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## To cite this version:

Monika Dörfler, Bruno Torrésani. Representation of operators by sampling in the time-frequency domain. SAMPTA'09, May 2009, Marseille, France. Special Session on Sampling and Communication. hal-00451424

**HAL Id: hal-00451424**

**https://hal.science/hal-00451424v1**

Submitted on 29 Jan 2010

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# Representation of operators by sampling in the time-frequency domain

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**Abstract:**

Gabor multipliers are well-suited for the approximation of certain time-variant systems. However, this class of systems is rather restricted. To overcome this restriction, multiple Gabor multipliers allowing for more than one synthesis windows are introduced. The influence of the choice of the various parameters involved on approximation quality is studied for both classical and multiple Gabor multipliers.

## 1. Introduction

In a recent paper [1], the authors describe the representation of operators in the time-frequency domain by means of a twisted convolution with the operator's spreading function. Although not suitable for direct discretization, the spreading representation provides a better understanding of certain operators' behavior: it reflects the operator's action in the time-frequency domain. This motivates an approach that uses the spreading representation of time-frequency multipliers [1], in order to optimize the parameters involved. More specifically, in the one-dimensional, continuous-time case, given an operator $H$ with integral kernel $\kappa_H$ and spreading function $\eta_H$:

$$\eta_H(b,\nu) = \int_{-\infty}^{\infty} \kappa_H(t, t-b) e^{-2i\pi\nu t}\, dt,$$

we aim at modeling the operator by its action on the sampled short-time Fourier transform (STFT) or Gabor coefficients, given for any $f \in \mathbf{L}^2(\mathbb{R})$ by

$$\mathscr{V}_g f(mb_0, n\nu_0) = \langle f, g_{mn} \rangle\,, \quad m,n \in \mathbb{Z} \qquad (1)$$

where the $g_{mn} = M_{n\nu_0} T_{mb_0} g$ denote the Gabor atoms associated to $g \in \mathbf{L}^2(\mathbb{R})$ and the lattice constants $b_0, \nu_0 \in \mathbb{R}^+$, see [3][1]. In the case of classical Gabor multipliers, the modification consists of a pure multiplication. Thus, the linear operator applied to the coefficients $\mathscr{V}_g f$ is diagonal, an approach that leads to accurate approximation for so-called underspread operators [5]. The restriction to diagonality may be relaxed in order to achieve better approximation for a wider class of operators at low cost. It also appears, that in certain approximation tasks it is more efficient, e.g. in the sense of sparsity, to use several side diagonals, but a lower redundancy in the Gabor system used.

The aim of this contribution is the description of error estimates for the approximation of operators by generalized Gabor multipliers, based on the operator's spreading function. From this description guidelines for the choice of good parameters for the approximation are deduced and illustrated by various numerical experiments.

## 2. Approximation in the time-frequency domain: the parameters

Throughout this paper, $\mathcal{H}$ denotes a (finite or infinite-dimensional) Hilbert space, equipped with an action of the Heisenberg group of time-frequency shifts.

### 2.1 Time-frequency multipliers

Let $\mathscr{V}_g^*$ denote the adjoint of $\mathscr{V}_g$. A Gabor multiplier [4] is defined as

$$\mathbb{M} : f \in \mathcal{H} \longmapsto \mathbb{M}f = \mathscr{V}_2^*(\mathbf{m} \cdot \mathscr{V}_1 f).$$

