HAL
open science

Capillary-Gravity Waves Generated by a Sudden Object Motion

Fabien Closa, Alexei Chepelianskii, Elie Raphael

To cite this version:

Fabien Closa, Alexei Chepelianskii, Elie Raphael. Capillary-Gravity Waves Generated by a Sudden Object Motion. 2010. hal-00449256

HAL Id: hal-00449256

https://hal.science/hal-00449256v1

Preprint submitted on 22 Jan 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Capillary-Gravity Waves Generated by a Sudden Object Motion

F. Closa,[1] A.D. Chepelianskii,[2] and E. Raphaël[1,*]
[1] *Laboratoire Physico-Chimie Théorique, UMR CNRS GULLIVER 7083, ESPCI, 10 rue Vauquelin, 75005 Paris, France and*
[2] *Laboratoire de Physique des Solides, UMR CNRS 8502, Bâtiment 510, Université Paris-Sud, 91405 Orsay, France*
(Dated: January 22, 2010)

We study theoretically the capillary-gravity waves created at the water-air interface by a small object during a sudden accelerated or decelerated rectilinear motion. We analyze the wave resistance corresponding to the transient wave pattern and show that it is nonzero even if the involved velocity (the final one in the accelerated case, the initial one in the decelerated case) is smaller than the minimum phase velocity $c_{min} = 23\,\mathrm{cm\,s^{-1}}$. These results might be important for a better understanding of the propulsion of water-walking insects where accelerated and decelerated motions frequently occur.

PACS numbers: 47.35.-i,68.03.-g

I. INTRODUCTION

If a body (like a boat or an insect), or an external pressure source, moves at the free liquid-air interface, it generates *capillary-gravity waves*. These are driven by a balance between the liquid inertia and its tendancy, under the action of gravity and under surface tension forces, to return to a state of stable equilibrium [1]. For an inviscid liquid of infinite depth, the dispersion relation of capillary-gravity waves, relating the angular frequency ω to the wavenumber k is given by $\omega^2 = gk + \gamma k^3/\rho$, where γ is the liquid-air surface tension, ρ the liquid density, and g the acceleration due to gravity [2]. The energy carried away by the waves is felt by the body (or the pressure source) as a drag R_w, called the *wave resistance* [3]. In the case of boats or ships, the wave resistance has been well studied in order to design hulls minimizing it [4]. The case of objects that are small compared to the capillary length $\kappa^{-1} = \sqrt{\gamma/(\rho g)}$ has been considered only recently [5–11].

In the case of a disturbance moving at constant velocity $V = |\mathbf{V}|$ on a rectilinear trajectory, the wave resistance R_w is zero for $V < c_{min}$ where $c_{min} = (4g\rho/\gamma)^{1/4}$ is the minimum of the wave velocity $c(k) = \omega(k)/k = \sqrt{g/k + \gamma k/\rho}$ for capillarity gravity waves [3, 5, 12]. Effectively, in the frame moving with that object, the problem must be stationary. That is true only if the radiated waves have a phase velocity $c(k)$ equal to the object's velocity V. If $V < c_{min}$, no solutions exist, hence no waves and the wave resistance is zero. For water with $\gamma = 73\,\mathrm{mN\,m^{-1}}$ and $\rho = 10^3\,\mathrm{kg\,m^{-3}}$, one has $c_{min} = 0.23\,\mathrm{m\,s^{-1}}$ (at room temperature). It was recently shown by Chepelianskii *et al.* [13] that no such velocity threshold exists for a steady circular motion, for which, even for small velocities, a finite wave drag is experienced by the object. Here we consider the case of a sudden accelerated or decelerated rectilinear motion and show that the transient wave pattern leads to a nonzero wave resistance even if the involved velocity (the final one for the accelerated case, the initial one for the decelerated case) is smaller than the minimum phase velocity $c_{min} = 23\,\mathrm{cm\,s^{-1}}$. The physical origin of these results is similar to the Cherenkov radiation emitted by accelerated (or decelerated) charged particles [14, 15].

II. EQUATIONS OF MOTION

We consider an inviscid, deep liquid with an infinitely extending free surface. To locate a point on the free surface, we introduce a vector $\mathbf{r} = (x, y)$ in the horizontal plane associated with the equilibrium state of a flat surface. The motion of the disturbance in this plane induces a vertical displacement $\zeta(\mathbf{r}, t)$ (Monge representation) of the free surface from its equilibrium position.

Assuming that the liquid equations of motion can be linearized (in the limit of small wave amplitudes), one has [13]

$$\frac{\partial^2 \hat{\zeta}(\mathbf{k}, t)}{\partial t^2} + \omega(k)^2 \hat{\zeta}(\mathbf{k}, t) = -\frac{k\hat{P}_{ext}(\mathbf{k}, t)}{\rho}, \qquad (1)$$

where $\hat{P}_{ext}(\mathbf{k}, t)$ and $\hat{\zeta}(\mathbf{k}, t)$ are the Fourier transforms of the pressure distribution and the displacement, respectively [16]. In what follows, we will assume that the pressure distribution is axisymmetric around the point $\mathbf{r}_0(t)$ (corresponding to the disturbance trajectory). $\hat{P}_{ext}(\mathbf{k}, t)$ can then be written as $\hat{P}_{ext}(k)e^{-i\mathbf{k}.\mathbf{r}_0(t)}$.

