



**HAL**  
open science

# Attractivity theory for second order non-smooth dynamical systems with application to dry friction

Samir Adly

► **To cite this version:**

Samir Adly. Attractivity theory for second order non-smooth dynamical systems with application to dry friction. *Journal of Mathematical Analysis and Applications*, 2006, 322 (2), pp.1055-1070. 10.1016/j.jmaa.2005.09.076 . hal-00450953

**HAL Id: hal-00450953**

**<https://hal.science/hal-00450953>**

Submitted on 18 Jul 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Attractivity theory for second order non-smooth dynamical systems with application to dry friction

Samir Adly

*LACO, University of Limoges, 123, Avenue A. Thomas, 87060 Limoges cedex, France*

In this paper, we study the attractivity properties of the set of stationary solutions for a general class of second order non-smooth dynamical system involving friction term. Sufficient conditions for the local attractivity of the set of stationary solutions are given in the case of dry friction and negative viscous damping. An estimation of the attraction domain is also given in this case. Applications can be found in unilateral mechanics.

*Keywords:* Variational inequalities; Unilateral dynamical systems; Non-smooth mechanics; Lyapunov stability; LaSalle invariance principle; Convex analysis; Friction problems in mechanics

---

## 1. Introduction

In recent year, the theory of stability (in the sense of Lyapunov) of stationary solutions of dynamical systems has been considerably developed. It is well known that this field is of major importance in both applied mathematics and engineering. With the emergence of many engineering disciplines, it is not surprising that the unilateral dynamical system has played a central role in the understanding of mechanical processes. The mathematical formulation of the unilateral dynamical system involved inequality constraints and necessarily contains natural non-smoothness. The non-smoothness could originate from the discontinuous control term, or from the environment (non-smooth impact), or from the dry friction. It is well known that dry friction generates

instabilities and consequently influences the performance and the behavior of mechanical systems. It seems that the formalism of evolution variational inequalities represents a large class of unilateral dynamical systems [3,4,8,9,12,13]. Due to the lack of smoothness, classical mathematical methods (see, e.g., [11,12]) are applicable only to a limited amount and require naturally extensions for both analytical and numerical methods. Recently, new analytical tools have been developed for the study of evolution variational inequalities [7–9].

Recently, S. Adly and D. Goeleven [1] have developed a LaSalle’s invariance theory applicable to a general class of first order non-linear evolution variational inequalities. This approach was applied to the study of the stability and the asymptotic properties of second order dynamical systems involving friction forces. Equally important is the study of the attractivity properties of the set of stationary solutions which correspond in general to a stationary mode where the friction elements are sticking.

It is well known that the dynamics of many mechanical systems could be influenced by the stability and the attractivity of the stationary sets (such as e.g. limit-cycling induced by friction). In the last decade, the stability and the attractivity of the set of stationary solutions have attracted many important research interests (see e.g. [7–9]). Inspired by a recent work by N. Van De Wouw and R.I. Leine [16] and motivated by the mechanical applications, we provide sufficient conditions for the global or local attractivity of the set of stationary solutions of a general class of second order dynamical systems with friction term. These results are obtained by applying the approach developed in [1] and could be seen as a unification and an extension of the results obtained by N. Van De Wouw and R.I. Leine [16] for the Coulomb friction in a general framework. A particular attention is made in the case of the non-positive definiteness of the viscous damping matrix (for the mechanical motivation we refer to [5,6,15,16] and references cited therein). In [16], it was proved that the presence of the Coulomb friction in such a situation can ensure the local attractivity of the set of stationary solutions. Moreover, an estimation of the attraction domain has been given. In this paper, we prove that these results are still valid for the general class of second order dynamical systems with friction term by using the techniques developed in [1].

The paper is organized as follows. In Section 2, we recall an existence and uniqueness result for the first order non-linear evolution variational inequalities. This result is a direct consequence of Kato’s theorem [10]. In Section 3, we study the attractivity of the set of stationary solutions. More precisely, we show that if the viscous damping matrix is symmetric and positive definite, then the set of stationary solutions is globally attractive. Moreover, if the viscous damping matrix is symmetric but not necessarily positive definite, then with (the presence of dry friction) the set of stationary solutions is only locally attractive. We give also an estimation of the attractivity domain. In Section 4, we discuss some examples.

Let us consider the following second order dynamic system: For  $(t_0, q_0, \dot{q}_0) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ , we consider the problem  $P(t_0, q_0, \dot{q}_0)$ : Find a function  $t \mapsto q(t)$  ( $t \geq t_0$ ) with  $q \in C^1([t_0, +\infty); \mathbb{R}^m)$  and such that:

$$\frac{d^2q}{dt^2} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^m), \quad (1)$$

$$\frac{dq}{dt} \text{ is right-differentiable on } [t_0, +\infty), \quad (2)$$

$$\left( q(t_0), \frac{dq}{dt}(t_0) \right) = (q_0, \dot{q}_0), \quad (3)$$

$$M \frac{d^2q}{dt^2}(t) + C \frac{dq}{dt}(t) + \Pi'(q(t)) \in -H_1 \partial \Phi \left( H_1^T \frac{dq}{dt}(t) \right), \quad \text{a.e. } t \geq t_0, \quad (4)$$

where  $\Phi : \mathbb{R}^l \rightarrow \mathbb{R}$  and  $\Pi : \mathbb{R}^m \rightarrow \mathbb{R}$  are functions satisfying the following assumptions:

- ( $\mathcal{H}_\Phi$ -i)  $\Phi$  is convex;
- ( $\mathcal{H}_\Phi$ -ii)  $\min_{x \in \mathbb{R}^l} \Phi(x) = \Phi(0) = 0$ ;
- ( $\mathcal{H}_\Pi$ -i)  $\Pi \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R})$  such that  $\Pi'$  is Lipschitz continuous on  $\mathbb{R}^m$ ;
- ( $\mathcal{H}_\Pi$ -ii)  $\Pi$  is coercive, i.e.,  $\Pi(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ ;
- ( $\mathcal{H}_M$ -i)  $M \in \mathbb{R}^{m \times m}$  is a symmetric and positive definite matrix;

and  $C \in \mathbb{R}^{m \times m}$  and  $H_1 \in \mathbb{R}^{m \times l}$  are given matrices. Here  $\partial\Phi$  denotes the convex subdifferential of  $\Phi$ .

The second order dynamical system (4) is useful for the study of many problems in unilateral mechanics. Indeed, the motion of various mechanical systems with frictional contact can be studied in the framework of Eq. (4). For such problems  $m$  is the number of degrees of freedom,  $M$  is the mass matrix,  $C$  is the viscous damping matrix and  $\Pi$  is the potential. The vector  $q \in \mathbb{R}^m$  is the vector of generalized coordinates. The term  $H_1 \partial\Phi(H_1^T \cdot)$  is used to model the unilaterality of the contact induced by friction forces.

