccurlyeq_i$-universality above is exactly the same notion as *intrinsic universality* defined in section 5 of [8] and has already been considered several times in the literature (see [32] for a survey). In fact, strong and general universality are the same notion for $\preccurlyeq_i$ and $\preccurlyeq_m$.

Theorem 5.1. *There exist strongly $\preccurlyeq_i$-universal CA and all $\preccurlyeq_i$-universal CA are strongly $\preccurlyeq_i$-universal. The same is true for $\preccurlyeq_m$.*

Proof. For the existence of strongly $\preccurlyeq_i$-universal CA, see [32]. The theorem follows by application of theorem 12 of [8]. ■

Of course, any $\preccurlyeq_i$-universal is also $\preccurlyeq_m$-universal. The converse is an open problem.

Open Problem 6. *Do the notions of $\preccurlyeq_i$-universality and $\preccurlyeq_m$-universality coincide?*

Concerning $\preccurlyeq_s$, the situation is different: no CA is strongly $\preccurlyeq_s$-universal[5].

Theorem 5.2. *There is no strongly $\preccurlyeq_s$-universal CA.*

[5]The proof of this fact was suggested by G. Richard.

PROOF. Suppose by contradiction that there is some strongly $\preccurlyeq_s$-universal $\mathcal{A}$. Consider a uniform configuration c of $\mathcal{A}$. There is n such that the orbit of c under $\mathcal{A}$ contains n different configurations (the orbit is ultimately periodic). Now consider $\mathcal{B}$ with $n+1$ states such that its uniform configurations are all in the same cycle of length $n+1$. By hypothesis, for any $\mathcal{B}$ there is some geometric transform α such that $\mathcal{B} \trianglelefteq_s \mathcal{A}^{\langle\alpha\rangle}$. Let d be the corresponding configuration of c for $\mathcal{A}^{\langle\alpha\rangle}$. The orbit of d contains at most n different configurations and it is therefore the same for the orbit of $s(d)$ under $\mathcal{B}$. But $s(d)$ is necessarily uniform and we get a contradiction with the choice of $\mathcal{B}$. ■

The theorem 12 of [8] don't apply for $\preccurlyeq_s$. However, we are not able either to construct a $\preccurlyeq_s$-universal CA, or to prove that there is no.

Open Problem 7. *Is there a $\preccurlyeq_s$-universal CA?*

For the rest of this section, we consider only $\preccurlyeq_i$ and $\preccurlyeq_m$.

5.1. On the Ways to Reach the Top

Universal CA are not hard to construct and the property of being universal is recursively enumerable since simulation relations considered here are recursively enumerable. However universality is not co-recursively enumerable as shown by the following theorem. The case of $\preccurlyeq_i$-universality was proven in [31]. Using theorem 4.10, the proof below is direct and includes the case of $\preccurlyeq_m$.

Theorem 5.3. *The set of $\preccurlyeq_i$-universal CA is not co-recursively enumerable and neither is the set of $\preccurlyeq_m$-universal CA.*

PROOF. There exists a CA which is $\preccurlyeq_i$-universal but not nilpotent over periodic configuration. To see this consider any universal CA and add a new state which is spreading: the resulting CA, say $\mathcal{A}$, contains at least two disjoint periodic orbits of periodic configurations and is thus not nilpotent over periodic configurations. The theorem follows by application of theorem 4.10 to $\mathcal{A}$ since $\mathcal{A}$ is by construction both $\preccurlyeq_i$-universal and $\preccurlyeq_m$-universal. ■

This result has some consequences on the structure of simulation quasi-orders 'near' the top. The following theorem indeed shows that a non-universal CA is always 'infinitely far' from the class of universal ones.

Theorem 5.4. *Let $\preccurlyeq$ be $\preccurlyeq_i$ or $\preccurlyeq_m$. And let $\mathfrak{U}$ be the set of $\preccurlyeq$-universal CA. Then we have:*

1. $\mathcal{A} \times \mathcal{B} \in \mathfrak{U} \iff \mathcal{A} \in \mathfrak{U}$ *or* $\mathcal{B} \in \mathfrak{U}$,
2. *if $\mathcal{A} \notin \mathfrak{U}$ then there is $\mathcal{B} \notin \mathfrak{U}$ with $\mathcal{A} \preccurlyeq \mathcal{B}$ but $\mathcal{B} \not\preccurlyeq \mathcal{A}$.*

PROOF.

1. By theorem 2.3 we have $\mathcal{A} \preccurlyeq \mathcal{A} \times \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A} \times \mathcal{B}$ which proves one direction. Moreover, there exists $\mathcal{C} \in \mathfrak{U}$ with 2 states only [2, 29]. If we suppose $\mathcal{A} \times \mathcal{B} \in \mathfrak{U}$ then, by theorem 5.1, it strongly simulates $\mathcal{C}$. Hence, by theorem 4.1, we have either $\mathcal{C} \preccurlyeq \mathcal{A}$ or $\mathcal{C} \preccurlyeq \mathcal{B}$ and thus either $\mathcal{A} \in \mathfrak{U}$ or $\mathcal{B} \in \mathfrak{U}$.
2. let $\mathcal{A} \notin \mathfrak{U}$. If $\mathcal{A}$ was such that $\mathcal{C} \preccurlyeq \mathcal{A}$ for all $\mathcal{C} \notin \mathfrak{U}$ then we would have $\mathfrak{U} = \{\mathcal{C} : \mathcal{C} \preccurlyeq \mathcal{A}\}$ and $\mathfrak{U}$ would be co-recursively enumerable contradicting theorem 5.3. So there is $\mathcal{C} \notin \mathfrak{U}$ with $\mathcal{C} \not\preccurlyeq \mathcal{A}$. To conclude the proof it is sufficient to choose $\mathcal{B} = \mathcal{A} \times \mathcal{C}$ (theorem 2.3). ■

5.2. Necessary But Not Sufficient Conditions

The purpose of this section is twofold. It compares the notions of universality defined above to other definitions of the literature and, by doing this, presents tools and techniques to prove non-universality of some CA (other proofs of non-universality for other purposes are developped in section 6).

One of the techniques we use to ensure that some CA is not universal yet achieving some behaviour B, is to add a spreading state and let the CA generate this state if it detects somewhere that the current configuration doesn't correspond to a 'legal' configuration, *i.e.* a configuration occuring normally when producing the behaviour B. Proofs of non-universality with this technique rely on the lemma below. Before stating and proving the lemma, we need to give some precision on spreading states and set of configurations 'supporting' a simulation.

First, the notion of spreading state is sensitive to the choice of the syntactical representation of the CA because it depends on the choice of the neihbourhood. In the sequel we say a CA $\mathcal{A}$ has a spreading state κ if any cell changes to state κ when κ appears in its minimal neighbourhood (*i.e.* the minimal set of cells upon which the local rule effectively depends).

