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Computing Chebyshev knots diagrams

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ABSTRACT

A Chebyshev curve $\mathcal{C}(a, b, c, \varphi)$ has a parametrization of the form $x(t) = T_a(t)$; $y(t) = T_b(t)$; $z(t) = T_c(t + \varphi)$, where a, b, c are integers, $T_n(t)$ is the Chebyshev polynomial of degree n and $\varphi \in \mathbf{R}$. When $\mathcal{C}(a, b, c, \varphi)$ has no double points, it defines a polynomial knot. We determine all possible knots when a, b and c are given.

Keywords

Zero dimensional systems, Chebyshev curves, Lissajous knots, polynomial knots, factorization of Chebyshev polynomials, minimal polynomial

1. INTRODUCTION

It is known that every knot may be obtained from a polynomial embedding $\mathbf{R} \rightarrow \mathbf{R}^3 \subset \mathbb{S}^3$, where \mathbb{S}^3 is the one-point compactification of \mathbf{R}^3 ([19, 5]).

In [10] we showed that any knot $K \subset \mathbf{R}^3 \subset \mathbb{S}^3$ is a Chebyshev knot, that is to say there exist positive integers a, b, c and a real φ such K is isotopic to the curve

$$\mathcal{C}(a, b, c, \varphi) : x = T_a(t), y = T_b(t), z = T_c(t + \varphi)$$

where T_n is the Chebyshev polynomial of degree n . This is our motivation for the study of curves $\mathcal{C}(a, b, c, \varphi)$, $\varphi \in \mathbf{R}$.

- The projection of $\mathcal{C}(a, b, c, \varphi)$ on the (x, y) -plane is the Chebyshev plane curve $\mathcal{C}(a, b) : x = T_a(t), y = T_b(t)$.
- The set $\mathcal{Z}_{a,b,c}$ of *critical values* φ such that $\mathcal{C}(a, b, c, \varphi)$ is singular, is finite and $|\mathcal{Z}_{a,b,c}| \leq \frac{1}{2}(a-1)(b-1)(c-1)$.
- Let $\mathcal{Z}_{a,b,c} = \{\varphi_1, \dots, \varphi_n\}$. The knot $\mathcal{C}(a, b, c, \varphi)$ is constant when $\varphi \in (\varphi_i, \varphi_{i+1})$.
- The type of $\mathcal{C}(a, b, c, \varphi)$ is given by its diagram, which is the plane projection $\mathcal{C}(a, b)$, and the (under/over) nature of the crossings over the double points.

From the algorithmic point of view, the description of the knots is strongly connected to the resolution of the zero-dimensional system:

$$\mathcal{V}_{a,b,c} = \left\{ (s, t, \varphi); \frac{T_a(t) - T_a(s)}{t-s} = 0, \frac{T_b(t) - T_b(s)}{t-s} = 0, \frac{T_c(t+\varphi) - T_c(s+\varphi)}{t-s} = 0 \right\} \quad (*)$$

Here the set of critical values $\mathcal{Z}_{a,b,c}$ is $\pi_\varphi(\mathcal{V}_{a,b,c}) = Z(R_{a,b,c})$, where $R_{a,b,c}$ is some polynomial in $\mathbf{Q}[\varphi]$ we have to determine. When $\varphi \in \mathbf{R} - \mathcal{Z}_{a,b,c}$, the knot diagram of $\mathcal{C}(a, b, c, \varphi)$ is deduced from the signs of $T_c(t + \varphi) - T_c(s + \varphi)$ when the parameters (t, s) correspond to a double point of $\mathcal{C}(a, b)$.

When $a \leq 4$, the knots are rational (or two-bridge) knots. They are classified by their Schubert fraction that can easily be deduced from their diagram ([15]). When $a \geq 5$, we also get diagrams that can be used for the computations of invariants. But the identification of the knot is a much more difficult problem.

In [12], we proposed a method to get $R_{a,b,c}$, based on resultant computations in $\mathbf{Z}[X, \varphi]$, where X is some indeterminate, for $a = 3$ or $a = 4$. We have classified all knots for a and b coprime integers ($a \leq 4, b \leq 15$) and $c \leq 300$. This method will not work for $a \geq 5$ or higher values of b and c because of the degrees of the system. We will give a direct way to determine a set r_0, \dots, r_n of rational numbers such that $\mathcal{C}(a, b, c, \varphi)$ is one of the $\mathcal{C}(a, b, c, r_i)$.

In the present paper we decompose the system by working on some (real cyclotomic) extension fields. We show that the system (*) is equivalent to the resolution of $\frac{1}{2}(a-1)(b-1)\lfloor \frac{c}{2} \rfloor$ second-degree polynomials in $\mathbf{Z}[\cos \frac{\pi}{a}, \cos \frac{\pi}{b}, \cos \frac{\pi}{c}]$. We show some properties of these extensions that allow to simplify the computations. We can represent the coefficients of the polynomials by intervals and certify the resolution. The main difficulty becomes here to compare the roots of the polynomials we obtain independently. More precisely

- We give an algorithm for the factorization of the Chebyshev polynomials T_n and V_n .
- We show that affine Chebyshev curves $T_n(x) - T_m(y) = 0$ has $\lfloor \frac{(n,m)}{2} \rfloor + 1$ irreducible components.
- This allows us:
 - To express $R_{a,b,c}$ as the product of second-degree polynomials in $\mathbf{Z}[\cos \frac{\pi}{a}, \cos \frac{\pi}{b}, \cos \frac{\pi}{c}][\varphi]$.
 - To get easily the minimal polynomials of $\cos \frac{\pi}{n}$.

We then easily and independently obtain the roots of the second-order polynomials but we need to make sure that they are distinct roots. A formal method would consist to compute their minimal polynomials over \mathbf{Q} , which is equivalent to the resolution of (*). We use interval arithmetic for coding the algebraic numbers $\cos \frac{k\pi}{n}$ as well as the solutions φ we get. If the two intervals are disjoint, the roots are

distinct. If not, we can certify whether the resultant of the two second-order polynomials equals 0 or not by Euclidean division.

In section 2, we first describe the Chebyshev polynomials and the link between their factorizations and the minimal polynomials of $\cos \frac{k\pi}{n}$. In section 3 we study the affine implicit Chebyshev curves $T_n(x) = T_m(y)$ and show that they have $\lfloor \frac{(n,m)}{2} \rfloor + 1$ irreducible components. $\lfloor \frac{(n,m)-1}{2} \rfloor$ are Lissajous curves. Lissajous curves admit a parametrization: $x = \cos at$, $y = \cos(bt + \varphi)$, where $\sin \varphi \neq 0$.

In section 4, we give a formula for $R_{a,b,c}$ as the product of second-degree polynomials $P_{\alpha,\beta,\gamma}$. We show how to get the roots of $R_{a,b,c}$ with their multiplicities in order to choose a rational number r in each component of $\mathbf{R} - \mathcal{Z}_{a,b,c}$. In section 6, we explain how to deduce the diagrams of the knots $\mathcal{C}(a, b, c, \varphi)$. This allows us to find an exhaustive and complete list of the minimal parametrization for the first 95 rational knots.

2. CHEBYSHEV POLYNOMIALS

Chebyshev polynomials and their algebraic properties play a central role here. The curves we will study are defined by Chebyshev polynomials. The algebraic extensions we will consider are spanned by their roots and we need to know their factors. In this section we recall some classical properties of Chebyshev polynomials. We will show also the link between their effective factorization in $\mathbf{Q}[t]$ and the minimal polynomial of $\cos \frac{k\pi}{n}$.

