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GIBBS MEASURE FOR THE PERIODIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

by

Laurent Thomann & Nikolay Tzvetkov

Abstract. — In this paper we construct a Gibbs measure for the derivative Schrödinger equation on the circle. The construction uses some renormalisations of Gaussian series and Wiener chaos estimates, ideas which have already been used by the second author in a work on the Benjamin-Ono equation.

1. Introduction

Denote by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the circle. The purpose of this work is to construct a Gibbs measure associated to the derivative nonlinear Schrödinger equation

$$(1.1) \quad \begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & (t, x) \in \mathbb{R} \times \mathbb{T}, \\ u(0, x) = u_0(x). \end{cases}$$

Many recent results (see the end of Section 1.2) show that a Gibbs measure is an efficient tool to construct global rough solutions of nonlinear dispersive equations. This is the main motivation of this paper: we hope that our result combined with a local existence theory for (1.1) (e.g. a result like Grünrock-Herr [5]) on the support of the measure will give a global existence result for irregular initial conditions. A second motivation is the fact that the Gibbs measure is an object which fits well in the study of recurrence properties of the flow of (1.1).

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For $f \in L^2(\mathbb{T})$, denote by $\int_{\mathbb{T}} f(x) dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$. The following quantities are conserved (at least formally) by the flow of the equation

- The mass

$$M(u(t)) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0).$$

- The energy

$$\begin{aligned} H(u(t)) &= \int_{\mathbb{T}} |\partial_x u|^2 dx + \frac{3}{2} \operatorname{Im} \int_{\mathbb{T}} |u|^2 u \partial_x \bar{u} dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6 dx \\ &= \int_{\mathbb{T}} |\partial_x u|^2 dx - \frac{3}{4} \operatorname{Im} \int_{\mathbb{T}} \bar{u}^2 \partial_x (u^2) dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6 dx \\ &= \int_{\mathbb{T}} |\partial_x u|^2 dx + \frac{3}{4} i \int_{\mathbb{T}} \bar{u}^2 \partial_x (u^2) dx + \frac{1}{2} \int_{\mathbb{T}} |u|^6 dx \\ &= H(u_0). \end{aligned}$$

The conservation of the energy can be seen by a direct computation, or using the Hamiltonian formulation of the equation (1.1) (see the appendix of this paper.)

Notice that the momentum

$$\begin{aligned} P(u(t)) &= \frac{1}{2} \int_{\mathbb{T}} |u|^4 dx + i \int_{\mathbb{T}} \bar{u} \partial_x u dx \\ &= \frac{1}{2} \int_{\mathbb{T}} |u|^4 dx - \operatorname{Im} \int_{\mathbb{T}} \bar{u} \partial_x u dx = P(u_0), \end{aligned}$$

is also formally conserved by (1.1), but the Gibbs measure won't depend on this quantity.

In order to avoid the 0 frequency, in the sequel we will work on the modified DNLS equation

$$(1.2) \quad \begin{cases} i\partial_t v + (\partial_x^2 - 1)v = i\partial_x(|v|^2 v), & (t, x) \in \mathbb{R} \times \mathbb{T}, \\ v(0, x) = u_0(x). \end{cases}$$

Indeed, if the function u is a solution of the equation (1.1) if and only if $v = e^{-it}u$ satisfies (1.2). This transformation preserves the mass and the energy, and the Hamiltonian of the equation (1.2) is

$$\tilde{H}(v) = \int_{\mathbb{T}} |\partial_x v|^2 dx + \int_{\mathbb{T}} |v|^2 dx - \frac{3}{4} \operatorname{Im} \int_{\mathbb{T}} \bar{v}^2 \partial_x (v^2) dx + \frac{1}{2} \int_{\mathbb{T}} |v|^6 dx = H(v) + M^2(v).$$

Let us define the complex vector space $E_N = \text{span}\left((e^{inx})_{-N \leq n \leq N}\right)$. Then we introduce the spectral projector Π_N on E_N by

$$(1.3) \quad \Pi_N\left(\sum_{n \in \mathbb{Z}} c_n e^{inx}\right) = \sum_{n=-N}^N c_n e^{inx}.$$

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $(g_n(\omega))_{n \in \mathbb{Z}}$ a sequence of independent complex normalised gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0, 1)$. We can write

$$(1.4) \quad g_n(\omega) = \frac{1}{\sqrt{2}}(h_n(\omega) + il_n(\omega)),$$

where $(h_n(\omega))_{n \in \mathbb{Z}}, (l_n(\omega))_{n \in \mathbb{Z}}$ are independent standard real Gaussians $\mathcal{N}_{\mathbb{R}}(0, 1)$.

1.1. Definition of the Gibbs measure for (1.2). —

We will show how to define a Gibbs measure for the equation (1.2).

In the sequel we will use the notation $\langle n \rangle = \sqrt{n^2 + 1}$.

Now write $c_n = a_n + ib_n$.

For $N \geq 1$, consider the probability measure on $\mathbb{R}^{2(2N+1)}$ defined by

$$d\mu_N = d_N \prod_{n=-N}^N e^{-\langle n \rangle^2 (a_n^2 + b_n^2)} da_n db_n,$$

where d_N is such that

$$(1.5) \quad \begin{aligned} \frac{1}{d_N} &= \prod_{n=-N}^N \int_{\mathbb{R}^2} e^{-\langle n \rangle^2 (a_n^2 + b_n^2)} da_n db_n \\ &= \pi^{2N+1} \left(\prod_{n=-N}^N \frac{1}{\langle n \rangle} \right)^2 = \pi^{2N+1} \left(\prod_{n=1}^N \frac{1}{\langle n \rangle} \right)^4. \end{aligned}$$

The measure μ_N defines a measure on E_N via the map

$$(a_n, b_n)_{n=-N}^N \longmapsto \sum_{n=-N}^N (a_n + ib_n) e^{inx},$$

which will still be denoted by μ_N . Then μ_N may be seen as the distribution of the E_N valued random variable

$$(1.6) \quad \omega \longmapsto \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle} e^{inx} \equiv \varphi_N(\omega, x),$$

where $(g_n)_{n=-N}^N$ are Gaussians as in (1.4).

Let $\sigma < \frac{1}{2}$. Then (φ_N) is a Cauchy sequence in $L^2(\Omega; H^\sigma(\mathbb{T}))$ which defines

$$(1.7) \quad \varphi(\omega, x) = \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx},$$

as the limit of (φ_N) . Indeed, the map

$$\omega \mapsto \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{\langle n \rangle} e^{inx},$$

defines a (Gaussian) measure on $H^\sigma(\mathbb{T})$ which will be denoted by μ .

For $u \in L^2(\mathbb{T})$, we will write $u_N = \Pi_N u$. Now define

$$f_N(u) = \operatorname{Im} \int_{\mathbb{T}} \overline{u_N^2(x)} \partial_x(u_N^2(x)) dx.$$

Let $\kappa > 0$, and let $\chi : \mathbb{R} \rightarrow \mathbb{R}$, $0 \leq \chi \leq 1$ be a continuous function with support $\operatorname{supp} \chi \subset [-\kappa, \kappa]$ and so that $\chi = 1$ on $[-\frac{\kappa}{2}, \frac{\kappa}{2}]$. We define the measure $\tilde{\rho}_N$ on E_N by

$$(1.8) \quad d\tilde{\rho}_N(u) = \chi(\|u_N\|_{L^2(\mathbb{T})}) e^{\frac{3}{4}f_N(u) - \frac{1}{2} \int_{\mathbb{T}} |u_N(x)|^6 dx} d\mu_N(u).$$

If A is a Borel set of $H^\sigma(\mathbb{T})$ then $A \cap E_N$ is a Borel set of E_N . Now, we define ρ_N which is the natural extension of $\tilde{\rho}_N$ to $H^\sigma(\mathbb{T})$, equipped with the Borel sigma algebra \mathcal{B} . More precisely for every $A \in \mathcal{B}$ which is a Borel set of $H^\sigma(\mathbb{T})$, we set

$$(1.9) \quad \rho_N(A) \equiv \tilde{\rho}_N(A \cap E_N).$$

1.2. Statement of the main result. —

Our main result which defines the Gibbs measure for (1.1) reads

Theorem 1.1. — *The sequence*

$$(1.10) \quad G_N(u) = \chi(\|u_N\|_{L^2(\mathbb{T})}) e^{\frac{3}{4}f_N(u) - \frac{1}{2} \int_{\mathbb{T}} |u_N(x)|^6 dx},$$

converges in measure, as $N \rightarrow \infty$, with respect to the measure μ . Denote by $G(u)$ the limit of (1.10) as $N \rightarrow \infty$. Then for every $p \in [1, \infty[$, there exist $\kappa_0 > 0$ so that for all $0 < \kappa \leq \kappa_0$, $G(u) \in L^p(d\mu(u))$ and if we set $d\rho(u) \equiv G(u)d\mu(u)$ then the sequence $d\rho_N$ converges weakly to $d\rho$ as N tends to infinity.