Here, $\mathbf{m}$ is the pointwise multiplication operator whose symbol, defined on the lattice $\Lambda$ will also be denoted by $\mathbf{m}$. We shall denote by $\Lambda^o$ the adjoint lattice, $\Box^o$ its fundamental domain, and $\Pi^o$ the corresponding periodization operator. In the infinite-dimensional situation $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$, and for a product lattice of the form $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$, we have $\Lambda^o = t_0\mathbb{Z} \times \xi_0\mathbb{Z}$ with $t_0 = 1/\nu_0$, $\xi_0 = 1/b_0$, and $\Pi^o f(\zeta) = \sum_{\lambda^o \in \Lambda^o} f(\zeta + \lambda^o)$, $\zeta \in \Box^o$. In a finite-dimensional setting $\mathcal{H} = \mathbb{C}^L$, with $\Lambda = \mathbb{Z}_{N_b} \times \mathbb{Z}_{N_\nu}$, with $N_b, N_\nu$ two divisors of $L$, we have $\Lambda^o = \mathbb{Z}_{N_\nu} \times \mathbb{Z}_{N_b}$, and the obvious form for the periodization operator.

In the definition of the multipliers, several parameters have to be fixed: the analysis and synthesis windows $g$ and $h$, the lattice $\Lambda$, and the symbol $\mathbf{m}$. For practical as well as theoretical reasons, the windows should be well-localized in time and frequency. As for the lattice, it is expected that denser lattices will lead to better results in approximation, but higher computational cost. However, it will be seen that too dense lattices are not suitable.

Finally, the symbol $\mathbf{m}$ can be optimized to best approximate a given operator. In [1], an explicit expression for the best approximation was obtainned in the spreading domain, yielding a very efficient algorithm (compare [2]).

[1] The finite dimensional case $\mathcal{H} = \mathbb{C}^L$ is obtained similarly, replacing integrals with finite sums, and letting $m = 0, \ldots N_b - 1$, $n = 0, \ldots N_\nu - 1$, where $N_b = L/b_0$, $N_\nu = L/\nu_0$ and $b_0, \nu_0$ divide $L$.

The spreading function of Gabor multipliers takes the form $\eta_{\mathbb{M}}(\zeta) = \mathscr{M}(\zeta) \cdot \mathscr{V}_g h(\zeta)$ , where $\mathscr{M}$ is the symplectic Fourier transform of $\mathbf{m}$. Note, that this leads to a periodic function with period $\square^o$. Hence, good approximation by a classical Gabor multiplier is possible, if the essential support of the spreading function is smaller than 1 and can then be contained in the fundamental domain $\square^o$ of the adjoint lattice for a dense enough lattice $\Lambda$. Also, to reduce aliasing as much as possible, the analysis and synthesis windows must be chosen such that $\mathscr{V}_g h$ is small outside $\square^o$ and positive on the support of the spreading function, also see Section 4.1.

### 2.2 Generalized Gabor multipliers

Multiple Gabor multipliers are sums of Gabor multipliers with different synthesis windows.

**Definition 1 (Multiple Gabor Multiplier)** *Let $g, h \in \mathcal{H}$ denote two window functions. Let $\Lambda$ be a time-frequency lattice. Let $\{\mu_j, j \in J\}$ denote a finite set of time-frequency shifts, and let $\{\mathbf{m}_j, j \in J\}$ be a family of bounded functions on $\Lambda$. Set $h^{(j)} = \pi(\mu_j)h$, then the associated generalized Gabor multiplier $\mathbb{M}$ is defined, for $f \in \mathcal{H}$, as*

$$\mathbb{M}f = \sum_{\lambda\in\Lambda}\sum_{j\in J} \mathbf{m}(\lambda, \mu_j)\langle f, \pi(\lambda)g\rangle \pi(\lambda)h^{(j)} \ .$$

It is immediately obvious that in addition to the parameters mentioned above, the window $h$ as well as the sampling points $J$ must be chosen.

