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A framework for adaptive Monte-Carlo procedures

Bernard Lapeyre* Jérôme Lelong†

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Abstract

Adaptive Monte Carlo methods are powerful variance reduction techniques. In this work, we propose a mathematical setting which greatly relaxes the assumptions needed by for the adaptive importance sampling techniques presented in [2, 1]. We establish the convergence and asymptotic normality of the adaptive Monte Carlo estimator under local assumptions which are easily verifiable in practice. We present one way of approximating the optimal importance sampling parameter using a randomly truncated stochastic algorithm. Finally, we apply this technique to the valuation of financial derivatives and our numerical experiments show that the computational time needed to achieve a given accuracy is divided by a factor up to 5.

1 A common parametric Monte-Carlo framework

Monte-Carlo methods aim at computing the expectation $\mathbb{E}(Z)$ of a real-valued random variable Z using samples along the law of Z . In this work, we focus on cases where there exists a parametric representation of the expectation

$$\mathbb{E}(Z) = \mathbb{E}(H(\theta, X)) \quad \text{for all } \theta \in \mathbb{R}^d, \quad (1)$$

where X is a random variable with values in \mathbb{R}^m and $H : \mathbb{R}^d \times \mathbb{R}^m \mapsto \mathbb{R}$ is a measurable function satisfying $\mathbb{E}|H(\theta, X)| < \infty$ for all $\theta \in \mathbb{R}^d$. We also impose that

$$\theta \mapsto v(\theta) = \text{Var}(H(\theta, X)) \text{ is finite for all } \theta \in \mathbb{R}^d, \quad (2)$$

We want to make the most of this free parameter θ to settle an automatic variance reduction method. It consists in first finding a minimiser θ^* of the variance $v(\theta)$ and then plugging it into the parametric estimator

$$\frac{1}{n} \sum_{i=1}^n H(\theta^*, X_i), \quad (3)$$

where $(X_n)_n$ is a sequence of independent and identically distributed random variables along the law of X . This technique heavily relies on the ability to find a parametric representation and to effectively minimise (probably using simulations) the variance $v(\theta)$. Many papers have

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been written on the ways of constructing the parametric representation $H(\theta, X)$ for several kinds of random variables Z . The first variance reduction technique that comes to mind is probably the use of control variates which can indeed be written as $H(\theta, X)$. Assume we want to compute the expectation of a random variable Z and we have at hand a d -dimensional random variable $Y = (Y_1, \dots, Y_d)$ of zero expectation and such that $\mathbb{E}(YY^*)$ is a positive definite matrix. Obviously, in this case $\mathbb{E}(Z) = \mathbb{E}(Z - \theta \cdot Y)$ for every $\theta \in \mathbb{R}^d$. So if $X = (Y, Z)$ and $H(\theta, (y, z)) = z - \theta \cdot y$, we obtain a parametric representation of $\mathbb{E}(Z)$. The second moment of the estimator is very easy to compute $v(\theta) = \mathbb{E}((Z - \theta \cdot Y)^2) = \mathbb{E}(Z^2) - 2\theta \cdot \mathbb{E}(YZ) + \theta^* \mathbb{E}(YY^*) \theta$. v is actually a quadratic form and the strong convexity comes from the definite positivity of $\mathbb{E}(YY^*)$. The minimiser θ^* of v is given by $\theta^* = \mathbb{E}(YY^*)^{-1} \mathbb{E}(ZY)$. In the context of control variates, several works have been carried out on how to devise an adaptive Monte Carlo method. A very natural idea is to estimate θ^* by $\theta_n = (\sum_{i=1}^n Y_i Y_i^*)^{-1} \sum_{i=1}^n Z_i Y_i$, where the r.v. (Y_n, Z_n) are i.i.d. samples along (Y, Z) . Kim and Henderson [13] and Glasserman [5] noticed that $\frac{1}{n} \sum_{i=1}^n Z_i - \theta_n \cdot Y_i$ is a convergent and asymptotically normal estimator of $\mathbb{E}(Z)$, even when θ_n is computed using the samples $(Y_i, Z_i)_{1 \leq i \leq n}$. Other representations $H(\theta, X)$ based on a change of measure have been proposed in [10, 11], [18] or [5]. Some of these examples are developed in Section 2.