Second, given a relation of the form $\mathcal{A} \sqsubseteq_i \mathcal{B}^{\langle m,1,t,z\rangle}$, there is an isomorphism between $(\mathcal{A}, S_{\mathcal{A}}^{\mathbb{Z}})$ and $\left(\mathcal{B}^{\langle m,1,t,z\rangle}, (i(S_{\mathcal{A}}))^{\mathbb{Z}}\right)$ as dynamical systems. At the level of $\mathcal{B}$, the configurations involved in this relation is the set X of configurations made of infinite concatenation of elements of $i(S_{\mathcal{A}}) \subseteq S_{\mathcal{B}}^m$ (viewed as words of length m over alphabet $S_{\mathcal{B}}$). This kind of sets are called *bloc subshifts* and discussed in more details in section 3.2 of [8]. In the sequel, such a set X is called the *support* of the simulation.

Lemma 5.1. *Let $\mathcal{A}$ be a CA without spreading state and $\mathcal{B}$ be a CA with a spreading state κ. If $\mathcal{B}$ strongly $\preccurlyeq_m$-simulates $\mathcal{A}$, then the support X of the simulation cannot contain κ.*

Proof. By hypothesis, there are parameters m, t, τ, z and a CA $\mathcal{C}$ such that

$$\mathcal{A} \trianglelefteq_\pi \mathcal{C} \sqsubseteq_i \mathcal{B}^{\langle m,\tau,t,z\rangle}.$$

By choice of $\mathcal{B}$, $\mathcal{B}^{\langle m,\tau,t\rangle}$ admits κ^m as spreading state. Moreover, by definition of $\trianglelefteq$, the minimal neighbourhood of $\mathcal{A}$ is included in the minimal neighbourhood of $\mathcal{B}^{\langle m,\tau,t\rangle}$. Thus, if κ appears in some configuration of X then the state $\pi(i^{-1}(\kappa^m))$ is a spreading state for $\mathcal{A}$ because κ^n also appears in X for arbitrarily large n. ■

We first study how embeddings of Turing machines into CA can relate the notions of universality for Turing machines to the notions of universality derived from quasi-orders as defined above.

An embedding of a Turing machine $\mathcal{M}$ into a CA $\mathcal{A}$ is an embedding of the instantaneous descriptions of $\mathcal{M}$ into configurations of $\mathcal{A}$ such that instantaneous descriptions of successive steps of $\mathcal{M}$ corresponds to successive steps of $\mathcal{A}$ via the embedding. We don't give any formal definition of embedding since we will never prove negative result (*i.e.* assertions of the form 'there is no embbeding of $\mathcal{M}$ such that...'). However, the embeddings we use in the sequel are classical and already appeared in the literature (see [35]).

Theorem 5.5. *For any Turing machine $\mathcal{M}$, there exists a CA $\mathcal{A}$ which embeds $\mathcal{M}$ but is not $\preccurlyeq_m$-universal.*

PROOF. Let $\mathcal{M} = (S_\mathcal{M}, Q_\mathcal{M}, \phi_\mathcal{M})$ where $S_\mathcal{M}$ is the set of states of $\mathcal{M}$, $Q_\mathcal{M}$ is the tape alphabet, and

$$\phi_\mathcal{M} : S_\mathcal{M} \times Q_\mathcal{M} \to S_\mathcal{M} \times Q_\mathcal{M} \times \{-1, 0, 1\}$$

is the transition function of $\mathcal{M}$. We construct a CA $\mathcal{A}$ over state set

$$S_\mathcal{A} = Q_\mathcal{M} \times \{\leftarrow, \rightarrow\} \cup Q_\mathcal{M} \times S_\mathcal{M} \cup \{\kappa\}$$

where $\rightarrow$ and $\leftarrow$ are states not already in $S_\mathcal{M}$. Each cell of $\mathcal{A}$ corresponds to a tape position of $\mathcal{M}$: it contains a letter from the tape alphabet and either a head with its current state or no head but an indication $\leftarrow$ or $\rightarrow$ telling in which direction to find the head. On configurations containing a single head, $\mathcal{A}$ mimics transitions of $\mathcal{M}$ step by step as expected. Thus, $\mathcal{A}$ embbeds $\mathcal{M}$. In addition, $\mathcal{A}$ checks that $\leftarrow$ a never occur to the left of a state from $S_\mathcal{M}$ or a $\rightarrow$ (and symmetrically for $\rightarrow$). If the check fails, then the state κ is generated and spreads.

This construction ensures that, for any initial configuration c, if the orbit of c never contains an occurrence of κ then it contains at most one head. Hence, these orbits are such that at any time step state changes occur on the neighbourhood of at most one position (a head move involves a state change in two adjacent cells).

Now suppose that $\mathcal{A}$ is $\preccurlyeq_m$-universal and consider the CA $\mathcal{B} = \sigma_1 \times \sigma_{-1}$. $\mathcal{A}$ strongly simulates $\mathcal{B}$ (theorem 5.1). Since $\mathcal{B}$ has no spreading state, then the set X of configuration of $\mathcal{A}$ on which the simulation occurs never contains κ. We deduce that all orbits of configuration from X have the property described above. This implied that $\mathcal{B}$ is such that on all its orbits, at most 2 cells change their state between two steps: this in contradiction with the choice of $\mathcal{B}$. ■

Turing-universality of cellular automata is a fairly vague notion in the literature. We don't give a formal definition here since we won't prove any negative result concerning Turing-universality. We just consider that a CA able to embed a universal Turing machine[6] is Turing-universal.

We can choose $\mathcal{M}$ to be universal in the previous theorem (theorem 5.5). In this case, since the embedding used in the proof ensures that $\mathcal{M}$ is simulated in real time by $\mathcal{A}$, we deduce that the following problem is P-complete:

Input: a state $q \in S_\mathcal{A}$, an integer $t \geq 1$, and a word $u \in S_\mathcal{A}^{2rt+1}$ where r is the radius of $\mathcal{A}$;

Query: do we have $\mathcal{A}^t(u) = q$?

This problem of finite triangle computation has been considered several times in the literature and it has been proven that it was P-complete for particular CA [12, 28]. This notion of complexity inherited from sequential computation theory fails to capture the notion of universality associated to simulation quasi-orders.

Corollary 5.1. *There exists a CA which is Turing-universal and P-complete but not $\preccurlyeq_m$-universal.*

[6] We don't give any formal notion of universality for Turing machine either. In fact, we only need to suppose the existence of at least one universal Turing machine.

6. Induced Orders

This sections aims at studying particular CA or sets of CA for the ordered structure they induce in the simulation quasi-orders. While studying various properties of the quasi-orders in the previous sections, we have already established the existence or several induced infinite structures.

For instance, theorem 5.4 allows to construct an infinite strictly increasing chain of non-universal CA starting from any non-universal CA for the quasi-orders associated to $\preccurlyeq_i$ and $\preccurlyeq_m$. Besides, theorem 3.4 implies the existence of inifinite chains in the 3 quasi-orders studied in this paper.