A. Uniform straight motion

Let us first recall the results previously obtained in the case of a uniform straight motion [3, 12, 17, 18]. Such a motion corresponds to $\mathbf{r}_0(t) = (-Vt, 0)$ where V is the constant velocity of the disturbance.

Equation (1) then becomes

$$\frac{\partial^2 \hat{\zeta}(\mathbf{k}, t)}{\partial t^2} + \omega(k)^2 \hat{\zeta}(\mathbf{k}, t) = -\frac{k\hat{P}_{ext}(k)e^{iVk_x t}}{\rho}. \qquad (2)$$

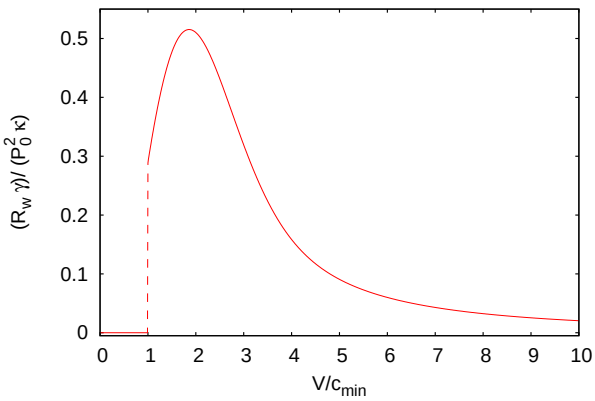

FIG. 1: Wave resistance R_w (in units of $P_0^2\kappa/\gamma$) as a function of the reduced velocity V/c_{min} for a uniform straight motion, see Eq.(5). The pressure disturbance is assumed to be a Lorentzian, with a Fourier transform $\hat{P}_{ext}(k) = P_0 e^{-bk}$, where b is the object size (set to $b = 0.1\kappa^{-1}$).

The above equation corresponds to the equation of an harmonic oscillator forced at angular frequency Vk_x. We can solve it by looking for solutions with a time-dependence of the form e^{iVk_xt}, leading to

$$\hat{\zeta}(\mathbf{k},t) \;=\; -\frac{k\hat{P}_{ext}(k)}{\rho(\omega(k)^2-(Vk_x)^2)}e^{iVk_xt}. \qquad (3)$$

Following Havelock [3], the wave resistance $\mathbf{R}_w$ experienced by the moving disturbance is given by

$$\mathbf{R}_w = -i\iint \frac{\mathrm{d}k_x}{2\pi}\,\frac{\mathrm{d}k_y}{2\pi}\,\mathbf{k}\;\hat{\zeta}(\mathbf{k},t)\,\hat{P}^*(k,t). \qquad (4)$$

This expression for the wave resistance represents as the total force exerted by the external pressure on the free surface $\mathbf{R}_w = \iint \mathrm{d}x\mathrm{d}y P_{ext}(\mathbf{r},t)\nabla\zeta(\mathbf{r},t)$ written in Fourier space. Using Eq.(3) and integrating over the angular variable one obtains [5]

$$\mathbf{R}_w \;=\; \int_0^\infty \frac{\mathrm{d}k}{2\pi}\,\frac{kP_0^2}{\rho}\,e^{-2bk}\frac{\theta(V-c(k))}{V^2\sqrt{1-(c(k)/V)^2}}\,\mathbf{u}_x, \qquad (5)$$

where $\theta(x)$ is the Heaviside step function and $\mathbf{u}_x$ the unit vector along the x-axis. The behavior of $\mathbf{R}_w$ as a function of the disturbance velocity V is illustrated in Fig. 1. There we have assumed a pressure disturbance of Lorentzian form with a Fourier transform $\hat{P}_{ext}(k) = P_0e^{-bk}$, where b is the object size (set to $b = 0.1\kappa^{-1}$). This choice will be taken for the figures throughout this work. The wave resistance is equal to zero for $V < c_{min}$ and presents a discontinuous behavior at $V = c_{min}$ (see reference [8] and [13] for a more complete discussion).

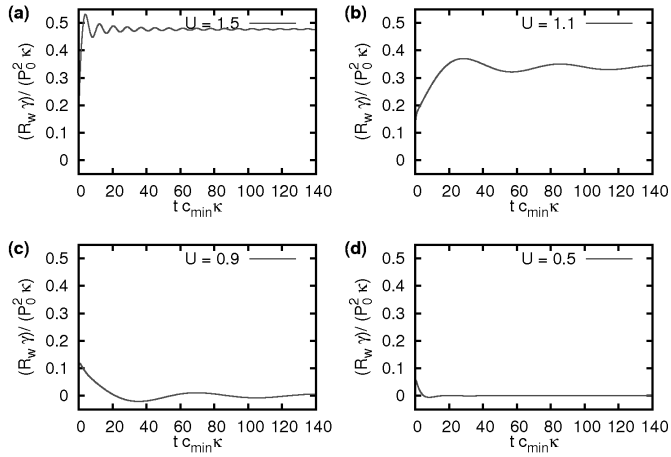

FIG. 2: The wave resistance R_w (in units of $P_0^2\kappa/\gamma$) is shown as a function of the reduced time $c_{min}\kappa t$ for an accelerated motion with different $U = V/c_{min}$ (see Eq.(7)). Respectively, panels (a), (b), (c) and (d) correspond to a reduced velocity $U = 1.5$, 1.1, 0.9 and 0.5.