The set of stationary solutions of (4) is given by

$$\mathcal{W} = \{\bar{q} \in \mathbb{R}^m : \Pi'(\bar{q}) \in -H_1 \partial\Phi(0)\}. \quad (5)$$

## 2. An existence and uniqueness result

Let us consider the following first order evolution variational inequality, denoted  $P(t_0, x_0)$ , and defined by

$$P(t_0, x_0) \quad \begin{cases} \text{Find } x \in C^0([t_0, +\infty[; \mathbb{R}^n), \frac{dx}{dt} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^n) \text{ such that} \\ \langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \rangle + \varphi(v) - \varphi(x(t)) \geq 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq t_0, \\ x(t_0) = x_0. \end{cases} \quad (6)$$

Here  $\langle \dots \rangle$  denotes the euclidean scalar product in  $\mathbb{R}^n$  and  $\|\cdot\|$  the corresponding norm.

We have the following existence and uniqueness result. For more details see [1,8].

**Theorem 2.1.** *Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a proper, convex and lower semicontinuous function and  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous operator such that for some  $\omega \geq 0$ , the operator  $F + \omega I$  is monotone, i.e.,*

$$\langle F(x) - F(y), x - y \rangle \geq -\omega|x - y|^2, \quad \forall x, y \in \mathbb{R}^n.$$

*Let  $t_0 \in \mathbb{R}$  and  $x_0 \in \text{Dom}(\partial\varphi)$ <sup>1</sup> be given. Then there exists a unique  $x \in C^0([t_0, +\infty); \mathbb{R}^n)$  such that*

$$\frac{dx}{dt} \in L_{\text{loc}}^\infty(t_0, +\infty; \mathbb{R}^n), \quad (7)$$

$$x \text{ is right-differentiable on } [t_0, +\infty), \quad (8)$$

$$x(t_0) = x_0, \quad (9)$$

$$x(t) \in \text{Dom}(\partial\varphi), \quad t \geq t_0, \quad (10)$$

$$\left\langle \frac{dx}{dt}(t) + F(x(t)), v - x(t) \right\rangle + \varphi(v) - \varphi(x(t)) \geq 0, \quad \forall v \in \mathbb{R}^n, \text{ a.e. } t \geq t_0. \quad (11)$$

<sup>1</sup> Here  $\text{Dom}(\partial\varphi)$  denotes the domain of the subdifferential of  $\varphi$ .

We will denote by  $x(\cdot; t_0, x_0)$  the unique solution of  $P(t_0, x_0)$  under the assumptions of Theorem 2.1.

Using Theorem 2.1, we have the following existence and uniqueness result for the second order dynamic system (4).

**Theorem 2.2** (*Existence and uniqueness*). *Suppose that assumptions  $(\mathcal{H}_\Phi$ -i, ii),  $(\mathcal{H}_\Pi$ -i) and  $(\mathcal{H}_M$ -i) are satisfied.*

*Let  $t_0 \in \mathbb{R}$ ,  $(q_0, \dot{q}_0) \in \mathbb{R}^m \times \mathbb{R}^m$ . Then there exists a unique  $q \in C^1([t_0, +\infty); \mathbb{R}^m)$  satisfying conditions (1)–(4).*

**Proof.** Since the matrix  $M$  is symmetric and positive definite, then problem (4) is equivalent to the following first order differential inclusion:

$$\begin{cases} \dot{x}(t) + F(x(t)) \in -\partial\varphi(x(t)), \\ x(t_0) = x_0, \end{cases} \quad (FO)$$

where the vector  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^n$  ( $n = 2m$ ) and the mapping  $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by

$$F(x) = \begin{pmatrix} -x_2 \\ M^{-\frac{1}{2}}CM^{-\frac{1}{2}}x_2 + M^{-\frac{1}{2}}\Pi'(M^{-\frac{1}{2}}x_1) \end{pmatrix}. \quad (12)$$

We note that

$$F(x) = Ax + \bar{F}(x),$$

where the matrix  $A \in \mathbb{R}^{n \times n}$  is defined by

$$A = \begin{pmatrix} 0_{m \times m} & -I_m \\ 0_{m \times m} & M^{-\frac{1}{2}}CM^{-\frac{1}{2}} \end{pmatrix},$$

and the mapping  $\bar{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is defined by

$$\bar{F}(x) = \begin{pmatrix} 0_{m \times 1} \\ M^{-\frac{1}{2}}\Pi'(M^{-\frac{1}{2}}x_1) \end{pmatrix}$$

the vector  $x_0 \in \mathbb{R}^n$  is given by

$$x_0 = \begin{pmatrix} M^{-\frac{1}{2}}q_0 \\ M^{-\frac{1}{2}}\dot{q}_0 \end{pmatrix}, \quad (13)$$

and the convex function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by

$$\varphi(x) = (\Phi \circ H_1^T M^{-\frac{1}{2}})(x_2). \quad (14)$$

In this case, the subdifferential of  $\varphi$  is given by

$$\partial\varphi(x) = \begin{pmatrix} 0 \\ \partial(\Phi \circ H_1^T M^{-\frac{1}{2}})(x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ M^{-\frac{1}{2}}H_1 \partial\Phi(H_1^T M^{-\frac{1}{2}}x_2) \end{pmatrix}. \quad (15)$$

It is clear that  $A$  is continuous and  $A + \omega I_{n \times n}$  is monotone provided that  $\omega \geq \sup_{\|x\|=1} \langle -Ax, x \rangle$ . By assumption  $(\mathcal{H}_\Pi$ -i), the mapping  $\bar{F}$  is Lipschitz continuous. The conclusion of Theorem 2.2 follows from Theorem 2.1.  $\square$

### 3. An attractivity result for second order dynamical systems

In this section, we will study some attractivity properties of the set of stationary solutions (5). Let us first recall some invariance properties.

We denote by  $\mathcal{S}(F, \varphi)$  the set of stationary solutions of the first order evolution variational inequalities (6), that is,

$$\mathcal{S}(F, \varphi) := \{z \in D(\partial\varphi) : \langle F(z), v - z \rangle + \varphi(v) - \varphi(z) \geq 0, \forall v \in \mathbb{R}^n\}. \quad (16)$$

Let  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  be given. We set

$$E(F, \varphi, V) := \{x \in D(\partial\varphi) : \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = 0\}. \quad (17)$$

For  $x_0 \in \mathbb{R}^n$ , we denote by  $\gamma(x_0)$  the *orbit*

$$\gamma(x_0) := \{x(\tau; t_0, x_0) : \tau \geq t_0\}$$

and by  $\Lambda(x_0)$  the *limit set*

$$\Lambda(x_0) := \{z \in \mathbb{R}^n : \exists\{\tau_i\} \subset [t_0, +\infty), \tau_i \rightarrow +\infty \text{ and } x(\tau_i; t_0, x_0) \rightarrow z\}.$$

We say that a set  $\mathcal{D} \subset \mathbb{R}^n$  is *invariant* if any solution of problem (6) starting in  $\mathcal{D}$  remains in  $\mathcal{D}$  for all  $t \geq t_0$ , i.e.,

$$x_0 \in \mathcal{D} \quad \Rightarrow \quad \gamma(x_0) \subset \mathcal{D}.$$

#### 3.1. Global attractivity: The damping matrix $C$ is positive definite

We recall the following result (see [1, Corollary 1]).

