HAL open science

Surface framed braids

Paolo Bellingeri, Sylvain Gervais

▶ To cite this version:

Paolo Bellingeri, Sylvain Gervais. Surface framed braids. Geometriae Dedicata, 2012, 159, pp.51-69. hal-00450226v2

HAL Id: hal-00450226
https://hal.science/hal-00450226v2

Submitted on 11 May 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Surface framed braids

Paolo Bellingeri and Sylvain Gervais

Abstract

In this paper we introduce the framed pure braid group on n strands of an oriented surface, a topological generalisation of the pure braid group P_n. We give different equivalents definitions for framed pure braid groups and we study exact sequences relating these groups with other generalisations of P_n, usually called surface pure braid groups. The notion of surface framed braid groups is also introduced.

1 Introduction

Let $\Sigma = \Sigma_{g,b}$ be an oriented surface of genus g with b boundary components (we will note $\Sigma_g := \Sigma_{g,0}$) and let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points (*punctures*) in the interior of Σ. Let $C_n(\Sigma) = \Sigma^n \setminus \Delta$, where Δ is the set of n-tuples $\underline{x} = (x_1, \ldots, x_n)$ for which $x_i = x_j$ for some $i \neq j$. The fundamental group $\pi_1(C_n(\Sigma), \underline{p})$ is called *pure braid group on n strands of the surface* Σ; it shall be denoted by $P_n(\Sigma)$.
The symmetric group $\mathfrak{S}_n$ acts freely on $C_n(\Sigma)$ by permutation of coordinates. We denote $\widehat{C_n}(\Sigma)$ the quotient space $C_n(\Sigma)/\mathfrak{S}_n$. The fundamental group of $\widehat{C_n}(\Sigma)$ is called *braid group on n strands of the surface* Σ; it shall be denoted by $B_n(\Sigma)$.
Since the projection map $C_n(\Sigma) \longrightarrow \widehat{C_n}(\Sigma)$ is a regular covering space with transformation group $\mathfrak{S}_n$, one has the following exact sequence:

$$1 \longrightarrow P_n(\Sigma) \longrightarrow B_n(\Sigma) \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

On the other hand, from the homotopy exact sequence associated to the fibration $C_{n+m}(\Sigma) \longrightarrow C_n(\Sigma)$, we get ([FN, Bir]) an exact sequence:

$$(SPB) \quad 1 \longrightarrow P_m\left(\Sigma \setminus n \text{ points }\right) \longrightarrow P_{n+m}(\Sigma) \longrightarrow P_n(\Sigma) \longrightarrow 1$$

when Σ has positive genus or genus equal to zero and non-empty boundary (in the case of the sphere such a sequence holds for $n + m \geq 4$). In the following we will denote this sequence by (SPB) (*Surface Pure Braids*).

2000 AMS Mathematics Subject Classification: 20F36, 57M05

When Σ has boundary, the (SPB) sequence has an evident geometric section, which corresponds to add m punctures "at the infinity" (see [B2, GG]); if Σ is closed and different from the sphere and the torus, the (SPB) sequence splits if and only if $n = 1$ (see [GG] for the general case and [BB] for an algebraical section in the case $n = 1$). In the case of the disk this sequence has some additional features [FR] and it is a powerful tool in the study of finite type invariants for links [P]. In the general case the (SPB) sequence can be used to find a group presentation for $P_n(\Sigma)$ (see for instance [B2]).

In this paper we introduce framed braid groups $FB_n(\Sigma)$ and $FP_n(\Sigma)$ of a surface Σ, which generalise respectively framed braid groups introduced in [KS] and framed pure braid groups considered in Theorem 5.1 of [MM]. These groups turn out also to be related to generalisations of Hilden groups introduced in [BC]. For surfaces of genus greater than 1, with boundary or closed, we give three equivalent definitions of these groups: in terms of configuration spaces, as subgroups of mapping class groups (Section 2) and as subgroups of braid groups of surfaces (Section 6 and 7). We prove that, when the surface is closed and of genus greater than 1, these groups are non trivial central extensions of surface pure braid groups (Section 4) and we provide a group presentation for $FP_n(\Sigma)$ (Section 5, Theorem 8) and therefore for $FB_n(\Sigma)$ (Theorem 13). We show also that the sequence (SPB) extends naturally to a sequence on framed braids, that we will call *framed surface pure braid sequence* (denoted by (FSPB)) and which splits even in the case of closed surfaces (Section 3, Theorem 4). In the case of the torus the proposed definitions are not equivalent and let arise different notions of framings: this case will treated separately in the last Section.

2 Framed braids: possible definitions

2.1 Framed braids via configuration spaces

Let $U\Sigma$ be the unit tangent bundle of Σ and $\pi : U\Sigma \longrightarrow \Sigma$ be the natural projection. We denote by $F_n(\Sigma)$ the subspace $(\pi^n)^{-1}(C_n(\Sigma))$ of $(U\Sigma)^n$ and fix a unit tangent vector v_i of Σ at p_i such that $F_n(\Sigma)$ is based at $\underline{v} = (p_i, v_i)_{i=1,\dots,n}$. The symmetric group $\mathfrak{S}_n$ acts freely on $F_n(\Sigma)$: we denote $\widehat{F_n}(\Sigma)$ the quotient space $F_n(\Sigma)/\mathfrak{S}_n$.

Definition 1 *The* pure framed braid group $FP_n(\Sigma)$ *on n strands of Σ is the fundamental group of $F_n(\Sigma)$. The* framed braid group *on n strands of Σ is the fundamental group of $\widehat{F_n}(\Sigma)$.*

Thus, a framed braid can be seen as a family of n continuous paths $b_i : [0,1] \longrightarrow U\Sigma$ for $i = 1, \ldots, n$ such that :

1) $b_i(0) = (p_i, v_i)$ for all $i \in \{1, \ldots, n\}$;

2) $\exists \sigma \in \mathfrak{S}_n$ such that $b_i(1) = (p_{\sigma(i)}, v_{\sigma(i)})$ for all $i \in \{1, \ldots, n\}$;

3) $\pi b_i(t) \neq \pi b_j(t)$ when $i \neq j$ for any $t \in [0,1]$.

Since the projection map $F_n(\Sigma) \longrightarrow \widehat{F_n}(\Sigma)$ is a regular covering space with transformation group $\mathfrak{S}_n$, the framed braid group and the pure framed braid group are related by the following exact sequence:

$$1 \longrightarrow FP_n(\Sigma) \longrightarrow FB_n(\Sigma) \longrightarrow \mathfrak{S}_n \longrightarrow 1 \tag{1}$$

2.2 Framed braids as mapping classes

In this section we will give an interpretation of framed braid groups of an oriented surface different from the sphere and the torus in terms of mapping classes.

2.2.1 Notations

Let $\mathrm{Diff}^+(\Sigma_{g,b})$ denote the group of orientation preserving diffeomorphisms of $\Sigma_{g,b}$ which are the identity on the boundary. Recall that the *mapping class group* of $\Sigma_{g,b}$, denoted $\Gamma_{g,b}$, is defined to be $\pi_0(\mathrm{Diff}^+(\Sigma_{g,b}))$, where $\mathrm{Diff}^+(\Sigma_{g,b})$ is equipped with the compact open topology. Note that we will denote the composition in the mapping class groups from left to right[1]. In the following, greek letters will be used to denote simple closed curves on $\Sigma_{g,b}$ and if α is such a curve, τ_α will denote the Dehn twist along α.
We shall also consider different subgroups of $\mathrm{Diff}^+(\Sigma_{g,b})$ and associated mapping class groups:

• $\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P}) = \{h \in \mathrm{Diff}^+(\Sigma_{g,b}) \,/\, \exists \sigma \in \mathfrak{S}_n,\ h(p_i) = p_{\sigma(i)}\}$, and the *punctured mapping class group* $\Gamma^n_{g,b} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P}))$;

• $\mathrm{Diff}^+(\Sigma_{g,b}, \underline{p}) = \{h \in \mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{P}) \,/\, h(p_i) = p_i\}$ and the *pure punctured mapping class group* $\mathrm{P}\Gamma^n_{g,b} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,b}, \underline{p}))$;

• $\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{V}) = \{h \in \mathrm{Diff}^+(\Sigma_{g,b}) \,/\, \exists \sigma \in \mathfrak{S}_n,\ h(p_i) = p_{\sigma(i)} \text{ and } d_{p_i}h(v_i) = v_{\sigma(i)}\}$ (where $\mathcal{V} = \{(p_1, v_1), \ldots, (p_n, v_n)\}$ is a set of n distinct points on $\Sigma_{g,b}$ equipped with n unit tangent vectors $v_1, \ldots, v_n$) and the *framed punctured mapping class group* $\mathrm{F}\Gamma^n_{g,b} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,b}, \mathcal{V}))$;

[1] We do this in order to have the same group-composition in braid groups and mapping class groups.

• $\mathrm{Diff}^+(\Sigma_{g,b},\underline{v}) = \{h \in \mathrm{Diff}^+(\Sigma_{g,b},\mathcal{V}) \,/\, h(p_i) = p_i \text{ and } d_{p_i}h(v_i) = v_i\}$, and the *pure framed punctured mapping class group* $\mathrm{PF}\Gamma^n_{g,b} = \pi_0(\mathrm{Diff}^+(\Sigma_{g,b},\underline{v}))$.

Remark 1 A preserving orientation diffeomorphism h of $\Sigma_{g,b}$ such that $h(p_i) = p_i$ and $d_{p_i}h(v_i) = v_i$ is isotopic to a diffeomorphism which is equal to identity on a small disc around p_i. Thus, the two groups $\mathrm{PF}\Gamma^n_{g,b}$ and $\Gamma_{g,b+n}$ are isomorphic.