B. Accelerated straight motion

We now turn to the case where the disturbance – initially at rest – is suddenly set to a uniform motion (characterized by a constant velocity V) at time $t = 0$. The corresponding trajectory is given by $\mathbf{r}_0(t) = -V\,t\,\theta(t)\,\mathbf{u}_x$. As long as the perturbation does not move (i.e. for $t < 0$), the wave resistance is equal to zero. In order to calculate the wave resistance for $t > 0$, we solve Eq.(2) along with the initial conditions $\hat{\zeta}(\mathbf{k}, t = 0) = 0$ and $\dfrac{\partial\hat{\zeta}(\mathbf{k},t=0)}{\partial t} = 0$, yielding

$$\hat{\zeta}(\mathbf{k},t) = -\int_0^t \mathrm{d}\tau\frac{k\,\hat{P}_{ext}\,(k)}{\omega\,(k)\,\rho}e^{-i\mathbf{k}\mathbf{r}_0(\tau)}\sin\left(\omega\,(k)\,(t-\tau)\right). \qquad (6)$$

Equation (4) then leads to the following expression for the wave resistance

$$\mathbf{R}_w(t) = \int_0^\infty\frac{\mathrm{d}k}{2\pi}\int_0^t \mathrm{d}u\frac{k^3|\hat{P}_{ext}(k)|^2}{\rho\omega(k)}\sin\left(\omega\,(k)\,u\right)J_1\left(kVu\right)\,\mathbf{u}_x, \qquad (7)$$

where $J_1(x)$ is the first Bessel functions of the first kind. In the long time limit, one has [19]

$$\lim_{t\to\infty}\int_0^t \sin(\omega\,(k)\,u)\,J_1(kVu)\,\mathrm{d}u = \frac{\omega(k)\theta(V-c(k))}{V^2k^2\sqrt{1-(c(k)/V)^2}} \qquad (8)$$

and therefore the wave resistance Eq. (7) converges to the uniform straight motion result, Eq. (5). The behavior of $\mathbf{R}_w(t)$ is represented on Fig.2 for different values of the

disturbance velocity and will be discussed in detail in section III.

C. Decelerated straight motion

Let us now consider the case of a disturbance moving with a constant velocity V for $t < 0$, and suddenly set and maintained at rest for $t > 0$. This corresponds to the following trajectory: $\mathbf{r}_0(t) = -V\,t\,\theta(-t)\,\mathbf{u}_x$. As long as $t < 0$, the wave resistance $\mathbf{R}_w$ is given by the uniform straight motion result Eq. (5). In order to calculate the wave resistance for $t > 0$, we solve Eq.(2) along with the initial conditions

$$\hat{\zeta}(\mathbf{k}, t = 0) \;=\; \frac{-k\hat{P}_{ext}(k)}{\rho(\omega(k)^2 - (Vk_x)^2)} \tag{9}$$

and

$$\frac{\partial\hat{\zeta}}{\partial t}(\mathbf{k}, t = 0) \;=\; \frac{-k\hat{P}_{ext}(k)(iVk_x)}{\rho(\omega(k)^2 - (Vk_x)^2)} \tag{10}$$

(where we have used Eq. (3)). This leads to

$$\begin{aligned}\hat{\zeta}(\mathbf{k}, t \geqslant 0) = &\\ &\left(\frac{k\hat{P}_{ext}(k)}{\rho\omega(k)^2} - \frac{k\,\hat{P}_{ext}(k)}{\rho(\omega(k)^2 - (Vk_x)^2)}\right)\cos\left(\omega\left(k\right)t\right)\\ &- \frac{k\left(iVk_x\right)\hat{P}_{ext}(k)}{\rho(\omega(k)^2 - (Vk_x)^2)}\frac{\sin\left(\omega\left(k\right)t\right)}{\omega(k)} - \frac{k\hat{P}_{ext}(k)}{\rho\omega(k)^2}.\end{aligned} \tag{11}$$

Equation (4) then leads to the following expression for the wave resistance:

$$\begin{aligned}\mathbf{R}_w(t) = &\\ &\int_0^\infty \frac{\mathrm{d}k}{2\pi}\frac{k|\hat{P}_{ext}\left(k\right)|^2}{\rho.V^2}\cos\left(\omega\left(k\right)t\right)\frac{\theta\left(V - c(k)\right)}{\sqrt{1 - \left(\frac{c(k)}{V}\right)^2}}\mathbf{u}_x\\ &+\int_0^\infty \frac{\mathrm{d}k}{2\pi}\frac{k|\hat{P}_{ext}\left(k\right)|^2}{\rho.V^2}\sin\left(\omega\left(k\right)t\right)\left(\frac{V}{c(k)} - \frac{\theta\left(c(k) - V\right)}{\sqrt{\left(\frac{c(k)}{V}\right)^2 - 1}}\right)\mathbf{u}_x.\end{aligned} \tag{12}$$

In the long time limit ($t \to \infty$), the Riemann-Lebesgue lemma [20], for a Lebesgue integrable function f

$$\lim_{t\to\infty}\int f(x)\,e^{i\,x\,t}\mathrm{d}x \;=\; 0 \tag{13}$$

permits to determine the limit: the wave resistance given by Eq. (12) converges to 0 for $t \to \infty$, as expected. The behavior of $\mathbf{R}_w(t)$ is represented in Fig.3 for different values of the disturbance velocity and will be discussed in detail in section the next section (Sec.III).