**Theorem 3.1.** *Suppose that the assumptions of Theorem 2.1 hold. Let  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  be a function such that*

- (1)  $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$  is lower semicontinuous on  $\text{Dom}(\partial\varphi)$ ;
- (2)  $\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \forall x \in \text{Dom}(\partial\varphi)$ ;
- (3)  $V(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty, x \in D(\partial\varphi)$ ;
- (4)  $D(\partial\varphi)$  is closed.

Let  $\mathcal{M}$  be the largest invariant subset of  $E(F, \varphi, V)$ . Then for each  $x_0 \in \text{Dom}(\partial\varphi)$ , the orbit  $\gamma(x_0)$  is bounded and

$$\lim_{\tau \rightarrow +\infty} \text{dist}(x(\tau; t_0, x_0), \mathcal{M}) = 0.$$

If the damping matrix  $C$  is symmetric and positive definite, then we have the following global attractivity result of the set  $\mathcal{W}$  of stationary solutions (5).

**Theorem 3.2** (Attractivity of  $\mathcal{W}$ ). *Suppose that the assumptions  $(\mathcal{H}_\Phi\text{-i, ii})$ ,  $(\mathcal{H}_\Gamma\text{-i, ii})$  and  $(\mathcal{H}_M\text{-i})$  hold. If the matrix  $C$  is symmetric and positive definite, then for any  $(q_0, \dot{q}_0) \in \mathbb{R}^m \times \mathbb{R}^m$ , we have the following asymptotic properties:*

$$\lim_{\tau \rightarrow +\infty} \text{dist}(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} \frac{dq}{dt}(\tau; t_0, q_0, \dot{q}_0) = 0.$$

**Proof.** Let us consider the following associated first order system to (4) defined by

$$\dot{x} + F(x) \in -\partial\varphi(x),$$

where  $F$  and  $\varphi$  are defined respectively in (12) and (14).

We will check that all assumptions of Theorem 3.1 are satisfied with the function  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  ( $n = 2m$ ) defined by

$$V(x) = \Pi(M^{-\frac{1}{2}}x_1) + \frac{1}{2}\|x_2\|_m^2. \quad (18)$$

The function  $V \in C^1(\mathbb{R}^n, \mathbb{R})$  and we have

$$V'(x) = \begin{pmatrix} M^{-\frac{1}{2}}\Pi'(M^{-\frac{1}{2}}x_1) \\ x_2 \end{pmatrix}.$$

Hence,

$$\varphi(x) - \varphi(x - V'(x)) = \Phi(H_1^T M^{-\frac{1}{2}}x_2) - \Phi(0) = (\Phi \circ H_1^T M^{-\frac{1}{2}})(x_2),$$

and the application  $x \mapsto \varphi(x) - \varphi(x - V'(x))$  is thus lower semicontinuous on  $\text{Dom}(\partial\varphi) = \mathbb{R}^n$ . It results that hypothesis (1) of Theorem 3.1 is satisfied.

We have

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = \langle M^{-\frac{1}{2}}CM^{-\frac{1}{2}}x_2, x_2 \rangle_m + (\Phi \circ H_1^T M^{-\frac{1}{2}})(x_2).$$

Using the fact that the matrix  $C$  is positive definite and  $(\mathcal{H}_\Phi\text{-ii})$ , we get hypothesis (2) of Theorem 3.1.

By  $(\mathcal{H}_\Pi\text{-ii})$ , hypothesis (3) of Theorem 3.1 is satisfied. Since  $\text{Dom}(\partial\varphi) = \mathbb{R}^n$ , it is clear that hypothesis (4) of Theorem 3.1 also holds.

Now, since

$$E(F, \varphi, V) = \{x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n: \langle M^{-\frac{1}{2}}CM^{-\frac{1}{2}}x_2, x_2 \rangle_m + \Phi(H_1^T M^{-\frac{1}{2}}x_2) = 0\},$$

using  $(\mathcal{H}_\Phi\text{-ii})$ ,  $(\mathcal{H}_M\text{-i})$  and the fact that  $C$  is positive definite, we get  $x_2 = 0$ , and hence,

$$E(F, \varphi, V) = \{(x_1, 0): x_1 \in \mathbb{R}^m\}. \quad (19)$$

Theorem 3.1 ensures that

$$\lim_{\tau \rightarrow +\infty} \text{dist}(x(\tau; t_0, x_0), \mathcal{M}) = 0,$$

where  $\mathcal{M}$  is the largest invariant subset of  $E(F, \varphi, V)$ . We show that  $\mathcal{M} = S(F, \varphi)$  (where  $S(F, \varphi)$  is defined in (16)).

Fix any  $z = (z_1, z_2) \in S(F, \varphi)$ . We have

$$\langle F(z), v - z \rangle + \varphi(v) - \varphi(z) \geq 0, \quad \forall v \in \mathbb{R}^n,$$

which yields for  $v = z - V'(z)$ ,

$$\langle F(z), V'(z) \rangle + \varphi(z) - \varphi(z - V'(z)) \leq 0,$$

and hence,

$$\langle M^{-\frac{1}{2}}CM^{-\frac{1}{2}}z_2, z_2 \rangle + \Phi(H_1^T M^{-\frac{1}{2}}z_2) \leq 0.$$

Invoking the positive definiteness of  $C$  and  $(\mathcal{H}_\Phi\text{-ii})$ , we obtain

$$\langle M^{-\frac{1}{2}}CM^{-\frac{1}{2}}z_2, z_2 \rangle + \Phi(H_1^T M^{-\frac{1}{2}}z_2) = 0,$$

that is

$$z \in E(F, \varphi, V).$$

Therefore,  $S(F, \varphi) \subset E(F, \varphi, V)$  and thus  $z_2 = 0$ , according to (19).

Consequently,

$$S(F, \varphi) = \{z = (z_1, 0): \Pi'(M^{-\frac{1}{2}}z_1) \in -H_1 \partial \Phi(0)\} = \mathcal{N} \times \{0\},$$

where  $\mathcal{N} = \{z_1 \in \mathbb{R}^n: \Pi'(M^{-\frac{1}{2}}z_1) \in -H_1 \partial \Phi(0)\}$ .

It is clear that the set of stationary solutions  $S(F, \varphi)$  is invariant. Let  $\mathcal{D}$  be any invariant subset of  $E(F, \varphi, V)$  and  $z = (z_1, z_2) \in \mathcal{D}$ . The function  $t \mapsto x(t; t_0, z)$  satisfies:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 + M^{-\frac{1}{2}}CM^{-\frac{1}{2}}x_2 + M^{-\frac{1}{2}}\Pi'(M^{-\frac{1}{2}}x_1) \in -M^{-\frac{1}{2}}H_1 \partial \Phi(H_1^T M^{-\frac{1}{2}}x_2). \end{cases} \quad (S)$$

Since  $\mathcal{D}$  is invariant, the orbit  $\gamma(z) \subset \mathcal{D} \subset E(F, \varphi, V)$ . Hence,  $x_2(t; t_0, z_2) = 0, \forall t \geq t_0$ . Thus  $x_1(t; t_0, z_1) = z_1, \forall t \geq t_0$  and  $z_2 = 0$ .