Let us also recall that if Σ' is a subsurface of Σ and $i : \Sigma' \hookrightarrow \Sigma$ is the inclusion map, there is a canonical morphism i_* from the mapping class group $\Gamma(\Sigma')$ of Σ' to the mapping class group $\Gamma(\Sigma)$ of Σ, which consists in extending each diffeomorphism h of Σ' by identity on $\Sigma \setminus \Sigma'$. When Σ is a genus g surface with b boundary components and $\Sigma \setminus \Sigma'$ is a collection of n disjoint discs (see Figure 1), we shall denote by $\lambda^n_{g,b} : \Gamma_{g,b+n} \longrightarrow \Gamma_{g,b}$ this morphism (λ^n_g when $b = 0$).

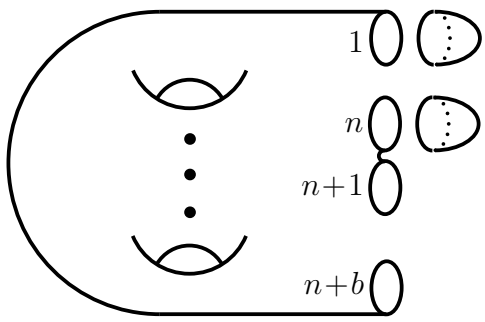

Figure 1: The embedding $\Sigma_{g,b+n} \hookrightarrow \Sigma_{g,b}$.

2.2.2 Braids groups as mapping class groups

Braid groups of surface are related to mapping class groups as follows (see [Bir, S1]):

Proposition 1 *Let $n \geq 1$. Let $\psi_n : \Gamma^n_{g,b} \longrightarrow \Gamma_{g,b}$ and $\varphi_n : \mathrm{P}\Gamma^n_{g,b} \longrightarrow \Gamma_{g,b}$ be the homomorphisms induced by the forgetting map $\mathrm{Diff}^+(\Sigma_{g,b},\mathcal{P}) \longrightarrow \mathrm{Diff}^+(\Sigma_{g,b})$ and $\mathrm{Diff}^+(\Sigma_{g,b},\underline{p}) \longrightarrow \mathrm{Diff}^+(\Sigma_{g,b})$.*
1) If $(g,b) \notin \{(0,0),(1,0)\}$, Ker($\psi_n$) and Ker($\varphi_n$) are respectively isomorphic to $B_n(\Sigma_{g,b})$ and $P_n(\Sigma_{g,b})$.
2) When $b = 0$ and $g \in \{0,1\}$, Ker(ψ_n) and Ker(φ_n) are respectively isomorphic to $B_n(\Sigma_g)/Z(B_n(\Sigma_g))$ and $P_n(\Sigma_g)/Z(P_n(\Sigma_g))$ (where $Z(G)$ is the center of the group G).

Remarks 2
1) Let $\mathbb{T}^2$ be a torus; one has $Z(B_n(\mathbb{T}^2)) = Z(P_n(\mathbb{T}^2)) = \mathbb{Z}^2$ (see [PR1]).

2) Let S^2 be a sphere; one has $B_1(S^2) = P_1(S^2) = 1$, $P_2(S^2) = 1$, $B_2(S^2) = \mathbb{Z}/2\mathbb{Z}$ and for $n \geq 3$, $Z(B_n(S^2)) = Z(P_n(S^2)) \approx \mathbb{Z}/2\mathbb{Z}$ (see [GV]).

We can provide a similar result for framed braids:

Proposition 2 *Let $n \geq 1$. Let $\Psi_n : \mathrm{F}\Gamma^n_{g,b} \longrightarrow \Gamma_{g,b}$ and $\Phi_n : \mathrm{PF}\Gamma^n_{g,b} \longrightarrow \Gamma_{g,b}$ be the homomorphism induced by the maps which forget the tangent vectors and the punctures. If $(g,b) \notin \{(0,0),(1,0)\}$, Ker$(\Psi_n)$ and Ker(Φ_n) are respectively isomorphic to $FB_n(\Sigma_{g,b})$ and $FP_n(\Sigma_{g,b})$.*

Proof. Following [Bir], we consider the evaluation map $Ev : \mathrm{Diff}^+(\Sigma_{g,b}) \longrightarrow F_n(\Sigma_{g,b})$ defined by $Ev(h) = \big(h(p_i), d_{p_i}h(v_i)\big)$. It is a locally trivial fibering with fiber $\mathrm{Diff}^+(\Sigma_{g,b}, \underline{v})$. The long exact sequence of homotopy groups of this fibration gives the following exact sequence:

$$\cdots \longrightarrow \pi_1\big(\mathrm{Diff}^+(\Sigma_{g,b})\big) \xrightarrow{Ev_*} \pi_1\big(F_n(\Sigma_{g,b})\big) \xrightarrow{\partial_*} \pi_0\big(\mathrm{Diff}^+(\Sigma_{g,b}, \underline{v})\big)$$
$$\xrightarrow{\Phi_n} \pi_0\big(\mathrm{Diff}^+(\Sigma_{g,b})\big) \xrightarrow{Ev_*} \pi_0\big(F_n(\Sigma_{g,b})\big).$$

Now, $\pi_0\big(F_n(\Sigma_{g,b})\big)$ is trivial, and if $\Sigma_{g,b}$ is not the sphere S^2 or the torus $\mathbb{T}^2$, $\mathrm{Diff}^+(\Sigma_{g,b})$ is contractible (see [S2]). Thus, we get the required result for Φ_n.
The proof for Ψ_n is analogous and is left to the reader; it suffices to consider the evaluation map $\widehat{E_v} : \mathrm{Diff}^+(\Sigma_{g,b}) \longrightarrow \widehat{F_n}(\Sigma_{g,b})$ defined by $\widehat{Ev}(h) = \overline{\big(h(p_i), d_{p_i}h(v_i)\big)}$. □

The definition of framed braid groups in terms of mapping classes and tangent vectors provided in Proposition 2 was introduced in [OS] as the definition of framed braid groups of surfaces with one boundary component.

We can provide an equivalent definition of pure framed braid groups as the kernel of the mapping induced by the inclusion of a surface with boundary components into another one.

Corollary 3 *For all (g,b) distinct from $(0,0)$ and $(1,0)$, the kernel of the morphism $\lambda^n_{g,b} : \Gamma_{g,b+n} \longrightarrow \Gamma_{g,b}$ induced by capping n boundary components is isomorphic to the pure framed braid group $FP_n(\Sigma_{g,b})$.*

Proof. Via the isomorphism $\mathrm{PF}\Gamma^n_{g,b} \approx \Gamma_{g,b+n}$, the homorphism Φ_n coincides with $\lambda^n_{g,b}$. □

3 The (FSPB) sequence

In this section, we prove the existence of a framed version of the (SPB) sequence. We also prove that it splits even in the case of closed surfaces of genus greater or equal than 2.

Theorem 4 *For $g \geq 2$, $b \geq 0$, $n \geq 0$ and $m \geq 0$, one has the following splitting exact sequence :*

$$(FSPB) \qquad 1 \longrightarrow FP_m(\Sigma_{g,b+n}) \longrightarrow FP_{n+m}(\Sigma_{g,b}) \xrightarrow{\alpha_{n,m}} FP_n(\Sigma_{g,b}) \longrightarrow 1$$

where $\alpha_{n,m}$ consists in forgetting the first m strands.

Proof. Using Proposition 2 and Corollary 3, one has the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & FP_m(\Sigma_{g,b+n}) \arrow[r] \arrow[d, equal] & FP_{n+m}(\Sigma_{g,b}) \arrow[r, "(\lambda^m_{g,b+n})_|"] \arrow[d] & FP_n(\Sigma_{g,b}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & FP_m(\Sigma_{g,b+n}) \arrow[r] & \Gamma_{g,b+n+m} \arrow[r, "\lambda^m_{g,b+n}"] \arrow[d, "\lambda^{n+m}_{g,b}"] & \Gamma_{g,b+n} \arrow[r] \arrow[d, "\lambda^n_{g,b}"] & 1 \\
 & & \Gamma_{g,b} \arrow[r, equal] \arrow[d] & \Gamma_{g,b} \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

where $(\lambda^m_{g,b+n})_|$ is the restriction of $\lambda^m_{g,b+n}$ to $FP_{n+m}(\Sigma_{g,b})$. Since, via the isomorphism $FP_{n+m}(\Sigma_{g,b}) \approx \mathrm{Ker}(\lambda^{n+m}_{g,b})$, forgetting m strands corresponds to capping m boundary components, the first row is the required exact sequence.
The splitting of the (FSBP) sequence is obtained considering an embedding ι of $\Sigma_{g,b+n}$ into $\Sigma_{g,b+n+m}$ as in figure 2. The induced morphism $\iota_* : \Gamma_{g,b+n} \longrightarrow \Gamma_{g,b+n+m}$ satisfies $\lambda^m_{g,b+n} \circ \iota_* = \mathrm{Id}_{\Gamma_{g,b+n}}$ and its restriction $\iota_{*|} : FP_n(\Sigma_{g,b}) \longrightarrow FP_{n+m}(\Sigma_{g,b})$ is a section for $(\lambda^m_{g,b+n})_|$.

□

Remark 3 Note that the splitting of (FSPB) sequence consists in cabling the first framed strand while the splitting of (SPB) sequence in the case of surfaces with boundary consists in adding m strands at the infinity. If Σ has boundary, adding m framed strands at the infinity gives another splitting of (FSPB) sequence.

4 Framed braids vs classical braids

The following Theorem shows that the group $FP_n(\Sigma_g)$ is a non trivial central extension of $P_n(\Sigma_g)$ by $\mathbb{Z}^n$.