The Chebyshev polynomials of the *first kind* are defined by the second-order linear recurrence

$$T_0 = 1, T_1 = t, T_{n+1} = 2tT_n - T_{n-1}. \quad (1)$$

$T_n \in \mathbf{Z}[t]$ and satisfies the identity $T_n(\cos \theta) = \cos n\theta$, and more generally $T_n \circ T_m = T_{nm}$. We have the decomposition

$$T_n = 2^{n-1} \prod_{k=0}^{n-1} (t - \cos \frac{(2k+1)\pi}{2n}).$$

Let V_n be the Chebyshev polynomials of the *second kind* defined by the second-order linear recurrence (the same as in (1))

$$V_0 = 0, V_1 = 1, V_{n+1} = 2tV_n - V_{n-1}.$$

$V_n \in \mathbf{Z}[t]$ and satisfies $V_n(\cos \theta) = \frac{\sin n\theta}{\sin \theta}$. We have easily

$$V_n = 2^{n-1} \prod_{k=1}^{n-1} (t - \cos \frac{k\pi}{n}).$$

Let us summarize some useful results in the following

LEMMA 1. *We have the following properties:*

- $T'_n(t) = 0 \Rightarrow T_n(t) = \pm 1$
- $T_n(t) = \pm 1 \Rightarrow T'_n(t) = 0$ or $t = \pm 1$.
- $T_n(t) = y$ has one real solution iff $|y| > 1$.
- $T_n(t) = y$ has n real solutions iff $|y| < 1$.
- $T_n(t) = 1$ has $\lfloor \frac{n}{2} \rfloor$ real solutions.
- $T_n(t) = -1$ has $\lfloor \frac{n-1}{2} \rfloor$ real solutions.

PROOF. From $T'_n = nV_n$, we deduce that $t \mapsto T_n(t)$ is monotonic when $|t| \geq \cos \frac{\pi}{n}$, that T_n has $n-1$ local extrema for $t_k = \cos \frac{k\pi}{n}$ and $T_n(t_k) = (-1)^k$. \square

Minimal Polynomial of $\cos \frac{k\pi}{n}$

Let us define $\psi_1 = 1$, $\psi_n = 2^{\varphi(n)} \prod_{(k,n)=1} (t - \cos \frac{k\pi}{n})$, where φ is the Euler function. We have

$$V_n = \prod_{d|n} \psi_d, \quad (2)$$

and we deduce by induction that $\psi_d \in \mathbf{Z}[t]$. As $\psi_2 = 2t$ and $V_n(-t) = (-1)^{n-1} V_n(t)$, we deduce that ψ_d is an even polynomial for $d \geq 2$.

These relations were already known and also their relations with the minimal polynomial of $\cos \frac{2\pi}{n}$ (see [20, 7, 17]). Note that the Relation (2) gives an effective way to compute the factors ψ_n (see algorithm 4). We will see that we deduce an effective way to compute the minimal polynomial of $\cos \frac{\pi}{n}$.

Let $\zeta_n = e^{\frac{2i\pi}{n}}$. We know that the minimal polynomial of ζ_n over \mathbf{Q} is of degree $\varphi(n)$. Let k be coprime with n . From $\zeta_n^{2k} - 2 \cos \frac{2k\pi}{n} \zeta_n^k + 1 = 0$ we deduce that $[\mathbf{Q}(\zeta_n^k) : \mathbf{Q}(\cos \frac{2k\pi}{n})] = 2$ because $\zeta_n^k \notin \mathbf{R}$ when $n > 1$. We have $[\mathbf{Q}(\cos \frac{2k\pi}{n}) : \mathbf{Q}] = \frac{1}{2} \varphi(n)$, when $n > 1$. From $\cos \frac{k\pi}{n} = \cos \frac{2k\pi}{2n}$ we deduce that $[\mathbf{Q}(\cos \frac{k\pi}{n}) : \mathbf{Q}] = \frac{1}{2} \varphi(2n)$ when $(k, 2n) = 1$ and $n > 1$. Consequently, the minimal polynomial M_n of $\cos \frac{\pi}{n}$ has degree $\frac{1}{2} \varphi(2n)$ when $n > 1$.

COROLLARY 2. *We have $M_{2n+1} = \psi_{2n+1}(\sqrt{\frac{1}{2} - \frac{1}{2}t})$, $M_{2n} = \psi_{2n}$, and $\psi_{2n+1} = M_{2n+1}(t)M_{2n+1}(-t)$.*

PROOF. $M_n | \psi_n$, $\deg \psi_{2n} = \deg M_{2n}$, and $\deg \psi_{2n+1} = 2 \deg M_{2n+1}$. We deduce that $\psi_{2n} = M_{2n}$ and $\psi_{2n+1} = M_{2n+1}N_{2n+1}$ where $\deg N_{2n+1} = \deg M_{2n+1}$.

Let $P(t) = \psi_{2n+1}(\sqrt{\frac{1}{2}(1+t)}) \in \mathbf{Q}[t]$. We have $P(\cos \frac{2k\pi}{2n+1}) = \psi_{2n+1}(\pm \cos \frac{k\pi}{2n+1}) = 0$ when $(k, 2n+1) = 0$. We thus deduce that

$$P(t) = 2^{\varphi(2n+1)/2} \prod_{(k,2n+1)=1} (t - \cos \frac{2k\pi}{2n+1}).$$

As $\cos \frac{\pi}{2n+1} = -\cos \frac{2n\pi}{2n+1}$, we deduce that

$$M_{2n+1}(t) = P(-t) = \psi_{2n+1}(\sqrt{\frac{1}{2}(1-t)}).$$

We thus have $\psi_{2n+1}(t) = M_{2n+1}(t)M_{2n+1}(-t)$. \square

Remark. The roots of M_n are $\cos \frac{k\pi}{n}$ where $(k, n) = 1$ and k is odd. $M_n(-t)$ is the minimal polynomial of $\cos \frac{2\pi}{n}$. The leading coefficient of M_n is $2^{\varphi(2n)/2}$.

Remark. $\cos \frac{k\pi}{n} \in \mathbf{Q}$ iff $\frac{1}{2} \varphi(2n) = 1$, that is $n = 1, 2, 3$. In this case we get $2 \cos \frac{k\pi}{n} \in \mathbf{Z}$.

We thus deduce the factorization of T_n and V_n in irreducible factors in $\mathbf{Z}[t]$.

PROPOSITION 3. **Factorization of T_n and V_n .**
We have

$$V_n(t) = \prod_{2d|n} M_{2d}(t) \prod_{2d+1|n} M_{2d+1}(t)M_{2d+1}(-t),$$

$$T_{2^k(2m+1)}(t) = \frac{1}{2} \prod_{d|2m+1} M_{2^{k+1}d}(t).$$

where M_n is the minimal polynomial of $\cos \frac{\pi}{n}$.

PROOF. The factorization of V_n is already proved. Let d be an odd divisor of n . We write $n = 2^k \cdot d_1 \cdot d$, where d_1 is odd. $\cos \frac{d_1 \pi}{2n} = \cos \frac{\pi}{2^{k+1}d}$ is a root of T_n so $M_{2^{k+1}d} | T_n$. Using the Euler formula and the fact that $\varphi(2^{k+1}d) = 2^k \varphi(d)$, we deduce the factorization. \square

ALGORITHM 4. **Minimal polynomial of $\cos \frac{\pi}{n}$.**
The minimal polynomial M_n of $\cos \frac{\pi}{n}$ can be computed recursively by Euclidean division.

PROOF. It is given by the following

Algorithm MINIMAL POLYNOMIAL	
Input:	n
Output:	M_n
Let $\psi_1 = 1$.	
Suppose we know $\psi_1, \dots, \psi_{n-1}$.	
Let $\psi_n = V_n \div \prod_{d n, d \neq n} \psi_d$.	
If n is even then $M_n = \psi_n$ else	
$M_n = \psi_n(\sqrt{\frac{1}{2}(1-t)})$.	