Therefore,

$$\Pi'(M^{-\frac{1}{2}}z_1) \in -H_1 \partial \Phi(0), \quad \text{i.e., } z_1 \in \mathcal{N}.$$

Hence,

$$z = (z_1, 0) \in S(F, \varphi).$$

Consequently  $\mathcal{D} \subset S(F, \varphi)$  and thus  $S(F, \varphi)$  is the largest invariant subset of  $E(F, \varphi, V)$ . Consequently,

$$\lim_{\tau \rightarrow +\infty} \text{dist}(x(\tau; t_0, x_0), S(F, \varphi)) = 0,$$

which means

$$\lim_{\tau \rightarrow +\infty} \text{dist}(x_1(\tau), \mathcal{N}) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} x_2(\tau) = 0.$$

Recalling that  $q = M^{-\frac{1}{2}}x_1$  and  $\dot{q} = M^{-\frac{1}{2}}x_2$ , we get

$$\lim_{\tau \rightarrow +\infty} \text{dist}(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} \dot{q}(\tau) = 0,$$

which completes the proof of Theorem 3.3.  $\square$

### 3.2. Local attractivity and dry friction: The damping matrix $C$ is not necessarily positive definite

Suppose now that the damping matrix  $C \in \mathbb{R}^{m \times m}$  is symmetric and not necessary positive definite. Suppose also that  $C$  has  $m_1$  non-positive eigenvalues and  $m_2$  positive ones such that  $m_1 + m_2 = m$ . Since  $C$  is symmetric, there exists an orthogonal matrix  $P_C \in \mathbb{R}^{m \times m}$  and a diagonal matrix  $D_C \in \mathbb{R}^{m \times m}$  such that  $C = P_C D_C P_C^T$ . We will assume the case of dry friction, i.e.,

$$(\mathcal{H}_\Phi\text{-iii}) \quad 0 \in \text{Int}(\partial \Phi(0)).$$

In the remaining of the paper, we will denote by  $(e_1, e_2, \dots, e_m)$  the canonical basis of  $\mathbb{R}^m$ . The following lemma will be needed in the proof of Theorem 3.4.



**Lemma 3.1.** *Suppose assumptions  $(\mathcal{H}_\Phi$ –i–iii). If*

$$(P_C)_i := P_C e_i \in \text{span}(H_1) \quad \text{for } i = 1, \dots, m_1. \quad (20)$$

*then there exists  $\alpha > 0$  such that*

$$\sum_{i=1}^{m_1} |y_i| \leq \alpha \Phi(H_1^T P_C y), \quad \forall y \in \mathbb{R}^m. \quad (21)$$

**Proof.** If  $(P_C)_i := P_C e_i \in \text{span}(H_1)$  for  $i = 1, \dots, m_1$ , then there exists  $\bar{x}^{(i)} \in \mathbb{R}^m \setminus \{0\}$  (depending on  $i$ ) such that  $P_C e_i = H_1 \bar{x}^{(i)}$ . Thus,  $e_i = P_C^T H_1 \bar{x}^{(i)}$  and for any  $y \in \mathbb{R}^m$ , we have

$$|y_i| = |\langle e_i, y \rangle| = |\langle \bar{x}^{(i)}, H_1^T P_C y \rangle| \leq \|\bar{x}^{(i)}\| \|H_1^T P_C y\|.$$

On the other hand, we have

$$0 \in \text{Int}(\partial\Phi(0)) \iff \exists \beta > 0: \quad \Phi(x) \geq \beta \|x\|, \quad \forall x \in \mathbb{R}^m.$$

In fact,

$$0 \in \text{Int}(\partial\Phi(0)) \iff \exists \beta > 0: \quad \beta \mathbb{B} := \mathbb{B}(0, \beta) \subset \partial\Phi(0).$$

Therefore,

$$\sigma_{\partial\Phi(0)}(x) \geq \sigma_{\beta\mathbb{B}}(x), \quad \forall x,$$

where  $\sigma_C(x) = \sup_{y \in C} \langle x, y \rangle$  is the support function to the closed convex set  $C$ .

Consequently,

$$\Phi'(0; x) \geq \beta \|x\|, \quad \forall x.$$

Since  $\Phi(0) = 0$  and  $\Phi$  is continuous, we get

$$\Phi(x) \geq \beta \|x\|, \quad \forall x.$$

Hence,

$$\|x\| \leq \frac{1}{\beta} \Phi(x), \quad \forall x \in \mathbb{R}^m.$$

Consequently,

$$|y_i| \leq \frac{1}{\beta} \|\bar{x}^{(i)}\| \Phi(H_1^T P_C y), \quad i = 1, \dots, m_1,$$

which yields

$$\sum_{i=1}^{m_1} |y_i| \leq \alpha \Phi(H_1^T P_C y), \quad \forall y \in \mathbb{R}^m,$$

with  $\alpha = \frac{1}{\beta} \sum_{i=1}^{m_1} \|\bar{x}^{(i)}\| > 0$ , and the proof of the lemma is thereby complete.  $\square$

To prove the local attractivity, we will use the following theorem (see [1, Theorem 5]).

**Theorem 3.3** (*Local Invariance Theorem*). *Suppose that the assumptions of Theorem 2.1 hold. Let  $\Psi \subset \mathbb{R}^n$  be a compact set and  $V \in C^1(\mathbb{R}^n; \mathbb{R})$  a function such that*

- (1)  $\varphi(\cdot) - \varphi(\cdot - V'(\cdot))$  is lower semicontinuous on  $\text{Dom}(\partial\varphi) \cap \Psi$ ;
- (2)  $\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \forall x \in \text{Dom}(\partial\varphi) \cap \Psi$ ;
- (3)  $\text{Dom}(\partial\varphi)$  is closed.

Let  $\mathcal{M}$  the largest invariant subset of  $E_\Psi(F, \varphi, V) := E(F, \varphi, V) \cap \Psi$ . Then for each  $x_0 \in \text{Dom}(\partial\varphi)$  such that  $\gamma(x_0) \subset \Psi$ , we have

$$\lim_{\tau \rightarrow +\infty} d(x(\tau; t_0, x_0), \mathcal{M}) = 0.$$

In the case of dry friction (i.e.,  $0 \in \text{Int}(\partial\Phi(0))$ ), we have the following result for the local attractivity.

**Theorem 3.4** (*Local Attractivity*). *Suppose that the assumptions  $(\mathcal{H}_\Phi\text{-i-iii})$ ,  $(\mathcal{H}_\Pi\text{-i, ii})$  and  $(\mathcal{H}_M\text{-i})$  hold. Suppose also that the damping matrix  $C$  is symmetric but not necessary positive definite and that (20) is satisfied. Then there exists a subset  $\mathcal{W}^* \subset \mathcal{W}$  which is locally attractive.*

**Proof.** As in the proof of Theorem 3.2, we consider the following associated first order system defined by

$$\dot{x} + F(x) \in -\partial\varphi(x),$$

where  $F$  and  $\varphi$  are defined respectively in (12) and (14).