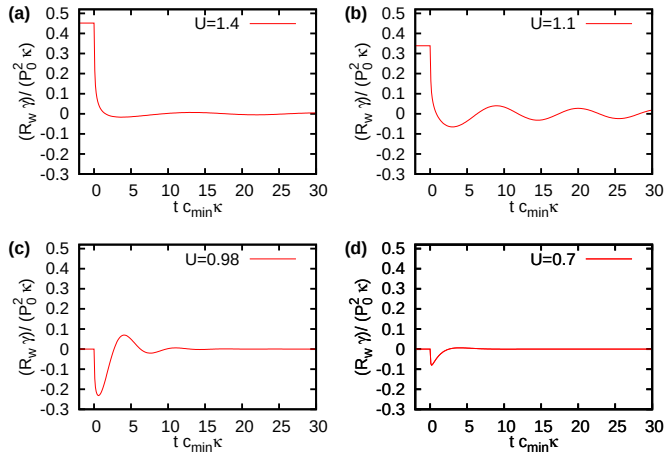

FIG. 3: The wave resistance R_w (in units of $P_0^2\kappa/\gamma$) is shown as a function of the reduced time $c_{min}\kappa t$ for a decelerated motion with different reduced velocities $U = V/c_{min}$ (see Eq.(12)). Respectively, panels (a), (b), (c) and (d) correspond to a reduced velocity $U = 1.4$, 1.1, 0.98 and 0.7.

III. RESULTS AND DISCUSSION

A. Accelerated straight motion

Figures 2 and 3 represent the behavior of the wave resistance for the accelerated and the decelerated cases, respectively. In order to get a better understanding of the behavior of $\mathrm{R}_w = \mathbf{R}_w \cdot \mathbf{u}_x$, we will perform analytic expansions of Eq.(7) and Eq.(12), respectively. Let us start with the accelerated case, Eq.(7). Since one has the product of two oscillating functions, $\sin\left(\omega\left(k\right)u\right)$ and $J_1\left(kVu\right)$, one can use a stationary phase approximation [21]. The sine function oscillates with a phase $\phi_1 = \omega(k)u$ whereas the Bessel function J_1 oscillates with a phase $\phi_2 = Vku$. Their product has thus two oscillating terms, one with a phase $\phi_- = \phi_1 - \phi_2$ and the other with a phase $\phi_+ = \phi_1 + \phi_2$. According to the stationary phase approximation, the important wavenumbers are given by $\frac{\mathrm{d}\phi_-}{\mathrm{d}k} = 0$ and $\frac{\mathrm{d}\phi_+}{\mathrm{d}k} = 0$. The latter equation does not admit any real solution and the corresponding contribution to the wave resistance decreases exponentially and can be neglected. The equation $\frac{\mathrm{d}\phi_-}{\mathrm{d}k} = 0$ leads to $(c_g(k) - V)u = 0$, where $c_g(k) = \frac{\mathrm{d}\omega(k)}{\mathrm{d}k}$ is the group velocity of the capillary-gravity waves [3]. This equation has solutions only if $V \geqslant \min(c_g(k)) = (3\sqrt{3}/2 - 9/4)^{1/4}c_{min} \approx 0.77c_{min}$. When this condition is satisfied, two wavenumbers are selected: k_g (mainly dominated by gravity) and k_c (mainly dominated by capillary forces), with $k_g < k_c$ (see Fig.4). If these two wavenumbers are sufficiently separated (that is, for velocities not too close to $0.77c_{min}$), one then finds that in the long time limit R_w oscillates around its final value as

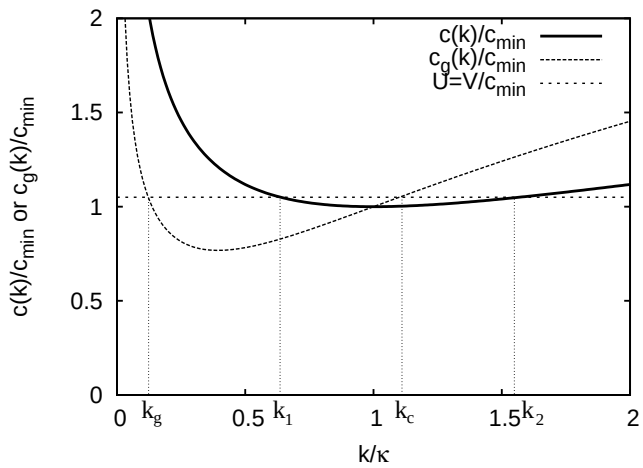

FIG. 4: Graphical representation of wavenumbers k_1, k_2, k_c and k_g in units of κ. k_1 and k_2 are the solutions of the equation $c(k) = V$ and correspond to the intersection between the curve $c(k)/c_{min}$ and the line $U = V/c_{min}$. k_g and k_c are the solutions of the equation $c_g(k) = V$ and correspond to the intersection between the curve $c_g(k)/c_{min}$ and the line $U = V/c_{min}$. Analytical expresions of k_c and k_g can be given but are rather lengthy.