Section 6.1 below gives a way to construct chains of length $\omega + \omega$ and an hint about the existence of chains of length $\omega \times \omega$. However, we leave open the question of the longest chain induced in any of the quasi-orders. We don't even know if one of them admits a dense chain.

Open Problem 8. *Does one of the quasi-orders admit a dense induced order?*

6.1. Limit Cartesian Product

We have seen in theorem 5.4 that if $\mathcal{A}$ is not universal, then $\mathcal{A} \times \mathcal{A}$ cannot be universal. Therefore, no finite Cartesian product of $\mathcal{A}$ with itself can be universal. Therefore, we have a chain of non-universal CA:

$$\mathcal{A} \preccurlyeq \mathcal{A} \times \mathcal{A} \preccurlyeq \mathcal{A} \times \mathcal{A} \times \mathcal{A} \preccurlyeq \cdots$$

For some $\mathcal{A}$, the chain collapses in a single equivalence class, *e.g.* if $\mathcal{A}$ is a translation (see lemma 3.4). However, the following theorem shows that for some $\mathcal{A}$, the chain is strictly increasing. Moreover, $\mathcal{A}$ can be chosen so that it embeds any Turing machine.

Theorem 6.1. *For any Turing machine $\mathcal{M}$, there is a CA $\mathcal{A}$ which embeds $\mathcal{M}$ and such that for any $1 \leq n < m$, one has:*

$$\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{m} \not\preccurlyeq_m \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n}.$$

Proof. Le $\mathcal{A}$ be the CA constructed in the proof of theorem 5.5. We can suppose that $\mathcal{M}$ is such that it can produce infinite sequences of left move of its head starting from a blank tape and leaving the tape blank, and the same for right moves (if $\mathcal{M}$ has not this property, just add some states to achieve this behaviour).

Denote by $\mathcal{B}_m$ the product of m copies of $\mathcal{A}$ and by $\mathcal{B}_n$ the product of n copies. We can construct for any set of positions $z_1, \ldots, z_m$ a configuration c of $\mathcal{B}_m$ such that for all i the ith component contains a correct instantaneous descritpion of $\mathcal{M}$ where the head is at position z_i in a state suitable to generate an infinite sequence of left or right moves. Now let c' be a onfiguration of $\mathcal{B}_n$ corresponding to c via simulation. First, if some component i of c' contains a spreading state, it will spread and, after some time t, will be present at some position where the configuration $G^t_{\mathcal{B}_m}(c)$ contains no head, but only a blank tape symbol on each component. This means that blocs of blank tape symbols in $\mathcal{B}_m$ can be simulated by blocs of $\mathcal{B}_n$ where the ith component is a bloc of spreading states. Considering again the orbit of c, we deduce that it can be simulated

by a configuration c'' where the ith component is everywhere a spreading state except at a finite number of positions. Thus after some time, the ith component will become uniform and constant. It is then straightforward to show that it is useless for the simulation and that in fact $\mathcal{B}_m$ on c can be simulated by only $n-1$ copies of $\mathcal{A}$.

Applying the reasoning inductively, we can therefore suppose that no spreading state appears on any component in the orbit of the configuration c' defined above. Since, the orbit of c is such that there are m distant position where some state change at each step, it must be the case in the orbit of c'. Since, $n < m$, there must be some component with two heads and therefore a spreading state must appear after the first step: this is in contradiction with what we have just supposed. ■

For the CA $\mathcal{A}$ of the previous theorem, we can ask if the infinite chain of Cartesian product is upper-bounded by some non-universal CA, or if any CA able to simulate each product of the chain is necessarily universal. One can imagine that for a sufficiently simple $\mathcal{A}$, there is some room above the chain of products of $\mathcal{A}$ and below the class of universal CA.

The rest of this section is devoted to the proof of a stronger result: for any $\mathcal{A}$, there is a CA $\mathcal{B}$ which is able to simulate any finite product of $\mathcal{A}$ and such that $\mathcal{B}$ is universal if and only if $\mathcal{A}$ is universal. Moreover, $\mathcal{B}$ can be obtained from $\mathcal{A}$ constructively. Because it extends property of Cartesian product given by theorem 5.4, this construction will be called *limit product* in the sequel. If $\mathcal{A}$ is a CA, its limit product is denoted by $\mathcal{A}_\infty$.

Note: In the rest of this section we only consider the simulation $\preccurlyeq_m$.

Without loss of generality, we can suppose that $\mathcal{A}$ has radius 1 (theorem 2.3). To be able to simulate the product $\mathcal{B}$ of n copies of $\mathcal{A}$, $\mathcal{A}_\infty$ is made of 3 layers (its state set is a Cartesian product union a single state, which is a spreading state as explain hereafter):

1. the *state* layer,
2. the *transport* layer, and
3. the *synchronisation* layer.

and proceeds as follows.

State layer. Each components of a cell of $\mathcal{B}$ is simulated by a bloc of 3 adjacent cells in the state layer of $\mathcal{A}_\infty$. More precisely, component i ($0 \leq i \leq n-1$) of cell z of $\mathcal{B}$ is simulated by the bloc of 3 cells of $\mathcal{A}_\infty$ beginning at position $3(nz+i)$. This bloc is referred to as $\mathtt{B}_{z,i}$ in the sequel. In $\mathtt{B}_{z,i}$, the center cell stores the ith component of the cell z of $\mathcal{B}$ and the two other are used to store temporarilly ith components of cell $z-1$ and $z+1$.

Transport layer. The role of the transport layer is precisely to bring states of ith components corresponding to cell $z-1$ and $z+1$ of $\mathcal{A}$ to the dedicated cells of $\mathcal{A}_\infty$ in $\mathtt{B}_{z,i}$. Then, the transition $f_{\mathcal{A}}(x_{z-1}, x_z, x_{z+1})$ of the ith component of $\mathcal{B}$ can be simulated locally by $\mathcal{A}_\infty$ in $\mathtt{B}_{z,i}$. Transport is done in parallel for any i and any z. To do this, the transport layer is made of a succession of particles (one every 3 cells), each one being able to carry a state of $\mathcal{A}$. Initially aligned with the center of blocs, the particules move in parallel according to a cycle of 5 steps:

1. move right by $3n$ cells and read the state seen on the state layer;
2. move left by $3n-1$ cells and write the memorized state on the state layer;
3. move left by $3n+1$ cells and read the state seen on the state layer;
4. move left by $3n-1$ cells and write the memorized state on the state layer;
5. move 1 cell right and apply local rule $f_{\mathcal{A}}$ on state layer at the current position;

Synchronisation layer. The role of the synchronisation layer is to orchestrate the cycle of particle moves and it must be able to do it for arbitrary large values of n (simulating arbitrarily large Cartesian products of $\mathcal{A}$ is sufficient to simulate all products of $\mathcal{A}$). It contains a flag that can take one of the 4 indications 'left', 'right', 'read', 'write' and 'transition'. The flag is changed everywhere synchronously according to a cycle suitable to ensure that particles of the transport layer produce the cycle described above when they follow the instruction given by the flag.