Obviously, when no closed form expression of the expectation $\mathbb{E}(H(\theta, X))$ is available, there are very few chances that $v(\theta)$ can be explicitly computed. Henceforth, it is needed to approach θ^* without being able to compute the variance itself. In this work, we recall the methodology developed by Arouna [1] to estimate θ^* using some stochastic gradient style algorithms. As Arouna, we aim at applying this methodology to the evaluation of financial derivatives and the main difficulty in approximating θ^* comes from the non-boundedness of the payoff functions usually considered and consequently the non-boundedness of the H functions. The fast growing behaviour of the H functions imposes to use randomly truncated stochastic algorithms. Once an approximation θ_n of θ^* is available, it can be plugged into Equation (3) to construct a convergent estimator of $\mathbb{E}(Z)$.

Another approach to the variance reduction problem, developed in the work of Arouna [2], is to use an *adaptive Monte Carlo method*, which means that the approximation of θ^* is computed on-line at the same time as the Monte Carlo estimator. Let $(X_n, n \geq 1)$ be the sequence of samples used in the Monte Carlo computation. We construct a convergent estimator $(\theta_n = f(X_k, k \leq n))_n$ of θ^* and use it to approximate the expectation $\mathbb{E}(Z)$ using some kind of adaptive empirical mean

$$S_n = \frac{1}{n} \sum_{i=1}^n H(\theta_{i-1}, X_i). \quad (4)$$

Because, θ_n is $\sigma(X_1, \dots, X_n)$ measurable, S_n is not a sum of independent terms but it has a martingale structure which can be exploited to prove convergence results. The aim of this work is to give a unified framework with easily verifiable assumptions under which both the almost sure convergence to $\mathbb{E}(Z)$ and a central limit theorem for the estimator defined by (4) can be established. As a corollary, we clarify in Section 6.2 the hypotheses needed to ensure the convergence in Arouna's framework, in particular we have got rid of the condition on the increasing rate of the compact sets appearing in the randomly truncated stochastic algorithm (see Remark 6.2).

First, we give in Section 2 some general ways of constructing a parametric estimator using importance sampling or other more elaborate transformations. Next in Section 3, we focus

on the mathematical foundation of the method and give both a strong law of large numbers (SLLN) and a central limit theorem (CLT) for the adaptive estimator (4) under very light assumptions; namely we only require that the random variable $H(\theta, X)$ has a finite second moment for all θ . Finally, we illustrate our theoretical results on examples coming from financial problems.

2 Examples of the Parametric Monte-Carlo Setting

In this section, we give various examples of cases in which a parametric representation of the expectation of interest is available

$$\mathbb{E}(Z) = \mathbb{E}(H(\theta, X)).$$

In each example, we highlight the strong convexity and the regularity of $\mathbb{E}(H^2(\theta, X))$ such that the minimiser θ^* is uniquely defined as the one root of $\theta \mapsto \nabla_{\theta} \mathbb{E}(H^2(\theta, X))$.

Importance sampling for normal random variables Let $G = (G_1, \dots, G_d)$ be a d -dimensional standard normal random vector. For any measurable function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\mathbb{E}(|h(G)|) < \infty$, one has for all $\theta \in \mathbb{R}^d$

$$\mathbb{E}(h(G)) = \mathbb{E}\left(e^{-\theta \cdot G - \frac{|\theta|^2}{2}} h(G + \theta)\right). \quad (5)$$

Assume we want to compute $\mathbb{E}(f(G))$ for a measurable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f(G)$ is integrable. By applying equality (5) to $h = f$ and $h(x) = f^2(x) e^{-\theta \cdot x + \frac{|\theta|^2}{2}}$, one obtains that the expectation and the variance of the random variable $f(G + \theta) e^{-\theta \cdot G - \frac{|\theta|^2}{2}}$ are respectively equal to $\mathbb{E}(f(G))$ and $v(\theta) - \mathbb{E}^2(f(G))$ where

$$v(\theta) = \mathbb{E}\left(f^2(G) e^{-\theta \cdot G + \frac{|\theta|^2}{2}}\right).$$

Proposition 2.1. *Assume that*

$$\mathbb{P}(f(G) \neq 0) > 0, \quad (6)$$

$$\exists \varepsilon > 0, \mathbb{E}(|f(G)|^{2+\varepsilon}) < \infty \quad (7)$$

Then, v is infinitely continuously differentiable and strongly convex.