We take  $V(x) = \Pi(M^{-\frac{1}{2}}x_1) + \frac{1}{2}\|x_2\|_2^2$ . We will show that all assumptions of Theorem 3.2 are satisfied.

It is clear that the function  $x \mapsto \varphi(x) - \varphi(x - V'(x)) = \Phi(H_1^T M^{-\frac{1}{2}}x_2)$  is lower semicontinuous. Further we have:

$$\begin{aligned} & \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \\ &= \langle M^{-\frac{1}{2}} C M^{-\frac{1}{2}} x_2, x_2 \rangle + \Phi(H_1^T M^{-\frac{1}{2}} x_2) \\ &= \langle C M^{-\frac{1}{2}} x_2, M^{-\frac{1}{2}} x_2 \rangle + \Phi(H_1^T M^{-\frac{1}{2}} x_2) \\ &= \langle P_C D_C P_C^T M^{-\frac{1}{2}} x_2, M^{-\frac{1}{2}} x_2 \rangle + \Phi(H_1^T M^{-\frac{1}{2}} x_2) \\ &= \langle D_C P_C^T M^{-\frac{1}{2}} x_2, P_C^T M^{-\frac{1}{2}} x_2 \rangle + \Phi(H_1^T P_C P_C^T M^{-\frac{1}{2}} x_2) \\ &= \langle D_C y, y \rangle + \Phi(H_1^T P_C y) \quad \text{by setting } y = P_C^T M^{-\frac{1}{2}} x_2 \\ &= \sum_{i=1}^m \lambda_i y_i^2 + \Phi(H_1^T P_C y). \end{aligned}$$

Therefore

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = \sum_{i=1}^{m_1} \lambda_i y_i^2 + \sum_{i=m_1+1}^m \lambda_i y_i^2 + \Phi(H_1^T P_C y),$$

and hence,

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq \sum_{i=1}^{m_1} \lambda_i y_i^2 + \Phi(H_1^T P_C y).$$

Using Lemma 3.1, there exists  $\alpha > 0$  such that

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq \sum_{i=1}^{m_1} \left( \lambda_i y_i^2 + \frac{1}{\alpha} |y_i| \right).$$

Since

$$\sum_{i=1}^{m_1} \left( \lambda_i y_i^2 + \frac{1}{\alpha} |y_i| \right) \geq 0, \quad \forall y_i \in \left[ \frac{1}{\alpha \lambda_i}, \frac{-1}{\alpha \lambda_i} \right], \quad i = 1, \dots, m_1,$$

it follows that

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \quad \forall x \in \mathcal{A},$$

where the set  $\mathcal{A}$  is defined by

$$\mathcal{A} = \bigcap_{i=1}^{m_1} \left\{ x = (x_1, x_2) \in \mathbb{R}^m : \left| \langle P_C^T M^{-\frac{1}{2}} x_2, e_i \rangle \right| \leq \frac{-1}{\alpha \lambda_i} \right\}. \quad (22)$$

Choose  $\rho \in \mathbb{R}$  such that the sublevel set  $\text{Lev}(V, \rho) = \{x \in \mathbb{R}^m : V(x) \leq \rho\} \subset \text{Int } \mathcal{A}$  and put  $\Psi = \text{Lev}(V, \rho)$ . By assumption ( $\mathcal{H}_{\Pi}$ -ii) it is clear that  $V$  is coercive. Hence  $\Psi$  is a compact set of  $\mathbb{R}^m$ . It is clear also that

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) \geq 0, \quad \forall x \in \Psi. \quad (23)$$

Let us show that  $\Psi$  is invariant, i.e., the orbit  $\gamma(x_0) \subset \Psi$  for any  $x_0 \in \Psi$ .

We recall that the orbital derivative is defined by

$$\dot{V}(x) = \langle V'(x), x'(t) \rangle.$$

Then we have

$$\begin{aligned} \dot{V}(x) &= \langle V'(x), x'(t) \rangle \leq -\langle M^{-\frac{1}{2}} C M^{-\frac{1}{2}} x_2, x_2 \rangle - \Phi(H_1^T M^{-\frac{1}{2}} x_2) \\ &= -\sum_{i=1}^{m_1} \lambda_i y_i^2 - \sum_{i=m_1+1}^m \lambda_i y_i^2 - \Phi(H_1^T M^{-\frac{1}{2}} x_2), \end{aligned}$$

hence  $\dot{V}(x) \leq 0, \forall x \in \Psi$ , which implies that  $V(x(\tau; t_0, x_0)) \leq V(x(t_0; t_0, x_0)) = V(x_0) \leq \rho$ . Consequently,  $\gamma(x_0) \subset \Psi$ .

Now, set

$$E_\Psi(F, \varphi, V) = \{x = (x_1, x_2) \in \Psi : x_2 = 0\},$$

and fix any invariant subset  $\mathcal{D}$  of  $E_\Psi(F, \varphi, V)$  and  $z \in \mathcal{D}$ .

The function  $x(\cdot; t_0, z)$  satisfies

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 + M^{-\frac{1}{2}} C M^{-\frac{1}{2}} x_2 + M^{-\frac{1}{2}} \Pi'(M^{-\frac{1}{2}} x_1) \in -M^{-\frac{1}{2}} H_1 \partial \Phi(H_1^T M^{-\frac{1}{2}} x_2). \end{cases} \quad (S)$$

Since  $\mathcal{D}$  is invariant in  $E_\Psi(F, \varphi, V)$ , we have  $\gamma(z) \subset \mathcal{D} \subset E_\Psi(F, \varphi, V)$ . Hence,  $x_2(t; t_0, z) = 0, \forall t \geq t_0$ . Therefore,  $x_1(t; t_0, z) = z, \forall t \geq t_0$ .

Consequently, according to (S) we obtain

$$-M^{-\frac{1}{2}} \Pi'(M^{-\frac{1}{2}} z) \in \partial(\Phi \circ H_1^T M^{-\frac{1}{2}})(0) = M^{-\frac{1}{2}} H_1 \partial \Phi(0),$$

which ensures that

$$-\Pi'(M^{-\frac{1}{2}}z) \in H_1 \partial \Phi(0).$$

Recall that  $q = M^{-\frac{1}{2}}x_1$  and that  $\mathcal{W} = \{\bar{q} \in \mathbb{R}^m: -\Pi'(\bar{q}) \in H_1 \partial \Phi(0)\}$ . It results that  $\mathcal{D} \subset \mathcal{S}(F, \varphi) \cap \Psi$ . Since the largest invariant subset of  $E_\Psi(F, \varphi, V)$  is

$$\mathcal{M}_{\rho^*} = \mathcal{S}(F, \varphi) \cap \text{Int Lev}(V, \rho^*),$$

where  $\rho^* = \max\{\rho: \text{Lev}(V, \rho^*) \subset \mathcal{A}\}$ , it follows that

$$\lim_{\tau \rightarrow +\infty} \text{dist}(q(\tau; t_0, q_0, \dot{q}_0), \mathcal{W}_{\rho^*}) = 0 \quad \text{and} \quad \lim_{\tau \rightarrow +\infty} \frac{dq}{dt}(\tau; t_0, q_0, \dot{q}_0) = 0,$$

where

$$\mathcal{W}_{\rho^*} = \mathcal{W} \cap \{(q, \dot{q}) \in \mathbb{R}^m \times \mathbb{R}^m: V(M^{\frac{1}{2}}q, M^{\frac{1}{2}}\dot{q}) \leq \rho^*\}. \quad (24)$$

The proof of the theorem is then complete.  $\square$

**Remark 3.1.** If the damping matrix  $C$  is not symmetric, then the result of Theorem 3.4 is still valid by replacing  $C$  with its symmetric part  $\frac{C+C^T}{2}$ .