We now describe in detail the synchronisation layer. Denote by u_n the flag sequence mentionned above in the simulation of a product of n copies of $\mathcal{A}$, and let E be the set of flag states.

Theorem 6.2. *There is a CA $\mathcal{C}$ with a spreading state κ and a map $\pi : S_{\mathcal{C}} \to E$ such that $\mathcal{C}$ is not $\preccurlyeq_m$-universal, and, for any configuration $c \in S_{\mathcal{C}}^{\mathbb{Z}}$, one of the following property is true:*

Cycle: *at each time in the orbit of c, all cells have the same image by π and the sequence with time of this common image is periodic of period u_n for some n;*

Frozen: *at each time in the orbit of c, all cells have the same image by π, but this common image remains constant after a certain time;*

Error: *the spreading states appears at some time in the orbit of c.*

Moreover $\mathcal{C}$ is such that there are configurations having the 'cycle' property above producing period u_n for arbitrarily large n.

Proof. First, notice that flag changes in the sequence u_n are separated by a number of steps which is either constant (independant of n), or of the form $3n+c$ with c a constant (we can suppose $c \geq 0$ without loss of generality). To simplify notations, we will suppose in this proof that u_n alternates between two values 0 and 1 every $3n$ steps. Adapting the proof for the real u_n is just a matter of adding a finite set of special states to deal with constants.

The proof is based on a reversible solution $\mathcal{B}$ to the firing squad synchronization problem proposed by K. Imai and K. Morita: in [16], they construct a reversible CA $\mathcal{B}$ with a subset of states F (the firing states) such that for any n, there is a periodic configuration c_n verifying[7]:

[7] In [16], the main concern is synchronisation of finite segments of cells surrounded by a quiescent state. To extend the property to infinite configurations, it is crucial that "garbage" (which must be conserved to ensure reversibility) do no spread outside the initial segment. The solution of K. Imai and K. Morita has precisely this property as it is explicitely mentionned in [16].

- $G_{\mathcal{B}}^{3n}(c_n) \in F^{\mathbb{Z}}$
- $G_{\mathcal{B}}^{t}(c_n) \in (S_{\mathcal{B}} \setminus F)^{\mathbb{Z}}$ for all t, $0 \leq t < 3n$.

Without loss of generality, we can suppose that $\mathcal{B}$ and its inverse are syntactically represented with the same radius. We now define a CA $\mathcal{C}_0$ of radius r, with states set $S_{\mathcal{C}_0} = S_{\mathcal{B}} \times S_{\mathcal{B}} \times \{0,1\}$, and with transition function:

$$f_{\mathcal{C}_0}\big((a_{-r},a'_{-r},b_{-r}),\dots,(a_r,a'_r,b_r)\big) = \begin{cases} \big(f_{\mathcal{B}}(a_{-r},\dots,a_r), f_{\mathcal{B}^{-1}}(a'_{-r'},\dots,a'_{r'}), \chi(a_0,b_0)\big) & \text{if } b_0 = 1, \\ \big(f_{\mathcal{B}^{-1}}(a_{-r},\dots,a_r), f_{\mathcal{B}}(a'_{-r},\dots,a'_r), \chi(a'_0,b_0)\big) & \text{if } b_0 = 0, \end{cases}$$

where $\chi(a,b)$ equals $1-b$ if $a \in F$ and b else. Intuitively, on configurations where the third component is uniform equal to b, $\mathcal{C}_0$ mimic $\mathcal{B}$ on the first component and $\mathcal{B}^{-1}$ on the second one if $b = 1$ or the converse if $b = 0$. Moreover, the value of b is switched each time the component playing $\mathcal{B}$ encounters a firing state. Hence, if we choose for π the projection on third component, $\mathcal{C}_0$ started from configurations c_n has the property 'cycle' and produce the periodic sequence u_n.

We now enrich $\mathcal{C}_0$ with a spreading state which is produced each time one of the following local checking fails:

- the third component $\{0,1\}$ must be uniform;
- for the two first components, a state from F (firing state) must always be surrounded by states from F only;
- states from F are forbidden on the second component if $b = 1$ and states from F in the first component are forbidden if $b = 0$.

The third condition ensures that in the case of a 'cycle' regime (firing states appearing infinitely often), the period is equally divided between steps where $b = 0$ and steps where $b = 1$. To ensure that such 'cycle' regime always produce an alternance of exactly $3n$ zeros and $3n$ ones, we add a component implementing a couter modulo 3: the value of this component is incremented modulo 3 at each step (whatever the context) and a spreading state is generated if a cell contains a firing state and the counter is not 0 modulo 3. Denote by $\mathcal{C}$ the CA obtained and consider any configuration c. If no spreading state appears in the orbit of c, then the third component is uniform. If it changes of state only a finite number of times, then we are in the 'frozen' regime. If there are infinitely many changes, it follows from the discussion above that the conditions of the 'cyclic' regime are fulfilled.

To conclude the theorem, it remains to prove that $\mathcal{C}$ is not $\preccurlyeq_m$-universal. Suppose by contradiction that it is and let $\mathcal{U}$ be any universal CA without spreading state and consider the set X of configurations of $\mathcal{C}$ which is the support of the strong simulation of $\mathcal{U}$ ($\mathcal{C}$ strongly simulmates $\mathcal{U}$ by theorem 5.1). X cannot contain any occurrence of the spreading state (by lemma 5.1), it implies that all configurations of X have a uniform third component. But, on such configuration, the dynamics of $\mathcal{C}$ is reversible. Hence $\mathcal{U}$ is reversible: this is a contradiction with its universality by theorem 4.3. ■

The synchronization layer of limit products are exactly the automaton $\mathcal{C}$ of the previous theorem, except that the spreading state of $\mathcal{C}$ now becomes a global spreading state. Before establishing the main result of this section, we give more details concerning the state layer and the transport layer of $\mathcal{A}_\infty$.

The state layer is made from state set $S_\mathcal{A} \times \{L, C, R\}$ where L, C and R are states to identify explicitely the role of each cell in each bloc $\mathtt{B}_{z,i}$: C for the cell storing the ith component of cell z of $\mathcal{A}$ and L and R to temporarilly store states of ith component of cells $z-1$ and $z+1$ respectively.

The transport layer is made from state set $S_\mathcal{A} \cup \{\perp\}$ where $\perp$ is the state used to separate particle carrying a state from $S_\mathcal{A}$.