Note that ψ_n is odd when $n \geq 2$, so $\psi_n(\sqrt{\frac{1}{2}(1-t)})$ can be evaluated only by $t^2 \mapsto t, t \mapsto \frac{t}{2}, t \mapsto 1-t$. \square

3. CHEBYSHEV AND LISSAJOUS CURVES

The following proposition will explain the notions of Lissajous and Chebyshev curves.

PROPOSITION 5. *The parametric curve*

$$\mathcal{C} : x = \cos(at), y = \cos(bt + \varphi), t \in \mathbf{C},$$

where a, b are coprime integers (a odd) and $\varphi \in \mathbf{R}$ admits the equation

$$T_b(x)^2 + T_a(y)^2 - 2 \cos(a\varphi) T_b(x) T_a(y) - \sin^2(a\varphi) = 0. \quad (3)$$

1. If $a\varphi \neq k\pi$, this equation is irreducible. \mathcal{C} is called a Lissajous curve. Its real part is 1-1 parametrized for $t \in [0, 2\pi]$.
2. If $a\varphi = k\pi$, this equation is equivalent to $T_b(x) = (-1)^k T_a(y)$. \mathcal{C} is called a Chebyshev curve. It can be parametrized by $x = T_a(t), y = (-1)^k T_b(t)$.

PROOF. Let $(x, y) \in \mathcal{C}$. We have $T_b(x) = \cos(bat), T_a(y) = \cos(bat + a\varphi)$. Let $\lambda = a\varphi, \theta = abt$. We get $T_a(y) = \cos(\theta + \lambda)$ so $(1 - \cos^2 \theta) \sin^2 \lambda = (\cos \theta \cos \lambda - T_a(y))^2$, that is $(1 - T_a(x))^2 \sin^2 \lambda = (T_a(x) \cos \lambda - T_a(y))^2$, and we deduce our Equation (3).

Conversely, suppose that (x, y) satisfies (3). Let $x = \cos(at)$ where $t \in \mathbf{C}$. We also have $x = \cos a(t + \frac{2k\pi}{a})$. We have $T_b(x) = \cos \theta$. $A = T_a(y)$ is solution of the second-degree equation

$$A^2 - 2 \cos(a\varphi) \cos \theta A - \sin^2(a\varphi) = 0.$$

Consequently, we get $T_a(y) = \cos(\theta \pm a\varphi) = T_a(\cos(\pm bt + \varphi))$. We deduce that $y = \cos(\pm bt + \varphi + \frac{2h\pi}{a}), h \in \mathbf{Z}$. Changing t by $-t$, we can suppose that

$$x = \cos at, y = \cos(bt + \varphi + \frac{2h\pi}{a}).$$

By choosing k such that $kb + h \equiv 0 \pmod{a}$, we get $x = \cos at', y = \cos(bt' + \varphi)$, where $t' = t + \frac{2k\pi}{a}$.

If $a\varphi \not\equiv 0 \pmod{\pi}$. Suppose that Equation (3) factors in $P(x, y)Q(x, y)$. We can suppose, for analyticity reasons, that $P(\cos at, \cos bt + \varphi) = 0$, for $t \in \mathbf{C}$. The curve \mathcal{C} intersects the line $y = 0$ in $2b$ distinct points so $\deg_x P \geq 2b$. Similarly, $\deg_y P \geq 2a$ so that Q is a constant which proves that the equation is irreducible.

If $\cos a\varphi = (-1)^k$, the equation becomes $T_b(x) - (-1)^k T_a(y) = 0$. In this case the curve admits the announced parametrization (see [6, 10] for more details). \square

Remark. If $a = b = 1$, we obtain the Lissajous ellipses. They are the first curves studied by Lissajous ([14]). Let $\mu \not\equiv 0 \pmod{\pi}$. The curve

$$\mathcal{E}_\mu : x^2 + y^2 - 2 \cos(\mu)xy - \sin^2(\mu) = 0$$

is an ellipse inscribed in the square $[-1, 1]^2$. It admits the parametrization $x = \cos t, y = \cos(t + \mu)$.

The following notation will be useful. Let $E_\mu(x, y) = x^2 + y^2 - 2 \cos(\mu)xy - \sin^2(\mu)$ when $\mu \not\equiv 0 \pmod{\pi}$ and $E_0 = x - y, E_\pi = x + y$. The Equation (3) is equivalent to $E_{a\varphi}(T_b(x), T_a(y)) = 0$. That shows that the real part of the Equation (3) is inscribed in the square $[-1, 1]^2$. Using Proposition 5 we recover the classical following result.

COROLLARY 6. *The Lissajous curve $x = \cos(at), y = \cos(bt + \varphi), (a\varphi \not\equiv 0 \pmod{\pi})$ has $2ab - a - b$ singular points which are real double points.*

PROOF. singular points of \mathcal{C} satisfy Equation (3) and the system

$$\begin{aligned} T'_b(x)(T_b(x) - T_a(y) \cos a\varphi) &= 0, \\ T'_a(y)(T_a(y) - T_b(x) \cos a\varphi) &= 0. \end{aligned}$$

Suppose that $T'_b(x) = T'_a(y) = 0$ then $T_a^2(y) = T_b^2(x) = 1$ and Equation 3 is not satisfied. Suppose that $T_b(x) - T_a(y) \cos a\varphi = T_a(y) - T_b(x) \cos a\varphi = 0$, then $T_b(x) = T_a(x) = 0$ and Equation 3 is not satisfied. We thus have either $T'_b(x) = 0$ and $T_a(y) - T_b(x) \cos a\varphi = 0$ that gives $(b-1) \times a$ real points because of the classical properties of Chebyshev polynomials, or $T'_a(y) = 0$ and $T_b(x) - T_a(y) \cos a\varphi = 0$ that gives $b \times (a-1)$ real double points. \square

Remark. The study of the double points of Lissajous curves is classical (see [2] for their parameters values). The study of the double points of Chebyshev curves is simpler (see [10]).

COROLLARY 7. *The affine implicit curve $T_n(x) = T_m(y)$ has $\lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor \lfloor \frac{m}{2} \rfloor$ singular points that are real double points.*

PROOF. The singular points satisfy either $T_n(x) = T_m(y) = 1$ or $T_n(x) = T_m(y) = -1$ and the announced results. \square

THEOREM 8. Factorization of $T_n(x) - T_m(y)$
Let $m = ad, n = bd, (a, b) = 1$ and a odd. We have the factorization

$$T_n(x) - T_m(y) = 2^{d-1} \prod_{k=0}^{\lfloor \frac{d}{2} \rfloor} C_k(x, y)$$

where

$$C_k(x, y) = E_{\frac{2ak\pi}{d}}(T_b(x), T_a(y))$$

is the irreducible equation of the curve $C_k : x = \cos at, y = \cos bt + \frac{2k\pi}{d}$, given in Proposition 5.

PROOF. Let \mathcal{C} be the curve $T_n(x) = T_m(y)$. We easily get $C_k \subset \mathcal{C}$ and $C_k \neq C_{k'}$. When $k = 0$, C_0 admits the equation $T_b(x) - T_a(y) = 0$. If $2k = d$, C_k admits the equation $T_b(x) + T_a(y) = 0$. In the other cases, the dominant term in x of C_k is $2^{2b-2}x^{2b}$. If d is even, we deduce that the dominant term of $\prod_{k=0}^{\lfloor \frac{d}{2} \rfloor} C_k(x, y)$ is $2^{(b-1)d}x^{2n}$ and we get our result in this case. If d is odd, we get the same result. \square

COROLLARY 9. Let $d = \gcd(a, b)$. The curve $T_b(x) = T_a(y)$ has $\lfloor \frac{d}{2} \rfloor + 1$ components. $\lfloor \frac{d-1}{2} \rfloor$ of them are Lissajous curves.