More precisely, for every continuous and bounded function $h : H^\sigma(\mathbb{T}) \rightarrow \mathbb{C}$, one has

$$\int_{H^\sigma(\mathbb{T})} h(u) d\rho(u) = \lim_{N \rightarrow +\infty} \int_{H^\sigma(\mathbb{T})} h(u) d\rho_N(u).$$

One can show that by varying the cut-off χ , the support of ρ describes the support of μ (see Lemma 4.3 below).

The main ideas of this paper come from the work of the second author [9] where a similar construction is made for the Benjamin-Ono equation using the pioneering work of Bourgain [2]. In [9], one of the main difficulties is that on the support of the measure μ , the L^2 norm is a.s. infinite, which is not the case in our setting, since for any $\sigma < \frac{1}{2}$, $\varphi(\omega) \in H^\sigma(\mathbb{T})$, for almost all $\omega \in \Omega$. Here the difficulty is to treat the term $\int_{\mathbb{T}} \bar{u}^2 \partial_x(u^2) dx$ in the Hamiltonian. Roughly speaking, it should be controlled by the $H^{\frac{1}{2}}$ norm, but this is not enough, since $\|u\|_{H^{\frac{1}{2}}(\mathbb{T})} = \infty$ on the support of $d\mu$. However, we will see in Section 2, that we can handle this term thanks to an adapted decomposition and thanks to the integrability properties of the Gaussians. This is the main new idea in this paper.

The result of Theorem 1.1 may be the first step to obtain almost sure global well-posedness for (1.1), with initial conditions of the form (1.7). To reach such a result, we will also need a suitable local existence theory on the statistical set, and prove the invariance of the measure $d\rho$ under this flow. For instance, this program was fruitful for Tzvetkov [10, 11] for NLS on the disc, Burq-Tzvetkov [4] for the wave equation, Oh [6, 7] for Schrödinger-Benjamin-Ono and KdV systems, and Burq-Thomann-Tzvetkov [3] for the one-dimensional Schrödinger equation. For the DNLS equation, we plan to pursue this issue in a subsequent work.

1.3. Notations and structure of the paper. —

Notations. — In this paper c, C denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters.

We denote by \mathbb{Z} (resp. \mathbb{N}) the set of the integers (resp. non negative integers), and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $x \in \mathbb{R}$, we write $\langle x \rangle = \sqrt{x^2 + 1}$. For $u \in L^2(\mathbb{T})$, we usually write $u_N = \Pi_N u$, where Π_N is the projector defined in (1.3).

The notation L^q stands for $L^q(\mathbb{T})$ and $H^s = H^s(\mathbb{T})$.

The paper is organised as follows. In Section 2 we give some large deviation bounds and some results on the Wiener chaos at any order. In Section 3 we study the term of the Hamiltonian containing the derivative, and Section 4 is devoted to the proof of Theorem 1.1.

In the appendix, we give the Hamiltonian formulation for the periodic DNLS equation.

2. Preliminaries : some stochastic estimates

2.1. Large deviation estimates. —

Lemma 2.1. — *Let $(\gamma_n(\omega))_{n \in \mathbb{Z}} \in \mathcal{N}_{\mathbb{R}}(0, 1)$ be a sequence of independent, normalised real Gaussians. Let $(c_n)_{n \in \mathbb{Z}}, (d_n)_{n \in \mathbb{Z}}$ be two bounded sequences of real numbers. Then there exist $c, C > 0$ so that for all $1 \leq N \leq \lambda$*

$$\mathbf{p}\left(\omega \in \Omega : \left| \sum_{|n_1|, |n_2| \leq N} \frac{c_{n_1} d_{n_2}}{\langle n_1 \rangle} \gamma_{n_1}(\omega) \gamma_{n_2}(\omega) \right| > \lambda\right) \leq C e^{-c\lambda}.$$

Proof. — We estimate

$$A_{\lambda, N} \equiv \mathbf{p}\left(\omega \in \Omega : \sum_{|n_1|, |n_2| \leq N} \frac{c_{n_1} d_{n_2}}{\langle n_1 \rangle} \gamma_{n_1}(\omega) \gamma_{n_2}(\omega) > \lambda\right).$$

For all $t > 0$ and all r.v. X we have, by the Tchebychev inequality

$$(2.1) \quad \mathbf{p}\left(\omega \in \Omega : X > \lambda\right) \leq e^{-\lambda t} \mathbb{E}\left[e^{tX}\right].$$

Thus we obtain that for all $\varepsilon > 0$

$$\begin{aligned} A_{\lambda, N} &\leq e^{-t\lambda} \mathbb{E}\left[\prod_{|n_1|, |n_2| \leq N} e^{t \frac{c_{n_1} d_{n_2}}{\langle n_1 \rangle} \gamma_{n_1} \gamma_{n_2}}\right] \\ &\leq e^{-t\lambda} \mathbb{E}\left[\prod_{|n_1|, |n_2| \leq N} e^{\frac{\varepsilon}{2} \frac{c_{n_1}^2}{\langle n_1 \rangle^2} \gamma_{n_1}^2 + \frac{1}{2\varepsilon} t^2 d_{n_2}^2 \gamma_{n_2}^2}\right] \\ (2.2) \quad &= e^{-t\lambda} \mathbb{E}\left[\left(\prod_{|n_1| \leq N} e^{\frac{\varepsilon}{2} \frac{c_{n_1}^2}{\langle n_1 \rangle^2} \gamma_{n_1}^2}\right) \left(\prod_{|n_2| \leq N} e^{\frac{1}{2\varepsilon} t^2 d_{n_2}^2 \gamma_{n_2}^2}\right)\right]. \end{aligned}$$

Now, the Cauchy-Schwarz inequality and the independence of the γ_n give

$$\begin{aligned}
 A_{\lambda,N} &\leq e^{-t\lambda} \mathbb{E} \left[\prod_{|n_1| \leq N} e^{\varepsilon \frac{c_{n_1}^2}{\langle n_1 \rangle^2} \gamma_{n_1}^2} \right]^{\frac{1}{2}} \mathbb{E} \left[\prod_{|n_2| \leq N} e^{t^2 d_{n_2}^2 \gamma_{n_2}^2 / \varepsilon} \right]^{\frac{1}{2}} \\
 (2.3) \quad &= e^{-t\lambda} \left(\prod_{|n_1| \leq N} \mathbb{E} \left[e^{\varepsilon \frac{c_{n_1}^2}{\langle n_1 \rangle^2} \gamma_{n_1}^2} \right] \prod_{|n_2| \leq N} \mathbb{E} \left[e^{t^2 d_{n_2}^2 \gamma_{n_2}^2 / \varepsilon} \right] \right)^{\frac{1}{2}}.
 \end{aligned}$$

Thanks to a change of variables we can compute explicitly the expectations in the right hand side of (2.3). In fact for $\mu < \frac{1}{2}$

$$\mathbb{E} \left[e^{\mu \gamma_n^2} \right] = \left(1 - 2\mu \right)^{-\frac{1}{2}}.$$

For $0 \leq x \leq \frac{1}{2}$, we have the inequality $(1-x)^{-1} \leq e^{2x}$, hence we deduce that for $\mu < \frac{1}{4}$

$$(2.4) \quad \mathbb{E} \left[e^{\mu \gamma_n^2} \right] \leq e^{2\mu}.$$

Recall that $(c_n), (d_n)$ are bounded. We now fix $\varepsilon > 0$ so that for all $|n| \leq N$, $\varepsilon \frac{c_n^2}{\langle n \rangle^2} \leq \frac{1}{2}$. Then the bound (2.4) implies

$$(2.5) \quad \prod_{|n_1| \leq N} \mathbb{E} \left[e^{\varepsilon \frac{c_{n_1}^2}{\langle n_1 \rangle^2} \gamma_{n_1}^2} \right] \leq \exp \left(2\varepsilon \sum_{|n_1| \leq N} \frac{c_{n_1}^2}{\langle n_1 \rangle^2} \right) \leq C.$$

With the previous choice of $\varepsilon > 0$ and $t > 0$ small enough we also have

$$(2.6) \quad \prod_{|n_2| \leq N} \mathbb{E} \left[e^{t^2 d_{n_2}^2 \gamma_{n_2}^2 / \varepsilon} \right] \leq \exp \left(2t^2 \sum_{|n_2| \leq N} d_{n_2}^2 / \varepsilon \right) \leq e^{Ct^2 N},$$

Finally, from (2.5), (2.6) and (2.3) we infer

$$A_{\lambda,N} \leq C e^{-t\lambda + Ct^2 N} \leq C e^{-c\lambda},$$

for some $c > 0$, if $t > 0$ is chosen small enough and $N \leq \lambda$.