Let us suppose now, that the potential  $\Pi$  is quadratic and given by the following formula:

$$\Pi(x) = \frac{1}{2} \langle Kx, x \rangle, \quad (25)$$

where  $K \in \mathbb{R}^{m \times m}$  is the stiffness matrix. We have the following result:

**Corollary 3.1.** *Suppose that assumptions of Theorem 3.4 are satisfied. Assume that  $\Pi$  is given by (25) and that the matrix  $K$  is symmetric and positive definite. Then the set  $\mathcal{W}_{\rho^*}$  in (24) is convex and compact in  $\mathbb{R}^m$ . Moreover, if  $\partial \Phi(0) \subset \overline{\mathbb{B}}_m(0, \gamma)$ ,<sup>2</sup> and  $\gamma < \frac{\sqrt{2\rho^*}}{\|K^{-1}\|_2 \|H_1\|_2 \sqrt{\|K\|_2}}$ , then  $\mathcal{W}_{\rho^*} = \mathcal{W}$ .*

**Proof.** If  $\Pi(x) = \frac{1}{2} \langle Kx, x \rangle$  with a symmetric and positive definite matrix  $K$ , then

$$\mathcal{W} = \{\bar{q} \in \mathbb{R}^m: -K\bar{q} \in H_1 \partial \Phi(0)\} = -K^{-1} H_1 \partial \Phi(0).$$

Therefore,

$$\mathcal{W}_{\rho^*} = (-K^{-1} H_1 \partial \Phi(0)) \cap \{(q, \dot{q}) \in \mathbb{R}^m \times \mathbb{R}^m: V(M^{\frac{1}{2}}q, M^{\frac{1}{2}}\dot{q}) \leq \rho^*\}.$$

Hence,  $\mathcal{W}_{\rho^*}$  is a convex and compact set in  $\mathbb{R}^m$ .

If  $\partial \Phi(0) \subset \overline{\mathbb{B}}_m(0, \gamma)$ , then  $\mathcal{W} \subset K^{-1} H_1 \overline{\mathbb{B}}_m(0, \gamma)$ . Fix any  $q_0 \in K^{-1} H_1 \partial \Phi(0)$  and note that

$$\frac{1}{2} |\langle Kq_0, q_0 \rangle| \leq \frac{1}{2} \|K\|_2 \|q_0\|_2^2 \leq \frac{1}{2} \gamma^2 \|K\|_2 \|K^{-1}\|_2^2 \|H_1\|_2^2.$$

Consequently, if  $\gamma < \frac{\sqrt{2\rho^*}}{\|K^{-1}\|_2 \|H_1\|_2 \sqrt{\|K\|_2}}$ , then

$$\frac{1}{2} |\langle Kq_0, q_0 \rangle| \leq \rho^*.$$

<sup>2</sup> Where  $\overline{\mathbb{B}}_m(0, \gamma)$  denotes the closed ball of center 0 and radius  $\gamma$  in  $\mathbb{R}^m$ .

Therefore,  $(q_0, 0) \in \{(q, \dot{q}) \in \mathbb{R}^m \times \mathbb{R}^m: V(M^{\frac{1}{2}}q, M^{\frac{1}{2}}\dot{q}) \leq \rho^*\}$ , which implies that

$$\mathcal{W} \subset \{(q, \dot{q}) \in \mathbb{R}^m \times \mathbb{R}^m: V(M^{\frac{1}{2}}q, M^{\frac{1}{2}}\dot{q}) \leq \rho^*\}.$$

Hence,  $\mathcal{W}_{\rho^*} = \mathcal{W}$ .  $\square$

**Remark 3.2.** We give an estimation of the attraction domain. For  $i = 1, \dots, m_1$  let  $\rho_i$  be defined such that

$$\sup_{V(x)=\rho_i} |\langle P_C^T M^{-\frac{1}{2}} x_2, e_i \rangle| = -\frac{1}{\alpha \lambda_i}.$$

By introducing a Lagrange multiplier  $\nu$  for the equality constraint and by setting the gradient of the corresponding Lagrangian function to 0, we obtain

$$\rho_i = \frac{1}{2\alpha^2 \lambda_i^2 \|M^{-\frac{1}{2}} P_C e_i\|_2^2}, \quad i = 1, 2, \dots, m_1. \quad (26)$$

Hence,  $\rho^* = \min_{i=1, \dots, m_1} \rho_i$ .

#### 4. Some examples in unilateral mechanics

In this section, we illustrate our theoretical results by simple examples in unilateral and non-smooth mechanics.

**Example 1.** Let us consider the following simple illustration (see Fig. 1):

$$m\ddot{x}(t) + c\dot{x}(t) + kx \in -\partial\Phi(\dot{x}),$$

where

$$\Phi(x) = \gamma|x| + \frac{\nu}{p}|x|^p, \quad (27)$$

with  $\gamma > 0$  the coefficient of friction,  $\nu \geq 0$  and  $p \in ]1, 2[$ . This is a combination of the Coulomb friction with the  $p$ -friction term  $\frac{\nu}{p}|x|^p$ ,  $p \in ]1, 2[$ .

If the damping coefficient  $c > 0$ , then by Theorem 3.2 we conclude to global attractivity of the set of stationary points  $\mathcal{W} = [-\frac{\gamma}{k}, \frac{\gamma}{k}]$ . Suppose now that  $c < 0$ .

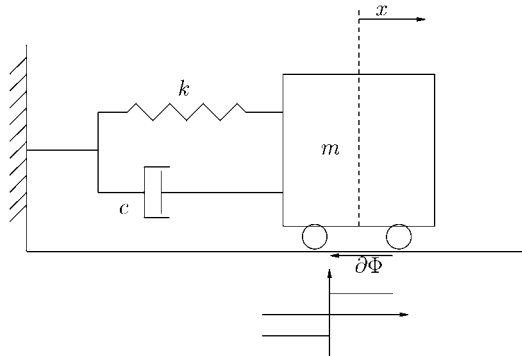


Fig. 1. Example 1.