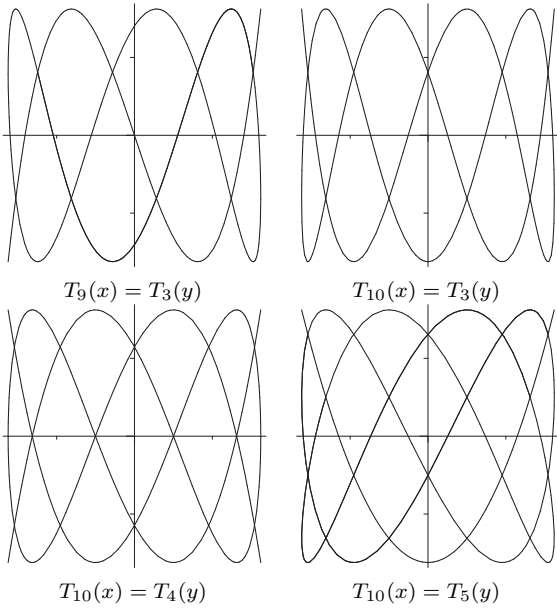


Figure 1: Implicit Chebyshev curves

Theorem 8 is particularly interesting when $m = n = d$ and $a = b = 1$. In this case the curve $T_n(x) = T_n(y)$ is a union of ellipses and some lines. It will be useful for the determination of the double points of Chebyshev space curves. We have

$$\frac{T_n(t) - T_n(s)}{t - s} = 2^{n-1} \prod_{k=1}^{\lfloor \frac{n}{2} \rfloor} E_{\frac{2k\pi}{n}}(s, t). \quad (4)$$

The curve $\frac{T_n(t) - T_n(s)}{t - s} = 0$ has $\lfloor \frac{n}{2} \rfloor$ irreducible components. Note that $\mathcal{E}_{\frac{2k\pi}{n}}$ and $\mathcal{E}_{\frac{2l\pi}{n}}$ intersect at the point $(t, s) = (\cos(\frac{k\pi}{n} + \frac{l\pi}{n}), \cos(\frac{k\pi}{n} - \frac{l\pi}{n}))$ and its symmetric with respect to the lines $s = -t$ and $s = t$. We recover the parametrization of the double points of $x = T_a(t), y = T_b(t)$ that will be very useful for the description of Chebyshev space curves.

PROPOSITION 10 ([10, 12]). Let a and b be nonnegative coprime integers, a being odd. Let the Chebyshev curve

\mathcal{C} be defined by $x = T_a(t), y = T_b(t)$. The pairs (t, s) giving a crossing point are

$$t = \cos(\frac{j\pi}{b} + \frac{i\pi}{a})\pi, \quad s = \cos(\frac{j\pi}{b} - \frac{i\pi}{a})\pi$$

where $1 \leq i \leq \frac{1}{2}(a-1), 1 \leq j \leq b-1$.

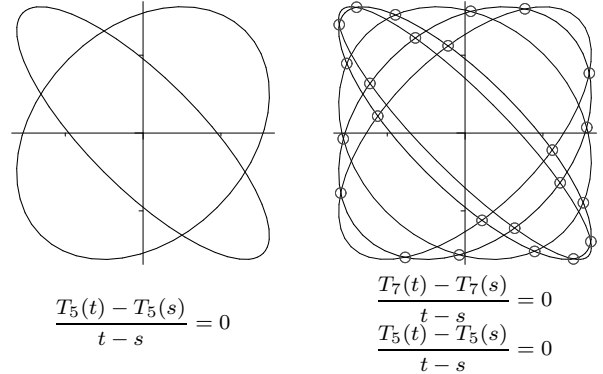


Figure 2: Double points in the parameters space

4. CHEBYSHEV KNOTS

Chebyshev knots are polynomial analogue to Lissajous knots that have been studied by many authors (see [1, 2, 8, 9, 13]). All knots are not Lissajous (for example the trefoil and the figure-eight knot). In [10], it is proved that every knot is a Chebyshev knot.

When a, b are coprime integers, c is an integer and φ is some real constant, the Chebyshev space curve $\mathcal{C}(a, b, c, \varphi)$ is defined by

$$x = T_a(t), y = T_b(t), z = T_c(t + \varphi).$$

Its projection on the (x, y) -plane is the Chebyshev curve $x = T_a(t), y = T_b(t)$. Because a and b are coprime integers, the curve $\mathcal{C}(a, b, c, \varphi)$ is singular if and only if it has some double points. We showed in [12], that the set $\mathcal{Z}_{a,b,c}$ of critical values φ for which the curve $\mathcal{C}(a, b, c, \varphi)$ is singular is finite. Consider the polynomials

$$P_n = \frac{T_n(t) - T_n(s)}{t - s}, \quad Q_n = \frac{T_n(t + \varphi) - T_n(s + \varphi)}{t - s}.$$

and the algebraic variety defined by

$$\mathcal{V}_{a,b,c} = \{P_a(s, t) = 0, P_b(s, t) = 0, Q_c(s, t, \varphi) = 0\}.$$

Because $\deg_{\varphi} Q_n = n - 1$ and the leading term of Q_n is $2^{n-1}n\varphi^{n-1}$, we showed that $\mathcal{V}_{a,b,c}$ is 0-dimensional and has at most $\frac{1}{2}(a-1)(b-1)(c-1)$ points. We deduced that $|\mathcal{Z}_{a,b,c}| \leq \frac{1}{2}(a-1)(b-1)(c-1)$.

Let $\mathcal{Z}_{a,b,c} = \{\varphi_1, \dots, \varphi_n\}$. The type of the knot $\mathcal{C}(a, b, c, \varphi)$ is constant when φ in $(\varphi_i, \varphi_{i+1})$. Our aim is to give a direct way to determine a set r_0, \dots, r_n of rational numbers such that $\mathcal{C}(a, b, c, \varphi)$ is one of the $\mathcal{C}(a, b, c, r_i)$.

We can determine a polynomial $R_{a,b,c} \in \mathbf{Z}[\varphi]$ for which $\mathcal{Z}_{a,b,c} = Z(R)$. It can be defined by $\langle R \rangle = \langle P_a, P_b, Q_c \rangle \cap \mathbf{Q}[\varphi]$ and may be obtained with Gröbner basis ([3]).

Example. When $a = 3, b = 4, c = 5$, we find that

$$R_{a,b,c} = (80\varphi^4 + 60\varphi^2 - 1) \cdot$$

$$(6400\varphi^8 - 3200\varphi^6 + 560\varphi^4 - 80\varphi^2 + 1).$$

There are exactly 6 critical values that are symmetrical about the origin. For these values of φ , the curve $Q_5(s, t, \varphi) =$

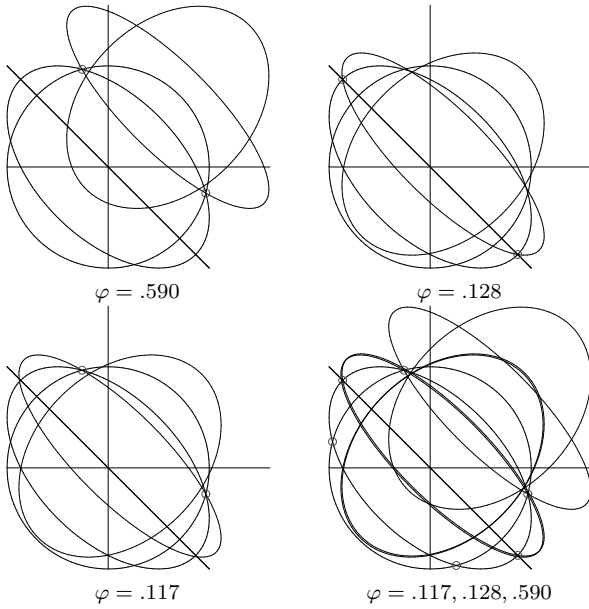


Figure 3: $P_3 = 0, P_4 = 0, Q_5 = 0$

0, which is translated from the curve $P_5(s, t) = 0$ by the vector (φ, φ) , meets the points $\{P_3 = 0, P_4 = 0\}$.