Similarly for $\lambda > 0$,

$$\mathbf{P} \left(\omega \in \Omega : \sum_{|n_1|, |n_2| \leq N} \frac{c_{n_1} d_{n_2}}{\langle n_1 \rangle} \gamma_{n_1}(\omega) \gamma_{n_2}(\omega) < -\lambda \right) \leq C e^{-c\lambda},$$

and this yields the result. \square

Lemma 2.2. — *Fix $\sigma < \frac{1}{2}$ and $p \in [2, \infty)$. Then*

$$\exists C > 0, \exists c > 0, \forall \lambda \geq 1, \forall N \geq 1,$$

$$\mu(u \in H^\sigma : \|\Pi_N u\|_{L^p(\mathbb{T})} > \lambda) \leq C e^{-c\lambda^2}.$$

Moreover there exists $\beta > 0$ such that

$$\exists C > 0, \exists c > 0, \forall \lambda \geq 1, \forall M \geq N \geq 1,$$

$$\mu(u \in H^\sigma : \|\Pi_M u - \Pi_N u\|_{L^p(\mathbb{T})} > \lambda) \leq C e^{-cN^\beta \lambda^2}.$$

Proof. — This result is consequence of the hypercontractivity of the Gaussian random variables : There exists $C > 0$ such that for all $r \geq 2$ and $(c_n) \in \ell^2(\mathbb{N})$

$$\left\| \sum_{n \geq 0} g_n(\omega) c_n \right\|_{L^r(\Omega)} \leq C \sqrt{r} \left(\sum_{n \geq 0} |c_n|^2 \right)^{\frac{1}{2}}.$$

See e.g. [3, Lemma 3.3] for the details of the proof. \square

2.2. Wiener chaos estimates. —

The aim of this subsection is to obtain $L^p(\Omega)$ bounds on Gaussian series. These are obtained thanks to the smoothing effects of the Ornstein-Uhlenbeck semi-group. The following considerations are inspired from [9]. See also [1, 8] for more details on this topic.

For $d \geq 1$, denote by L the operator

$$L = \Delta - x \cdot \nabla = \sum_{j=1}^d \left(\frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial x_j} \right).$$

This operator is self adjoint on $\mathcal{K} = L^2(\mathbb{R}^d, e^{-|x|^2/2} dx)$ with domain

$$\mathcal{D} = \left\{ u : u(x) = e^{|x|^2/4} v(x), \ v \in \mathcal{H}^2 \right\},$$

where $\mathcal{H}^2 = \{u \in L^2(\mathbb{R}^d), x^\alpha \partial_x^\beta v(x) \in L^2(\mathbb{R}^d), \forall (\alpha, \beta) \in \mathbb{N}^{2d}, |\alpha| + |\beta| \leq 2\}$. Denote by $\mathbf{k} = k_1 + \dots + k_d$ and by $(P_n)_{n \geq 0}$ the Hermite polynomials defined by

$$P_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}).$$

Then a Hilbertian basis of eigenfunctions of L on \mathcal{K} is given by

$$P_{\mathbf{k}}(x_1, \dots, x_d) = P_{k_1}(x_1) \dots P_{k_d}(x_d),$$

with eigenvalue $-\mathbf{k} = -(k_1 + \dots + k_d)$.
 Finally define the measure γ_d on \mathbb{R}^d by

$$d\gamma_d(x) = (2\pi)^{-d/2} e^{-|x|^2/2} dx.$$

The next result is a direct consequence of [9, Proposition 3.1]

Lemma 2.3. — *Let $d \geq 1$ and $k \in \mathbb{N}$. Assume that \tilde{P}_k is an eigenfunction of L with eigenvalue $-k$. Then for all $p \geq 2$*

$$\|\tilde{P}_k\|_{L^p(\mathbb{R}^d, d\gamma_d)} \leq (p-1)^{\frac{k}{2}} \|\tilde{P}_k\|_{L^2(\mathbb{R}^d, d\gamma_d)}.$$

Thanks to Lemma 2.3, we will prove the following L^p smoothing effect for some stochastic series.

Proposition 2.4 (Wiener chaos). —

Let $d \geq 1$ and $c(n_1, \dots, n_k) \in \mathbb{C}$. Let $(g_n)_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0, 1)$ be complex L^2 -normalised independent Gaussians.

For $k \geq 1$ denote by $A(k, d) = \{(n_1, \dots, n_k) \in \{1, \dots, d\}^k, n_1 \leq \dots \leq n_k\}$ and

$$(2.7) \quad S_k(\omega) = \sum_{A(k, d)} c(n_1, \dots, n_k) g_{n_1}(\omega) \cdots g_{n_k}(\omega).$$

Then for all $d \geq 1$ and $p \geq 2$

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1} (p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}.$$

Proof. — Let $g_n \in \mathcal{N}_{\mathbb{C}}(0, 1)$. Then we can write $g_n = \frac{1}{\sqrt{2}}(\gamma_n + i\tilde{\gamma}_n)$ with $\gamma_n, \tilde{\gamma}_n \in \mathcal{N}_{\mathbb{R}}(0, 1)$ mutually independent Gaussians. Hence, up to a change of indexes (and with d replaced with $2d$) we can assume that the random variables in (2.7) are real valued. Thus in the following we assume that $g_n \in \mathcal{N}_{\mathbb{R}}(0, 1)$ and are independent.

Denote by

$$\Sigma_k(x_1, \dots, x_d) = \sum_{A(k, d)} c(n_1, \dots, n_k) x_{n_1} \cdots x_{n_k}.$$

Then obviously for all $p \geq 1$,

$$(2.8) \quad \|S_k\|_{L^p(\Omega)} = \|\Sigma_k\|_{L^p(\mathbb{R}^d, d\gamma_d)}.$$

Let $(n_1, \dots, n_k) \in A(k, d)$. Then we can write

$$x_{n_1} \cdots x_{n_k} = x_{m_1}^{p_1} \cdots x_{m_l}^{p_l},$$

where $l \leq k$, $p_1 + \dots + p_l = k$ and $n_1 = m_1 < \dots < m_l \leq n_k$. Now, each monomial $x_{m_j}^{p_j}$ can be expanded on the Hermite polynomials $(P_n)_{n \geq 0}$

$$x_{m_j}^{p_j} = \sum_{k_j=0}^{p_j} \alpha_{j,k_j} P_{k_j}(x_{m_j}).$$

Therefore there exists $\beta(k_1, \dots, k_l) \in \mathbb{C}$ so that

$$x_{n_1} \cdots x_{n_k} = \sum_{j=0}^k \sum_{\substack{k_1 + \dots + k_l = j \\ 0 \leq k_i \leq p_i}} \beta(k_1, \dots, k_l) P_{k_1}(x_{m_1}) \cdots P_{k_l}(x_{m_l}),$$

and we have

$$(2.9) \quad \Sigma_k(x_1, \dots, x_d) = \sum_{j=0}^k \tilde{P}_j(x_1, \dots, x_d),$$

where the polynomial \tilde{P}_j is given by

$$\begin{aligned} \tilde{P}_j(x_1, \dots, x_d) &= \sum_{A(k,d)} \sum_{\substack{k_1 + \dots + k_l = j \\ 0 \leq k_i \leq p_i}} c(n_1, \dots, n_k) \beta(k_1, \dots, k_l) P_{k_1}(x_{m_1}) \cdots P_{k_l}(x_{m_l}). \end{aligned}$$

For $0 \leq k_i \leq p_i$ so that $k_1 + \dots + k_l = j$, the polynomial \tilde{P}_j is an eigenfunction of L with eigenvalue $-j$, hence by Lemma 2.3 we have that for all $p \geq 2$

$$\|\tilde{P}_j\|_{L^p(\mathbb{R}^d, d\gamma_d)} \leq (p-1)^{\frac{j}{2}} \|\tilde{P}_j\|_{L^2(\mathbb{R}^d, d\gamma_d)}.$$