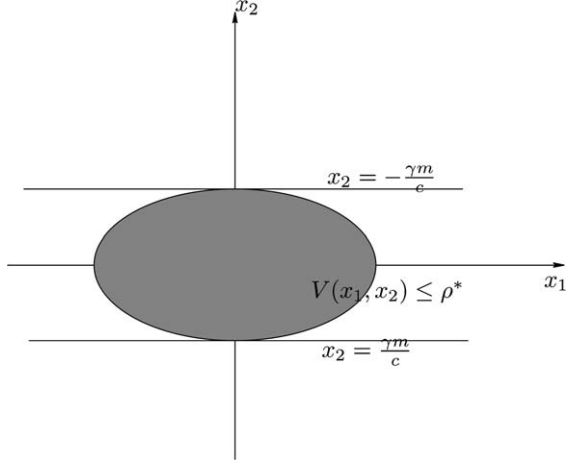


Fig. 2. Example 1.

The associated first order system is given by

$$\dot{x}(t) + A(x(t)) \in -\partial\varphi(x(t)),$$

where

$$A = \begin{pmatrix} 0 & -1 \\ \frac{k}{m} & \frac{c}{m} \end{pmatrix},$$

and  $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,

$$x = (x_1, x_2) \mapsto \varphi(x) = \Phi\left(\frac{x_2}{\sqrt{m}}\right) = \frac{\gamma}{\sqrt{m}}|x_2| + \frac{\nu}{p}\left|\frac{x_2}{\sqrt{m}}\right|^p.$$

We take  $V(x_1, x_2) = \frac{1}{2}\frac{k}{m}x_1^2 + \frac{1}{2}x_2^2$ . We have

$$\begin{aligned} \langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) &= \frac{c}{m}x_2^2 + \Phi\left(\frac{1}{\sqrt{m}}x_2\right) \\ &= \frac{c}{m}x_2^2 + \frac{\gamma}{\sqrt{m}}|x_2| + \frac{\nu}{pm^{p/2}}|x_2|^p. \end{aligned}$$

Here Lemma 3.1 is satisfied with  $\alpha = \frac{\sqrt{m}}{\gamma}$  and  $\lambda_i = \frac{c}{m}$ .

Using (26), we have (see Fig. 2)

$$\rho^* = \frac{m^2\gamma^2}{2c^2}.$$

Using Corollary 3.1, we see that if  $c \in ]-m\sqrt{k}, 0[$ , then  $\mathcal{W}^* = [m\gamma/(c\sqrt{k}), -m\gamma/(c\sqrt{k})]$  and if  $c \in ]-\infty, -m\sqrt{k}[$ , then  $\mathcal{W}^* = \mathcal{W} = [-\gamma/k, \gamma/k]$ .

**Example 2.** A coupled system of rotational oscillators with two friction elements.

Consider a system of two moving masses respectively attached to rotational springs (see Fig. 3). The two masses are coupled by a viscous damper, characterized by the coefficient  $c < 0$ .

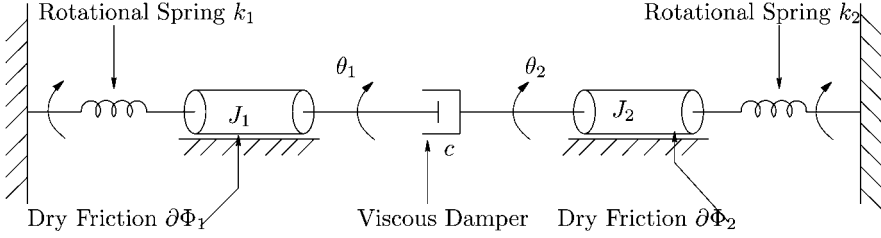


Fig. 3. Example 2.

The angular displacements of the masses are measured by  $\theta_1$  and  $\theta_2$ , respectively. The angular velocities and accelerations are denoted respectively by  $\dot{\theta}_i$  and  $\ddot{\theta}_i$ ,  $i = 1, 2$ . Without loss of generality, the moments of inertia of the respective masses are assumed to be equal to 1. The forces exerted by the two springs are supposed to derive from the potentials  $f_i(x) = \frac{1}{2}k_i x^2$ ,  $i = 1, 2$ . We assume moreover that the contact of each mass with its support generates a dry friction characterized by  $-\partial\Phi_i(\dot{\theta}_i)$ ,  $i = 1, 2$ .

By applying Newton's second law for rotational systems, we get the following torque equations:

$$\ddot{\theta}_1 + c(\dot{\theta}_1 - \dot{\theta}_2) + k_1\theta_1 \in -\partial\Phi_1(\dot{\theta}_1), \quad (28)$$

$$\ddot{\theta}_2 + c(\dot{\theta}_2 - \dot{\theta}_1) + k_2\theta_2 \in -\partial\Phi_2(\dot{\theta}_2). \quad (29)$$

Equations (28)–(29) can be written in the following form:

$$M\ddot{\theta} + C\dot{\theta} + \Pi'(\theta) \in -H_1\partial\Phi(H_1^T\dot{\theta}),$$

with  $\theta = (\theta_1, \theta_2)^T$ ,  $M = I_2$ ,  $C = \begin{pmatrix} c & -c \\ -c & c \end{pmatrix}$ ,  $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto \Phi(x, y) = \Phi_1(x) + \Phi_2(y)$ ,  $H_1 = I_2$  and  $\Pi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto \frac{1}{2}k_1x^2 + \frac{1}{2}k_2y^2$ .

It is clear that

$$\partial\Phi(x, y) = \begin{pmatrix} \partial\Phi_1(x) \\ \partial\Phi_2(y) \end{pmatrix}, \quad DC = \begin{pmatrix} 0 & 0 \\ 0 & 2c \end{pmatrix} \quad \text{and} \quad PC = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since  $\text{span}(H_1) = \mathbb{R}^2$ , it is clear that the assumption in Lemma 3.1 is satisfied.

For  $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^4$  (with  $(x_1) = (\theta_1, \theta_2)^T$  and  $x_2 = (\dot{\theta}_1, \dot{\theta}_2)$ ), we consider the following Lyapunov function:

$$V(\mathbf{x}) = \frac{1}{2}\langle Kx_1, x_2 \rangle + \frac{1}{2}\|x_2\|_2^2,$$

where  $K = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ .

We have

$$\langle F(x), V'(x) \rangle + \varphi(x) - \varphi(x - V'(x)) = \langle Cx_2, x_2 \rangle + \Phi(x_2).$$

If we take for example  $\Phi_1$  and  $\Phi_2$  of the form (27), i.e.,  $\Phi_1(x) = \gamma_1|x| + \frac{\nu_1}{p}|x|^p$  and  $\Phi_2(x) = \gamma_2|x| + \frac{\nu_2}{p}|x|^p$  with  $\gamma_1, \gamma_2 > 0$ ,  $\nu_1, \nu_2 \geq 0$  and  $p \in ]1, 2[$ . Then it is clear that Lemma 3.1 is

satisfied with  $\alpha = \frac{\sqrt{2}}{\min(\gamma_1, \gamma_2)}$ .