Proof. By Hölder's inequality and Equation (7),

$$\forall \theta \in \mathbb{R}^d, \mathbb{E}\left(f^2(G) e^{-\theta \cdot G}\right) < \infty. \quad (8)$$

The function $\theta \mapsto f^2(G) e^{-\theta \cdot G + \frac{|\theta|^2}{2}}$ is infinitely continuously differentiable. Since,

$$\sup_{|\theta| \leq M} |\partial_{\theta_j} f^2(G) e^{-\theta \cdot G + \frac{|\theta|^2}{2}}| \leq e^{\frac{M^2}{2}} f^2(G) \left(M + (e^{G^j} + e^{-G^j})\right) \prod_{k=1}^d (e^{MG^k} + e^{-MG^k})$$

where the right-hand-side is integrable by (8), Lebesgue's theorem ensures that v^f is continuously differentiable with $\frac{\partial}{\partial \theta^j} v^f(\theta) = \mathbb{E} \left(f^2(G) (\theta^j - G^j) e^{-\theta \cdot G + \frac{|\theta|^2}{2}} \right)$. Higher order differentiability properties are obtained by similar arguments and in particular $\frac{\partial^2}{\partial \theta^j \partial \theta^i} v^f(\theta) = \mathbb{E} \left((\mathbf{1}_{\{i=j\}} + (\theta^j - G^j)(\theta^i - G^i)) f^2(G) e^{-\theta \cdot G + \frac{|\theta|^2}{2}} \right)$.

$$\mathbb{E}(f^2(G) e^{-\theta \cdot G + \frac{|\theta|^2}{2}}) = \mathbb{E}(f^2(G) e^{-\theta \cdot G}) \mathbb{E}(e^{\theta \cdot G}) \geq (\mathbb{E}(f^2(G)))^2.$$

Assumption (6) ensures that $\mathbb{E}(f^2(G)) > 0$. Then, the Hessian matrix is uniformly bounded from below by positive definite matrix $(\mathbb{E}(f^2(G)))^2 I_d$. This yields the strong convexity of the function v^f . \square

Proposition 2.1 implies that v has a unique minimiser θ^* characterised by $\nabla v(\theta^*) = 0$, i.e. $\mathbb{E} \left((\theta^* - G) e^{-\theta^* \cdot G + \frac{|\theta^*|^2}{2}} f^2(G) \right) = 0$. The idea of using Equality (5) is owed to Arouna [1]. He proposed to estimate θ^* using randomly truncated stochastic algorithms. This will also be our point of view in the following.

2.1 Importance sampling for processes

Equality (5) can actually be extended to the Brownian motion framework using Girsanov's theorem. Let $(W_t, 0 \leq t \leq T)$ be a d -dimensional Brownian motion and \mathcal{F} its natural filtration. For any measurable and \mathcal{F} -predictable process $(\theta_t, 0 \leq t \leq T)$ such that $\mathbb{E} \left(e^{\frac{1}{2} \int_0^T |\theta_t|^2 dt} \right) < \infty$, one has

$$\mathbb{E}(f(W_t, 0 \leq t \leq T)) = \mathbb{E} \left(e^{-\int_0^T \theta_t \cdot dW_t - \frac{1}{2} \int_0^T |\theta_t|^2 dt} f \left(W_t + \int_0^t \theta_s ds, 0 \leq t \leq T \right) \right).$$

Assume $\theta_t = \theta \in \mathbb{R}^d$, for all $t \in [0, T]$. The variance of $e^{-\theta \cdot W_T - \frac{\theta^2 T}{2}} f(W_t, 0 \leq t \leq T)$ writes down $v(\theta) - \mathbb{E} \left(f^2(W_t, 0 \leq t \leq T) \right)$ with

$$v(\theta) = \mathbb{E} \left(e^{-\theta \cdot W_T + \frac{|\theta|^2}{2} T} f^2(W_t + \theta t, 0 \leq t \leq T) \right).$$

A similar result to Proposition 2.1 holds; in particular v is infinitely continuously differentiable, strictly convex and goes to infinity at infinity.

For more general processes $(\theta_t, 0 \leq t \leq T)$, we refer the reader to the work of [18].

2.2 The exponential change of measure

The idea of tilting some probability measure to find the ones that minimises the variance is a very common idea which can be also be applied to a wide range of distribution, see for instance the recent results of Kawai [12, 11] in which he applied an exponential change of measure to Lévy processes, also known as the Esscher transform.