Therefore, by (2.9) and by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\Sigma_k\|_{L^p(\mathbb{R}^d, d\gamma_d)} &\leq (p-1)^{\frac{k}{2}} \sum_{j=0}^k \|\tilde{P}_j\|_{L^2(\mathbb{R}^d, d\gamma_d)} \\ &\leq \sqrt{k+1} (p-1)^{\frac{k}{2}} \left(\sum_{j=0}^k \|\tilde{P}_j\|_{L^2(\mathbb{R}^d, d\gamma_d)}^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{k+1} (p-1)^{\frac{k}{2}} \|\Sigma_k\|_{L^2(\mathbb{R}^d, d\gamma_d)}, \end{aligned}$$

where in the last line we used that the polynomials \tilde{P}_j are orthogonal. This concludes the proof by (2.8) \square

We will need the following lemma which is proved in [9, Lemma 4.5]

Lemma 2.5. — *Let $F : H^\sigma(\mathbb{T}) \rightarrow \mathbb{R}$ be a measurable function. Assume that there exist $\alpha > 0, N > 0, k \geq 1$, and $C > 0$ so that for every $p \geq 2$*

$$\|F\|_{L^p(d\mu)} \leq CN^{-\alpha} p^{\frac{k}{2}}.$$

Then there exist $\delta > 0, C_1$ independent of N and α such that

$$\int_{H^\sigma(\mathbb{T})} e^{\delta N^{\frac{2\alpha}{k}} |F(u)|^{\frac{2}{k}}} d\mu(u) \leq C_1.$$

As a consequence, for all $\lambda > 0$,

$$\mu(u \in H^\sigma(\mathbb{T}) : |F(u)| > \lambda) \leq C_1 e^{-\delta N^{\frac{2\alpha}{k}} \lambda^{\frac{2}{k}}}.$$

3. Study of the sequence $(f_N(u))_{N \geq 1}$

Recall that $f_N(u)$ is defined by $f_N(u) = \text{Im} \int_{\mathbb{T}} \overline{u_N^2(x)} \partial_x(u_N^2(x)) dx$.

The main result of this section is the following

Proposition 3.1. — *The sequence $(f_N)_{N \geq 1}$ is a Cauchy sequence in $L^2(H^\sigma(\mathbb{T}), \mathcal{B}, d\mu)$. Indeed for all $0 < \varepsilon < \frac{1}{2}$ there exists $C > 0$ so that for all $M > N \geq 1$*

$$(3.1) \quad \|f_M(u) - f_N(u)\|_{L^2(H^\sigma(\mathbb{T}), \mathcal{B}, d\mu)} \leq \frac{C}{N^{\frac{3}{2}-\varepsilon}}.$$

Moreover, for all $p \geq 2$ and $M > N \geq 1$

$$(3.2) \quad \|f_M(u) - f_N(u)\|_{L^p(H^\sigma(\mathbb{T}), \mathcal{B}, d\mu)} \leq \frac{C(p-1)^2}{N^{\frac{3}{2}-\varepsilon}}.$$

Then a combination of the estimate (3.2) and Lemma 2.5 yields the following large deviation estimate

Corollary 3.2. — *For every $\alpha < \frac{3}{2}$, there exist $C, \delta > 0$ such that for all $M > N \geq 1$ and $\lambda > 0$*

$$\mu(u \in H^\sigma(\mathbb{T}) : |f_M(u) - f_N(u)| > \lambda) \leq C e^{-\delta(N^\alpha \lambda)^{\frac{1}{2}}}.$$

Thanks to Proposition 3.1, we are able to define the limit in L^2 of the sequence $(f_N)_{N \geq 1}$, which is

$$f(u) = \text{Im} \int_{\mathbb{T}} \overline{u^2(x)} \partial_x(u^2(x)) dx,$$

Notice that Corollary 3.2 implies in particular the convergence in measure

$$(3.3) \quad \forall \varepsilon > 0, \quad \lim_{N \rightarrow \infty} \mu \left(u \in \mathcal{H}^{-\sigma} : |f_N(u) - f(u)| > \varepsilon \right) = 0.$$

For the proof of Proposition 3.1, we have to put $\int_{\mathbb{T}} \overline{\varphi_N^2(\omega)} \partial_x(\varphi_N^2(\omega)) dx$ in a suitable form.

Recall the notation (1.6), then

$$(3.4) \quad \varphi_N^2(\omega) = \sum_{|n_1|, |n_2| \leq N} \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{\langle n_1 \rangle \langle n_2 \rangle} e^{i(n_1+n_2)x}.$$

Therefore we deduce that

$$(3.5) \quad \partial_x(\varphi_N^2(\omega)) = \sum_{|m_1|, |m_2| \leq N} i(m_1 + m_2) \frac{g_{m_1}(\omega) g_{m_2}(\omega)}{\langle m_1 \rangle \langle m_2 \rangle} e^{i(m_1+m_2)x}.$$

Now, by (3.4), (3.5) and the fact that $(e^{inx})_{n \in \mathbb{Z}}$ is an orthonormal family in $L^2(\mathbb{T})$ (endowed with the scalar product $\langle f, g \rangle = \int_{\mathbb{T}} f(x) \overline{g(x)} dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx$), we obtain

$$(3.6) \quad \int_{\mathbb{T}} \overline{\varphi_N^2(\omega)} \partial_x(\varphi_N^2(\omega)) dx = \sum_{A_N} i(n_1 + n_2) \frac{g_{m_1}(\omega) g_{m_2}(\omega) \overline{g_{n_1}(\omega)} \overline{g_{n_2}(\omega)}}{\langle m_1 \rangle \langle m_2 \rangle \langle n_1 \rangle \langle n_2 \rangle},$$

where

$$A_N = \{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \text{ s.t. } |m_1|, |m_2|, |n_1|, |n_2| \leq N \text{ and } m_1+m_2 = n_1+n_2\}.$$

We now split the sum (3.6) in two parts, by distinguishing the cases $m_1 = n_1$ and $m_1 \neq n_1$ in A_N and write

$$\int_{\mathbb{T}} \overline{\varphi_N^2(\omega)} \partial_x(\varphi_N^2(\omega)) dx = S_N^1 + S_N^2,$$

with

$$(3.7) \quad S_N^1 = \sum_{B_N} i(n_1 + n_2) \frac{g_{m_1}(\omega) g_{m_2}(\omega) \overline{g_{n_1}(\omega)} \overline{g_{n_2}(\omega)}}{\langle m_1 \rangle \langle m_2 \rangle \langle n_1 \rangle \langle n_2 \rangle},$$

where $B_N = A_N \cap \{m_1 = n_1 \text{ or } m_1 = n_2\}$, and

$$(3.8) \quad S_N^2 = \sum_{\substack{A_N, m_1 \neq n_1 \\ m_1 \neq n_2}} i(n_1 + n_2) \frac{g_{m_1}(\omega) g_{m_2}(\omega) \overline{g_{n_1}(\omega)} \overline{g_{n_2}(\omega)}}{\langle m_1 \rangle \langle m_2 \rangle \langle n_1 \rangle \langle n_2 \rangle}.$$

3.1. Study of S_N^1 . —

Lemma 3.3. — *Let S_N^1 be defined by (3.7). Then there exists $C > 0$ so that for all $M > N > 0$,*

$$\|S_M^1 - S_N^1\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac{3}{2}}}.$$

Proof. — Let $(m_1, m_2, n_1, n_2) \in B_N$. Then as $m_1 + m_2 = n_1 + n_2$, we have $(m_1, m_2) = (n_1, n_2)$ or $(m_1, m_2) = (n_2, n_1)$, and deduce that

$$S_N^1 = \sum_{|n_1|, |n_2| \leq N} 2i(n_1 + n_2) \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} = X_N + Y_N,$$

where

$$X_N = \sum_{|n| \leq N} 4in \frac{|g_n(\omega)|^4}{\langle n \rangle^4},$$

and

$$Y_N = \sum_{\substack{|n_1|, |n_2| \leq N, \\ n_1 \neq n_2}} 2i(n_1 + n_2) \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1 \rangle^2 \langle n_2 \rangle^2}.$$

♠ First we will show that there exists $C > 0$ so that for all $M > N > 0$,

$$(3.9) \quad \|X_M - X_N\|_{L^2(\Omega)} \leq \frac{C}{N^2}.$$

Let $M > N \geq 1$. Then

$$|X_M - X_N|^2 = \sum_{N < |n_1|, |n_2| \leq M} 16n_1 n_2 \frac{|g_{n_1}(\omega)|^4 |g_{n_2}(\omega)|^4}{\langle n_1 \rangle^4 \langle n_2 \rangle^4}.$$

Thus

$$\|X_M - X_N\|_{L^2(\Omega)}^2 \leq C \sum_{N < |n_1|, |n_2| \leq M} \frac{1}{\langle n_1 \rangle^3 \langle n_2 \rangle^3} \leq \frac{C}{N^4},$$

which proves (3.9).