In [12], we developed a method for $a = 3$ or $a = 4$ based on resultant calculations in $\mathbf{Z}[X, t]$. We have computed all possible values for $a = 3$ and $a = 4$, $b \leq 25$ and $c \leq 300$. But this method would not work for $a > 4$ or for bigger values of c . Here we give an explicit formula for the polynomial R .

PROPOSITION 11. *Let a, b be nonnegative coprime integers and c be an integer. Suppose that a is odd. Let $R_{a,b,c}(\varphi)$ be the polynomial*

$$\prod_{i=1}^{\frac{a-1}{2}} \prod_{j=1}^{b-1} Q_c(\cos(\frac{j}{b} + \frac{i}{a})\pi, \cos(\frac{j}{b} - \frac{i}{a})\pi, \varphi). \quad (5)$$

$R_{a,b,c} \in \mathbf{Z}[\varphi]$ and $\mathcal{C}(a, b, c, \varphi)$ is singular iff $R_{a,b,c}(\varphi) = 0$.

PROOF. $\varphi \in \mathcal{Z}_{a,b,c}$ iff there exists (s, t) such that $P_a(s, t) = P_b(s, t) = 0$ and $Q_c(s, t, \varphi) = 0$. This conditions are equivalent to have $s = \cos(\frac{j}{b} + \frac{i}{a})$ and $t = \cos(\frac{j}{b} - \frac{i}{a})$ and $Q_c(s, t, \varphi) = 0$, for some $1 \leq i \leq \frac{a-1}{2}$ and $1 \leq j \leq b-1$, from Proposition 10.

$Q_c(s, t, \varphi)$ is a symmetric polynomial of $\mathbf{Z}[\varphi][t, s]$. Let $\alpha_i = \frac{i\pi}{a}$, $\beta_j = \frac{j\pi}{b}$ and $s = \cos(\alpha_i + \beta_j)$, $t = \cos(\alpha_i - \beta_j)$. From $s + t = 2 \cos \alpha_i \cos \beta_j$ and $st = \cos^2 \alpha_i + \cos^2 \beta_j - 1$, we deduce that $Q_c(s, t, \varphi)$ belongs to $\mathbf{Z}[\varphi, \cos \alpha_i][\cos \beta_j]$.

$$P_i = \prod_{j=1}^{b-1} Q_c(\cos(\alpha_i + \beta_j), \cos(\alpha_i - \beta_j), \varphi)$$

belongs to $\mathbf{Z}[\varphi, \cos \alpha_i]$ because the $\cos \beta_j$ are the roots of $V_b \in \mathbf{Z}[t]$. From $Q_c(-s, -t, \varphi) = Q_c(s, t, -\varphi)$ we deduce

that $\prod_{i=1}^{\frac{a-1}{2}} P_i(-\varphi)P_i(\varphi) = \prod_{i=1}^{a-1} P_i(\varphi) \in \mathbf{Z}[\varphi]$. We thus have $R_{a,b,c}^2 \in \mathbf{Z}[\varphi]$ and so it is for $R_{a,b,c}$. \square

Let $s = \cos(\alpha + \beta)$, $t = \cos(\alpha - \beta)$. Using Theorem 8 and Formula (4), we get

$$Q_c(s, t, \varphi) = 2^{c-1} \prod_{k=1}^{\lfloor \frac{c}{2} \rfloor} E_{\frac{2k\pi}{c}}(s, t).$$

Let us consider $P_{\alpha,\beta,\gamma} = \frac{1}{\sin^2 \gamma} E_{2\gamma}(s + \varphi, t + \varphi)$. For $\gamma \neq \frac{\pi}{2}$, $P_{\alpha,\beta,\gamma}$ is

$$\varphi^2 + 2\varphi \cos \alpha \cos \beta + \frac{(\cos^2 \alpha - \cos^2 \gamma)(\cos^2 \beta - \cos^2 \gamma)}{\sin^2 \gamma}$$

and

$$P_{\alpha,\beta,\frac{\pi}{2}} = \varphi + \cos \alpha \cos \beta.$$

We therefore obtain

$$Q_c(\cos(\alpha + \beta), \cos(\alpha - \beta), \varphi) = K \prod_{k=1}^{\lfloor \frac{c}{2} \rfloor} P_{\alpha,\beta,\frac{k\pi}{c}}(\varphi)$$

with $K = 2^{c-1} \prod_{k=1}^c 2 \sin \frac{k\pi}{c} = c2^{c-1}$. We get therefore

$$R_{a,b,c}(\varphi) = K^{\frac{1}{2}(a-1)(b-1)} \prod_{k=1}^{\lfloor \frac{c}{2} \rfloor} \prod_{i=1}^{\frac{a-1}{2}} \prod_{j=1}^{b-1} P_{\frac{i\pi}{a}, \frac{j\pi}{b}, \frac{k\pi}{c}}(\varphi).$$

We have written $R_{a,b,c}$ as the product of second or first-degree polynomials in $\mathbf{Q}(\cos \frac{\pi}{a}, \cos \frac{\pi}{b}, \cos \frac{\pi}{c})$.

5. COMPUTING THE CRITICAL VALUES

The goal is to get compute the real roots of $R_{a,b,c}$ to get the diagram of $\mathcal{C}(a, b, c, \varphi)$ (we basically need sample points for $\mathbf{R} - Z(R_{a,b,c})$). As $R_{a,b,c}$ factorizes as a product of polynomials $P_{\alpha,\beta,\gamma}$ where $\alpha = \frac{i\pi}{a}$, $\beta = \frac{j\pi}{b}$ and $\gamma = \frac{k\pi}{c}$.

Our strategy consists in first computing separately the real roots of each $P_{\alpha,\beta,\gamma}$ and then combining these roots to get those of $R_{a,b,c}$. A straightforward approach would be to use interval arithmetic to approximate the various trigonometric expressions, but this will fail when $R_{a,b,c}$ has multiple roots.

5.1 Real roots of $P_{\alpha,\beta,\gamma}$

Let $\alpha = \frac{i\pi}{a}$, $\beta = \frac{j\pi}{b}$ and $\gamma = \frac{k\pi}{c}$ with $1 \leq i \leq \frac{a-1}{2}$, $1 \leq j \leq b-1$, $1 \leq k \leq \lfloor \frac{c-1}{2} \rfloor$. We obtain $a_1 \leq \cos \alpha \leq a_2$, $b_1 \leq \cos \beta \leq b_2$, $c_1 \leq \cos \gamma \leq c_2$.

If $\gamma = \frac{\pi}{2}$, the unique root of $P_{\alpha,\beta,\frac{\pi}{2}}$ is $-\cos \alpha \cos \beta$.

If $\gamma \neq \frac{\pi}{2}$, the discriminant of $P_{\alpha,\beta,\gamma}$ is $D = 4 \cos^2 \gamma \left(1 - \frac{\sin^2 \alpha \sin^2 \beta}{\sin^2 \gamma}\right)$, and it has the same sign as

$$\sin^2 \gamma - \sin^2 \alpha \sin^2 \beta \quad (6)$$

The knowledge of the sign of D then gives explicit formulas for the distinct real root(s) of $P_{\alpha,\beta,\gamma}$.