Consider a random variable X with values in \mathbb{R}^d and cumulative generating function $\psi(\theta) = \log \mathbb{E}(e^{\theta \cdot X})$. We assume that $\psi(\theta) < \infty$ for all $\theta \in \mathbb{R}^d$. Let p denote the density of X . We define the density p_θ by

$$p_\theta(x) = p(x) e^{\theta \cdot x - \psi(\theta)}, \quad x \in \mathbb{R}^d.$$

Let $X^{(\theta)}$ have p_θ as a density, then

$$\mathbb{E}(f(X)) = \mathbb{E} \left[f(X^{(\theta)}) \frac{p(X^{(\theta)})}{p_\theta(X^{(\theta)})} \right].$$

The variance of $f(X^{(\theta)}) \frac{p(X^{(\theta)})}{p_\theta(X^{(\theta)})}$ writes $v(\theta) = \mathbb{E} \left(f(X^{(\theta)})^2 \frac{p(X^{(\theta)})^2}{p_\theta(X^{(\theta)})^2} \right)$ with

$$v(\theta) = \mathbb{E} \left(\left| f(X^{(\theta)}) \frac{p(X^{(\theta)})}{p_\theta(X^{(\theta)})} \right|^2 \right) = \mathbb{E} \left(f(X)^2 e^{-\theta \cdot X + \psi(\theta)} \right).$$

Obviously, this change of measure is only valuable as a variance reduction technique if $X^{(\theta)}$ can be simulated at approximately the same cost as X .

Proposition 2.2. *Assume that*

$$\exists \varepsilon > 0, \mathbb{E}(|f(G)|^{2+\varepsilon}) < \infty \quad (9)$$

$$\lim_{|\theta| \rightarrow \infty} p_\theta(x) = 0 \text{ for all } x \text{ in } \mathbb{R}^d \quad (10)$$

Then, v is infinitely continuously differentiable, convex and $\lim_{|\theta| \rightarrow \infty} v(\theta) = \infty$.

Proof. To prove the differentiability of v , it suffices to reproduce the first part of the proof of Proposition 2.1. The convexity of v comes from the log-convexity of ψ . Moreover,

$$v(\theta) = \mathbb{E} \left(f(X)^2 e^{-\theta \cdot X + \psi(\theta)} \right) = \mathbb{E} \left(f(X)^2 \frac{p(X)}{p_\theta(X)} \right)$$

Combining Equation (10) with Fatou's Lemma yields that $\lim_{|\theta| \rightarrow \infty} v(\theta) = \infty$. \square

Remark 2.3. *If X is a random standard normal vector, $p_\theta(x) = p(x-\theta)$ and $X^{(\theta)}$ is a random normal vector with mean θ and identity covariance matrix. Hence, we recover Equality (5).*

3 Mathematical foundations of the method

In this section, $(X_n)_{n \geq 1}$ is an i.i.d. sequence following the law of X and we introduce the σ -algebra \mathcal{F}_n it generates $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$.

3.1 An adaptive strong law of large numbers

Theorem 3.1 (Adaptive strong law of large numbers). *Assume Equation (1) and (2) hold. Let $(\theta_n)_{n \geq 0}$ be a (\mathcal{F}_n) -adapted sequence with values in \mathbb{R}^d such that for all $n \geq 0$, $\theta_n < \infty$ a.s and for any compact subset $K \subset \mathbb{R}^d$, $\sup_{\theta \in K} \mathbb{E}(|H(\theta, X)|^2) < \infty$. If*

$$\inf_{\theta \in \mathbb{R}^n} v(\theta) > 0 \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^n v(\theta_k) < \infty \quad \text{a.s.}, \quad (11)$$

then S_n converges a.s. to $\mathbb{E}(Z)$.