## 3. Error analysis in $\mathbf{L}^2(\mathbb{R})$

In [1], it was shown that the symbol $\mathbf{m}(\lambda, \mu_j) := \mathbf{m}_j(\lambda)$ of the best approximation of a Hilbert-Schmidt operator by a multiple Gabor multiplier with fixed sets $\Lambda$, $J$ and windows, is given by the symplectic Fourier transform of the $\square^o$-periodic functions $\mathscr{M}_j$ obtained via the vector equation

$$\mathscr{M}(\zeta) = \mathcal{U}(\zeta)^{-1} \cdot \mathcal{B}(\zeta) \ , \quad \zeta \in \square^o \ , \tag{2}$$

where the matrix and vector valued functions $\mathcal{U}$ and $\mathcal{B}$ are given by the $\Lambda^o$-periodizations

$$\mathcal{U}_{jj'} = \Pi^o\Big(\mathscr{V}_g h^{(j')}\overline{\mathscr{V}_g h^{(j)}}\Big) \ , \ \mathcal{B}_j = \Pi^o\Big(\eta_H \overline{\mathscr{V}_g h^{(j)}}\Big) \ ,$$

provided $\mathcal{U}$ is invertible a.e.
The case of one synthesis windows may be immediately obtained from the above formula. Note that formula (2) allows for an efficient implementation of the otherwise expensive calculation of the best approximation by multiple Gabor multipliers.
We may now give an expression for the error in the approximation given above, in the case $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$

**Proposition 1** *Let $\mathscr{M}$ denote the vector-valued function obtained as in* (2) *and set, for the Hilbert-Schmidt operator $H$, $\Gamma_H = \Pi^o(|\eta_H|^2)$. Then the approximation error $E = \|\eta_H - \sum_j \mathscr{M}_j \mathscr{V}_j\|^2$ is given by*

$$E = \int_{\square^o} |\Gamma_H(\zeta)| \Big(1 - \frac{\sum_{i,j}(\mathcal{U}^{-1})_{ij}(\zeta)\mathcal{B}_i(\zeta)\overline{\mathcal{B}_j}(\zeta)}{|\Gamma_H(\zeta)|}\Big) d\zeta$$

Notice that this covers the multiplier case obtained in [1]. Notice also that this immediately yields

$$E \le \|\eta_H\|^2 \left\| 1 - \frac{\sum_{i,j}(\mathcal{U}^{-1})_{ij}\mathcal{B}_i\overline{\mathcal{B}_j}}{|\Gamma_H|} \right\|_\infty$$

The finite-dimensional situation is similar, replacing the integral over $\square^o$ with a finite sum over the finite fundamental domain $\{0, \dots t_0 - 1\} \times \{0, \dots \xi_0 - 1\}$.

## 4. Choosing the parameters

For simlicity, we specialize the following discussion to the infinite-dimensional case $\mathcal{H} = \mathbf{L}^2(\mathbb{R})$, and rectangular lattice $\Lambda = b_0\mathbb{Z} \times \nu_0\mathbb{Z}$. The finite-dimensional situation is handled similarly.

### 4.1 Gabor Multipliers

If an operator with known spreading function is to be approximated by a Gabor multiplier, the lattice may be adapted to the eccentricity of the spreading function according to the error expression obtained in Proposition 1, which may be considerably simplified for the case of only one synthesis window, see [1]. In order to choose the eccentricity of the lattice accordingly and adapt the window to the chosen lattice as to avoid aliasing, assume, that we may find $b_0, \nu_0$, with $b_0 \cdot \nu_0 < 1$, such that $\mathrm{supp}(\eta_H) \subseteq T_z\square^o$, where $\square^o = [0, \frac{1}{\nu_0}] \times [0, \frac{1}{b_0}]$. In this case, the error resulting from best approximation by a Gabor multiplier with respect to the lattice $b_0\mathbb{Z} \times \nu_0\mathbb{Z}$ is bounded by $C_e \cdot \|\eta_H\|_2^2$, with

$$C_e = 1 - \inf_{t,\xi\in\square_H^o} \frac{|\mathscr{V}_g h(t,\xi)|^2}{\sum_{k,l}|\mathscr{V}_g h(t + kt_0, \xi + l\xi_0)|^2}, \tag{3}$$

with $\square_H^o = \square^o \cap \mathrm{Supp}(\eta_H)$, and becomes minimal for a window that is optimally concentrated inside $\square^o$. Heuristically as well as from numerical experiments we know, that the tight window, [3], corresponding to the given lattice is usually a good choice to fulfill this requirement.