So the states set of $\mathcal{A}_\infty$ is:

$$\underbrace{S_\mathcal{A} \times \{L, C, R\}}_{\text{state}} \times \underbrace{S_\mathcal{A} \cup \{\perp\}}_{\text{transport}} \times \underbrace{S_\mathcal{C} \setminus \{\kappa\}}_{\text{synchronization}} \cup \{\kappa\}.$$

In addition to the behaviour described above, $\mathcal{A}_\infty$ do the following local checkings and generates the spreading state κ if one of them fails:

- the second component of transport layer must be periodic of period LCR;
- the transport layer must contain an alternance of 1 state from $S_\mathcal{A}$ and 2 states $\perp$;
- when doing read and write operations, the particles of the transport layer must be aligned with the right type of state in the state layer:
  - type C when reading,
  - type R when writing at step 2,
  - type L when writing at step 4;
- when the synchronisation layer says 'transition', check that the particules are aligned with cell of type C in the state layer.

All those checking ensure the following property: if no spreading state is generated and if the component layer produce a correct cycle of instructions, then the behaviour of the state layer is equivalent to the behaviour of some Cartesian product of $\mathcal{A}$ (up-to some rescaling).

Before stating the main theorem, we establish a simple yet useful lemma saying that if $\mathcal{A}$ simulates $\mathcal{B}$ with support X, then everything $\mathcal{A}$ can simulate using a support included in X can also be simulated by $\mathcal{B}$.

Lemma 6.1. *Let $\preccurlyeq$ be either $\preccurlyeq_i$ or $\preccurlyeq_m$. Let $\mathcal{A}$ and $\mathcal{B}$ be such that the simulation $\mathcal{A} \preccurlyeq \mathcal{B}$ occur on a support X of configurations of $\mathcal{B}$. If $\mathcal{B}$ $\preccurlyeq$-simulate $\mathcal{C}$ on a support included in X, then $\mathcal{A}$ $\preccurlyeq$-simulates $\mathcal{C}$.*

Proof. We consider the case where $\preccurlyeq$ is $\preccurlyeq_i$. By hypothesis, we have $\mathcal{A}^{\langle\alpha\rangle} \sqsubseteq_i \mathcal{B}^{\langle\beta_1\rangle}$ on support X and $\mathcal{C}^{\langle\gamma\rangle} \sqsubseteq_j \mathcal{B}^{\langle\beta_2\rangle}$ on support $Y \subseteq X$. Now, let m_α, m_{β_1}, m_{β_2} and m_γ be the packing parameters of transforms α, β_1, β_2 and γ respectively. The injective maps i and j induce two injective maps i_{β_2} and j_{β_1} with the following domains and ranges:

$$i_{\beta_2} : S_\mathcal{A}^{m_\alpha m_{\beta_2}} \to S_\mathcal{B}^{m_{\beta_1} m_{\beta_2}}$$
$$j_{\beta_1} : S_\mathcal{C}^{m_\gamma m_{\beta_1}} \to S_\mathcal{B}^{m_{\beta_1} m_{\beta_2}}$$

Therefore $\phi = i_{\beta_2}^{-1} \circ j_{\beta_1}$ is a well-defined injective map from $S_{\mathcal{C}}^{m_\gamma m_{\beta_1}}$ into $S_{\mathcal{A}}^{m_\alpha m_{\beta_2}}$. Now define the transforms η_a and η_c to be the composition of α and β_2, and of γ and β_1 respectively. Then we have $\mathcal{C}^{\langle \eta_c \rangle} \sqsubseteq_\phi \mathcal{A}^{\langle \eta_a \rangle}$.

The extension of the previous reasoning to $\preccurlyeq_m$ is straightforward. ■

This lemma together with lemma 5.1 is the key to a kind of 'self-checking' simulation used in the construction of the limit product (and re-used in section 6.2). A 'self-checking' simulation of $\mathcal{B}$ by $\mathcal{A}$ is standard simulation of $\mathcal{B}$ by $\mathcal{A}$ on some support X with the additional property that $\mathcal{A}$ 'checks' locally on any configuration that it belongs to X, and triggers some pathological behaviour (typically a spreading state) in case of check failure. Hence any possible strong simulation of some $\mathcal{C}$ by $\mathcal{A}$ is such that:

- either it has a support included in X in which case $\mathcal{B}$ can also simulate $\mathcal{C}$ by lemma 6.1,
- or it must contain some $c \notin X$ in its support in which case a spreading state is genereated and lemma 5.1 give some limitation on $\mathcal{C}$.

To show that a spreading state is generated in the second case above, a crucial property is that the support of any simulation is by definition always irreducible: if u_1 and u_2 are words occuring in two configurations of the support, there exist a third configuration of the support where u_1 and u_2 both appear (see section 3.2 of [8] for a more detailed discussion on supports of simulations).

We now state the main theorem of this section.

Theorem 6.3. *For any $\mathcal{A}$, its limit product $\mathcal{A}_\infty$ is such that:*

- $\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n} \preccurlyeq_m \mathcal{A}_\infty$ *for all* $n \geq 1$,
- $\mathcal{A}_\infty$ *is $\preccurlyeq_m$-universal if and only if $\mathcal{A}$ is $\preccurlyeq_m$-universal.*

Proof. The first assertion follows from the construction of $\mathcal{A}_\infty$ and the detailed discussion above. Now suppose that $\mathcal{A}_\infty$ is $\preccurlyeq_m$-universal and let $\mathcal{U}$ be any universal CA without spreading state. By theorem 5.1, $\mathcal{A}_\infty$ strongly simulates $\mathcal{U}$: $\mathcal{U} \trianglelefteq\!\!\sqsubseteq \mathcal{A}_\infty{}^{\langle \alpha \rangle}$ for some geometrical transform α. Let X denote the support of the simulation. By choice of $\mathcal{U}$, the spreading state κ cannot appear in any orbit of any configuration of X (by lemma 5.1). We deduce from theorem 6.2 that the synchronization component is in the same regime (either 'cycle' for a fixed value n or 'frozen') for all the configurations of X because otherwise, we could construct a configuration in X producing a spreading (by irreducibility of X).

In the case where all configurations are in the frozen regime, the flag of the synchronization layer becomes constant after some time t_0, so the transport layer has the behaviour of a translation (or identity) and the state layer remains constant. t_0 is identical for all configurations of X (because otherwise, we could once more combine two configurations to produce a spreading state, by irreducibility of X). Then, consider a CA $\mathcal{U}_+$ with state set $S_\mathcal{U} \times \{0, \ldots, t_0\}$ which has the following behaviour:

- the second component is decreased by one until it reaches 0;

- on the first component, the local rule of $\mathcal{U}$ is applied, but only if the second component is 0.