Proof. For any $p \geq 0$, we define $\tau_p = \inf\{k \geq 0; |\theta_k| \geq p\}$. The sequence $(\tau_p)_p$ is an increasing sequence of (\mathcal{F}_n) -stopping times such that $\lim_{p \rightarrow \infty} \tau_p \uparrow \infty$ a.s.. Let $M_n = \sum_{i=0}^{n-1} H(\theta_i, X_{i+1}) - \mathbb{E}(Z)$. We introduce $M_n^{\tau_p} = M_{\tau_p \wedge n}$ defined by

$$M_n^{\tau_p} = \sum_{i=0}^{n-1 \wedge \tau_p} H(\theta_i, X_{i+1}) - \mathbb{E}(Z) = \sum_{i=0}^{n-1} (H(\theta_i, X_{i+1}) - \mathbb{E}(Z)) \mathbf{1}_{\{i \leq \tau_p\}}.$$

$\mathbb{E}(|H(\theta_i, X_{i+1}) - \mathbb{E}(Z)|^2 \mathbf{1}_{\{i \leq \tau_p\}}) \leq \mathbb{E}(\mathbf{1}_{\{i \leq \tau_p\}} \mathbb{E}(|H(\theta, X) - \mathbb{E}(Z)|^2)_{\theta=\theta_i})$. On the set $\{i \leq \tau_p\}$, the conditional expectation is bounded from above by $\sup_{|\theta| \leq p} v(\theta)$. Hence, the sequence $(M_n^{\tau_p})_n$ is square integrable and it is obvious that $(M_n^{\tau_p})_n$ is a martingale, which means that the sequence $(M_n)_n$ is a locally square integrable martingale (i.e. a local martingale which is locally square integrable).

$$\langle M \rangle_n = \sum_{i=0}^{n-1} \mathbb{E}((H(\theta_i, X_{i+1}) - \mathbb{E}(Z))^2 | \mathcal{F}_i) = \sum_{i=0}^{n-1} v(\theta_i).$$

By Condition (11), we have a.s. $\limsup_n \frac{1}{n} \langle M \rangle_n < \infty$ and $\liminf_n \frac{1}{n} \langle M \rangle_n > 0$. Applying the strong law of large numbers for locally \mathbb{L}^2 martingales (see [19]) yields the result. \square

The sequence $(\theta_n)_n$ can be any sequence adapted to $(X_n)_{n \geq 1}$ convergent or not. For instance, $(\theta_n)_n$ can be an ergodic Markov chain distributed around the minimizer θ^* such as Monte Carlo Markov Chain algorithms. When the sequence $(\theta_n)_{n \geq 0}$ converges a.s. to a deterministic constant θ_∞ , it is sufficient to assume that v is continuous at θ_∞ and $v(\theta_\infty) > 0$ to ensure that Condition (11) is satisfied. Note that there is no need to impose that $\theta_\infty = \theta^*$ although it is undoubtedly wished in practice. For instance, θ_∞ can be an approximation of θ^* obtained either by heuristic arguments such as large deviations or by stochastic approximation as explained in Section 4.

3.2 A Central limit theorem for the adaptive strong law of large numbers

To derive a central limit theorem for the adaptive estimator S_n , we need a central limit theorem for locally square integrable martingales, whose convergence rate has been extensively studied. We refer to the works of Rebolledo [22], Jacod and Shiryaev [8], Hall and Heyde [7] and Whitt [24] to find different statements of central limit theorems for locally square integrable càdlàg martingales in continuous time, from which theorems can easily be deduced for discrete time locally square integrable martingales.

Theorem 3.2. *Assume Equation (1) and (2) hold. Let $(\theta_n)_{n \geq 0}$ be a \mathcal{F}_n -adapted sequence with values in \mathbb{R}^d such that for all $n \geq 0$, $\theta_n < \infty$ a.s and converging to some deterministic value θ_∞ . Assume there exists $\eta > 0$ such that the function $s_{2+\eta} : \theta \in \mathbb{R}^d \mapsto \mathbb{E}(|H(\theta, X)|^{2+\eta})$ is finite for all $\theta \in \mathbb{R}^d$ and continuous at θ_∞ . Then, $\sqrt{n}(S_n - \mathbb{E}(Z)) \xrightarrow{\text{law}} \mathcal{N}(0, v(\theta_\infty))$.*

Proof. We know from the proof of Theorem 3.1 that $M_n = \sum_{i=0}^{n-1} H(\theta_i, X_{i+1}) - \mathbb{E}(Z)$ is a locally square integrable martingale and that $\frac{1}{n} \langle M \rangle_n$ converges a.s. to $v(\theta_\infty)$.