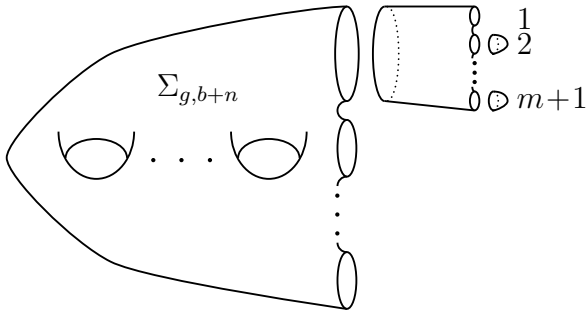

Figure 2: ι is a section of λ^m_{b+n}.

THEOREM 5 *Let $\Sigma_{g,b}$ a surface of genus $g \geq 2$ with b boundary components.*
1) If $\Sigma_{g,b}$ has boundary, the framed pure braid group $FP_n(\Sigma_{g,b})$ is isomorphic to $\mathbb{Z}^n \times P_n(\Sigma_{g,b})$.
2) If Σ_g is a closed surface, there is a non-splitting central extension

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow FP_n(\Sigma_g) \xrightarrow{\beta_n} P_n(\Sigma_g) \longrightarrow 1 \tag{2}$$

*where β_n is the morphism induced by the projection map $F_n(\Sigma_g) \longrightarrow C_n(\Sigma_g)$ (*i.e. *β_n consists in forgetting the framing).*
3) In the two cases, $F_n(\Sigma_{g,b})$ is an Eilenberg-Maclane space of type $(FP_n(\Sigma_{g,b}), 1)$.

Proof. If $\Sigma_{g,b}$ has boundary, $\Sigma_{g,b}$ is parallelizable and the unit tangent bundle $U\Sigma_{g,b}$ is homeomorphic to $S^1 \times \Sigma_{g,b}$. Thus, $F_n(\Sigma_{g,b})$ is homeomorphic to $(S^1)^n \times C_n(\Sigma_{g,b})$ and $FP_n(\Sigma_{g,b})$ is isomorphic to $\mathbb{Z}^n \times P_n(\Sigma_{g,b})$. Furthermore, since S^1 and $C_n(\Sigma_{g,b})$ are Eilenberg-Maclane spaces (see [PR1]), $F_n(\Sigma_{g,b})$ is also one.

Now, suppose that Σ_g is a closed surface of genus greater than 1 and consider the exact sequence of homotopy groups of the locally trivial fibrations (with fiber $(S^1)^n$) $F_n(\Sigma_g) \longrightarrow C_n(\Sigma_g)$:

$$\cdots \longrightarrow \pi_n\big((S^1)^n\big) \longrightarrow \pi_n\big(F_n(\Sigma_{g,b})\big) \longrightarrow \pi_n\big(C_n(\Sigma_{g,b})\big) \longrightarrow \pi_{n-1}\big((S^1)^n\big) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_2\big(C_n(\Sigma_{g,b})\big) \longrightarrow \pi_1\big((S^1)^n\big) \longrightarrow \pi_1\big(F_n(\Sigma_{g,b})\big) \xrightarrow{\beta_n} \pi_1\big(C_n(\Sigma_{g,b})\big) \longrightarrow 1$$

Since $C_n(\Sigma_g)$ is an Eilenberg-Maclane spaces (see [PR1]), this sequence leads to the sequence (2) and to the third point of the Theorem.
Now, using Propositions 1 and 2 one has the following commutative diagram with

exact rows and columns:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, equal] & FP_n(\Sigma_g) \arrow[r] \arrow[d] & P_n(\Sigma_g) \arrow[r] \arrow[d] & 1 & (2) \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] & \Gamma_{g,n} \arrow[r] \arrow[d, "\Phi_n"] & \mathrm{P}\Gamma_g^n \arrow[r] \arrow[d, "\varphi_n"] & 1 & (3) \\
 & & \Gamma_g \arrow[r, equal] \arrow[d] & \Gamma_g \arrow[d] & & \\
 & & 1 & 1 & &
\end{tikzcd}$$

where the exact sequence (3) is obtained by capping n boundary components of $\Sigma_{g,n}$ by n punctured discs (see [PR2]). In this sequence, $\mathbb{Z}^n$ can be seen as a subgroup of $\Gamma_{g,n}$ generated by Dehn twists along curves parallel to boundary components, thus (3) is a central extension of $\mathrm{P}\Gamma_g^n$. Therefore, (2) is a central extension of $P_n(\Sigma_g)$.

To conclude, we shall prove that this sequence does not split. First we remark that for $n = 1$ we obtain the canonical projection of the fundamental group of the unit tangent space into the fundamental group of the surface which splits if and only if the surface is not closed (we are considering the case $g \geq 2$). Thus, let $n \geq 2$ and let us consider the following diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, equal] & FP_{n+1}(\Sigma_g) \arrow[r, "\beta_{n+1}"] \arrow[d, shift left, "(\lambda^1_{g,n})_|"] & P_{n+1}(\Sigma_g) \arrow[r] \arrow[d, "p_*"] & 1 \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] & FP_n(\Sigma_g) \arrow[r, "\beta_n"] \arrow[u, shift left, "(\iota_*)_|"] & P_n(\Sigma_g) \arrow[r] & 1
\end{tikzcd}$$

where ι_* is the section of $\lambda^1_{g,n}$ described in figure 2 and p_* the map wich consists in forgetting one strand. One can easily verify that the diagram is commutative. Suppose that there is a section $s_n : P_n(\Sigma_g) \longrightarrow FP_n(\Sigma_g)$ for β_n.

The composition $\beta_{n+1} \circ \iota_* \circ s_n : P_n(\Sigma_g) \longrightarrow P_{n+1}(\Sigma_g)$ is therefore a section for p_*: in fact $p_* \circ \beta_{n+1} \circ \iota_* \circ s_n = \beta_n \circ \lambda^n_g \circ \iota_* \circ s_n = Id_{P_n(\Sigma_g)}$. This is impossible, since p_* has no section when $n \geq 2$ and $g > 1$ [GG]. $\square$

Remark 4 The splitting of sequence (2) in the case of surfaces with boundary was proven in Lemma 19 of [BGG] in a combinatorial way.

Proposition 6 *Let $\Sigma_{g,b}$ be a surface different from the sphere S^2 and the torus $\mathbb{T}^2$.*
1) there is an exact sequence

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow FB_n(\Sigma_{g,b}) \xrightarrow{\widehat{\beta}_n} B_n(\Sigma_{g,b}) \longrightarrow 1 \ , \tag{4}$$

where $\widehat{\beta}_n$ consists in forgetting the framing. This sequence splits if $\Sigma_{g,b}$ has non-empty boundary.
2) $\widehat{F_n}(\Sigma_{g,b})$ is an Eilenberg-Maclane space of type $(FB_n(\Sigma_{g,b}), 1)$.

Proof. Using local trivialisations of the fibering $F_n(\Sigma_{g,b}) \longrightarrow C_n(\Sigma_{g,b})$ and the covering spaces $C_n(\Sigma_{g,b}) \longrightarrow \widehat{C_n}(\Sigma_{g,b})$ and $F_n(\Sigma_{g,b}) \longrightarrow \widehat{F_n}(\Sigma_{g,b})$, one can easily see that $\widehat{F_n}(\Sigma_{g,b}) \longrightarrow \widehat{C_n}(\Sigma_{g,b})$ is a locally trivial bundle with fibre $(S^1)^n$. The exact sequence of homotopy groups of this fibration is

$$\cdots \longrightarrow \pi_n\big((S^1)^n\big) \longrightarrow \pi_n\big(\widehat{F_n}(\Sigma_{g,b})\big) \longrightarrow \pi_n\big(\widehat{C_n}(\Sigma_{g,b})\big) \longrightarrow \pi_{n-1}\big((S^1)^n\big) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_2\big(\widehat{C_n}(\Sigma_{g,b})\big) \longrightarrow \pi_1\big((S^1)^n\big) \longrightarrow \pi_1\big(\widehat{F_n}(\Sigma_{g,b})\big) \xrightarrow{\widehat{\beta}_n} \pi_1\big(\widehat{C_n}(\Sigma_{g,b})\big) \longrightarrow 1$$

Since $\widehat{C_n}(\Sigma_{g,b})$ is an Eilenberg-Maclane space (see [PR1]), we get the required exact sequence and the second point of the proposition. If $\Sigma_{g,b}$ is parallelizable, any section $s : C_n(\Sigma_{g,b}) \longrightarrow F_n(\Sigma_{g,b})$ induces a section $\hat{s} : \widehat{C_n}(\Sigma_{g,b}) \longrightarrow \widehat{F_n}(\Sigma_{g,b})$ which gives the splitting of (4). □

5 Presentations of pure framed braid groups

Now, let us look for a presentation of pure framed braid groups. By Theorem 5, the group $FP_n(\Sigma_{g,b})$ is isomorphic to the direct product $\mathbb{Z}^n \times P_n(\Sigma_{g,b})$ if $b \geq 1$. Thus, with the following theorem (see [B2]), we get a presentation of $FP_n(\Sigma_{g,b})$.

Theorem 7 *Let $\Sigma_{g,b}$ be a compact, connected, orientable surface of genus $g \geq 1$ with b boundary components, $b \geq 1$. The group $P_n(\Sigma_{g,b})$ admits the following presentation:*

Generators: $\{A_{i,j} \mid 1 \leq i \leq 2g+b+n-2, 2g+b \leq j \leq 2g+b+n-1, i < j\}$.