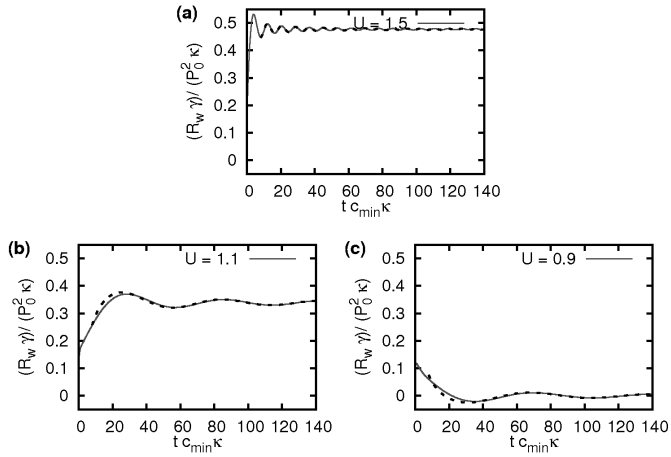

FIG. 5: The wave resistance R_w (in units of $P_0^2\kappa/\gamma$) is shown as a function of the reduced time $c_{min}\kappa t$ for an accelerated motion with different $U = V/c_{min}$ (see Eq.(7)). Respectively, panels (a), (b) and (c) correspond to a reduced velocity $U = 1.5$, 1.1 and 0.9. The solid lines are obtained by a numerical integration of Eq.(7). The dashed lines correspond to the asymptotic expansion given by Eq. (14).

$$\mathrm{R}_w(t) = \mathrm{R}_w(\infty) + \frac{1}{2\pi\rho\sqrt{V}} \frac{k_c^{5/2}|\hat{P}_{ext}(k_c)|^2}{\omega(k_c)\sqrt{|\frac{d^2\omega}{dk^2}(k_c)|}} \frac{\cos(\Omega_c t)}{\Omega_c t} + \frac{1}{2\pi\rho\sqrt{V}} \frac{k_g^{5/2}|\hat{P}_{ext}(k_g)|^2}{\omega(k_g)\sqrt{|\frac{d^2\omega}{dk^2}(k_g)|}} \frac{\sin(\Omega_g t)}{\Omega_g t}, \quad (14)$$

where $\mathrm{R}_w(\infty)$ is given by Eq.(5) (and is equal to zero if $V < c_{min}$), $\Omega_c = (c(k_c) - c_g(k_c))k_c$ and $\Omega_g = (c(k_g) - c_g(k_g))k_g$.

Therefore, even if the disturbance velocity V is smaller than c_{min}, there exists a transient nonzero wave resistance [22] decreasing as $1/t$ (for $V > 0.77\, c_{min}$).

We obtain a good agreement between the numerical calculation of Eq.(7) and the analytical approximation Eq.(14) as shown on Fig.5. Note that the oscillations displayed by the wave resistance are characterized by a period $2\pi/\Omega_c$ or $2\pi/\Omega_g$ that diverges as V approaches c_{min}. In the particular case where $V \gg c_{min}$, Eq.(14) reduces to

$$\mathrm{R}_w(t) = \mathrm{R}_w(\infty) + \frac{1}{2^{9/2}\pi} \frac{\kappa c_{min}^2 |\hat{P}_{ext}(k_g)|^2}{\rho V^5 t} \sin\left(\frac{c_{min}^2 \kappa t}{8V}\right) - \frac{2^{3/2}}{\pi} \frac{\kappa |\hat{P}_{ext}(k_c)|^2}{c_{min}^2 \rho V t} \cos\left(\frac{8V^3 \kappa t}{27 c_{min}^2}\right). \quad (15)$$

Let us now give a physical interpretation for the wave resistance behavior as described by Eq.(14): during the sudden acceleration of the disturbance (taking place at $t = 0$), a large range of wavenumbers is emitted. Waves with wavenumbers k such that $c_g(k) > V$ will move faster than the disturbance (which moves with the velocity V), whereas waves with wavenumbers k such that $c_g(k) < V$ will move slower. The main interaction with the moving disturbance will therefore correspond to wavenumbers satisfying $c_g(k) = V$, that is to k_c and k_g (hence their appearance in Eq.(14)). Due to the Doppler effect, the wave resistance R_w (which is the force exerted by the fluid on the moving disturbance) oscillates with an angular frequency $(c(k) - V)k$, hence the appearance of Ω_c and Ω_g in Eq.(14).

Note that in the case $V < 0.77 c_{min}$, the wave resistance, Eq.(7), is nonzero and decreases exponentially with time (see Fig.2(d)).