♠ To complete the proof of Lemma 3.3, it remains to check that there exists $C > 0$ so that for all $M > N > 0$,

$$(3.10) \quad \|Y_M - Y_N\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac{3}{2}}}.$$

For $M \geq N \geq 1$ we write

$$\begin{aligned} Y_N &= \sum_{\substack{|n_1|, |n_2| \leq N, \\ n_1 \neq n_2}} i(n_1 + n_2) \frac{|g_{n_1}(\omega)|^2 |g_{n_2}(\omega)|^2}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \\ &= Y_N^1 + Y_N^2 + Y_N^3, \end{aligned}$$

with

$$(3.11) \quad Y_N^1 = \sum_{\substack{|n_1|, |n_2| \leq N, \\ n_1 \neq n_2}} 2i(n_1 + n_2) \frac{(|g_{n_1}(\omega)|^2 - 1)(|g_{n_2}(\omega)|^2 - 1)}{\langle n_1 \rangle^2 \langle n_2 \rangle^2},$$

$$(3.12) \quad Y_N^2 = \sum_{\substack{|n_1|, |n_2| \leq N, \\ n_1 \neq n_2}} 2i(n_1 + n_2) \frac{(|g_{n_1}(\omega)|^2 - 1) + (|g_{n_2}(\omega)|^2 - 1)}{\langle n_1 \rangle^2 \langle n_2 \rangle^2},$$

and

$$Y_N^3 = \sum_{\substack{|n_1|, |n_2| \leq N, \\ n_1 \neq n_2}} 4i(n_1 + n_2) \frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^2}.$$

By the symmetry $(n_1, n_2) \mapsto (-n_1, -n_2)$, we have that $Y_N^3 = 0$. For $n \in \mathbb{Z}$, denote by

$$(3.13) \quad G_n(\omega) = |g_n(\omega)|^2 - 1.$$

By the definition (1.4) of $g_n(\omega)$, we infer that for $n \neq m$,

$$(3.14) \quad \mathbb{E}[G_n(\omega) G_m(\omega)] = 0$$

• First we analyse (3.11). We compute

$$|Y_M^1 - Y_N^1|^2 = \sum_{C_{M,N}} 4(n_1 + n_2)(m_1 + m_2) \frac{G_{m_1}(\omega) G_{m_2}(\omega) G_{n_1}(\omega) G_{n_2}(\omega)}{\langle m_1 \rangle^2 \langle m_2 \rangle^2 \langle n_1 \rangle^2 \langle n_2 \rangle^2},$$

where

$$C_{M,N} = \{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \text{ s.t. } N < |m_1|, |m_2|, |n_1|, |n_2| \leq M \\ \text{and } m_1 \neq m_2, n_1 \neq n_2\}.$$

We compute $\mathbb{E}[|Y_M^1 - Y_N^1|^2]$, and thanks to (3.14) we see that only the terms $(n_1 = m_1 \text{ and } n_2 = m_2)$ or $(n_1 = m_2 \text{ and } n_2 = m_1)$ give some contribution,

hence

$$\begin{aligned}
\|Y_M^1 - Y_N^1\|_{L^2(\Omega)}^2 &\leq C \sum_{N < |n_1|, |n_2| \leq M} \frac{(n_1 + n_2)^2}{\langle n_1 \rangle^4 \langle n_2 \rangle^4} \\
(3.15) \quad &\leq C \sum_{N < |n_1|, |n_2| \leq M} \left(\frac{1}{\langle n_1 \rangle^2 \langle n_2 \rangle^4} + \frac{1}{\langle n_1 \rangle^4 \langle n_2 \rangle^2} \right) \leq \frac{C}{N^4}.
\end{aligned}$$

• We now turn to (3.12). Similarly, we get

$$|Y_M^2 - Y_N^2|^2 = \sum_{C_{M,N}} 4(n_1 + n_2)(m_1 + m_2) \frac{(G_{m_1}(\omega) + G_{m_2}(\omega))(G_{n_1}(\omega) + G_{n_2}(\omega))}{\langle m_1 \rangle^2 \langle m_2 \rangle^2 \langle n_1 \rangle^2 \langle n_2 \rangle^2},$$

and using the symmetries in (n_1, n_2, m_1, m_2) , and with (3.14) we obtain

$$(3.16) \quad \|Y_M^2 - Y_N^2\|_{L^2(\Omega)}^2 \leq C \left| \sum_{\substack{C_{M,N}, \\ m_1 = n_1}} \frac{(n_1 + n_2)(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^4 \langle n_2 \rangle^2} \right|.$$

Observe that we have

$$\begin{aligned}
(3.17) \quad \sum_{\substack{N < |n_2| \leq M, \\ n_2 \neq n_1}} \frac{(n_1 + n_2)(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^4 \langle n_2 \rangle^2} &= \\
&= \left(\sum_{N < |n_2| \leq M} \frac{n_1(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^4 \langle n_2 \rangle^2} \right) - \frac{2n_1(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^6} \\
&= s_{M,N} \frac{n_1(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^4} - \frac{2n_1(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^6} \\
&= \frac{n_1}{\langle n_1 \rangle^4} \left(s_{M,N} - \frac{2}{\langle n_1 \rangle^2} \right) \frac{n_1 + m_2}{\langle m_2 \rangle^2},
\end{aligned}$$

with $s_{M,N} = \sum_{N < |n_2| \leq M} \frac{1}{\langle n_2 \rangle^2} \leq \frac{C}{N}$. Similarly

$$(3.18) \quad \sum_{\substack{N < |m_2| \leq M, \\ m_2 \neq n_1}} \frac{n_1 + m_2}{\langle m_2 \rangle^2} = \left(\sum_{N < |m_2| \leq M} \frac{n_1 + m_2}{\langle m_2 \rangle^2} \right) - \frac{2n_1}{\langle n_1 \rangle^2} = n_1 s_{M,N} - \frac{2n_1}{\langle n_1 \rangle^2}.$$

Then, from (3.17) and (3.18), we deduce

$$\left| \sum_{\substack{N < |m_2| \leq M, \\ m_2 \neq n_1}} \sum_{\substack{N < |n_2| \leq M, \\ n_2 \neq n_1}} \frac{(n_1 + n_2)(n_1 + m_2)}{\langle m_2 \rangle^2 \langle n_1 \rangle^4 \langle n_2 \rangle^2} \right| \leq C \left(\frac{1}{N^2 \langle n_1 \rangle^2} + \frac{1}{\langle n_1 \rangle^6} \right),$$

and from (3.16),

$$(3.19) \quad \|Y_M^2 - Y_N^2\|_{L^2(\Omega)}^2 \leq \frac{C}{N^3}.$$

Finally, (3.15) and (3.19) yield the estimate (3.10). \square

3.2. Study of S_N^2 . —

We first state the elementary lemma

Lemma 3.4. — *Let $n \in \mathbb{Z}$ and $N \geq 1$. Then for all $0 < \varepsilon \leq \frac{1}{2}$*

$$\sum_{\substack{n_1 \in \mathbb{Z} \\ |n_1|, |n-n_1| \geq N}} \frac{1}{\langle n_1 \rangle^2 \langle n-n_1 \rangle^2} \leq \frac{C}{N^{\frac{3}{2}-\varepsilon} \langle n \rangle^{\frac{3}{2}+\varepsilon}}.$$

Proof. — Let $N \geq 1$. For $\alpha > 1$ we have the inequalities

$$\langle n \rangle^\alpha \leq C(\langle n_1 \rangle^\alpha + \langle n-n_1 \rangle^\alpha),$$

and

$$\langle n_1 \rangle^2 \langle n-n_1 \rangle^2 \geq CN^{4-2\alpha} \langle n_1 \rangle^\alpha \langle n-n_1 \rangle^\alpha \text{ for } |n_1|, |n-n_1| \geq N.$$