Relations:

(PR1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ *if* $(i < j < r < s)$ *or* $(r+1 < i < j < s)$,
or $(i = r+1 < j < s$ *for even* $r < 2g$ *or* $r > 2g)$;
(PR2) $A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1}$ *if* $(i < j < s)$;
(PR3) $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ *if* $(i < j < s)$;

$$\textit{(PR4)}\ A_{i,j}^{-1}A_{r,s}A_{i,j} = A_{i,s}A_{j,s}A_{i,s}^{-1}A_{j,s}^{-1}A_{r,s}A_{j,s}A_{i,s}A_{j,s}^{-1}A_{i,s}^{-1} \quad \textit{if} \ \ (i+1<r<j<s)$$
$$\textit{or} \ \ (i+1=r<j<s \ \ \textit{for odd} \ \ r<2g \ \ \textit{or} \ \ r>2g)\,;$$
$$\textit{(ER1)}\ A_{r+1,j}^{-1}A_{r,s}A_{r+1,j} = A_{r,s}A_{r+1,s}A_{j,s}^{-1}A_{r+1,s}^{-1} \quad \textit{if} \ \ r \ \ \textit{odd and} \ \ r<2g\,;$$
$$\textit{(ER2)}\ A_{r-1,j}^{-1}A_{r,s}A_{r-1,j} = A_{r-1,s}A_{j,s}A_{r-1,s}^{-1}A_{r,s}A_{j,s}A_{r-1,s}A_{j,s}^{-1}A_{r-1,s}^{-1}$$
$$\textit{if} \ \ r \ \ \textit{even and} \ \ r<2g\,.$$

As a representative of the generator $A_{i,j}$, we may take a geometric braid whose only non-trivial (non-vertical) strand is the $(j-2g-b+1)$th one. In Figure 3, we illustrate the projection of such braids on the surface $\Sigma_{g,1}$.

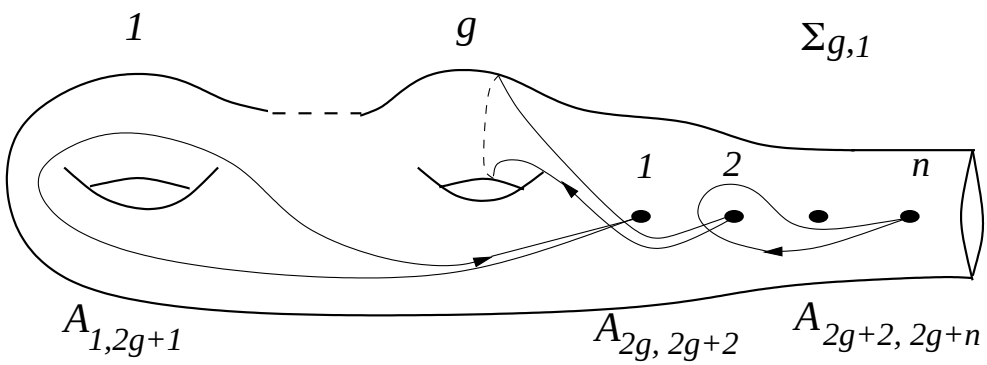

Figure 3: Projection of representatives of the generators $A_{i,j}$. We represent $A_{i,j}$ by its only non-trivial strand.

Recall that pure (framed) braid groups can be seen as subgroups of mapping class groups (see Proposition 1 and 2). The isomorphism $P_n(\Sigma_{g,b}) \approx \mathrm{Ker}(\varphi_n)$ (where φ_n forgets the punctures) is defined as follows: to an element h of $\mathrm{P}\Gamma_{g,b}^n$ which is isotopic to the identity in $\Sigma_{g,b}$ (*i.e.* $h \in \mathrm{Ker}(\varphi_n)$), we associate the braid $t \mapsto \big(H_t(p_i)\big)_{1\le i\le n}$ where $H : \Sigma_{g,b} \times I \longrightarrow \Sigma_{g,b}$ is an isotopy between Id and h. From this point of view, one can easily see that the $A_{i,j}$'s correspond to the following elements of $\mathrm{P}\Gamma_{g,b}^n$:

$$\begin{cases} \tau_{\beta_r}\tau_{\beta_{r,s}}^{-1} & \text{if } \ i=2r-1, 1\le r\le g,\ j=2g+b+s-1,\ 1\le s\le n, \\ \tau_{\alpha_r}\tau_{\alpha_{r,s}}^{-1} & \text{if } \ i=2r, 1\le r\le g,\ j=2g+b+s-1,\ 1\le s\le n, \\ \tau_{\delta_r}\tau_{\delta_{r,s}}^{-1} & \text{if } \ i=2g+r, 1\le r\le b-1,\ j=2g+b+s-1,\ 1\le s\le n, \\ \tau_{\delta_{r,s}}^{-1} & \text{if } \ i=2g+b+r-1,\ j=2g+b+s-1,\ 1\le r<s\le n, \end{cases}$$

where curves are those described by figure 4.

Now, let us loot at the closed case. The group $P_n(\Sigma_g)$ is the quotient of $P_n(\Sigma_{g,1})$ by the following relations (see [B2]):

$$\text{(TR)}\ [A_{2g,2g+k}^{-1}, A_{2g-1,2g+k}] \cdots [A_{2,2g+k}^{-1}, A_{1,2g+k}] =$$
$$= A_{2g+1,2g+k} \cdots A_{2g+k-1,2g+k} A_{2g+k,2g+k+1} \cdots A_{2g+k,2g+n}$$
$$(1 \leq k \leq n, \text{ with the notation } A_{2g+1,2g+1} = A_{2g+n,2g+n} = 1)$$

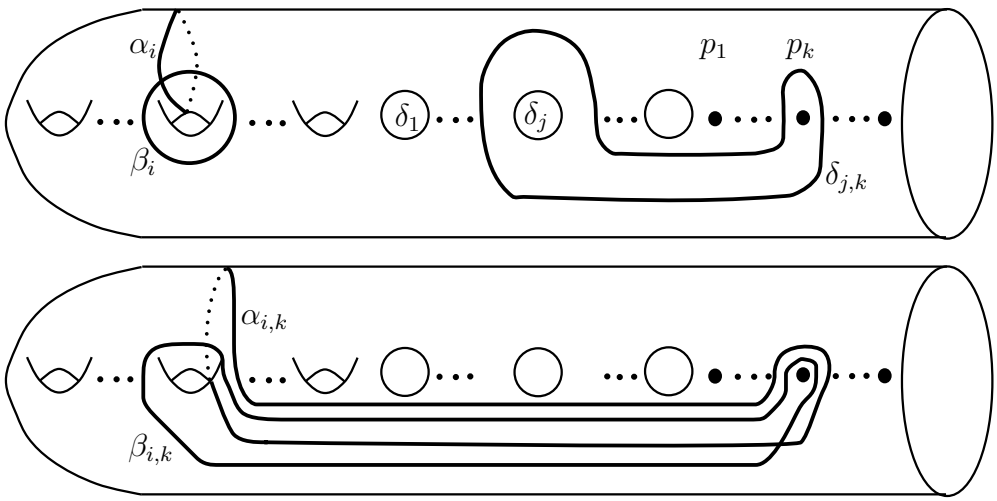

Figure 4: curves on $\Sigma_{g,b}$

From this result, we can prove the following:

THEOREM 8 *Let Σ_g be a compact, connected, closed, orientable surface of genus $g \geq 2$. The framed pure braid group $FP_n(\Sigma_g)$ admits the following presentation:*

Generators: $\{B_{i,j},\ f_k \mid 1 \leq i \leq 2g+n-1, 2g+1 \leq j \leq 2g+n,\ i < j,\ 1 \leq k \leq n\}$.
Relations: *relations (PR1-4) and (ER1-2) together with the following:*

(C) *the f_k's are central;*

(FTR) $[B_{2g,2g+k}^{-1}, B_{2g-1,2g+k}] \cdots [B_{2,2g+k}^{-1}, B_{1,2g+k}] =$
$B_{2g+1,2g+k} \cdots B_{2g+k-1,2g+k} B_{2g+k,2g+k+1} \cdots B_{2g+k,2g+n}\, f_k^{2(g-1)}$

$(1 \leq k \leq n,$ *with the notation* $B_{2g+1,2g+1} = B_{2g+n,2g+n} = 1)$

Proof. Consider the sequence (2):

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow FP_n(\Sigma_g) \longrightarrow P_n(\Sigma_g) \longrightarrow 1.$$

In terms of mapping class groups, $\mathbb{Z}^n$ is generated by $\tau_{\delta_1}, \ldots, \tau_{\delta_n}$ where δ_k is a curve parallel to the kth-boundary component. As shown in [J], a presentation of $FP_n(\Sigma_{g,b})$ can be established as follow. Take as generators

$$\{\tau_{\delta_1}, \ldots, \tau_{\delta_n}\} \cup \{B_{i,j},\ 1 \leq i \leq 2g+n-1, 2g+1 \leq j \leq 2g+n, i < j\}$$

where $B_{i,j}$ is a representative of $A_{i,j}$. Relations are of three types: the first corresponds to relations between the τ_{δ_k}'s, the second to lifting of each relations in $P_n(\Sigma_g)$. The last one comes from the action under conjugation of each $B_{i,j}$ on the τ_{δ_k}'s. In order to define the $B_{i,j}$'s, consider the curves in Figure 4 where the boundary is capped by a disk and the marked points are replaced by holes. Then, we put

$$B_{i,j} = \begin{cases} \tau_{\beta_r}\tau_{\beta_{r,s}}^{-1}\tau_{\delta_s} & \text{if } \ i = 2r-1, 1 \leq r \leq g,\ j = 2g+s,\ 1 \leq s \leq n, \\ \tau_{\alpha_r}\tau_{\alpha_{r,s}}^{-1}\tau_{\delta_s} & \text{if } \ i = 2r, 1 \leq r \leq g,\ j = 2g+s,\ 1 \leq s \leq n, \\ \tau_{\delta_r}\tau_{\delta_{r,s}}^{-1}\tau_{\delta_s} & \text{if } \ i = 2g+r,\ j = 2g+s,\ 1 \leq r < s \leq n. \end{cases} \tag{5}$$

When capping each boundary components of $\Sigma_{g,n}$, the τ_{δ_k}'s are sent to identity, thus $B_{i,j}$ is indeed a representative of $A_{i,j}$.