### 4.2 Generalized Gabor Multipliers

The main additional task in the generalized situation is the choice of the sampling points $\mu_j$ for the synthesis windows. A good choice will again be guided by the behavior of the spreading function. The relevant areas in the spreading domain should be covered as well as possible with the smallest possible overlap by the cross-ambiguity functions of the different synthesis windows with respect to a given reference-window localized at $(0, 0)$ e.g. the Gaussian window. Motivated by the results from the Gabor multiplier situation, we choose a tight window with respect to the analysis lattice and look for the most appropriate sampling points for the synthesis windows. Examples will be given in Section 5.2.

## 5. Examples

We now turn to numerical experiments, in the finite case $\mathcal{H} = \mathbb{C}^L$. In the following examples, the relative approximation error for the best approximation $\tilde{H}$ of $H$ is given

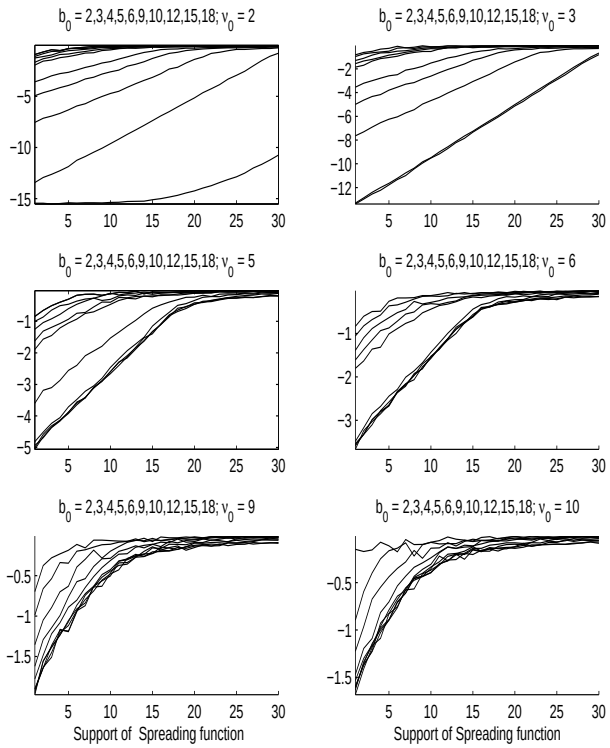


Figure 1: *Approximation error for different bandwidth of spreading function and different values of* $b_0, \nu_0$.

by

$$E = \|\tilde{H} - H\| / \|H\| \ ,$$

the logarithm of which is represented in the next plots. We display here the Fröbenius norm, the plots obtained with the operator norm are almost identical.

## 5.1 Classical Gabor Multipliers

We generate operators with compact support in the spreading domain, in a square of side size between $3$ and $61$, symmetric about $0$. The values are random, the signal length is $L = 180$. We then investigate the approximation quality for various pairs of lattice constants, with $b_0$ varying between $2$ and $18$ and $\nu_0$ between $2$ and $10$. The results are presented in Figure 1. Note the two distinct regimes: the error grows exponentially up to a certain value of the support size, depending on the lattice density, and slower thereafter. A possible explanation for this effect, to be further investigated, is the fact, that the error (see the bound in (3)) is comprised of an aliasing error and the inherent inaccuracy of Gabor multiplier approximation, even for very high sampling density, of overspread operators.

In order to emphasize the importance of lattice adaptation to eccentricity, we show the results for different lattice constants resulting in the same redundancy $(5)$ in Figure 2. The solid lines show the results for $b_0 = \nu_0 = 6$, leading to far better results than the lattice constants not adapted to the (symmetric) support of the spreading function.