Using (26), we have

$$\rho^* = \frac{\min(\gamma_1, \gamma_2)^2}{16c^2}.$$

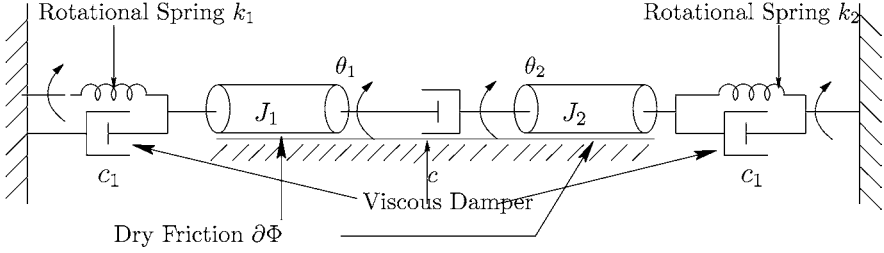


Fig. 4. Example 3.

**Example 3.** A coupled system of rotational oscillators with one friction elements.

Consider a system of two moving masses attached to rotational springs as depicted in Fig. 4. By applying Newton's second law for rotational systems, we get the following torque equations:

$$\begin{aligned} \ddot{\theta}_1 + c_1 \dot{\theta}_1 + c(\dot{\theta}_1 - \dot{\theta}_2) + k_1 \theta_1 &\in -\partial\Phi(\dot{\theta}_1 - \dot{\theta}_2), \\ \ddot{\theta}_2 + c_1 \dot{\theta}_2 + c(\dot{\theta}_2 - \dot{\theta}_1) + k_2 \theta_2 &\in -\partial\Phi(\dot{\theta}_2 - \dot{\theta}_1). \end{aligned}$$

The equation of motion of this system can be rewritten in the form

$$M\ddot{\theta} + C\dot{\theta} + \Pi'(\theta) \in -H_1 \partial\Phi(H_1^T \dot{\theta}),$$

with  $\theta = (\theta_1, \theta_2)^T$ ,  $M = I_2$ ,  $C = \begin{pmatrix} c+c_1 & -c \\ -c & c+c_1 \end{pmatrix}$ ,  $H_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ,  $\Pi: \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $(x, y) \mapsto \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2$  and  $\Phi: \mathbb{R} \rightarrow \mathbb{R}$  of the form (27), i.e.  $x \mapsto \Phi(x) = \gamma|x| + \frac{\nu}{p}|x|^p$ , with  $\gamma > 0$ ,  $\nu \geq 0$  and  $p \in ]1, 2[$ .

It is clear that

$$D_C = \begin{pmatrix} c_1 & 0 \\ 0 & c_1 + 2c \end{pmatrix} \quad \text{and} \quad P_C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

If  $c_1 > 0$  and  $c_1 + 2c > 0$ , then the matrix  $C$  is positive definite and the global attractivity of the stationary points  $\mathcal{W}$  is assured due to Theorem 3.2.

If  $c_1 > 0$  and  $c_1 + c < 0$ , then the assumption of Lemma 3.1 is satisfied and the local attractivity of the stationary points  $\mathcal{W}$  is guaranteed by Theorem 3.4.

## 5. Conclusion

In this paper we have provided some conditions ensuring the global or the local attractivity (depending whether the viscous damping matrix is positive definite or not) of the set of stationary solution of a non-smooth second order dynamical system with friction. An estimation of the domain of attraction was also discussed. These results extend and unify some existing result [2,16].

It is well known that the Coulomb friction is not sufficient if we are looking for the stick-slip phenomena (see, for example, E. Rabinowicz [14]). However, with a slip velocity dependent coefficient, this leads to a non-monotone differential inclusion, so it would be interesting to obtain some similar results for the non-convex case. This is out of the scope of this paper and will be probably the subject of another paper.



## Acknowledgments

The author thanks Professor Lionel Thibault for some stimulating discussions, which contributed to improve the presentation of the manuscript and the anonymous referees for their remarks which helped to improve the clarity of this paper.

## References

- [1] S. Adly, D. Goeleven, A stability theory for second order non-smooth dynamical systems with application to friction problems, *J. Math. Pures Appl.* 83 (2004) 17–51.
- [2] J. Alvarez, I. Orlov, L. Acho, An invariance principle for discontinuous dynamic systems with applications to a Coulomb friction oscillator, *J. Dynam. Systems Measurement Control* 122 (2000) 687–690.
- [3] B. Brogliato, On the control of nonsmooth complementarity dynamical systems, in: *Philos. Trans. Roy. Soc. Ser. A*, vol. 359, 2001, pp. 2369–2384.
- [4] B. Brogliato, *Nonsmooth Mechanics*, second ed., *Commun. Control Engrg. Ser.*, Springer, London, 1999.
- [5] G. Genta, *Vibration of Structures and Machines, Practical Aspects*, Springer, New York, 1998.
- [6] B. Goodwine, G. Stépán, Controlling unstable rolling phenomena, *J. Vibration Control* 6 (2000) 137–158.
- [7] D. Goeleven, B. Brogliato, Necessary conditions of asymptotic stability for unilateral dynamical systems, 2005, submitted for publication.
- [8] D. Goeleven, D. Motreanu, V. Motreanu, On the stability of stationary solutions of first order evolution variational inequalities, *Adv. Nonlinear Var. Inequal.* 6 (2003) 1–30.
- [9] D. Goeleven, B. Brogliato, Stability and unstability matrices for linear evolution variational inequalities, 2002, submitted for publication.
- [10] T. Kato, Accretive operators and nonlinear evolutions equations in Banach spaces, in: *Nonlinear Functional Analysis*, in: *Proc. Sympos. Pure Math.*, vol. 18(1), 1970.
- [11] R.K. Miller, A.N. Michel, *Ordinary Differential Equations*, Academic Press, 1982.
- [12] P. Quittner, On the principle of linearized stability for variational inequalities, *Math. Ann.* 283 (1989) 257–270.
- [13] P. Quittner, An instability criterion for variational inequalities, *Nonlinear Anal.* 15 (1990) 1167–1180.
- [14] E. Rabinowicz, The intrinsic variables affecting the stick-slip process, *Proc. Roy. Phys. Soc.* 71 (1958) 668–675.
- [15] H. Troger, A. Steindl, *Nonlinear Stability and Vibration Theory*, Springer, Berlin, 1991.
- [16] N. Van De Wouw, R.I. Leine, Attractivity of equilibrium sets of systems with dry friction, *Nonlinear Dynam.* 35 (2004) 19–39.

## Further reading

- [17] J.A.C. Martins, S. Barbarin, M. Raous, A. Pinto da Costa, Dynamic stability of finite linearly elastic systems with unilateral contact and Coulomb friction, *Comput. Methods Appl. Mech. Engrg.* 177 (1999) 289–328.
- [18] D.R. Merkin, *Introduction to the Theory of Stability*, Springer, New York, 1996.
- [19] N. Rouche, J. Mawhin, *Equations Différentielles Ordinaires*, Tome 2, Masson and Cie, Paris, 1973.