Now, the τ_{δ_k}'s are central in $\Gamma_{g,n}$, thus the first and third type of relations correspond clearly to relations (C). Let us look at relations (PR1-4), (ER1-2) and (TR) of $P_n(\Sigma_g)$. Relations (PR1-4) – (ER1-2) have been verified in [BGG, Lemma 19] to hold in $\Gamma_{g,n}$ and therefore in $FP_n(\Sigma_g)$. Then it suffices to lift (TR) relation from $P_n(\Sigma_g)$ to $FP_n(\Sigma_g)$.

Since $\tau_{\beta_{i,k}}\tau_{\beta_i}^{-1}(\alpha_i) = \gamma_{i,k}$ and $\tau_{\beta_{i,k}}\tau_{\beta_i}^{-1}(\alpha_{i,k}) = \alpha_i$ where $\gamma_{i,k}$ is the curve descibed in figure 5, one has (we omit the $\tau'_{\delta_l}s$ because they are central)[2]:

$$\begin{aligned}[B_{2i,2g+k}^{-1}, B_{2i-1,2g+k}] &= \tau_{\alpha_i}^{-1}\tau_{\alpha_{i,k}}\tau_{\beta_i}\tau_{\beta_{i,k}}^{-1}\tau_{\alpha_i}\tau_{\alpha_{i,k}}^{-1}\tau_{\beta_i}^{-1}\tau_{\beta_{i,k}} \\ &= \tau_{\alpha_i}^{-1}\tau_{\alpha_{i,k}}\tau_{\gamma_{i,k}}\tau_{\alpha_i}^{-1}.\end{aligned}$$

Considering the curves described in figure 5, one has the following lantern relations:

$$\tau_{\alpha_i}^2\tau_{\mu_{i,k}}\tau_{\delta_k} = \tau_{\mu_i}\tau_{\alpha_{i,k}}\tau_{\gamma_{i,k}}$$

and we get

$$[B_{2i,2g+k}^{-1}, B_{2i-1,2g+k}] = \tau_{\mu_i}^{-1}\tau_{\mu_{i,k}}\tau_{\delta_k}.$$

[2]Recall that we denote composition of applications from left to right. Thus, if $\gamma = h(\alpha)$, one has $\tau_\gamma = h^{-1}\tau_\alpha h$.

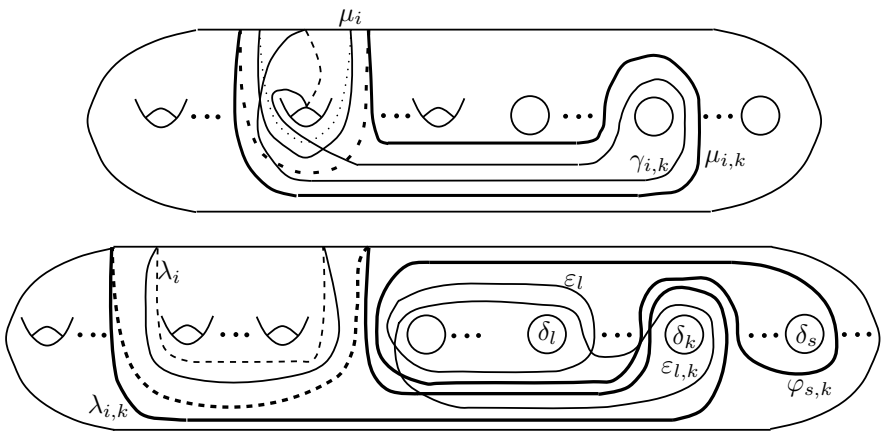

Figure 5: curves on Σ_g

Then, using lantern relations

$$\tau_{\mu_{i-1}}\tau_{\lambda_i}\tau_{\lambda_{i-1,k}}\tau_{\delta_k} = \tau_{\lambda_{i-1}}\tau_{\lambda_{i,k}}\tau_{\mu_{i-1,k}},$$

it is easy to check by induction on i that[3]

$$[B_{2g,2g+k}^{-1}, B_{2g-1,2g+k}]\cdots[B_{2i,2g+k}^{-1}, B_{2i-1,2g+k}] = \tau_{\lambda_i}^{-1}\tau_{\lambda_{i,k}}\tau_{\delta_k}^{2(g-i)+1}. \tag{6}$$

On the other hand, by definition of the $B_{i,j}$'s, on has, for $1 \leq k \leq n$:

$$B_{2g+1,2g+k}\cdots B_{2g+k-1,2g+k}B_{2g+k,2g+k+1}\cdots B_{2g+k,2g+n} = \\ \tau_{\delta_1}\tau_{\delta_{1,k}}^{-1}\cdots\tau_{\delta_{k-1}}\tau_{\delta_{k-1,k}}^{-1}\tau_{\delta_{k+1}}\tau_{\delta_{k,k+1}}^{-1}\cdots\tau_{\delta_n}\tau_{\delta_{k,n}}^{-1}\tau_{\delta_k}^{n-1}$$

Now, with the curves described in figure 5, one has the lantern relation

$$\tau_{\varepsilon_{i-1}}\tau_{\delta_i}\tau_{\delta_k}\tau_{\varepsilon_{i,k}} = \tau_{\varepsilon_i}\tau_{\delta_{i,k}}\tau_{\varepsilon_{i-1,k}},$$

from which one can check by induction on $i < k$ that ($\varepsilon_1 = \delta_1$ and $\varepsilon_{1,k} = \delta_{1,k}$)

$$\tau_{\delta_{1,k}}^{-1}\tau_{\delta_{2,k}}^{-1}\cdots\tau_{\delta_{i,k}}^{-1} = \tau_{\varepsilon_i}\tau_{\varepsilon_{i,k}}^{-1}\tau_{\delta_1}^{-1}\tau_{\delta_2}^{-1}\cdots\tau_{\delta_i}^{-1}\tau_{\delta_k}^{1-i}.$$

Then, with the lantern relation (where $s > k$ and $\varphi_{k,k} = \varepsilon_{k-1}$)

$$\tau_{\varepsilon_s}\tau_{\delta_s}\tau_{\delta_k}\tau_{\varphi_{s-1,k}} = \tau_{\varphi_{s,k}}\tau_{\delta_{k,s}}\tau_{\varepsilon_{s-1}},$$

[3]Note that $\lambda_g = \mu_g$ and $\lambda_{g,k} = \mu_{g,k}$.

one obtains by induction on s

$$\tau_{\delta_{1,k}}^{-1}\cdots\tau_{\delta_{k-1,k}}^{-1}\tau_{\delta_{k,k+1}}^{-1}\cdots\tau_{\delta_{k,s}}^{-1}=\tau_{\varphi_{s,k}}\tau_{\varepsilon_s}^{-1}\tau_{\delta_s}^{-1}\cdots\tau_{\delta_{k+1}}^{-1}\tau_{\delta_1}^{-1}\cdots\tau_{\delta_{k-1}}^{-1}\tau_{\delta_k}^{2-s}$$

and finally

$$\begin{aligned}B_{2g+1,2g+k}\cdots B_{2g+k-1,2g+k}B_{2g+k,2g+k+1}\cdots B_{2g+k,2g+n}&=\tau_{\varphi_{n,k}}\tau_{\varepsilon_n}^{-1}\tau_{\delta_k}^{2-n}\tau_{\delta_k}^{n-1}\\&=\tau_{\varphi_{n,k}}\tau_{\varepsilon_n}^{-1}\tau_{\delta_k}.\end{aligned}\tag{7}$$

To conclude, one has just to compare equation (6) with $i=1$ and equation (7), and to observe that $\varepsilon_n=\lambda_1$ and $\varphi_{n,k}=\lambda_{1,k}$. □

Remark 5 This presentation gives another proof of the fact that the central extension (2) does not split for all $g\geq 2$. Indeed, if it splits, the group $FP_n(\Sigma_g)$ is the direct product of $\mathbb{Z}^n$ and $P_n(\Sigma_g)$: it contredicts the relation *(FTR)*.

6 Framed pure braids as centralizers of Dehn twists

We give a third possible definition of framed pure braid groups in terms of centralizers of Dehn twists: since in the case of surfaces with boundary the corresponding framed pure braid groups are trivial central extension of pure braid groups, we will focus on the case of closed surfaces of genus greater than one. This interpretation of framed pure braid groups will allow us to introduce another possible definition of framed braid groups (Section 7). All following Definitions and Propositions can be easily extended to the case with boundary.

Let us consider a surface $\Sigma_{g,n}$ of genus $g\geq 2$ and n boundary components and cap each boundary component with a disk with 2 marked points (see Figure 6). We will denote by $\chi_n:\Gamma_{g,n}\longrightarrow \mathrm{P}\Gamma_g^{2n}$ the morphism induced by the inclusion of $\Sigma_{g,n}$ into Σ_g^{2n}, a surface of genus g with $2n$ marked points.

One has the following commutative diagram at the level of mapping classes:

$$\begin{tikzcd}
1 \arrow[r] & FP_n(\Sigma_g) \arrow[r] \arrow[d] & \Gamma_{g,n} \arrow[r, "\lambda_g^n"] \arrow[d, "\chi_n"] & \Gamma_g \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & P_{2n}(\Sigma_g) \arrow[r] & \mathrm{P}\Gamma_g^{2n} \arrow[r, "\varphi_{2n}"] & \Gamma_g \arrow[r] & 1
\end{tikzcd}$$

Since χ_n is injective (see Proposition 4.1 in [PR1]), we can consider $FP_n(\Sigma_g)$ as a subgroup of $P_{2n}(\Sigma_g)$.

Moreover, we can provide a characterization of $FP_n(\Sigma_g)$ as subgroup of $\mathrm{P}\Gamma_g^{2n}$.