In order to get a better physical picture of the generated wave patterns, we have also calculated numerically the transient vertical displacement of the free surface $\zeta(\mathbf{r}, t)$ in the accelerated case (Eq.(6)). The corresponding patterns (as seen in the frame of the moving object), are presented on Figs. 6, 7 and 8 for different reduced times $c_{min}\kappa t$ (1, 10 and 50, respectively). At $c_{min}\kappa t = 1$, the perturbation of the free surface is very localized around the disturbance (close to the same one obtained by a stone's throw). At $c_{min}\kappa t = 10$, some capillary waves can already be observed at the front of the disturbance. At $c_{min}\kappa t = 50$, on can see a V-shaped pattern that prefigure the steady pattern of the uniform straight motion (obtained at very long times). These predictions concerning the wave pattern might be compared to experimental data using the recent technique of Moisy, Rabaud, and Salsac [23].

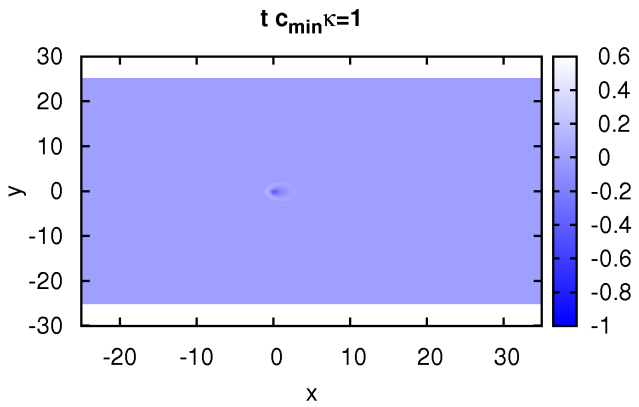

FIG. 6: Accelerated straight motion: Transient vertical displacement (in units of $P_0\kappa/(\rho c_{min}^2)$) of the free surface at $tc_{min}\kappa = 1$ obtained by inverse Fourier transform of Eq.(6). Note that the surface disturbance is localized around the object and close to the one obtained by a stone thrown in water.

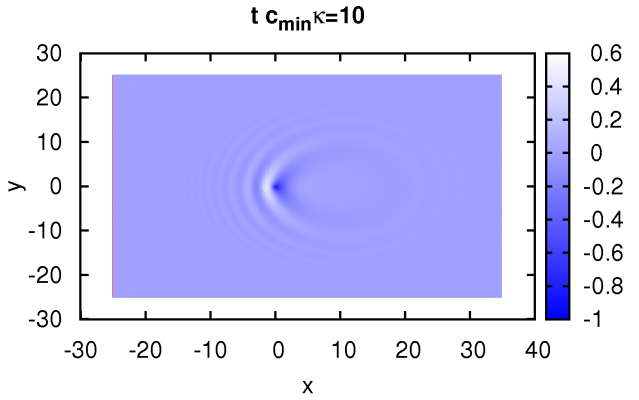

FIG. 7: Accelerated straight motion: Transient vertical displacement (in units of $P_0\kappa/(\rho c_{min}^2)$) of the free surface at $tc_{min}\kappa = 10$ obtained by inverse Fourier transform of Eq.(6). One can already see ahead of the disturbance the waves associated with k_1, while the ones associated to k_2 are less well formed.

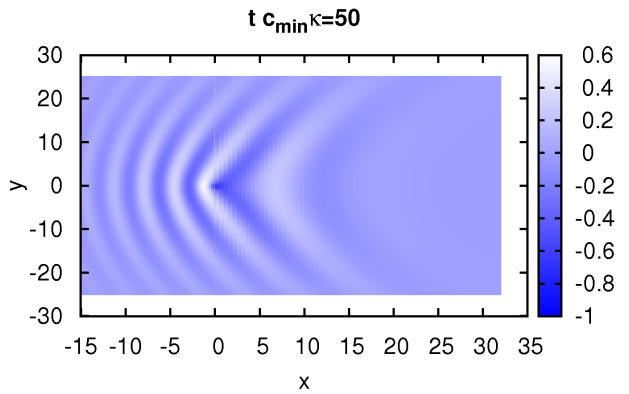

FIG. 8: Accelerated straight motion: Transient vertical displacement (in units of $P_0\kappa/(\rho c_{min}^2)$) of the free surface at $tc_{min}\kappa = 50$ obtained by inverse Fourier transform of Eq.(6).

B. Decelerated straight motion

Let us now turn to the case of the decelerated motion, described by Eq.(12). We first will assume that $V > c_{min}$. In that case, the denominators appearing in the integrals vanish for k such that $c(k) = V$, that is for $k_1 = \kappa(U^2 - \sqrt{U^4 - 1})$ (mainly dominated by gravity) and $k_2 = \kappa(U^2 + \sqrt{U^4 - 1}))$ (mainly dominated by capillary forces) as shown in Fig.4, where $U = V/c_{min}$. Both wavenumbers contribute to the wave resistance. Using the stationary phase approximation one then finds that in the long time limit R_w oscillates as $\cos\left(Vk_1t + \pi/4\right)/\sqrt{t}$. More precisely, one has (for large t)

$$R_w(t) = \frac{1}{\sqrt{\pi}\,V\,\sqrt{\rho\gamma}}\frac{k_1^{3/2}|\hat{P}_{ext}(k_1)|^2}{\sqrt{(k_2 - k_1)\,c_g(k_1)\,t}}\cos(Vk_1t + \pi/4). \tag{16}$$