$$\frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}(|H(\theta_i, X_{i+1}) - \mathbb{E}(Z)|^{2+\eta} | \mathcal{F}_i) \leq c \left(\frac{1}{n} \sum_{i=0}^{n-1} s_{2+\eta}(\theta_i) + \mathbb{E}(Z)^{2+\eta} \right).$$

The term on the r.h.s is bounded thanks to the continuity of $s_{2+\eta}$ at θ_∞ . Hence, the local martingale $(M_n)_n$ satisfies Lindeberg's condition. The result ensues from the central limit theorem for locally \mathbb{L}^2 martingales. \square

Theorem 3.3. *Assume Equation (1) and (2) hold. Let $(\theta_n)_{n \geq 0}$ be a \mathcal{F}_n -adapted sequence with values in \mathbb{R}^d such that for all $n \geq 0$, $\theta_n < \infty$ a.s and converging to some deterministic value θ_∞ . Assume there exists $\eta > 0$ such that the function $s_{4+\eta} : \theta \in \mathbb{R}^d \mapsto \mathbb{E}(|H(\theta, X)|^{4+\eta})$ is finite for all $\theta \in \mathbb{R}^d$ and continuous at θ_∞ . Then, $\sigma_n^2 = \frac{1}{n} \sum_{i=0}^{n-1} H(\theta_i, X_{i+1})^2 - S_n^2 \xrightarrow{a.s.} v(\theta_\infty)$. If moreover $v(\theta_\infty) > 0$, then $\frac{\sqrt{n}}{\sigma_n}(S_n - \mathbb{E}(Z)) \xrightarrow[n \rightarrow +\infty]{law} \mathcal{N}(0, 1)$.*

Remark 3.4. *Even if $v(\theta_\infty) > 0$, σ_n may take negative values for n small. This corollary is really essential from a practical point of view because it proves that confidence intervals can be built as in the case of a crude Monte Carlo procedure. The only difference lies in the way of approximating the asymptotic variance.*

The assumptions of Theorem 3.2 are fairly easy to check in practice since they are formulated independently of the sequence $(\theta_n)_n$. When $\theta_\infty = \theta^*$, which is nonetheless not required, the limiting variance is optimal in the sense that a crude Monte Carlo computation with the optimal parameter θ^* would have lead to the same limiting variance. These assumptions are satisfied in the frameworks introduced in Section 2.

4 Estimation of the optimal variance parameter

From Theorem 3.1 and Theorem 3.2, we know that if we can construct a convergent estimator $(\theta_n)_n$ of θ^* , the adaptive estimator M_n is a convergent and asymptotically normal estimator of the expectation $\mathbb{E}(Z)$. The challenging issue is now to propose an automatic way of approximating the minimiser θ^* of $v(\theta) = \mathbb{E}(H(\theta, X)^2) - \mathbb{E}(Z)^2$. In the following, we will assume that v is strictly convex, goes to infinity at infinity and is continuously differentiable. Moreover, we assume that ∇v admits a representation as an expectation

$$\nabla v(\theta) = \mathbb{E}(U(\theta, X)),$$

where $U : \mathbb{R}^d \times \mathbb{R}^m \mapsto \mathbb{R}$ is a measurable and integrable function. We could see in the examples developed in Section 2 that these conditions are very easily satisfied. Stochastic algorithms such as the Robbins Monro (see [23]) algorithm are perfectly well suited to estimate quantities defined as the root of an expectation. First, we will present the pioneer Robbins Monro algorithm. Meanwhile, we will see that for our purpose a more robust algorithm is needed. This will naturally lead us to consider randomly truncated stochastic algorithms as introduced by Chen et al. [3]. When dealing with stochastic approximations, the idea of averaging the iterates comes out quite naturally. We will present this improvement in Section 4.3. Once we have an efficient algorithm to construct a convergent estimator of θ^* , we will turn to the Monte Carlo computation itself and explain two different strategies to implement the computation of the expectation itself : the adaptive strategy and the sequential one.