5.2 Multiplicity of 0

PROPOSITION 12. *The multiplicity of $\varphi = 0$ in $R_{a,b,c}$ is*

$$\frac{a-1}{2}((b, c) - 1) + \lfloor \frac{b}{2} \rfloor((a, c) - 1).$$

PROOF. We have to examine whenever $\varphi = 0$ is a root of $P_{\alpha,\beta,\gamma}$ where $\alpha = \frac{i\pi}{a}$, $\beta = \frac{j\pi}{b}$ and $\gamma = \frac{k\pi}{c}$. Here a is odd so $\cos \alpha \neq 0$. Thus, $\varphi = 0$ is a root of $P_{\alpha,\beta,\frac{\pi}{2}}$ if and only if

$$\cos \alpha \cos \beta \quad (7)$$

is null and when $\gamma \neq \frac{\pi}{2}$, $\varphi = 0$ is a root of $P_{\alpha,\beta,\gamma}$ if and only if the following expression is null

$$(\cos^2 \alpha - \cos^2 \gamma)(\cos^2 \beta - \cos^2 \gamma) \quad (8)$$

- If $\gamma = \beta = \frac{\pi}{2}$, $\varphi = 0$ is a root for $i = 1, \dots, \frac{a-1}{2}$.
- If $\gamma \neq \frac{\pi}{2}$, $\varphi = 0$ is a root of $P_{\alpha,\beta,\gamma}$ if and only if $\sin^2 \gamma = \sin^2 \alpha$ or $\sin^2 \gamma = \sin^2 \beta$, that is $ic = ka$ or $jc = kb$ or $(b-j)c = kb$. The root $\varphi = 0$ is obtained for $i = \lambda \frac{a}{(a,c)}$, $k = \lambda \frac{c}{(a,c)}$, $\lambda = 1, \dots, \frac{(a,c)-1}{2}$ and it is double when $\beta = \frac{\pi}{2}$. It is also obtained for $j = \mu \frac{b}{(b,c)}$, $k = \mu \frac{c}{(a,c)}$, $\mu = 1, \dots, (b,c) - 1$. We obtain $\lfloor \frac{b}{2} \rfloor ((a,c) - 1) + ((b,c) - 1)(a-1)/2$.

We thus obtain the result. \square

Remark. We find that 0 is not a critical value if and only if a, b and c are pairwise coprime integers. This result was first proved by Comstock ([4], 1897), who found the number of crossing points of the harmonic curve parametrized by $x = T_a(t), y = T_b(t), z = T_c(t)$.

5.3 Non null multiple roots of $R_{a,b,c}$

It may happen that $R_{a,b,c}$ has multiple root φ . Several cases may occur.

1. $P_{\alpha,\beta,\gamma}$ has a double root $\varphi = -\cos \alpha \cos \beta$.
2. $P_{\alpha,\beta,\gamma_1}$ and $P_{\alpha,\beta,\gamma_2}$ have a common root.
3. $P_{\alpha_1,\beta_1,\gamma_1}$ and $P_{\alpha_2,\beta_2,\gamma_2}$ have a common root.

► $P_{\alpha,\beta,\gamma}$ has a double root if and only if $\text{Disc}(P_{\alpha,\beta,\gamma}) = 0$, this discriminant having the same sign as

$$\sin^2 \gamma - \sin^2 \alpha \sin^2 \beta \quad (9)$$

PROPOSITION 13. Let $\alpha = \frac{i\pi}{a}$, $\beta = \frac{j\pi}{b}$ and $\gamma = \frac{k\pi}{c}$, where $(i, a) = (j, b) = (k, c) = 1$. If a, b, c are pairwise coprime integers, then $\text{Disc}(P_{\alpha,\beta,\gamma}) \neq 0$.

PROOF. We have $\mathbf{Q}(\sin^2 \alpha, \sin^2 \beta) = \mathbf{Q}(\cos 2\alpha, \cos 2\beta) = \mathbf{Q}(\zeta_a, \zeta_b) \cap \mathbf{R}$. We have $\zeta_{ab}^b = \zeta_a$ and $\zeta_{ab}^a = \zeta_b$. Let u, v such that $au + bv = 1$, we get $\zeta_a^v \zeta_b^u = \zeta_{ab}$. We thus deduce $\mathbf{Q}(\zeta_a, \zeta_b) = \mathbf{Q}(\zeta_{ab})$. So $\mathbf{Q}(\sin^2 \alpha, \sin^2 \beta) = \mathbf{Q}(\cos \frac{2\pi}{ab})$.

If $\sin^2 \gamma = \sin^2 \alpha \sin^2 \beta$ then $\cos^2 \gamma \in \mathbf{Q}(\cos \frac{2\pi}{ab})$ and

$$\cos 2\gamma \in \mathbf{Q}(\cos \frac{2\pi}{ab}) \cap \mathbf{Q}(\cos \frac{2\pi}{c}) = \mathbf{Q}.$$

Consequently $2 \cos \gamma \in \mathbf{Z}$, then $2 \cos 2\alpha, 2 \cos 2\beta \in \mathbf{Z}$. We therefore deduce that $\alpha, \beta, \gamma \in \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$. There is no solution with a, b, c relatively coprime integers. \square

► The resultant of $P_{\alpha,\beta,\gamma_1}$ and $P_{\alpha,\beta,\gamma_2}$ is

$$4 \cos^2 \gamma_1 \cos^2 \gamma_2 \frac{\sin^2 \gamma_1 \sin^2 \gamma_2 - \sin^2 \alpha \sin^2 \beta}{\sin^2 \gamma_1 \sin^2 \gamma_2}$$

PROPOSITION 14. Let $\alpha = \frac{i\pi}{a}$, $\beta = \frac{j\pi}{b}$ and $\gamma_1 = \frac{k_1\pi}{c}$, $\gamma_2 = \frac{k_2\pi}{c}$, where $(i, a) = (j, b) = 1$. If a, b, c are relatively coprime integers, then $\text{Res}(P_{\alpha,\beta,\gamma_1}, P_{\alpha,\beta,\gamma_2}) \neq 0$.

PROOF. $\sin^2 \gamma_1 \sin^2 \gamma_2 \in \mathbf{Q}(\cos \frac{2\pi}{c})$ and following the proof of Proposition 13, we have $\cos 2\alpha, \cos 2\beta, \cos \frac{2\pi}{c} \in \frac{1}{2}\mathbf{Z}$ when $\text{Res}(P_{\alpha,\beta,\gamma_1}, P_{\alpha,\beta,\gamma_2}) = 0$. We thus have $\cos 2\gamma_1$ and $\cos 2\gamma_2$ in $\frac{1}{2}\mathbf{Z}$. and $\alpha, \beta, \gamma_1, \gamma_2 \in \{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}\}$. It is not possible to have $(c, ab) = 1$. \square

If $(c, ab) > 1$, it may occur that the resultant is null, which occurs if and only if the following expression is null

$$\sin^2 \alpha \sin^2 \beta - \sin^2 \gamma_1 \sin^2 \gamma_2 \quad (10)$$

For example with $\alpha = \frac{13\pi}{35}$, $\beta = \frac{\pi}{6}$, $\gamma_1 = \frac{11\pi}{35}$, $\gamma_2 = \frac{13\pi}{70}$. In this example we have $\gamma_1 + \gamma_2 = \frac{\pi}{2}$ and $\alpha = 2\gamma_2$. It seems that all the example of double values come from the same identity.