Now choose $\alpha = \frac{3}{2} + \varepsilon \leq 2$ to get

$$(3.20) \quad \frac{\langle n \rangle^{\frac{3}{2}+\varepsilon}}{\langle n_1 \rangle^2 \langle n-n_1 \rangle^2} \leq \frac{C}{N^{1-2\varepsilon}} \left(\frac{1}{\langle n_1 \rangle^{\frac{3}{2}+\varepsilon}} + \frac{1}{\langle n-n_1 \rangle^{\frac{3}{2}+\varepsilon}} \right).$$

We sum up (3.20), thus

$$\begin{aligned} & \sum_{\substack{n_1 \in \mathbb{Z} \\ |n_1|, |n-n_1| \geq N}} \frac{1}{\langle n_1 \rangle^2 \langle n-n_1 \rangle^2} \leq \\ & \leq \frac{C}{N^{1-2\varepsilon} \langle n \rangle^{\frac{3}{2}+\varepsilon}} \sum_{\substack{n_1 \in \mathbb{Z} \\ |n_1|, |n-n_1| \geq N}} \left(\frac{1}{\langle n_1 \rangle^{\frac{3}{2}+\varepsilon}} + \frac{1}{\langle n-n_1 \rangle^{\frac{3}{2}+\varepsilon}} \right) \leq \frac{C}{N^{\frac{3}{2}-\varepsilon} \langle n \rangle^{\frac{3}{2}+\varepsilon}}, \end{aligned}$$

which was the claim. \square

We are now able to prove

Lemma 3.5. — *Let S_N^2 be defined by (3.8). For all $0 < \varepsilon \leq \frac{1}{2}$, there exists $C > 0$ so that for all $M > N > 0$,*

$$\|S_M^2 - S_N^2\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac{3}{2}-\varepsilon}}.$$

Proof. — We compute

$$|S_M^2 - S_N^2|^2 = \sum_{D_{M,N} \times D_{M,N}} (n_1 + n_2)(p_1 + p_2) \frac{g_{m_1} g_{m_2} \overline{g_{n_1}} \overline{g_{n_2}} g_{p_1} g_{p_2} \overline{g_{q_1}} \overline{g_{q_2}}}{\langle m_1 \rangle \langle m_2 \rangle \langle n_1 \rangle \langle n_2 \rangle \langle p_1 \rangle \langle p_2 \rangle \langle q_1 \rangle \langle q_2 \rangle},$$

where

$$D_{M,N} = \{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \text{ s.t. } N < |m_1|, |m_2|, |n_1|, |n_2| \leq M \\ \text{and } m_1 + m_2 = n_1 + n_2, m_1 \neq n_1, m_1 \neq n_2\}.$$

The expectation of each term of the previous sum vanishes, unless $((m_1, m_2) = (q_1, q_2)$ or $(q_2, q_1))$ and $((n_1, n_2) = (p_1, p_2)$ or $(p_2, p_1))$. Hence

$$\|S_M^2 - S_N^2\|_{L^2(\Omega)}^2 \leq C \left| \sum_{D_{M,N}} \frac{(n_1 + n_2)(m_1 + m_2)}{\langle n_1 \rangle^2 \langle n_2 \rangle^2 \langle m_1 \rangle^2 \langle m_2 \rangle^2} \right|.$$

Write $n = n_1 + n_2 = m_1 + m_2$, therefore

$$\begin{aligned} \|S_M^2 - S_N^2\|_{L^2(\Omega)}^2 &\leq C \sum_{n \in \mathbb{Z}} \sum_{\substack{|n_1|, |n-n_1| > N, \\ |m_1|, |n-m_1| > N}} \frac{n^2}{\langle n_1 \rangle^2 \langle n - n_1 \rangle^2 \langle m_1 \rangle^2 \langle n - m_1 \rangle^2} \\ &= C \sum_{n \in \mathbb{Z}} n^2 \left(\sum_{|n_1|, |n-n_1| > N} \frac{1}{\langle n_1 \rangle^2 \langle n - n_1 \rangle^2} \right)^2 \\ &\leq \frac{C}{N^{3-2\varepsilon}} \sum_{n \in \mathbb{Z}} \frac{n^2}{\langle n \rangle^{3+2\varepsilon}} \leq \frac{C}{N^{3-2\varepsilon}}, \end{aligned}$$

by Lemma 3.4. □

The results of Lemmas 3.3 and 3.5 imply (3.1).

To complete the proof of Proposition 3.1, it remains to show (3.2). But this is a direct consequence of (3.1) and Proposition 2.4

We are now able to define the density $G : H^\sigma(\mathbb{T}) \rightarrow \mathbb{R}$ (with respect to the measure μ) of the measure ρ . By (3.3) and Proposition 3.1 and 2.2, we have the following convergences in the μ measure : $f_N(u)$ converges in to $f(u)$ and $\|u_N\|_{L^6(\mathbb{T})}$ to $\|u\|_{L^6(\mathbb{T})}$. Then, by composition and multiplication of continuous functions, we obtain

$$(3.21) \quad \chi(\|u_N\|_{L^2(\mathbb{T})}) e^{\frac{3}{4} f_N(u) - \frac{1}{2} \int_{\mathbb{T}} |u_N(x)|^6 dx} \rightarrow \chi(\|u\|_{L^2(\mathbb{T})}) e^{\frac{3}{4} f(u) - \frac{1}{2} \int_{\mathbb{T}} |u(x)|^6 dx} \equiv G(u),$$

in measure, with respect to the measure μ . As a consequence, G is measurable from $(H^\sigma(\mathbb{T}), \mathcal{B})$ to \mathbb{R} .

4. Integrability of the density of $d\rho$

We now state a result which will be useful for the L^p estimates in Theorem 1.1.

Proposition 4.1. — *There exist $\kappa_0 > 0$ and $c, C > 0$ so that for all $0 < \kappa \leq \kappa_0$, $\lambda \geq 2$ and $1 \leq N \leq \lambda$*

$$\mu(u \in H^\sigma(\mathbb{T}) : \|\partial_x(u_N^2)\|_{L^\infty(\mathbb{T})} > \lambda, \|u_N\|_{L^2(\mathbb{T})} \leq \kappa) \leq Ce^{-c\lambda}.$$

Proof. — We can follow the main lines of the proof of [9, Proposition 4.1].

For $j \in \{0, \dots, [\lambda^5]\}$, we define the points $x_j \in \mathbb{T}$ by $x_j = \frac{2\pi j}{\lambda^5}$.

Denote by dist the distance on \mathbb{T} . Then by construction, $\text{dist}(x_j, x_{j+1}) \leq \frac{2\pi}{\lambda^5}$, with $x_{[\lambda^5]+1} \equiv x_0$. We define the set A_λ by

$$A_\lambda \equiv \left\{ u \in H^\sigma(\mathbb{T}) : \|\partial_x(u_N^2)\|_{L^\infty(\mathbb{T})} \geq \lambda, \|u_N\|_{L^2(\mathbb{T})} \leq \kappa \right\},$$

and the sets $A_{\lambda,j}$ by

$$A_{\lambda,j} \equiv \left\{ u \in H^\sigma(\mathbb{T}) : |\partial_x(u_N^2)(x_j)| \geq \frac{\lambda}{2}, \|u_N\|_{L^2(\mathbb{T})} \leq \kappa \right\}.$$

• As in [9] we will show that

$$(4.1) \quad A_\lambda \subset \bigcup_{j=0}^{[\lambda^5]} A_{\lambda,j}.$$

Let $u \in A_\lambda$, and denote by $v_N = \partial_x(u_N^2)$. Let $x^* \in \mathbb{T}$ be such that

$$|v_N(x^*)| = \max_{x \in \mathbb{T}} |v_N(x)|.$$

Thus $|v_N(x^*)| \geq \lambda$. Then there exists $j_0 \in \{0, \dots, [\lambda^5]\}$ such that

$$(4.2) \quad |x^* - x_{j_0}| \leq \frac{2\pi}{\lambda^5}.$$

Then thanks to the Taylor formula, we have

$$(4.3) \quad \begin{aligned} |v_N(x^*) - v_N(x_{j_0})| &= \left| \int_{x_{j_0}}^{x^*} (v_N)'(tx^* + (1-t)x_{j_0}) dt \right| \\ &\leq |x^* - x_{j_0}|^{\frac{1}{2}} \|(v_N)'\|_{L^2(\mathbb{T})}. \end{aligned}$$

Now by the Sobolev embeddings we obtain the bound (with $N \leq \lambda$)

$$(4.4) \quad \begin{aligned} \|(v_N)'\|_{L^2(\mathbb{T})} &\leq CN \|v_N\|_{L^2(\mathbb{T})} \leq CN \|u_N\|_{L^2(\mathbb{T})} \|\partial_x u_N\|_{L^\infty(\mathbb{T})} \\ &\leq CN^{\frac{5}{2}} \|u_N\|_{L^2(\mathbb{T})}^2 \leq C\lambda^{\frac{5}{2}} \kappa^2. \end{aligned}$$

Therefore, from (4.2), (4.3) and (4.4) we deduce that for $\kappa > 0$ small enough

$$|v_N(x^*) - u_N(x_{j_0})| \leq C\kappa^2 \leq \frac{1}{2}\lambda,$$

Thus, by the triangle inequality

$$|v_N(x_{j_0})| \geq |v_N(x^*)| - |v_N(x^*) - v_N(x_{j_0})| \geq \lambda - \frac{1}{2}\lambda = \frac{1}{2}\lambda,$$

we can conclude that $u \in A_{\lambda, j_0}$, which proves (4.1).