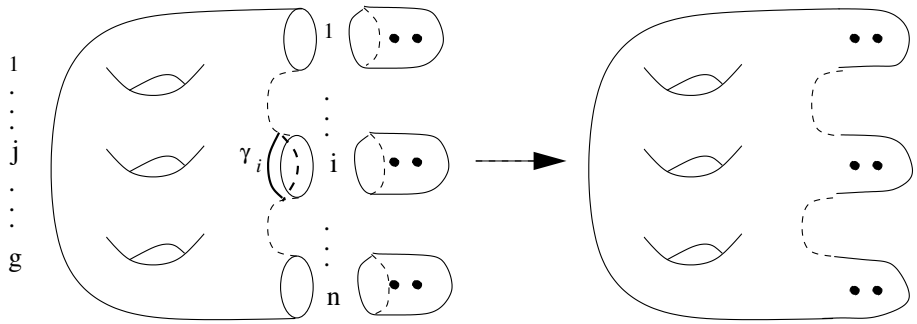

Figure 6: Inclusion of $\Sigma_{g,n}$ into Σ_g^{2n}

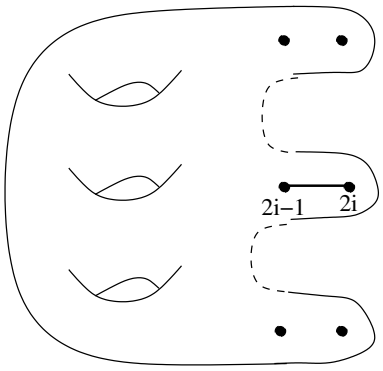

Figure 7: The arc $[2i-1, 2i]$.

Proposition 9 *Let $g \geq 2$, $\chi_n : \Gamma_{g,n} \longrightarrow \mathrm{P}\Gamma_g^{2n}$ be the map defined above and γ_i be the boundary of the ith capping punctured disk (see Figure 6). The group $\chi_n(FP_n(\Sigma_g))$ coincides with $\bigcap_{j=1}^{n} C_{P_{2n}(\Sigma_g)}(\tau_{\gamma_j})$, where $C_{P_{2n}(\Sigma_g)}(\tau_{\gamma_j})$ is the centralizer of the Dehn twist τ_{γ_j} in $P_{2n}(\Sigma_g)$.*

Before proving Proposition 9 we need to introduce some definitions and to recall some results. Let Σ be a surface with a finite set $\mathcal{P}$ of n marked points. An arc is an embedding $A : [0,1] \longrightarrow \Sigma$ such that $A(0), A(1)$ are in $\mathcal{P}$ and $A(x)$ is in $\Sigma \setminus \mathcal{P}$ for all x in $(0,1)$. A (j,k)-arc is an arc such that $A(0) = j$ and $A(1) = k$. Note that two (j,k)-arcs are isotopic if and only if they can be connected by a continuous family of (j,k)-arcs.

For any $i = 1, \ldots, n$ we fix an arc with end points $2i-1$ and $2i$ in the interior of the ith capping disk. In a disk with 2 punctures all arcs are isotopic; let us choose a representative as in Figure 7 that we will denote by $[2i-1, 2i]$.

Consider a disk $\mathbb{D}_{2n}$ in Σ_g^{2n} which contains all $2n$ punctures. The embedding of $\mathbb{D}_{2n}$ into Σ_g^{2n} induces an embedding of B_{2n} into $B_{2n}(\Sigma_g)$ (see for instance [PR1]): the usual generator σ_j of B_{2n} can be considered therefore as an element of $B_{2n}(\Sigma_g)$;

in particular the generator σ_{2j-1} for $j = 1, \ldots, n$ corresponds to the *braid twist* (see [LP] for a definition) associated to the arc $[2j-1, 2j]$.

The group $B_{2n}(\Sigma_g)$ acts on Σ_g up to isotopy. In the following we adopt the convention that, for any β in $B_{2n}(\Sigma_g)$, $*\beta : \Sigma_g \longrightarrow \Sigma_g$ corresponds to a mapping $\Sigma_g \times \{0\} \longrightarrow \Sigma_g \times \{1\}$ and defines an action on the right, whereas $\beta* : \Sigma_g \longrightarrow \Sigma_g$ corresponds to a mapping $\Sigma_g \times \{1\} \longrightarrow \Sigma_g \times \{0\}$ and defines an action on the left.

In particular $B_{2n}(\Sigma_g)$ acts on the right and on the left on $\mathcal{A}_{2n}$, the set of arcs up to isotopy.

THEOREM 10 *[B1] Let $g > 1$. For each β in $B_{2n}(\Sigma_g)$ the following properties are equivalent:*

(a) $\sigma_{2j-1}\beta = \beta\sigma_{2k-1}$,

(b) $\sigma_{2j-1}^r\beta = \beta\sigma_{2k-1}^r$ *for any integer r,*

(c) $\sigma_{2j-1}^r\beta = \beta\sigma_{2k-1}^r$ *for some nonzero integer r,*

(d) $[2j-1, 2j] * \beta = [2k-1, 2k]$.

REMARK 6 Theorem 10 is a weaker version of Theorem 2.2 in [B1], which statement concerns all braid groups $B_n(\Sigma_g)$ and all braid generators $\sigma_1, \ldots, \sigma_{n-1}$.

We recall that γ_i denotes the boundary of the ith capping disk. From the fact that $\sigma_{2i-1,2i}^2 = \tau_{\gamma_i}$ we can therefore deduce the following result:

COROLLARY 11 *Let $g > 1$. For each β in $P_{2n}(\Sigma_g)$ the following properties are equivalent:*

(a) $\tau_{\gamma_i}\beta = \beta\tau_{\gamma_i}$,

(b) $\tau_{\gamma_i}^r\beta = \beta\tau_{\gamma_i}^r$, *for any integer r,*

(c) $\tau_{\gamma_i}^r\beta = \beta\tau_{\gamma_i}^r$, *for some nonzero integer r,*

(d) $[2i-1, 2i] * \beta = [2i-1, 2i]$.

REMARK 7 The equivalences between a) and d) are in the folkrore even for the case $g = 1$.

Proof of Proposition 9. We recall that a multitwist is a product of twists along pairwise disjoint curves and that the map $\chi_n : \Gamma_{g,n} \longrightarrow \mathrm{P}\Gamma_g^{2n}$ defined above is injective and sends $FP_n(\Sigma_g)$ into $P_{2n}(\Sigma_g)$. Any generator g of $FP_n(\Sigma_g)$ is a multitwist (see (5)), as well as its image $\chi_n(g)$. Since two twists τ_γ and τ_δ commute if and only if γ and δ are disjoint up to isotopy (see for instance [PR2]), we deduce that any element $\chi_n(g)$ commutes with τ_{γ_j} for $j = 1, \ldots, n$ and therefore

$\chi_n(FP_n(\Sigma_g))$ is a subgroup of $\bigcap_{j=1}^{n} C_{P_{2n}(\Sigma_g)}(\tau_{\gamma_j})$. On the other hand, if g is an element of $\bigcap_{j=1}^{n} C_{P_{2n}(\Sigma_g)}(\tau_{\gamma_j})$, from Corollary 11 it follows that $[2j-1,2j]*g=[2j-1,2j]$ for $j=1,\dots,n$ and then up to isotopy we can suppose that $g=Id$ on each capping disk. Therefore we can consider g as the image by χ_n of an element $\tilde{g}$ in $\Gamma_{g,n}$. Since $\lambda_g^n(\tilde{g})=\varphi_{2n}(\chi_n(\tilde{g}))=\varphi_{2n}(g)=1$ we deduce that $\tilde{g}\in FP_n(\Sigma_g)$ and hence that $FP_n(\Sigma_g)$ maps onto $\bigcap_{j=1}^{n} C_{P_{2n}(\Sigma_g)}(\tau_{\gamma_j})$.

□

Let $A_{k,l}$ be the generator of $P_{2n}(\Sigma_g)$ depicted in Theorem 7 for $1\le k\le 2g+2n-1$ and $2g+1\le l\le 2g+2n$. Now, set:

$$C_{i,j}=A_{i,2(j-g)-1}A_{i,2(j-g)}A_{2(j-g)-1,2(j-g)} \quad \text{for} \quad i=1,\dots,2g \quad \text{and} \quad j=2g+1,\dots,2g+n,$$

$$C_{i,j}=A_{2(i-g)-1,2(j-g)-1}A_{2(i-g),2(j-g)-1}A_{2(i-g)-1,2(j-g)}A_{2(i-g),2(j-g)}A_{2(j-g)-1,2(j-g)} \quad \text{for} \quad 2g+1\le i<j\le 2g+n$$

and finally $F_k=A_{2g+2k-1,2g+2k}$ for $k=1,\dots,n$.

Roughly speaking, the element F_k corresponds to $\sigma^2_{2k-1,2k}$ in $P_{2n}(\Sigma_g)$ and the $C_{i,j}$'s correspond to pure braids on $P_{2n}(\Sigma_g)$ where the only non trivial strands are the $(2(j-g)-1)$th and the $2(j-g)$th one, which are "parallel", meaning that they bound an annulus in $\Sigma_g\setminus\{p_1,\dots,p_{2(j-g-1)},p_{2(j-g)+1},\dots,p_{2n}\}$.

We recall that it is possible to embed P_n into P_{2n} (or B_n into B_{2n}) "doubling" any strand (see for instance [FRZ]); one can remark that in the case of braid groups on closed surfaces of genus $g\ge 2$ such embeddings are not well defined because of Theorem 5.

From Theorem 8 and Proposition 9 one can therefore deduce the following group presentation for $FP_n(\Sigma_g)$ as subgroup of $FP_{2n}(\Sigma_g)$.

Proposition 12 *Let Σ_g be a compact, connected, closed, orientable surface of genus $g>1$. The framed braid group $FP_n(\Sigma_g)$ as subgroup of $P_{2n}(\Sigma_g)$ admits the following presentation:*

Generators: $\{C_{i,j},F_k \mid 1\le i\le 2g+n-1, 2g+1\le j\le 2g+n, \; i<j,\; 1\le k\le n\}$.