The wave resistance displays oscillations that are characterized by a period $2\pi/(Vk_1)$. Figure 9 shows the good agreement between the numerical calculation of Eq.(12) and the analytical approximation Eq.(16) for large times ($c_{min}\kappa t > 10$). Again one can give a simple physical interpretation of the wave resistance, Eq.(16). The period of the oscillations is given by $2\pi/(Vk_1)$ and depends only on the wavenumber k_1 (mainly dominated by gravity), even in the case of an object with a size b much smaller than the capillary length. This can be understood as follows. For $t < 0$, the disturbance moves with a constant velocity V and emits waves with wavenumber k_1 and waves with wavenumber k_2 (mainly dominated by capillary forces). The k_1-waves lag behind the disturbance, while the k_2-waves move ahead of it. When the disturbance stops at $t = 0$, the k_2-waves keep moving forward and do not interact with the disturbance. However, the k_1-waves will encounter the disturbance and interact with it. Hence the period $2\pi/(Vk_1)$ of the wave resistance Eq.(16). The $1/\sqrt{t}$ decrease of the magnitude of wave resistance in Eq.(16) can be understood as follows. At time $t > 0$, the disturbance (which is at rest at $x = 0$) is hit by a k_1-wave that has previously been emitted distance $c_g(k_1)t$ away from it. The vertical amplitude ζ of this wave is inversely proportional to the square root of this distance. Indeed, as the liquid is inviscid, the energy has to be conserved and one thus has $\zeta \propto 1/\sqrt{c_g(k_1)t}$. Since the wave resistance R_w is proportional to vertical amplitude of the wave ζ, on recovers that R_w decreases with time as $1/\sqrt{c_g(k_1)t}$.

In the particular case where $V \gg c_{min}$, Eq.(16) reduces to:

$$R_w(t) = \frac{1}{2\sqrt{\pi}}\frac{\kappa^{3/2}\,|\hat{P}_{ext}(k_c)|^2\,c_{min}^3}{\rho\,V^{11/2}\,\sqrt{t}}\cos\left(\frac{\kappa c_{min}^2}{2V}t + \pi/4\right). \tag{17}$$

Let us now discuss the case $V < c_{min}$ (still considering the decelerated motion Eq.(12)). In that case, the wave resistance oscillates (in the long time limit) approxima-

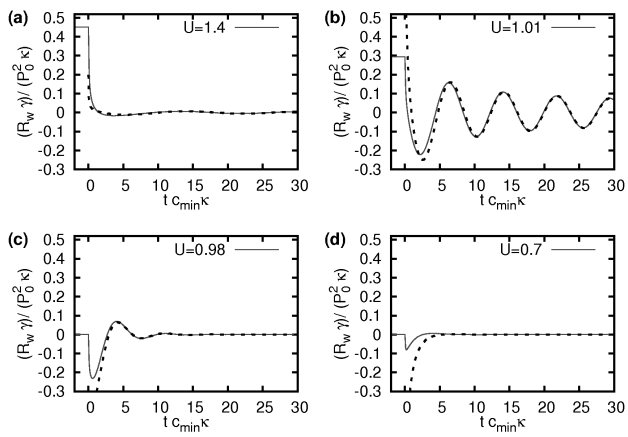

FIG. 9: The wave resistance R_w (in units of $P_0^2\kappa/\gamma$) is shown as a function of the reduced time $c_{min}\kappa t$ for a decelerated motion with different reduced velocities $U = V/c_{min}$ (see Eq.(12)). Respectively, panels (a), (b), (c) and (d) correspond to a reduced velocity $U = 1.5$, 1.01, 0.98 and 0.7. The dashed lines correspond to the asymptotic expansion given by Eq.(16).

tively as $\left(e^{-U\sqrt{1-U^4}t\kappa c_{min}}/\sqrt{t\kappa c_{min}}\right)\sin(U^3 t\kappa c_{min}+\chi)$. More precisely, one has (for large t)

$$\mathrm{R}_w(t) = \sqrt{\frac{2}{\pi}}\frac{\kappa^2|\hat{P}_{ext}(\tilde{k})|^2}{\rho c_{min}^2}\frac{\sin(U^3\, c_{min}\,\kappa\, t+\chi)}{\sqrt{\kappa\, c_{min}\, t}} \frac{e^{-U\sqrt{1-U^4}t\kappa c_{min}}}{\sqrt{2}\pi U^{1/2}(1-U^4)^{1/4}(3U^4+1)^{1/4}}, \quad (18)$$

where

$$\chi = (3/2)\arctan\left(\sqrt{1-U^4}/U^2\right) - (1/2)\arctan\left(\sqrt{1-U^4}/(2U^2)\right) \quad (19)$$

and $\tilde{k} = \kappa\left(U^2 + i\sqrt{1-U^4}\right)$. Note that in the case $V < c_{min}$, the wave resistance Eq.(18) also displays some oscillations, although no waves were emitted at $t < 0$. These oscillations - that have a exponential decay in time - might be due to the sudden arrest of the disturbance. Such oscillations could also be present in the case $V > c_{min}$ but are hidden by the interaction between the k_1-waves and the disturbance which lead to a much more slower $1/\sqrt{t}$ decay.