• We now estimate $\mu(A_{\lambda, j})$.

As in [9], we can forget the L^2 constraint and write

$$\mu(A_{\lambda, j}) \leq \mathbf{P}(\omega \in \Omega : |\partial_x(\varphi_N^2)(x_j)| \geq \frac{\lambda}{2}).$$

First observe that

$$\begin{aligned} & \{ \omega \in \Omega : |\partial_x(\varphi_N^2)(x_j)| \geq \frac{\lambda}{2} \} \subset \\ & \{ \omega \in \Omega : |\operatorname{Re} \partial_x(\varphi_N^2)(x_j)| \geq \frac{\lambda}{4} \} \cup \{ \omega \in \Omega : |\operatorname{Im} \partial_x(\varphi_N^2)(x_j)| \geq \frac{\lambda}{4} \}. \end{aligned}$$

Indeed we can describe the previous sets by the following way. Write

$$\frac{1}{2}\partial_x(\varphi_N^2)(x_j) = \sum_{|n_1|, |n_2| \leq N} in_2 \frac{g_{n_1}(\omega) g_{n_2}(\omega)}{\langle n_1 \rangle \langle n_2 \rangle} e^{i(n_1+n_2)x_j},$$

and use that $g_n(\omega) = \frac{1}{\sqrt{2}}(h_n(\omega) + il_n(\omega))$ where $(h_n)_{n \in \mathbb{Z}}, (l_n)_{n \in \mathbb{Z}} \in \mathcal{N}_{\mathbb{R}}(0, 1)$ are independent. Then a straightforward computation enables us to put $\operatorname{Re} \partial_x(\varphi_N^2)(x_j)$ and $\operatorname{Im} \partial_x(\varphi_N^2)(x_j)$ in the form

$$\sum_{|n_1|, |n_2| \leq N} \frac{c_{n_1} d_{n_2}}{\langle n_1 \rangle} \gamma_{n_1}(\omega) \gamma_{n_2}(\omega),$$

with $|c_n|, |d_n| \leq C$ and where $(\gamma_n)_{n \in \mathbb{Z}} \in \mathcal{N}_{\mathbb{R}}(0, 1)$ is an independent family of real Gaussians (indeed $\gamma_n = h_n$ or $\gamma_n = l_n$). Therefore we can apply the Lemma 2.1 to get

$$(4.5) \quad \mu(A_{\lambda, j}) \leq Ce^{-c\lambda}.$$

Finally by (4.1) and (4.5) we deduce that

$$\mu(A_\lambda) \leq \sum_{j=0}^{[\lambda^5]} \mu(A_{\lambda, j}) \leq C\lambda^5 e^{-c\lambda} \leq Ce^{-\frac{c}{2}\lambda},$$

which was the claim. \square

Proposition 4.2. — For all $1 \leq p < \infty$, there exists $\kappa_0 > 0$ so that for all $0 < \kappa \leq \kappa_0$ there exists $C > 0$ such that for every $N \geq 1$.

$$\left\| \chi(\|u_N\|_{L^2(\mathbb{T})}) e^{\frac{3}{4}f_N(u) - \frac{1}{2}\int_{\mathbb{T}} |u_N(x)|^6 dx} \right\|_{L^p(d\mu(u))} \leq C.$$

Proof. — Here we can follow the proof of [9, Proposition 4.9]. To prove the proposition, it is sufficient to show that the integral

$$(4.6) \quad \int_0^\infty \lambda^{p-1} \mu(A_{\lambda,N}) d\lambda,$$

is convergent uniformly with respect to N for $\kappa > 0$ small enough and where

$$A_{\lambda,N} = \left\{ u \in H^\sigma : \chi(\|u_N\|_{L^2(\mathbb{T})}) e^{\frac{3}{4}f_N(u) - \frac{1}{2}\int_{\mathbb{T}} |u_N(x)|^6 dx} > \lambda \right\}.$$

We set $N_0 = \ln \lambda$.

- Assume that $N_0 \geq N$.

On the support of χ , $\|u_N\|_{L^2(\mathbb{T})} \leq \kappa$, thus we have

$$\begin{aligned} |f_N(u)| &= \left| \operatorname{Im} \int_{\mathbb{T}} \overline{u_N(x)}^2 \partial_x (u_N(x)^2) dx \right| \leq C \|u_N\|_{L^2(\mathbb{T})}^2 \|\partial_x (u_N^2)\|_{L^\infty(\mathbb{T})} \\ &\leq C \kappa^2 \|\partial_x (u_N^2)\|_{L^\infty(\mathbb{T})}. \end{aligned}$$

Then by Proposition 4.1 (which can be applied, since $N \leq N_0 = \ln \lambda \leq \frac{c_1}{\kappa^2} \ln \lambda$ for $\kappa > 0$ small enough), we obtain

$$\begin{aligned} \mu(A_{\lambda,N}) &\leq \mu\left(u \in H^\sigma : |f_N(u)| > \frac{4}{3} \ln \lambda, \ \|u_N\|_{L^2(\mathbb{T})} \leq \kappa\right) \\ &\leq \mu\left(u \in H^\sigma : \|\partial_x (u_N^2)\|_{L^\infty(\mathbb{T})} > \frac{c_1}{\kappa^2} \ln \lambda, \ \|u_N\|_{L^2(\mathbb{T})} \leq \kappa\right) \\ &\leq C e^{-\frac{c_2}{\kappa^2} \ln \lambda} = C \lambda^{-\frac{c_2}{\kappa^2}}, \end{aligned}$$

where c_2 is independent of κ . Hence the integral (4.6) is convergent if $\kappa = \kappa(p) > 0$ is small enough.

- Assume now $N > N_0$.

Thanks to the triangle inequality $A_{\lambda,N} \subset B_{\lambda,N} \cup C_{\lambda,N}$, where

$$B_{\lambda,N} \equiv \left\{ u \in H^\sigma : |f_{N_0}(u)| > \frac{1}{2} \ln \lambda, \ \|u_N\|_{L^2(\mathbb{T})} \leq \kappa \right\},$$

and

$$C_{\lambda,N} \equiv \left\{ u \in H^\sigma : |f_N(u) - f_{N_0}(u)| > \frac{1}{2} \ln \lambda, \ \|u_N\|_{L^2(\mathbb{T})} \leq \kappa \right\}.$$

The measure of $B_{\lambda,N}$ can be estimated exactly as we did in the analysis of the case $N_0 \geq N$. Finally, by Corollary 3.2, as $N_0 = \ln \lambda$, we obtain that for all

$$1 < \alpha < \frac{3}{2}$$

$$\mu(C_{\lambda,N}) \leq C e^{-\delta(\ln \lambda)^{\frac{1+\alpha}{2}}} \leq C_L \lambda^{-L},$$

for all $L \geq 1$. This completes the proof of the proposition. \square

Proof of Theorem 1.1. — Recall (3.21). Let $p \in [1, +\infty)$ and choose $\kappa_0 > 0$ so that Proposition 4.2 holds. Then there exists a subsequence $G_{N_k}(u)$ so that $G_{N_k}(u) \rightarrow G(u)$, μ a.s. Then by Fatou's lemma,

$$\int_{H^\sigma(\mathbb{T})} |G(u)|^p d\mu(u) \leq \liminf_{k \rightarrow \infty} \int_{H^{-\sigma}(\mathbb{T})} |G_{N_k}(u)|^p d\mu(u) \leq C,$$

thus $G(u) \in L^p(d\mu(u))$.