Since $\mathcal{U}$ is $\preccurlyeq_m$-universal, it can strongly simulate $\mathcal{U}_+$ (by theorem 5.1): precisely, $\mathcal{U}_+ \trianglelefteq \mathcal{U}^{\langle m,1,t,z\rangle}$. Consider the set Y of configurations of $\mathcal{U}$ corresponding via simulation to the set of configurations of $\mathcal{U}_+$ uniformly equal to t_0 on the second component. Denote by $X_Y \subseteq X$ the corresponding set of configurations of $\mathcal{A}_\infty$. By choice of $\mathcal{U}_+$, we know that $\mathcal{U}$ simulates itself on the set of configurations $G_{\mathcal{U}}^{tt_0}(Y)$. This implies that for some $t' \geq t_0$, $\mathcal{A}_\infty$ can simulate U using as support the set of configuration $G_{\mathcal{A}_\infty}^{t'}(X_Y)$. By hypothesis, starting from such configurations, $\mathcal{A}_\infty$ has a behaviour of translation or identity on the state and transport layers. Since the synchronizing component evolves independantly of the others, we deduce by lemma 6.1 that there is some CA $\mathcal{B}$ which is a product of translation (corresponding to state and transport layers) such that $\mathcal{B} \times \mathcal{C}$ simulates $\mathcal{U}$: this is a contradiction by theorem 5.4 since neither $\mathcal{B}$ (theorem 3.4), nor $\mathcal{C}$ (theorem 6.2) is universal.

Hence, we are necessarily in the case where the synchronization layers produce a valid cycle. Since no spreading state can be generated in the orbit of any configuration of X, the state layer always behave like a Cartesian product of n copies of $\mathcal{A}$. The value of n is in fact common to all configurations of X (as shown above), so we deduce by lemma 6.1 that $\underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n}$ simulates $\mathcal{U}$ and $\mathcal{A}$ is therefore universal by theorem 5.4. ■

Of course, we can consider $\mathcal{A}_\infty$ itself as a new candidate for taking its finite Cartesian products and applying the limit product construction. In fact, the process can be repeated forever with the guarantee that no CA ever produced in this chain will be universal, provided the initial CA is not. However, there is no reason why this infinite chain should be strictly increasing. In particular, even if

$$\mathcal{A} \preccurlyeq \mathcal{A} \times \mathcal{A} \preccurlyeq \mathcal{A} \times \mathcal{A} \times \mathcal{A} \preccurlyeq \cdots$$

is a strictly increasing chain, it might be the case that $\mathcal{A}_\infty$ is equivalent to $\mathcal{A}_\infty \times \mathcal{A}_\infty$. Therefore we have only proven that one of the following properties is true:

- there is a strictly increasing chain of length $\omega \times \omega$ in the quasi-order $(AC, \preccurlyeq_m)$,
- for any non-universal CA $\mathcal{A}$, there is a non-universal CA $\mathcal{B}$ such that $\mathcal{A} \preccurlyeq_m \mathcal{B}$ and $\mathcal{B} \times \mathcal{B}$ is equivalent to $\mathcal{B}$.

6.2. Sub-Families of Cellular Automata

Theorem 2.3 shows that any equivalence class in any quasi-order contains some CA with radius 1. This fact is a direct consequence of a well-known transformation of CA with large radius into CA of smaller radius with more states (this transformation is called 'higher block presentation' in symbolic dynamics, see [21]).

It is also sometimes invoked in the literature that considering CA with 2 states only is not restrictive since there is a converse transformation that transforms a CA with many states into a CA with only 2 states but a larger radius.

However, the situation is different since there are equivalence classes without CA having 2 states: *e.g.* $\mathcal{Z}_p$ for any prime $p \neq 2$ as shown by theorem 3.2 and lemma 3.2. Note that the same is true for any fixed state set of cardinal n: the equivalence class of $\mathcal{Z}_p$ contains no such CA provided p is prime and do not divide n.

Hence, this transformation introduces a bias: the transformed CA may be inequivalent to the original one. Meanwhile, we know that CA with 2 states can be as powerful as CA in general since there are universal CA with 2 states only [2, 29] (for simulation realtions $\preccurlyeq_i$ and $\preccurlyeq_m$). More precisely, as we will see below, the transformation applied to a universal CA always yields a universal CA because the transformed CA simulates the original one. Since the original and the transformed CA are not always in the same equivalence class, one question that naturally arises is: what CA can be simulated by the transformed CA but not by the original one? Although it provides only partial answers, this section is devoted to that kind of questions, for CA with 2 states and for other families.

Formally, given a familly $\mathfrak{F}$ of CA, we say a map $\phi : AC \to \mathfrak{F}$ is a $\preccurlyeq$-encoding of CA into familly $\mathfrak{F}$ if

$$\forall \mathcal{A}, \mathcal{A} \preccurlyeq \phi(\mathcal{A}).$$

We will only consider simulation relations $\preccurlyeq_i$ and $\preccurlyeq_m$ in the sequel, thus an encoding into $\mathfrak{F}$ implies that there are universal CA in $\mathfrak{F}$. A trivial example of such an encoding is given by $\mathfrak{F} = \{\mathcal{U}\}$ where $\mathcal{U}$ is a universal CA and ϕ is the map sending any CA to $\mathcal{U}$. We are interested in using this notion of encoding with families which are more 'representative' of the diversity of behaviours in the whole set of CA. To express this we introduce the following notion of *faithfulness*.

Given a $\preccurlyeq$-encoding $\phi : AC \to \mathfrak{F}$ and a set E of CA, we say that ϕ is *faithful* for E if:

$$\forall \mathcal{B} \in E : \mathcal{B} \preccurlyeq \mathcal{A} \iff \mathcal{B} \preccurlyeq \phi(\mathcal{A}).$$

An encoding is faithful for E if the original CA and its image by the encoding simulate exactly the same CA in E. So, to give some evidence that a family $\mathfrak{F}$ is 'representative' of CA in general, we can exhibit an encoding of CA into $\mathfrak{F}$ which is faithful for a set E of CA as large as possible. When E is the whole set of CA, the faithfulness implies that there is a CA of family $\mathfrak{F}$ in any equivalence class: this is the case for CA with radius 1.

The next theorem gives 4 encodings which are faithful for U, the set of $\preccurlyeq_m$-universal CA. The families corresponding to these encodings were already defined in this paper except one: captive CA.

Captive CA where introduced in [36] and are defined by a simple restriction on the transition rule. A CA $\mathcal{A}$, of states set $S_\mathcal{A}$, radius r and local rule $f_\mathcal{A}$ is captive if:

$$\forall a_{-r}, \ldots, a_r \in S_\mathcal{A} : f_\mathcal{A}(a_{-r}, \ldots, a_r) \in \{a_{-r}, \ldots, a_r\}.$$

In the following theorem, encodings are different but their faithfulness rely on the same idea of 'self-checking' simulation explained above which uses lemma 5.1 and lemma 6.1.