4.1 The Robbins Monro algorithm

The Robbins Monro algorithm is probably the most famous stochastic approximation method to find the root of an expectation which has no closed form expression. It can be seen a randomised Newton algorithm. For $\tilde{\theta}_0 \in \mathbb{R}^d$, the algorithm writes down

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n - \gamma_{n+1}U(\tilde{\theta}_n, X_{n+1}), \quad (12)$$

where $(X_n)_{n \geq 1}$ is an i.i.d sequence of random variables following the law of X and $(\gamma_n)_{n \geq 1}$ is a decreasing sequence of a positive real numbers satisfying

$$\sum_n \gamma_n = \infty \quad \text{and} \quad \sum_n \gamma_n^2 < \infty. \quad (13)$$

The sequence $(\gamma_n)_n$ is often called the gain sequence or the step sequence. We define the σ -field $\mathcal{F}_n = \sigma(X_k, k \leq n)$. Note that because U is measurable, \mathcal{F}_n is also the σ -field generated by the random vectors $(\tilde{\theta}_0, \tilde{\theta}_1, \dots, \tilde{\theta}_n)$. The sequence $(\tilde{\theta}_n)_{n \geq 0}$ is known to converge to θ^* under assumptions on the asymptotic behaviour of $\mathbb{E}(U^2(\theta, X))$. This convergence was first established by Robbins and Monro (see [23]). A proof of the following theorem can also be found in [4] or [15].

Theorem 4.1 (Robbins Monro). *Assume that*

(A1) ∇v is continuous and there exists a unique θ^* s.t. $\nabla v(\theta^*) = 0$ and $\forall \theta \neq \theta^*$, $(\nabla v(\theta) | \theta - \theta^*) > 0$.

(A2) $\exists K > 0$ s.t. $\forall n \geq 0$, $\mathbb{E}(|U(\tilde{\theta}_n, X_{n+1})|^2 | \mathcal{F}_n) \leq K(1 + |\tilde{\theta}_n|^2)$ p.s.

Then, the sequence $(\tilde{\theta}_n)_{n \geq 0}$ converges a.s. to θ^* .

Remark 4.2 (Comments on the hypotheses). *Hypothesis (A1) is a Lyapounov type assumption to ensure the existence and uniqueness of the root of the function ∇v . When the function whose root we are looking for can be written as the gradient of a strictly convex function as it is the case here, Hypothesis (A1) is automatically satisfied. The most restrictive hypothesis in this theorem is undoubtedly Hypothesis (A2) which somehow imposes that the function U has a sub-linear behaviour on “average”. This condition dramatically narrows the range of applications. In particular, it is not satisfied in our problem. To deal with this restriction, an other algorithm has been proposed by Chen and Zhu (see [3]). The next section is devoted to the presentation of the main properties of this improved algorithm.*

4.2 Randomly truncated stochastic algorithms

In this section, we concentrate on the randomly truncated stochastic algorithm introduced in [3]. First, we present this new algorithm which essentially consists in a truncation of the Robbins Monro algorithm on an increasing sequence of compact sets. Then, we give some results concerning its asymptotic behaviour. We introduce an increasing sequence of compact sets $(K_j)_j$ of \mathbb{R}^d

$$\bigcup_{j=0}^{\infty} K_j = \mathbb{R}^d \quad \text{and} \quad K_j \subsetneq K_{j+1} \quad (14)$$

We still consider the sequences $(X_n)_{n \geq 1}$ and $(\gamma_n)_{n \geq 1}$ as introduced in Section 4.1. For $\theta_0 \in K_0$ and $\lambda_0 = 0$, we define the sequences of random variables $(\theta_n)_n$ and $(\lambda_n)_n$ by

$$\begin{cases} \theta_{n+\frac{1}{2}} = \theta_n - \gamma_{n+1}U(\theta_n, X_{n+1}), \\ \text{if } \theta_{n+\frac{1}{2}} \in \mathcal{K}_{\lambda_n} & \theta_{n+1} = \theta_{n+\frac{1}{2}} \quad \text{and} \quad \lambda_{n+1} = \lambda_n, \\ \text{if } \theta_{n+\frac{1}{2}} \notin \mathcal{K}_{\lambda_n} & \theta_{n+1} = \theta_0 \quad \text{and} \quad \lambda_{n+1} = \lambda_n + 1. \end{cases} \quad (15)$$

where $\theta_{n+\frac{1}{2}}$ is the new sample we draw, either we accept it and set $\theta_{n+1} = \theta_{n+\frac{1}{2}}$ or we reject it and reset the algorithm to θ_0 when it tries to jump too far ahead in a single step. Note that $\theta_{n+\frac{1}{2}}$ is actually drawn along the dynamics of the Robbins Monro algorithm defined by Equation (12). The use of truncations enables to relax the hypotheses required to ensure the convergence. From the recent results of Lelong [17], we can state the following convergence result