Now it remains to check that for any continuous bounded function $f : H^\sigma(\mathbb{T}) \rightarrow \mathbb{R}$ we have

$$(4.7) \quad \lim_{N \rightarrow +\infty} \int_{H^\sigma(\mathbb{T})} f(u) G_N(u) d\mu(u) = \int_{H^\sigma(\mathbb{T})} f(u) G(u) d\mu(u),$$

which will be implied by

$$(4.8) \quad \lim_{N \rightarrow +\infty} \int_{H^\sigma(\mathbb{T})} |f(u)(G_N(u) - G(u))| d\mu(u) = 0.$$

As in [9], for $N \geq 0$ and $\varepsilon > 0$, we introduce the set

$$A_{N,\varepsilon} = \{u \in H^\sigma(\mathbb{T}) : |G_N(u) - G(u)| \leq \varepsilon\},$$

and denote by $\overline{A_{N,\varepsilon}}$ its complementary.

Firstly, as f is bounded, there exists $C > 0$ so that for all $N \geq 0$, $\varepsilon > 0$

$$\left| \int_{A_{N,\varepsilon}} f(u)(G_N(u) - G(u)) d\mu(u) \right| \leq C\varepsilon.$$

Secondly, by Cauchy-Schwarz, Proposition 4.2 and as $G(u) \in L^p(d\mu(u))$, we obtain

$$\begin{aligned} \left| \int_{\overline{A_{N,\varepsilon}}} f(u)(G_N(u) - G(u)) d\mu(u) \right| &\leq \|G_N - G(u)\|_{L^2(d\mu)} \mu(\overline{A_{N,\varepsilon}})^{\frac{1}{2}} \\ &\leq C \mu(\overline{A_{N,\varepsilon}})^{\frac{1}{2}}. \end{aligned}$$

By (3.21), we deduce that for all $\varepsilon > 0$,

$$\mu(\overline{A_{N,\varepsilon}}) \rightarrow 0, \quad N \rightarrow +\infty,$$

which yields (4.8). This ends the proof of Theorem 1.1. Notice that (4.8) with $f = 1$ gives

$$(4.9) \quad \lim_{N \rightarrow \infty} \rho_N(E_N) = \rho(H^\sigma(\mathbb{T})).$$

□

Lemma 4.3. — *The measure ρ is not trivial*

Proof. — First observe that for all $\kappa > 0$

$$\mu(u \in H^\sigma(\mathbb{T}) : \|u\|_{L^2(\mathbb{T})} \leq \kappa) = \mathbf{p}(\omega \in \Omega : \sum_{n \in \mathbb{Z}} \frac{1}{\langle n \rangle^2} |g_n(\omega)|^2 \leq \kappa^2) > 0.$$

Then, by Lemma 2.2 and Proposition 3.1, the quantities $\|u\|_{L^6(\mathbb{T})}$ and $f(u)$ are μ almost surely finite. Hence, the density of ρ does not vanish on a set of positive μ measure. In other words, ρ is not trivial. □

A

Hamiltonian structure of DNLS

In this section we give the Hamiltonian structure of the equation (1.1) First we define the projection Π on the 0-mean functions :

$$\Pi(f) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \alpha_n e^{inx}, \quad \text{for } f(x) = \sum_{n \in \mathbb{Z}} \alpha_n e^{inx},$$

then we introduce the integral operator

$$\partial^{-1} : f(x) = \sum_{n \in \mathbb{Z}} \alpha_n e^{inx} \longmapsto \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{\alpha_n}{in} e^{inx}.$$

Notice that we have

$$\partial^{-1}(f') = \Pi f = f - \int_{\mathbb{T}} f(x) dx$$

Next we define the operator

$$K(u, v) = \begin{pmatrix} -u\partial^{-1}u & -i + u\partial^{-1}v \\ i + v\partial^{-1}u & -v\partial^{-1}v \end{pmatrix}.$$

Lemma A.1. — *For u, v , the operator $K(u, v)$ is skew symmetric : $K(u, v)^* = -K(u, v)$.*

Proof. — This is a straightforward computation. We only have to use that $(\partial^{-1})^* = -\partial^{-1}$. □

Define

$$H(u, v) = \int_{\mathbb{T}} \partial_x u \partial_x v + \frac{3}{4} i \int_{\mathbb{T}} v^2 \partial_x (u^2) + \frac{1}{2} \int_{\mathbb{T}} u^3 v^3.$$

Notice that we also have the expressions

$$\begin{aligned} H(u, v) &= - \int_{\mathbb{T}} \partial_x^2 u v + \frac{3}{4} i \int_{\mathbb{T}} v^2 \partial_x (u^2) + \frac{1}{2} \int_{\mathbb{T}} u^3 v^3 \\ &= - \int_{\mathbb{T}} u \partial_x^2 v - \frac{3}{4} i \int_{\mathbb{T}} u^2 \partial_x (v^2) + \frac{1}{2} \int_{\mathbb{T}} u^3 v^3, \end{aligned}$$

therefore, we can deduce the variational derivatives

$$(A.1) \quad \frac{\delta H}{\delta u}(u, v) = -\partial_x^2 v - \frac{3}{2} i u \partial_x (v^2) + \frac{3}{2} u^2 v^3$$

$$(A.2) \quad \frac{\delta H}{\delta v}(u, v) = -\partial_x^2 u + \frac{3}{2} i v \partial_x (u^2) + \frac{3}{2} u^3 v^2.$$

We consider the Hamiltonian system

$$(A.3) \quad \begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = K(u, v) \begin{pmatrix} \frac{\delta H}{\delta u}(u, v) \\ \frac{\delta H}{\delta v}(u, v) \end{pmatrix}.$$

Denote by

$$F_u(t) = 2 \operatorname{Im} \int_{\mathbb{T}} u \partial_x \bar{u} + \frac{3}{2} \int_{\mathbb{T}} |u|^4,$$

and notice that for all $t \in \mathbb{R}$, $F_u(t) \in \mathbb{R}$

Proposition A.2. — *The system (A.3) is a Hamiltonian formulation of the equation*

$$(A.4) \quad i \partial_t u + \partial_x^2 u = i \partial_x (|u|^2 u) + F_u(t) u,$$

in the coordinates $(u, v) = (u, \bar{u})$.

As a consequence, if we set

$$(A.5) \quad v(t, x) = e^{i \int_0^t F_u(s) ds} u(t, x),$$

then v is the solution of the equation

$$(A.6) \quad \begin{cases} i \partial_t v + \partial_x^2 v = i \partial_x (|v|^2 v), & (t, x) \in \mathbb{R} \times \mathbb{T}, \\ v(0, x) = u_0(x). \end{cases}$$

Moreover, if u and v are linked by (A.5), we have $F_u = F_v$.

Proof. — We have

$$u \partial_x^2 v = \partial_x^2 u v + (u \partial_x v)' - (\partial_x u v)',$$

therefore

$$(A.7) \quad \partial^{-1}(u \partial_x^2 v) = \partial^{-1}(\partial_x^2 u v) + u \partial_x v - \partial_x u v - \int_{\mathbb{T}} (u \partial_x v - \partial_x u v).$$

Similarly we obtain the relation

$$(A.8) \quad \partial^{-1}(u^2 \partial_x(v^2)) = -\partial^{-1}(v^2 \partial_x(u^2)) + u^2 v^2 - \int_{\mathbb{T}} u^2 v^2.$$

By (A.1), (A.2), using (A.7) and (A.8), a straightforward computation gives

$$\begin{aligned} \partial_t u &= -u \partial^{-1}\left(u \frac{\delta H}{\delta u}\right) - i \frac{\delta H}{\delta v} + u \partial^{-1}\left(v \frac{\delta H}{\delta v}\right) \\ &= i \partial_x^2 u + \partial_x(u^2 v) - \frac{1}{2\pi} u \int_{\mathbb{T}} (u \partial_x v - v \partial_x u) - \frac{3}{2} i u \int_{\mathbb{T}} u^2 v^2, \end{aligned}$$

and

$$\begin{aligned} \partial_t v &= i \frac{\delta H}{\delta u} + v \partial^{-1}\left(u \frac{\delta H}{\delta u}\right) - v \partial^{-1}\left(v \frac{\delta H}{\delta v}\right) \\ &= -i \partial_x^2 v + \partial_x(u v^2) - \frac{1}{2\pi} v \int_{\mathbb{T}} (v \partial_x u - u \partial_x v) - \frac{3}{2} i v \int_{\mathbb{T}} u^2 v^2. \end{aligned}$$

Now assume that $v = \bar{u}$. This yields the result, as

$$\int_{\mathbb{T}} (u \partial_x \bar{u} - \bar{u} \partial_x u) = 2i \operatorname{Im} \int_{\mathbb{T}} u \partial_x \bar{u}.$$

□

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