► In case when $\alpha_1 \neq \alpha_2$ or $\beta_1 \neq \beta_2$, $P_{\alpha_1,\beta_1,\gamma_1}$ and $P_{\alpha_2,\beta_2,\gamma_2}$ have a common root if $\text{Res}_\varphi(P_{\alpha_1,\beta_1,\gamma_1}, P_{\alpha_2,\beta_2,\gamma_2}) = 0$. This resultant can be expanded and its sign is the one of:

$$\begin{aligned} & ((\cos^2 \alpha_1 - \cos^2 \gamma_1)(\cos^2 \beta_1 - \cos^2 \gamma_1) \sin^2 \gamma_2 - \\ & (\cos^2 \alpha_2 - \cos^2 \gamma_2)(\cos^2 \beta_2 - \cos^2 \gamma_2) \sin^2 \gamma_1)^2 - \\ & 4(\cos \alpha_1 \cos \beta_1 - \cos \alpha_2 \cos \beta_2) \sin^2 \gamma_1 \sin^2 \gamma_2 \times \\ & ((\cos^2 \alpha_1 - \cos^2 \gamma_1)(\cos^2 \beta_1 - \cos^2 \gamma_1) \cos \alpha_2 \cos \beta_2 \sin^2 \gamma_2 - \\ & (\cos^2 \alpha_2 - \cos^2 \gamma_2)(\cos^2 \beta_2 - \cos^2 \gamma_2) \cos \alpha_1 \cos \beta_1 \sin^2 \gamma_1). \end{aligned} \quad (11)$$

5.4 The Algorithm

From the above results, one can see that computing the exact expression of all the real roots of $R_{a,b,c}$, requires the computation of the sign of expressions $P(f_j^2(\mu_i), i = 1 \dots 2, j = 1 \dots 6)$ where P is a polynomial with rational coefficients, where $f_j \in \{\sin, \cos\}$ and $\mu_i \in \{\frac{i_1\pi}{a}, \frac{i_2\pi}{a}, \frac{j_1\pi}{b}, \frac{j_2\pi}{b}, \frac{k_1\pi}{c}, \frac{i_2\pi}{c}\}$.

Precisely, one may need to compute, at some stages, the signs of 6,7,8,10 and 11.

Isolate and Refine. One straightforward way for testing the sign of such expressions is to replace the various values of $\cos \frac{k*\pi}{m}$ or $\sin \frac{k*\pi}{m}$ by some accurate interval with rational bounds and to perform interval arithmetic to get an interval that contains the value (and thus the sign). This may not give the sign in every situation but this will give a filter that is easy to perform.

Such intervals can be computed by performing algorithms based on Descartes's rule of signs (see for example [18]) on the used Chebyshev polynomials.

Note that we know in advance which polynomials to consider, thanks to the above study, and note also that algorithms like [18] can easily solve such polynomials for very high degrees (several thousands) with a large accuracy. The computation of the required isolating intervals can then be performed as a pre-processing for the global algorithm.

From now, we denote by $Isolate(P, acc)$ the function that isolates the roots of a univariate polynomial P with rational coefficients by means of intervals with rational bounds for a given accuracy acc (maximal length of the intervals). This function provides non overlapping intervals that contains a unique real root of P (and such that each real root of P is contained in one of the intervals).

Note also that if more accuracy for some intervals are required, is is easy, from the isolating intervals provided by the function $Isolate$ to refine them (decreasing their length) efficiently (just evaluating P at some points, without running again the $Isolate$ function with a higher value for acc). We name by $Refine(interval, P, acc)$ the function that decreases the length of the interval $interval$ to get an accuracy $\leq acc$ knowing that the interval isolates a real root of P .

FilterSignTest. We first use a filter (named *FilterSignTest* in the sequel) which consists in using interval arithmetic for the evaluations. Given Isolating intervals with rational bounds that contain the values of the required sin/cos, the function *FilterSignTest* straightforwardly evaluates our expressions. If the resulting interval is $[0, 0]$ or do not contain 0, one can decide the sign of the input, other else, the function returns *FAIL*.

FormalNullTest. In case of failure of *FilterSignTest*, one has to decide if the input is null or not, which is the goal of the function *FormalNullTest* we now describe.

Let m be the smallest common multiple of a, b and c . According to the definitions of V_n and T_n ,

$$\cos \frac{k\pi}{m} = T_k(\cos \frac{\pi}{m}), \quad \sin \frac{k\pi}{m} = V_k(\cos \frac{\pi}{m}) \sin \frac{\pi}{m},$$

$$2 \sin^2 \frac{k\pi}{m} = 1 - T_{2k}(\cos \frac{\pi}{m}), \quad 2 \cos^2 \frac{k\pi}{m} = 1 + T_{2k}(\cos \frac{\pi}{m}).$$

Also, testing if $P(f_i(\mu_j), i = 1 \dots 2, j = 1 \dots 6) = 0$ is equivalent to testing if the expression $P(F_i(\cos \frac{\pi}{m}), i = 1 \dots 2, j = 1 \dots 6) = 0$ where the F_i 's are Chebyshev polynomials of first order of maximal index $2 * k$ or of second order of maximal index k , where k is the max of $\{i_1, i_2, j_1, j_2, k_1, k_2\}$. Since M_m is the minimal polynomial of $\cos \frac{\pi}{m}$, this is equivalent to testing if $P(F_i(T), i = 1 \dots 2, j = 1 \dots 6) = 0$ in $\mathbf{Q}[T]/\langle M_m(T) \rangle$.

SignTest. Our function first performs the *FilterSignTest*. If it returns an interval with bounds of same sign, then the sign of the tested expression is the sign of the two bounds of the interval. Otherwise, we run the *FormalNullTest*. If this test returns 0 then the expression is null. Otherwise, we decrease the lengths of the intervals that represent the values of $\cos \frac{k\pi}{m}, \sin \frac{k\pi}{m}$ by calling the function *Refine* until the *FilterSignTest* do not FAIL (the fact that the sign of the expression to be tested is known not to be 0 ensures that this process will end).

The global algorithm. We proceed in four steps :

(0) We isolate the roots of some Chebyshev polynomials using the *Isolate* black-box with an arbitrary accuracy.

(1) We compute separately and symbolically the roots of the $P_{\alpha, \beta, \gamma}$: the formulas used depends on the sign of some elements that belong to $\mathbf{Q}(\cos \frac{\pi}{a}, \cos \frac{\pi}{b}, \cos \frac{\pi}{c})$ (expressions 6, 7 and 8).

(2) We then take the union of these symbolic expressions and remove the duplicate roots, which remains to deciding if the difference of two elements of $\mathbf{Q}(\cos \frac{\pi}{a}, \cos \frac{\pi}{b}, \cos \frac{\pi}{c})$ is null or not (expressions 10 and 11).

(3) We then use interval arithmetic to evaluate the formal expressions obtained at step (2) which are simple expressions depending the roots which were isolated at step (0). Since step (2) ensures that the roots are all distinct, there exists an accuracy for the isolation intervals so that the evaluation of all these formal expressions give disjoint intervals. When the current accuracy is not sufficient, we refine the intervals we got at step (0) using the *Refine* black box and perform again the step (3).

From the these disjoint intervals with rational bounds, we straightforwardly get the sample points we need.