## 5.2 Generalized Gabor Multipliers

In order to illustrate the influence of additional synthesis windows on the approximation quality, we first consider

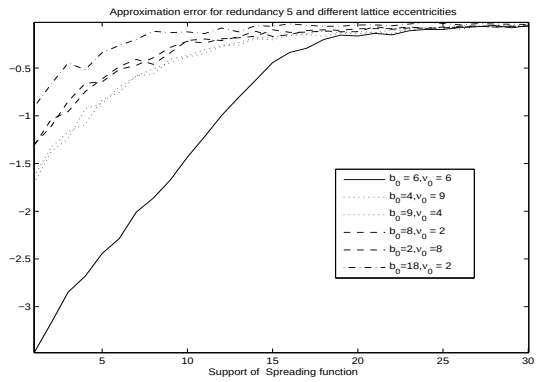


Figure 2: *Approximation error for different lattice-eccentricity* .

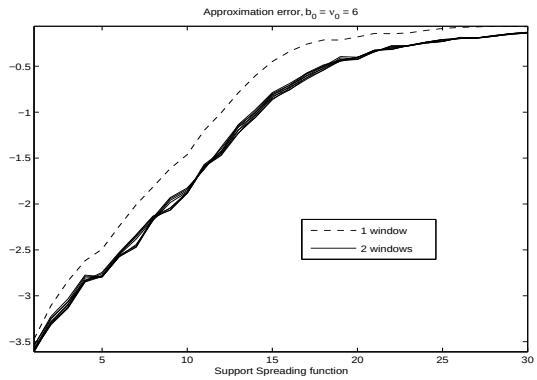


Figure 3: *Spreading function of operator and best approximation with one or two synthesis windows, approximation error for growing support of spreading function.*

the same operators as in the previous section, but allow for one additional synthesis window. Here, and in the subsequent examples, one window will always be a window centered about $0$, as above, with a time-shifted version of the original window as additional window. Hence, only the shift-parameter of the additional window has to be considered. Figure 3 shows the improvement in approximation quality for shift-parameters of the additional window between $-5$ and $5$ (solid), as opposed to the single window approximation.

Next, we investigate the following situation: an operator with two effectively disjoint components in the spreading domain is, again, approximated by a multiple Gabor multiplier with 2 synthesis windows. For better comparison, the two components are the component from the previous examples plus a shifted version (by $90$ samples) thereof. Figure 4 shows the spreading functions of one of the operators and its best approximation with two synthesis windows, for the optimal additional window. Note the aliasing effect. In this situation, using two appropriate synthesis windows, the obtained results are similar to those in the case of one spreading function component and one synthesis window, as discussed in the previous section. In Figure 5, we display the results for $3$ symmetric pairs of lattice constants, the optimal window's result being represented by the solid line, while the dashed lines show the results of close but suboptimal synthesis windows. As the operator was generated by a translation by $90$ samples, the

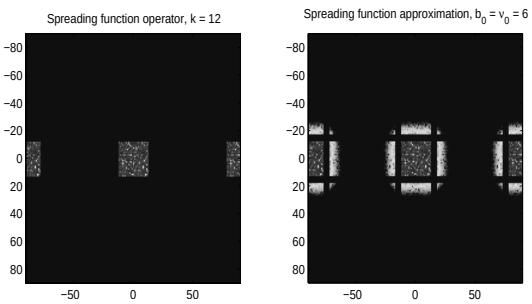


Figure 4: *Spreading function of operator and best approximation.*

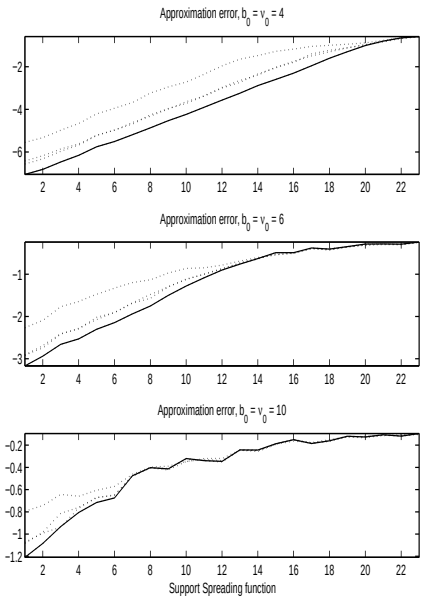


Figure 5: *Approximation error for varying support of two components of spreading function and two synthesis windows.*