Relations: *relations (PR1-4) and (ER1-2) from Theorem 7 replacing $A_{i,j}$ with $C_{i,j}$ together with the following:*

(C) *the F_k's are central;*

(FTR) $[C^{-1}_{2g,2g+k},C_{2g-1,2g+k}]\cdots[C^{-1}_{2,2g+k},C_{1,2g+k}]=$

$$C_{2g+1,2g+k}\cdots C_{2g+k-1,2g+k}C_{2g+k,2g+k+1}\cdots C_{2g+k,2g+n}\,F_k^{2(g-1)}$$

$$(1\leq k\leq n,\ \textit{with the notation}\ C_{2g+1,2g+1}=C_{2g+n,2g+n}=1)$$

Proof. One has to check that χ_n sends the generators $B_{i,j}$ and f_k respectively into $C_{i,j}$ and F_k. □

7 Framed braids as $2n$-strands braids

The definition of framed pure braid groups in terms of centralizers of Dehn twists given in Proposition 9 allows us to give another equivalent definition for the framed braid group of a closed surface introduced in Definition 1.

Definition 2 *Let Σ be a surface of genus $g>1$ with a finite set $\mathcal{P}$ of $2n$ marked points and let $\mathcal{I}_n$ be the set of arcs $\{[1,2],[3,4],\ldots,[2n-1,2n]\}$. We say that an element β of $B_{2n}(\Sigma_g)$ preserves with orientation $\mathcal{I}_n$ if for any $[2i-1,2i]$ there exists j such that $[2i-1,2i]*\beta=[2j-1,2j]$ with $\{2i-1\}*\beta=\{2j-1\}$ (and therefore $\{2i\}*\beta=\{2j\}$). The set of braids preserving with orientation $\mathcal{I}_n$ forms a subgroup of $B_{2n}(\Sigma_g)$ that we will denote by $\widetilde{FB}_n(\Sigma_g)$.*

Let us recall that the group $B_{2n}(\Sigma_g)$ is generated (see [B2]) by the usual generators $\sigma_1,\ldots,\sigma_{2n-1}$ of B_{2n} plus the braids $a_1,b_1,\ldots,a_g,b_g$ which are the pure braids $A_{1,2g+1},A_{2,2g+1},\ldots,A_{2g,2g+1}$ depicted in Theorem 7. Let us define

$$A_i=A_{2i-1,2g+1}A_{2i-1,2g+2}A_{2g+1,2g+2}\ \text{ and }\ B_i=A_{2i,2g+1}A_{2i,2g+2}A_{2g+1,2g+2}$$

for $i=1,\ldots,g$ and let $\tau_j=\sigma_{2j}\sigma_{2j-1}\sigma_{2j+1}\sigma_{2j}$ for $j=1,\ldots,n-1$. The elements $A_1,B_1,\ldots,A_g,B_g,\ \tau_1,\ldots,\tau_{n-1},\ F_1,\ldots,F_n$, belong to $\widetilde{FB}_n(\Sigma_g)$. Moreover we have the following result:

Theorem 13 *Let Σ_g be a compact, connected, closed, orientable surface of genus $g>1$. The group $\widetilde{FB}_n(\Sigma_g)$ admits the following presentation:*

Generators: $A_1,B_1,\ldots,A_g,B_g,\tau_1,\ldots,\tau_{n-1},F_1,\ldots,F_n$.

Relations:

$$\tau_iF_j=F_j\tau_i\ \textit{for}\ j\neq i,i+1 \tag{8}$$

$$\tau_iF_i=F_{i+1}\tau_i \tag{9}$$

$$\tau_iF_{i+1}=F_i\tau_i \tag{10}$$

$$\tau_i\tau_j=\tau_j\tau_i\ \textit{if}\ |i-j|\geq 2 \tag{11}$$

$$\tau_i\tau_{i+1}\tau_i=\tau_{i+1}\tau_i\tau_{i+1}\ \textit{for all}\ 1\leq i\leq n-2 \tag{12}$$

$$c_i\tau_j = \tau_j c_i \text{ for all } j \geq 2,\ c_i = A_i \text{ or } B_i \text{ and } i = 1, \ldots, g \tag{13}$$

$$c_i\tau_1 c_i\tau_1 = \tau_1 c_i\tau_1 c_i \text{ for } c_i = A_i \text{ or } B_i \text{ and } i = 1, \ldots, g \tag{14}$$

$$A_i\tau_1 B_i = \tau_1 B_i\tau_1 A_i\tau_1 \text{ for } i = 1, \ldots, g \tag{15}$$

$$c_i\tau_1^{-1}c_j\tau_1 = \tau_1^{-1}c_j\tau_1 c_i \text{ for } c_i = A_i \text{ or } B_i,\ c_j = A_j \text{ or } B_j \text{ and } 1 \leq j < i \leq g \tag{16}$$

$$\prod_{i=1}^{g}[A_i^{-1}, B_i] = \tau_1 \cdots \tau_{n-2}\tau_{n-1}^2\tau_{n-2} \cdots \tau_1 F_1^{2(g-1)}. \tag{17}$$

Proof. If $\pi : \widetilde{FB}_n(\Sigma_g) \longrightarrow \mathfrak{S}_n$ is the map which associates to any $\beta \in \widetilde{FB}_n(\Sigma_g)$ the corresponding permutation on the set $\mathcal{I}_n$, we have the following exact sequence:

$$1 \longrightarrow FP_n(\Sigma_g) \longrightarrow \widetilde{FB}_n(\Sigma_g) \longrightarrow \mathfrak{S}_n \longrightarrow 1.$$

Then, the statement follows by Proposition 12 and, as already done in Theorem 8, by another application of the technique from [J] that we leave to the reader; we simply remark that $\tau_1, \ldots, \tau_{n-1}$ are coset representative of the usual generators of $\mathfrak{S}_n$ and the other generators of $FP_n(\Sigma_g)$ can be deleted since they are conjugated by words in $\tau_1, \ldots, \tau_{n-1}$.

□

Before proving that $\widetilde{FB}_n(\Sigma_g)$ is isomorphic to the framed braid group $FB_n(\Sigma_g)$ defined in Definition 1 let us recall few additional notations and results from [B1].

Definition 3 *We define a* ribbon *as an embedding*

$$R : [0,1] \times [0,1] \longrightarrow \Sigma \times [0,1]\,,$$

such that $R(s,t)$ is in $\Sigma \times \{t\}$.

Let A be a (j,k)-arc in $\Sigma \times \{0\}$. Then the isotopy corresponding to $\beta \in B_n(\Sigma_g)$ moves A through a ribbon which is proper *for β, meaning that*

- *$R(0,t)$ and $R(1,t)$ trace out the strands j and k of the braid β, while the rest of the ribbon is disjoint from β;*
- *$R([0,1] \times \{0\}) = A$ and $R([0,1] \times \{1\}) = A * \beta$.*

Definition 4 *We say that the braid β in $B_{2n}(\Sigma)$ has a $(2j-1, 2k-1)$-band if there exists a ribbon proper for β and connecting $[2j-1, 2j] \times \{0\}$ to $[2k-1, 2k] \times \{1\}$.*

Proposition 14 *[B1, Proposition 2.2] Let $g > 1$ and $1 \leq j < k \leq n$. For each β in $B_{2n}(\Sigma_g)$, the following properties are equivalent:*

(a) β has a $(2j-1, 2k-1)$-band,

(b) $[2j-1, 2j] * \beta = [2k-1, 2k]$.

Proposition 15 *The group $\widetilde{FB}_n(\Sigma_g)$ is isomorphic to the framed braid group $FB_n(\Sigma_g)$ defined in Definition 1.*

Proof. By its definition an element $\beta \in FB_n(\Sigma_g)$ can be seen as an element $\beta' \in B_{2n}(\Sigma_g)$ such that for any $i = 1, \ldots, n$ there exists a $(2i-1, 2k-1)$-band and $\{2i-1\} * \beta' = \{2k-1\}$. This defines a morphism from $FB_n(\Sigma_g)$ to $\widetilde{FB}_n(\Sigma_g)$ wich is an isomorphism by Proposition 14. □

8 Framed braids on the torus

In the case of the torus the proposed definitions are not equivalent and let arise different notions of framings.

Let $F_n(\Sigma)$ and $\widehat{F_n}(\Sigma)$ be the spaces defined at the beginning of the Section 2. We recall Definition 1 in the case of $\Sigma = \mathbb{T}^2$.

Definition 5 *The* pure framed braid group $FP_n(\mathbb{T}^2)$ *on n strands of $\mathbb{T}^2$ is the fundamental group of $F_n(\mathbb{T}^2)$. The* framed braid group *on n strands of $\mathbb{T}^2$ is the fundamental group of $\widehat{F_n}(\mathbb{T}^2)$.*

Definition 6 *We denote by $\widetilde{FP}_n(\mathbb{T}^2)$ the kernel of the morphism $\lambda_1^n : \Gamma_{1,n} \longrightarrow \Gamma_1$ induced by capping n boundary components.*

In what follows we prove that $\widetilde{FP}_n(\mathbb{T}^2)$ is a quotient of $FP_n(\mathbb{T}^2)$ and we examine exact sequences for $FP_n(\mathbb{T}^2)$ and $\widetilde{FP}_n(\mathbb{T}^2)$.

Theorem 16 *The group $FP_n(\mathbb{T}^2)$ is isomorphic to $\mathbb{Z}^n \times P_n(\mathbb{T}^2)$ and $F_n(\mathbb{T}^2)$ is an Eilenberg-Maclane space of type $(FP_n(\mathbb{T}^2), 1)$.*

Proof. Since $\mathbb{T}^2$ is parallelizable, the proof of Theorem 16 is the same as in the case of surfaces with boundary given in Theorem 5. □

Theorem 17 *For $n \geq 0$ and $m \geq 0$, one has the following splitting exact sequence:*

$$(FSPB) \qquad 1 \longrightarrow FP_m\left(\mathbb{T}^2 \setminus n \text{ discs}\right) \longrightarrow FP_{n+m}(\mathbb{T}^2) \xrightarrow{\alpha_{n,m}} FP_n(\mathbb{T}^2) \longrightarrow 1$$

where $\alpha_{n,m}$ consists in forgetting the first m strands.