IV. CONCLUSIONS

In this article, we have shown that a disturbance undergoing a rectilinear accelerated or decelerated motion at a liquid-air interface emits waves even if its velocity V (the final one in the accelerated case and the initial one in the decelerated case, respectively) is smaller than c_{min}. This corroborates the results of Ref.[13]. For this purpose, we treat the wave emission problem by a linearized theory in a Monge representation. Then, we derive the analytical expression of the wave resistance and solve it by numerical integration. Asymptotic expansions permit to extract the predominant behavior of the wave resistance. Some vertical displacement patterns are also calculated in order to shown how the waves invade the free surface.

The results presented in this paper should be important for a better understanding of the propulsion of water-walking insects [24–27], like whirligig beetles, where accelerated and decelerated motions frequently occurs (e.g., when hunting a prey or escaping a predator [28]). Even in the case where the insect motion appears as rectilinear and uniform, one has to keep in mind that the rapid leg strokes are accelerated and might produce a wave drag even below c_{min}. The predictions concerning the wave patterns might be compared to experimental data using the recent technique of Moisy, Rabaud, and Salsac [23].

It will be interesting to take in our model some nonlinear effects [29] because the waves radiated by whirligig beetles [28] have a large amplitude. Recently Chepelianskii *et al.* derived a self-consistent integral equation describing the flow velocity around the moving disturbance [30]. It would be interesting to incorporate this approach into the present study.

Acknowledgments

We thank Jonathan Voice and Frédéric Chevy for useful discussions and Falko Ziebert for a critical reading of the manuscript.

[1] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, New York, 1987), 2nd ed
[2] D.J. Acheson, *Elementary Fluid Dynamics* (Clarendon Press, Oxford, 1990)
[3] J. Lighthill, *Waves in Fluids* (Cambridge University Press, Cambridge, 1979), 6th ed.; T.H. Havelock, Proc. R. Soc. A **95**, 354 (1918)
[4] J.H. Milgram, Annu. Rev. Fluid Mech. **30**, 613 (1998).
[5] E. Raphaël and P.-G. de Gennes, Phys. Rev. E **53**, 3448 (1996).
[6] D. Richard and E. Raphaël, Europhys. Lett. **48**, 53 (1999).
[7] S.-M. Sun and J. Keller, Phys. Fluids. **13**, 2146 (2001).
[8] F. Chevy and E. Raphaël, Europhys. Lett. **61**, 796

(2003).
[9] J. Browaeys, J.-C. Bacri, R. Perzynski, and M. Shliomis, Europhys. Lett. **53**, 209 (2001).
[10] T. Burghelea and V. Steinberg, Phys. Rev. Lett. **86**, 2557 (2001)
[11] T. Burghelea and V. Steinberg, Phys. Rev. E **66**, 051204 (2002).
[12] H. Lamb, *Hydrodynamics* (Cambridge University Press, Cambridge, 1993), 6th ed
[13] A.D. Chepelianskii, F. Chevy and E. Raphaël, Phys. Rev. L **100**, 074504 (2008).
[14] P.A. Cherenkov, Doklady Akad. Nauk. SSSR **2**, 451 (1934).
[15] J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1998), 3rd ed.
[16] The Fourier transform $\hat{f}(\mathbf{k},t)$ is related with the function $f(\mathbf{r},t)$ through $f(\mathbf{r},t) = \int \frac{\mathrm{d}^2k}{(2\pi)^2} e^{i\mathbf{k}.\mathbf{r}} \hat{f}(\mathbf{k},t)$
[17] Lord Rayleigh, Proc. London Math. Soc. **s1-15**, 69 (1883).
[18] Lord Kelvin, Proc. London Math. Soc. **15**, 80 (1887).
[19] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, USA, 1979), Corrected and enlarged edition
[20] H.J, Weber and G.B. Arkfen *Mathematical Methods for Physicists* (Academic Press, New York, 2000), 5th ed.
[21] K. F. Riley, M. P. Hobson and S. J. Bence *Mathematical Methods of Physics and Engineering* (Cambridge University Press, Cambridge), 3rd ed.
[22] In the particular case of a Lorentzian pressure disturbance with a Fourier transform $\hat{P}_{ext}(k) = P_0 e^{-bk}$, where b is the object size (set to $b = 0.1\kappa^{-1}$), we have numerically checked that up to $V \approx 8c_{min}$ the main contribution to the wave resistance is due to k_c.
[23] F. Moisy, M. Rabaud, and K. Salsac, Exp. Fluids **46**, 1021 (2009).
[24] O. Buhler, J. Fluid Mech. L **573**, 211 (2007).
[25] R. MacNeill Alexander, *Principle of Animal Locomotion* (Princeton University Press, Princeton, 2002).
[26] J.W. Bush and D. L. Hu, Annu. Rev. Fluid Mech. **38**, 339 (2006).
[27] M.W. Denny, J. Exp. Biol. **207**, 1601 (2004); F. Fish and A. Nicastro, J. Exp. Biol. **206**, 1649 (2003)
[28] J. Voise and J. Casas, J. R. Soc. Interface, **7**, 343-352 (2010)
[29] F. Dias and C. Kharif, Annu. Rev. Fluid Mech, **31**, 301-346 (1999)
[30] A. D. Chepelianskii, M. Schindler, F. Chevy and E. Raphal Phys. Rev. E **81**, 016306 (2010).