Theorem 6.4. *Let $\preccurlyeq$ be $\preccurlyeq_i$ or $\preccurlyeq_m$. For any family of CA below, there is a $\preccurlyeq$-encoding from CA into $\mathfrak{F}$ wich is faithful for the set U of $\preccurlyeq_m$-universal CA:*

- *CA with 2 states,*
- *CA in $\mathcal{T}_2$,*
- *CA in $\mathcal{T}_3$,*
- *captive CA.*

PROOF. To describe the encoding for each family, we suppose $\mathcal{A}$ is a CA with states set $S_\mathcal{A} = \{a_1, \dots, a_n\}$, with radius r and location rule $f_\mathcal{A}$.

2 states CA. Let m be an integer large enough and ψ be an injective map from $S_\mathcal{A}$ to $\{0,1\}^m$ such that no word $\psi(a)$ contains an occurrence of 11. Now define the injective map $i : S_\mathcal{A} \to \{0,1\}^{m+4}$ by $i(a) = 0110\psi(a)$. Let $r' = (r+1)(m+4)$. $\phi(\mathcal{A})$ is a CA of radius r' and states set $\{0,1\}$ defined as follows:

- on the set X of configurations made of infinite concatenation of words from $i(S_\mathcal{A})$, $\phi(\mathcal{A})$ is isomorphic to $\mathcal{A}$ so that $\mathcal{A} \sqsubseteq_i \phi(\mathcal{A})$;
- everywhere else, $\phi(\mathcal{A})$ generates a 1.

The map ϕ is thus an encoding of CA into 2-states CA. Now suppose that $\phi(\mathcal{A})$ is universal and let $\mathcal{U}$ be a universal CA with 2 states and no spreading state which is strongly simulated by $\phi(\mathcal{A})$ on support Y (theorem 5.1). If there is some $y \in Y$ with $y \notin X$ then

- either there are two occurrences of 0110 in y which are not correctly spaced,
- or there is a word $0110u0110$ occurring in y with $u \notin \psi(S_\mathcal{A})$.

In any case, the image of y will contain an occurrence of 111 (because the above error must be seen by at least 3 consecutive cells) and 1's will propagate like a spreading state which is impossible by lemma 5.1 because otherwise $\phi(\mathcal{A})^{\langle 3,1,1,0\rangle}$ could simulate $\mathcal{U}$ on a support where it possesses the spreading state 111. So $Y \subseteq X$ and lemma 6.1 shows that $\mathcal{A}$ simulates $\mathcal{U}$. Hence $\mathcal{A}$ is universal if and only if $\phi(\mathcal{A})$ is.

Captive CA. The encoding technique for captive CA is very similar and already appeared in a non-faithful form in [36]. Let u be the word $a_1 \cdots a_n$, let $\#$ be a state not in $S_\mathcal{A}$ and denote $Q = S_\mathcal{A} \cup \{\#\}$. We define the injective map $i : S_\mathcal{A} \to Q^{n+3}$ by $i(a) = \#u\#a$. We then define $\phi(\mathcal{A})$ in a way similar to the case above. Its radius is $r' = (r+1)(n+3)$, its states set is Q and its local rule is such that:

- on the set X of configurations made of infinite concatenation of words from $i(S_\mathcal{A})$, $\phi(\mathcal{A})$ is isomorphic to $\mathcal{A}$ so that $\mathcal{A} \sqsubseteq_i \phi(\mathcal{A})$;
- everywhere else, $\phi(\mathcal{A})$ take as new state the maximum of its neighbours for some fixed ordering of Q such that $\#$ is the maximum.

First, $\phi(\mathcal{A})$ is captive and ϕ defines an encoding of CA into captive CA. Second, notice that for any support of simulation Y of $\phi(\mathcal{A})$, if there is some $y \in Y$ with $y \notin X$ then, by irreducibility of Y, either there is $y' \in Y$ with $y' \notin X$ and y' contains a $\#$, or $\#$ never appears in Y. In the second

case, $\phi(\mathcal{A})$ always applies a max as local rule and therefore possesses a spreading state when restricted to Y. In the first case, consider the configuration y' and $z \in \mathbb{Z}$ such that positions z and $z+1$ both see a # in their neighbourhood and a local pattern not in X (such a z must exist by choice of y' and definition of X). Then $\phi(\mathcal{A})(y')$ contains the pattern ## which is spreading by definition of $\phi(\mathcal{A})$. In any case we can apply the usual reasoning with lemma 5.1 and lemma 6.1: any CA without spreading state strongly simulated by $\phi(\mathcal{A})$ is also simulated by $\mathcal{A}$. So the encoding ϕ is faithful for universal CA.

$\mathcal{T}_2$ and $\mathcal{T}_3$. For $\mathcal{T}_2$, the encoding is simple: $\phi(\mathcal{A})$ is just $\mathcal{A}$ with an additional state κ which is spreading. The resulting CA $\phi(\mathcal{A})$ is always in $\mathcal{T}_2$ since κ^{2r} is a blocking word (see [19]). Lemma 5.1 is then sufficient to prove that it is an encoding from AC to $\mathcal{T}_3$ which is faithful for universal CA.

For $\mathcal{T}_3$, the proof is even simpler: $\phi(\mathcal{A}) = \mathcal{A} \times \sigma_1 \times \sigma_{-1}$ is always in $\mathcal{T}_3$ since $\sigma_1 \times \sigma_{-1} \in \mathcal{T}_3$ and an equicontinuous point in a Cartesian product induce equicontinuous points for each component. Theorem 5.4 concludes for the faithfulness.

■

These encodings allow to transport some properties of general CA concerning the top of quasi-orders into order structures induced by each family[8].

Corollary 6.1. *Let $\preccurlyeq$ be $\preccurlyeq_i$ or $\preccurlyeq_m$ and let $\mathfrak{F}$ be a family of CA among: CA with 2 states, $\mathcal{T}_2$, $\mathcal{T}_3$, captive CA. Then we have the following properties:*

- *the set of $\preccurlyeq$-universal CA in $\mathfrak{F}$ is not co-r.e.*
- *for any non-universal $\mathcal{A} \in \mathfrak{F}$, there is a non universal $\mathcal{B} \in \mathfrak{F}$ with $\mathcal{A} \preccurlyeq \mathcal{B}$ but $\mathcal{B} \not\preccurlyeq \mathcal{A}$.*

Proof. The first property is a direct corollary of theorem 6.4 and 5.3 by definition of faithful encodings.

For the second property, consider the encoding ϕ established in theorem 6.4 and let $\mathcal{A} \in \mathfrak{F}$ be any non-universal CA. By theorem 5.4, there is some non-universal CA $\mathcal{B}$ such that $\mathcal{A} \preccurlyeq \mathcal{B}$ but $\mathcal{B} \not\preccurlyeq \mathcal{A}$. By faithfulness of ϕ, $\phi(\mathcal{B}) \in \mathfrak{F}$ is not universal and by the definition of encoding it simulates $\mathcal{A}$ without being simulated by $\mathcal{A}$. ■

The families considered above induce structures sharing some properties with the general quasi-orders 'near the top'. However, the complete characterisation of equivalence classes occupied by some CA of these families is more challenging.