6. KNOTS DIAGRAMS

We shall study the diagram of the curve $\mathcal{C}(a, b, c, \varphi)$, that is to say the plane projection $\mathcal{C}(a, b)$ onto the (x, y) -plane and the nature (under/over) of the crossings over the double points of $\mathcal{C}(a, b)$. There are two cases of crossing: the right

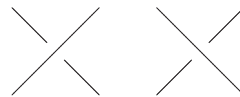


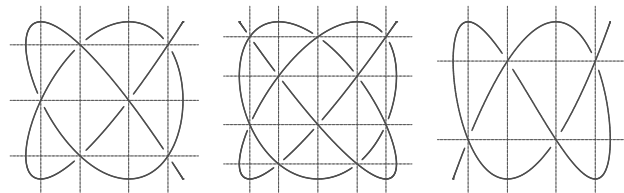
Figure 4: The right twist and the left twist

twist and the left twist (see [15] and Figure 4)) In [12], we showed that the nature of the crossing over the double point $A_{\alpha, \beta}$ corresponding to parameters $(t = \cos \alpha + \beta, s = \cos \alpha - \beta), \alpha = \frac{i\pi}{a}, \beta = \frac{j\pi}{b}$ is given by the sign of

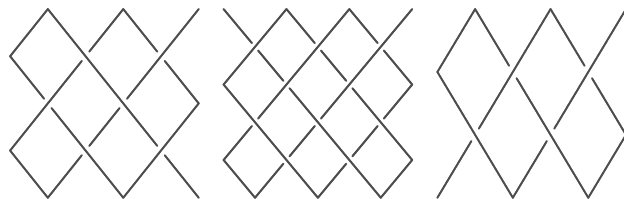
$$\begin{aligned} D(s, t, \varphi) &= Q_c(s, t, \varphi) T_{b-a}(s, t, \varphi) \\ &= (-1)^{i+j} Q_c(s, t, \varphi) \sin \frac{ib\pi}{a} \sin \frac{ib\pi}{b} \\ &= (-1)^{i+j} Q_c(s, t, \varphi) (-1)^{ib \div a} (-1)^{ja \div b}. \end{aligned}$$

$D(s, t, \varphi) > 0$ if and only if the crossing is a right twist.

Note that crossing points of the Chebyshev curve $\mathcal{C}(a, b)$: $x = T_a(t), y = T_b(t)$ lie on the $(b-1)$ vertical lines $T'_b(x) = 0$ and on the $(a-1)$ horizontal lines $T'_a(y) = 0$. We can therefore represent the knot diagram of $\mathcal{C}(a, b, c, r)$ by a billiard knot (see [10]), which is a pure combinatoric objet. As an example, consider the knots $\bar{5}_2 = \mathcal{C}(4, 5, 7, 0), \bar{5}_2 = \mathcal{C}(5, 6, 7, 0), \bar{4}_1 = \mathcal{C}(3, 5, 7, 0)$. We can represent their diagrams by the



following billiard knots: When $a = 3$ or $a = 4$, the knot is



rational and is classified by its Schubert fraction that we deduce easily from its diagram (see [11, 12]). When $b \geq a \geq 5$, the knot might be rational (for example $\mathcal{C}(5, 6, 7, 0)$) but the problem of classification is much more difficult. Nevertheless, the knowledge of the diagrams allows the computation of invariants like the Conway, Alexander and Jones polynomials (see [15]).

Computing the knots diagram

Let $\varphi \in \mathbf{R}$. φ may be a rational number $r \in \mathbf{Q} - \mathcal{Z}_{a, b, c}$ or an algebraic number given by a polynomial whom it is a root and an isolating interval. The main step is the computation of the crossing nature at the double point $A_{\alpha, \beta}$, where $\alpha = \frac{i\pi}{a}, \beta = \frac{j\pi}{b}$. There are two cases to consider.

1. We know the roots $\varphi_1 \leq \dots \leq \varphi_m$ of $Q_c(s, t, \varphi)$.
If $\varphi < \varphi_1$ then $n = 0$ otherwise let $n = \max\{k, \varphi > \varphi_k\}$. We have $\text{sign}(Q_c(s, t, \varphi)) = (-1)^n$.
2. We do not know the roots of $Q_c(s, t, \varphi)$.

We compute $Q_c(s, t, \varphi)$ using the recurrence formula:

$$\begin{aligned} Q_0 &= 0, Q_1 = 1, Q_2 = 2S + 4\varphi, \\ Q_3 &= -4T + 12\varphi S + 4S^2 + 12\varphi^2 - 3. \\ Q_{n+4} &= 2(S + 2\varphi)(Q_{n+3} + Q_{n+1}) \\ &\quad - 2(2\varphi^2 + 2T + 2\varphi S + 1)Q_{n+2} - Q_n. \end{aligned}$$

where $S = s + t = 2 \cos \alpha \cos \beta$ and $T = st = \cos^2 \alpha + \cos^2 \beta - 1$ (see [12]). We work formally in $\mathbf{Q}[u, v]/\langle M, N \rangle$ where M, N are the minimal polynomials of $u = \cos \alpha$, $v = \cos \beta$.

The sign of the crossing is $(-1)^{i+j+ib \div a + ja \div b} Q_c(s, t, \varphi)$.

7. EXPERIMENTS

In [12], we gave parametrizations of every rational knot as $\mathcal{C}(3, b, c, \varphi)$ and $\mathcal{C}(4, b, c, \varphi)$ where (b, c) were minimal for the lexicographic order ($c \leq 300$). For 6 knots we knew the minimal b and that $c > 300$. With the methods we developed here, we know now that these knots admit the following minimal Chebyshev parametrizations.

$$\begin{aligned} 9_5 &= \mathcal{C}(3, 13, 326, 1/85), & 10_3 &= \mathcal{C}(4, 13, 348, 1/138), \\ 10_{30} &= \mathcal{C}(4, 13, 306, 1/738), & 10_{33} &= \mathcal{C}(4, 13, 856, 1/328), \\ 10_{36} &= \mathcal{C}(3, 14, 385, 1/146), & 10_{39} &= \mathcal{C}(3, 14, 373, 1/182). \end{aligned}$$

$R_{3,14,385}$ has degree 4992. It has 2883 real roots. All are simple roots except 0 that is of multiplicity 6.

$R_{4,13,856}$ has degree 15390 and 9246 real roots (0 has multiplicity 18). We get 2050 non trivial knots, 83 of them are distinct, and 63 have less than 10 crossings.

Outside the intrinsic combinatorial aspects of the problem, the complexity of our algorithm essentially depends on the *FormalNullTest*. In the worst case $d = abc$ and $\deg M_d = \frac{1}{2}(a-1)(b-1)(c-1)$, when a, b, c are prime integers and the most difficult computation consists in deciding if the expression 11 is null or not which is equivalent to testing if a univariate polynomial of degree $4 * d$ is null modulo M_d or not.

These computations can be speed up a lot since they can be performed modulo a prime integer : all the considered polynomials have a power of two as leading coefficient and we just need to test if one polynomial is null modulo another one.

In this challenging experiments, we never had to run the *FormalNullTest*, the *FilterSignTest* being always sufficient, thanks to the filters given by propositions 13 and 14 and to a good (experimental) choice initial choice of accuracy when computing the prerequisites running the *Isolate* algorithm.

The main difference with the algorithm described in [12] and the computation of $R_{a,b,c}$ as a polynomial of degree $\frac{1}{2}(a-1)(b-1)(c-1)$, is that it came as a resultant of a polynomial of degree $(c-1)$ in (X, φ) and a polynomial of degree $\frac{1}{2}(a-1)(b-1)$ in X with coefficients in a unique field extension. The example described in this section can be considered as the extremal case, in terms of degree, to be solved using methods from the state of the art when running [12] while it can be solved in few minutes with the method proposed in this article.

8. CONCLUSION

The method we developed in this paper allows us to compute Chebyshev knot diagrams for high values of a, b and c . Our experience with small a and b shows that the difficult

cases (multiple roots of $R_{a,b,c}$ we found were predictable). There are certainly some specific reasons connected with arithmetic properties and the structure of cyclic extensions.

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