tight window, shifted by $90$ samples itself, is expected to be the optimal additional window. This is confirmed by the experiments.

In a last experiment, the two components in the spreading domain are close and, for growing bandwidth, overlapping. Figure 6 shows, as before, the results of approximation for growing support of both spreading function components, with $b_0 = \nu_0 = 6$ and various additional synthesis windows. The additional window with shift-parameter $0$ is, of course, the original window and yields the approximation result obtained for a single synthesis window. For the optimal window, the result is close to the single window/single component case for the same lattice.

## 6. Discussion and conclusions

The examples given in the previous section show that the choice of various parameters has considerable influence on the performance of approximation by (generalized) Gabor multipliers. While the situation is rather easily understood in the case of classical Gabor multipliers, it is much more intricate in the generalized case. It should be noted that, while yielding better results in the approximation, using a small number of additional synthesis windows does not dramatically increase the computational cost: in (2),

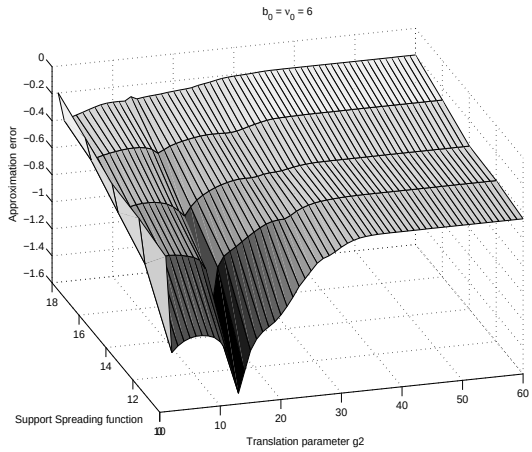


Figure 6: *Approximation error for growing support of spreading function and various additional synthesis windows.*

going from $|J| = 1$ to larger index sets $J$ involves inverting (generally small) matrices instead of computing a point-wise ratio. Higher redundancy of the Gabor system involved is more expensive in the sense of coefficients. In many cases, using an additional window may be more favorable in improving approximation quality than a denser lattice. Future work on this topic will include systematic numerical experiments as well as the analytical investigation of the approximation quality of generalized and classical Gabor multipliers. Another goal is the development of a method to determine an adapted sampling scheme for the synthesis windows from an operator's spreading function.

## 7. Acknowledgments

The first author was funded by project MA07-025 of WWTF Austria. The second author was partly supported by the CNRS programme PEPS/ST2I *MTF&Sons*.

## References:

[1] Monika Dörfler and Bruno Torrésani. Representation of operators in the time-frequency domain and generalized Gabor multipliers. *arXiv:0809.2698*, 2008, to appear in *Journal of Fourier Anal. and Appl.*

[2] Hans G. Feichtinger, Mario Hampejs, and Günther Kracher. Approximation of matrices by Gabor multipliers. IEEE Signal Proc. Letters, 11(11):883– 886, 2004.

[3] Hans G. Feichtinger and Thomas Strohmer. *Gabor Analysis and Algorithms. Theory and Applications.* Birkhäuser, 1998.

[4] Hans Georg Feichtinger and Kristof Nowak. A first survey of Gabor multipliers. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Boston, 2002. Birkhauser.

[5] Werner Kozek. Adaptation of Weyl-Heisenberg frames to underspread environments. In [3], 1998.