Open Problem 9. *What are the quivalence classes of the simulation quasi-orders containing a 2-states CA? a captive CA? a CA from $\mathcal{T}_2$? a CA from $\mathcal{T}_3$?*

[8] A stronger result concerning captive CA appears in [38]: $\preccurlyeq_i$-universality is undecidable even if we restrict to captive CA with a fixed (but sufficiently large) radius.

References

[1] S. Amoroso, Y. Patt, Decision Procedures for Surjectivity and Injectivity of Parallel Maps for Tesselation Structures, Journal of Computer and System Sciences 6 (1972) 448–464.

[2] E. R. Banks, Universality in cellular automata, in: Eleventh Annual Symposium on Switching and Automata Theory, IEEE, Santa Monica, California, 1970.

[3] N. Boccara, J. Nasser, M. Roger, Particlelike structures and their interactions in spatiotemporal patterns generated by one-dimensional deterministic cellular-automaton rules, Phys. Rev. A 44 (2) (1991) 866–875.

[4] G. Cattaneo, E. Formenti, L. Margara, Topological definitions of deterministic chaos: applications to cellular automata dynamics, in: Cellular automata (Saissac, 1996), Kluwer Acad. Publ., Dordrecht, 1999, pp. 213–259.

[5] G. Cattaneo, E. Formenti, L. Margara, J. Mazoyer, A new shift-invariant metric on S^Z inducing a non-trivial topology, in: I. Privara, P. Rusika (eds.), Mathematical Foundations of Computer Science (MFCS'97), vol. 1295 of Lecture Notes in Computer Science, Springer-Verlag, 1997.

[6] K. Čulik, II, J. Pachl, S. Yu, On the limit sets of cellular automata, SIAM Journal on Computing 18 (4) (1989) 831–842.

[7] B. A. Davey, H. A. Priestley, Introduction to Lattices and Order, Cambridge University Press, 2002.

[8] M. Delorme, J. Mazoyer, N. Ollinger, G. Theyssier, Bulking I: an Abstract Theory of Bulking, *unpublished* (2008).

[9] J. Durand-Lose, Intrinsic universality of a 1-dimensional reversible cellular automaton, in: Symposium on Theoretical Aspects of Computer Science, 1997.

[10] R. H. Gilman, Classes of linear automata, Ergodic Theory and Dynamical Systems 7 (1) (1987) 105–118.

[11] E. Goles, A. Maass, S. Martinez, On the limit set of some universal cellular automata, Theoretical computer science 110 (1) (1993) 53.

[12] D. Griffeath, C. Moore, Life without death is **P**-complete, Complex Systems 10 (6) (1996) 437–447.

[13] G. A. Hedlund, Endormorphisms and automorphisms of the shift dynamical system, Mathematical Systems Theory 3 (1969) 320–375.

[14] J. Hopcroft, J. Ullman, Introduction to Automata Theory, Languages, and Computation, Addison Wesley, 1979.

[15] L. P. Hurd, Formal language characterizations of cellular automaton limit sets, Complex Systems 1 (1987) 69–80.

[16] K. Imai, K. Morita, Firing squad synchronization problem in reversible cellular automata, Theoretical Computer Science 165.

[17] N. Israeli, N. Goldenfeld, Computational irreducibility and the predictability of complex physical systems, Phys. Rev. Lett. 92 (7) (2004) 074105.

[18] J. Kari, The Nilpotency Problem of One-dimensional Cellular Automata, SIAM Journal on Computing 21 (1992) 571–586.

[19] P. Kůrka, Languages, equicontinuity and attractors in cellular automata, Ergodic Theory and Dynamical Systems 17 (1997) 417–433.

[20] P. Kůrka, Topological and symbolic dynamics, Socit Mathmatique de France, 2003.

[21] D. Lind, B. Marcus, An introduction to symbolic dynamics and coding, Cambridge University Press, Cambridge, 1995.

[22] J. Mazoyer, I. Rapaport, Additive cellular automata over Z_p and the bottom of $(\mathrm{ca}, \leq)$, in: Mathematical foundations of computer science (Brno, 1998), Springer, Berlin, 1998, pp. 834–843.

[23] J. Mazoyer, I. Rapaport, Global fixed point attractors of circular cellular automata and periodic tilings of the plane: undecidability results, Discrete Mathematics 199 (1999) 103–122.

[24] J. Mazoyer, I. Rapaport, Inducing an order on cellular automata by a grouping operation, Discrete Applied Mathematics 91 (1-3) (1999) 177–196.

[25] J. Mazoyer, V. Terrier, Signals in one-dimensional cellular automata, Theor. Comput. Sci. 217 (1) (1999) 53–80.

[26] E. F. Moore, Machine Models of Self-Reproduction, in: Proceedings of Symposia in Applied Mathematics, vol. 14, American Mathematical Society, 1962, pp. 17–33.

[27] J. Myhill, The Converse of Moore's Garden-of-Eden Theorem, in: Proceedings of the American Mathematical Society, vol. 14, American Mathematical Society, 1963, pp. 658–686.

[28] T. Neary, D. Woods, P-completeness of cellular automaton rule 110, in: ICALP (1), 2006, pp. 132–143.

[29] N. Ollinger, Two-states bilinear intrinsically universal cellular automata, in: Fundamentals of Computation Theory, 13th International Symposium, FCT 2001, vol. 2138, Lecture Notes in Computer Science, 2001, pp. 396–399.

[30] N. Ollinger, Automates cellulaires : structures, Ph.D. thesis, École Normale Supérieure de Lyon (2002).

[31] N. Ollinger, The intrinsic universality problem of one-dimensional cellular automata, in: Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Computer Science, 2003, pp. 632–641.

[32] N. Ollinger, Universalities in cellular automata: a (short) survey, in: B. Durand (ed.), Symposium on Cellular Automata Journées Automates Cellulaires (JAC'08), MCCME Publishing House, Moscow, 2008, pp. 102–118.

[33] M. Sablik, Directional dynamics for cellular automata: a sensitivity to initial condition approach, Theoretical Computer Science 400 (2008) 1–18.

[34] A. R. Smith, III, Simple computation-universal cellular spaces, Journal of the ACM 18 (1971) 339–353.

[35] K. Sutner, Cellular automata and intermediate degrees, Theor. Comput. Sci. 2 (296) (2003) 365–375.

[36] G. Theyssier, Captive cellular automata, in: MFCS, 2004, pp. 427–438.

[37] G. Theyssier, Automates cellulaires : un modèle de complexités, Ph.D. thesis, École Normale Supérieure de Lyon (2005).

[38] G. Theyssier, How common can be universality for cellular automata?, in: STACS, 2005, pp. 121–132.

[39] B. Weiss, Subshifts of finite type and sofic systems, Monatshefte für Mathematik 77 (1973) 462–474.

[40] S. Wolfram, Computation theory of cellular automata, Communications in Mathematical Physics 96 (1) (1984) 15–57.