Proof. Because of Theorem 16 the (FSPB) sequence reduces to the exact sequence induced by Fadell-Neuwirth fibration

$$(SPB) \quad 1 \longrightarrow P_m\left(\mathbb{T}^2 \setminus n \text{ points}\right) \longrightarrow P_{n+m}(\mathbb{T}^2) \longrightarrow P_n(\mathbb{T}^2) \longrightarrow 1$$

which is a splitting sequence ([GG]). □

Let us denote $\widetilde{P}_n(\mathbb{T}^2)$ the kernel of the morphism $\varphi_n : \mathrm{P}\Gamma_1^n \longrightarrow \Gamma_1$ induced by forgetting the n marked points. We recall that, according to Proposition 1, one has that $\widetilde{P}_n(\mathbb{T}^2) \simeq P_n(\mathbb{T}^2)/Z(P_n(\mathbb{T}^2))$.

THEOREM 18 *The group $\widetilde{P}_n(\mathbb{T}^2)$ admits the following presentation:*
Generators: $\{A_{i,j} \mid 1 \leq i \leq n+1, 3 \leq j \leq n+2, i < j\}$.
Relations:

(PR1) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{r,s}$ *if* $(i < j < r < s)$ *or* $(r+1 < i < j < s)$,
or $(i = r+1 < j < s \quad or \quad r > 2)$;
(PR2) $A_{i,j}^{-1} A_{j,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1}$ *if* $(i < j < s)$;
(PR3) $A_{i,j}^{-1} A_{i,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ *if* $(i < j < s)$;
(PR4) $A_{i,j}^{-1} A_{r,s} A_{i,j} = A_{i,s} A_{j,s} A_{i,s}^{-1} A_{j,s}^{-1} A_{r,s} A_{j,s} A_{i,s} A_{j,s}^{-1} A_{i,s}^{-1}$ *if* $(i+1 < r < j < s)$
or $(i+1 = r < j < s$ *for odd* $\quad or \quad r > 2)$;
(ER1) $A_{2,j}^{-1} A_{1,s} A_{2,j} = A_{1,s} A_{2,s} A_{j,s}^{-1} A_{2,s}^{-1}$
(ER2) $A_{1,j}^{-1} A_{2,s} A_{1,j} = A_{1,s} A_{j,s} A_{1,s}^{-1} A_{2,s} A_{j,s} A_{1,s} A_{j,s}^{-1} A_{1,s}^{-1}$
(TR) $[A_{2,2+k}^{-1}, A_{1,2+k}] = A_{3,2+k} \cdots A_{1+k,2+k} A_{2+k,3+k} \cdots A_{2+k,2+n}$ $(1 \leq k \leq n)$
$(1 \leq k \leq n,$ *with the notation* $A_{3,3} = A_{2+n,2+n} = 1)$
(QR1) $A_{1,3} \cdots A_{1,2+n} = 1$
(QR2) $A_{2,3} \cdots A_{2,2+n} = 1$

Proof. It suffices to remark that the previous presentation is the presentation for the group $P_n(\mathbb{T}^2)$ given in Theorem 7, quotiented by (QR1) and (QR2) relations, where $A_{1,3} \cdots A_{1,2+n}, A_{2,3} \cdots A_{2,2+n}$ generate the center of $P_n(\mathbb{T}^2)$ (see for instance [PR1]). □

One has the following commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d, equal] & \widetilde{FP}_n(\mathbb{T}^2) \arrow[r] \arrow[d] & \widetilde{P}_n(\mathbb{T}^2) \arrow[r] \arrow[d] & 1 & (2) \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r] & \Gamma_{1,n} \arrow[r] \arrow[d, "\Phi_n"] & \mathrm{P}\Gamma_1^n \arrow[r] \arrow[d, "\varphi_n"] & 1 & (17) \\
 & & \Gamma_1 \arrow[r, equal] \arrow[d] & \Gamma_1 \arrow[d] & & \\
 & & 1 & 1 & &
\end{tikzcd}
$$

One can repeat word by word the proof of Theorem 8 and verify that (QR1) and (QR2) relations lift to $\widetilde{FP}_n(\mathbb{T}^2)$. Since $g = 1$ one deduces then also (TR) relation lifts to $\widetilde{FP}_n(\mathbb{T}^2)$ and therefore that the natural map from $\widetilde{P}_n(\mathbb{T}^2)$ to $\widetilde{FP}_n(\mathbb{T}^2)$ is actually a section. Therefore the fact that the group $\mathbb{Z}^n$ generated by Dehn twists around boundary components is central implies the following result.

Proposition 19 *The group $\widetilde{FP}_n(\mathbb{T}^2)$ is isomorphic to $\mathbb{Z}^n \oplus \widetilde{P}_n(\mathbb{T}^2)$.*

Proposition 20 *The group $\widetilde{FP}_n(\mathbb{T}^2)$ is isomorphic to $\dfrac{FP_n(\mathbb{T}^2)}{\mathbb{Z} \oplus \mathbb{Z}}$.*

Proof. It follows from Proposition 19 and Theorem 16. One can easily deduce the result also considering the long exact sequence for $\Sigma_{g,b} = \mathbb{T}^2$ in the proof of Proposition 2 and remarking that $\pi_1\big(\mathrm{Diff}^+(\mathbb{T}^2)\big) = \pi_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ (see [S2]). □

Finally, let $\chi_n : \Gamma_{1,n} \longrightarrow \mathrm{P}\Gamma_1^{2n}$ be the injective map defined in Section 6.

Proposition 21 *The group $\chi_n(\widetilde{FP}_n(\mathbb{T}^2))$ coincides with $\bigcap_{j=1}^{n} C_{\widetilde{P}_{2n}(\mathbb{T}^2)}(\tau_{\gamma_j})$, where $C_{\widetilde{P}_{2n}(\mathbb{T}^2)}(\tau_{\gamma_j})$ is the centralizer of the Dehn twist τ_{γ_j} in $\widetilde{P}_{2n}(\mathbb{T}^2)$.*

Proof. The proof is the same as the proof of Proposition 9 using Remark 7. □

References

[BB] V. Bardakov and P. Bellingeri, On residual properties of pure braid groups of closed surfaces, Comm. Algebra **37** (2009), 1481–1490.

[B1] P. Bellingeri, Centralisers of surface braid groups, Comm. Algebra **32** (2004), 4099–4116.

[B2] P. Bellingeri, On presentations of surface braid groups, J. Algebra **274** (2004), 543–563.

[BC] P. Bellingeri and A. Cattabriga, Hilden Braid Groups, ArXiv 0909.4845.

[BGG] P. Bellingeri, S. Gervais and J. Guaschi, Lower central series of Artin Tits and surface braid groups, J. Algebra **319** (2008) 1409–1427.

[Bir] J. S. Birman, Braids, Links, and Mapping Class Groups, Ann. Math. Stud., Princeton Univ. Press, vol.82, 1974.

[FN] E. Fadell and L. Neuwirth, Configuration spaces, Math. Scand. **10** (1962), 111-118.

[FR] M. Falk and R. Randell, The lower central series of a fiber type arrangement, Invent. Math. **82** (1985), 77–88.

[FRZ] R. Fenn, D. Rolfsen and J. Zhu, Centralisers in the braid group and singular braid monoid, Enseign. Math. **42** (1996), 75–96.

[GG] D. Gonçalves and J. Guaschi, On the structure of surface pure braid groups, J. Pure Appl. Algebra **186** (2004), 187-218.

[GV] R. Gillette and J. Van Buskirk, The word problem and its consequences for the braid groups and mapping class groups of the 2-sphere, Trans. Amer. Math. Soc. **131** (1968), 277–296.

[J] D.L. Johnson, Presentation of Groups, London Math. Soc. Lectures Notes, Vol 22, Cambridge University Press, 1976.

[KS] H. Ko and L. Smolinsky, The Framed Braid Group and 3-manifolds, Proc. Amer. Math. Soc. **115** (1992) 541–551.

[LP] C. Labruère and L. Paris, Presentations for the punctured mapping class groups in terms of Artin groups, Alg. Geom. Topol. **1** (2001) 73–114.

[MM] D. Margalit and J. McCammond, Geometric presentations for pure braid group, J. Knot Theory and Ramif. **18** (2009) 1–20.

[OS] A. Oancea and D. A. Salamon, Lefschetz fibrations, intersection numbers, and representations of the braid group, ArXiv 0708.2051v3

[P] S. Papadima, The universal finite-type invariant for braids, with integer coefficients, Topology Appl. **118** (2002), 169-185.

[PR1] L. Paris and D. Rolfsen, Geometric subgroups of surface braid groups, Ann. Inst. Fourier **49** (1999), n°2, 417–472.

[PR2] L. Paris and D. Rolfsen, Geometric subgroups of mapping class groups, J. Reine Angew. Math. **521** (2000), 47-83

[S1] G. P. Scott, Braid groups and the group of homeomorphisms of a surface, Proc. Cambridge Philos. Soc. **68** (1970), 605–617.

[S2] G. P. Scott, The space of homeomorphisms of a 2-manifold, Topology **9** (1970), 97–109

PAOLO BELLINGERI, Université de Caen, CNRS UMR 6139, LMNO, Caen, 14000 (France). Email: paolo.bellingeri@math.unicaen.fr

SYLVAIN GERVAIS, Université de Nantes, CNRS-UMR 6629, Laboratoire Jean Leray, 2, rue de la Houssinière, F-44322 NANTES cedex 03 (France)