Theorem 4.3. *Assume Condition (13), Assumption (A1) and that the function $\theta \mapsto \mathbb{E}(|U(\theta, Z)|^2)$ is locally bounded. Then, the sequence $(\theta_n)_n$ defined by (15) converges a.s. to θ^* for any sequence of compact sets satisfying (14) and moreover the sequence $(\lambda_n)_n$ is a.s. finite.*

Remark 4.4. *When $\theta_{n+\frac{1}{2}} \notin \mathcal{K}_{\lambda_n}$, one can set θ_{n+1} to any measurable function of $(\theta_0, \dots, \theta_n)$ with values in a fixed compact set. This existence of such a compact set is definitely essential to prove the stability of the sequence $(\theta_n)_n$.*

Note that the assumptions required to ensure the convergence are very weak and are formulated independently of the algorithm trajectories, which makes them easy to check. Since the variance reduction technique we settle here aims at being automatic in the sense that it does not require any fiddling of the gain sequence depending on the function U , it is quite natural to average the procedure defined by Equation (15).

4.3 Averaging a stochastic algorithm

This section is based on the remark that Cesaro type averages tend to smooth the behaviour of convergent estimators at least from a theoretical point of view. Such averaging techniques have already been studied and proved to provide asymptotically efficient estimators (see for instance [21], [14] or [20]).

At the same time, it is well known that true Cesaro averages are not so efficient from a practical point of view because the rate at which the impact of the first iterates vanishes in the average is too slow and it induces some kind of a numerical bias which in turn dramatically slows down the convergence. Combining these two facts has led us to consider a moving window average of Algorithm (15).

In this section, we restrict to gain sequences of the form $\gamma_n = \frac{\gamma}{(n+1)^\alpha}$ with $\frac{1}{2} < \alpha < 1$. Let $t > 0$ be the length of the window used for averaging. We introduce

$$\hat{\theta}_n(t) = \frac{\gamma_n}{t} \sum_{i=n}^{n+\lfloor t/\gamma_n \rfloor} \theta_i. \quad (16)$$

The almost sure convergence of $(\hat{\theta}_n(t))_n$ can easily be deduced from Theorem 4.3. The asymptotic normality of the sequence $(\hat{\theta}_n(t))_n$ has been studied by Lelong in [16].

5 How to couple the Monte Carlo and the stochastic approximation procedures

There are two strategies to implement the variance reduction procedure presented above. Either one uses a first set of samples to compute an approximation of θ^* using Equations (15) or (16) and a new set of samples to compute the desired expectation afterwards using a Monte Carlo method with the approximation of θ^* found before; or one uses an adaptive strategy which means that the same samples are used to compute the approximation of θ^* and the Monte Carlo summation.

5.1 The non adaptive algorithm

Let $\theta_0 \in \mathbb{R}^d$ and $\xi_0 = 0$.

Algorithm 5.1. *Let n be the number of samples used for the Monte Carlo computation.*

1. *Draw a first set of samples following the law of X to compute an estimate θ_n of θ^* using Equations (15) or (16).*
2. *Draw a second set (X_1, \dots, X_n) of n samples independent of the ones used to compute θ_n .*

$$\xi_n = \frac{1}{n} \sum_{i=1}^n H(t_n, X_i).$$

Once the convergence of the sequence $(\theta_n)_n$ is established, the convergence of $(\xi_n)_n$ to $\mathbb{E}(Z)$ ensues from the strong law of large numbers. Obviously, the sequence $(\xi_n)_n$ satisfies a central limit theorem

$$\sqrt{n}(\xi_n - \mathbb{E}(Z)) \xrightarrow[n \rightarrow \infty]{law} \mathcal{N}(0, \text{Var}(H(\theta^*, X))).$$

5.2 The adaptive algorithm

Let $\theta_0 \in \mathbb{R}^d$ and $S_0 = 0$.

Algorithm 5.2 (Adaptive Importance Sampling (ADIS)). *Let n be the number of samples used for the Monte Carlo computation.*

For each i in $0, \dots, n-1$, do

1. *Draw X_{i+1} according to the law of X and independently of $\{X_j; j \leq i\}$,*
2. *Compute S_{i+1} defined by*

$$S_{i+1} = \frac{i}{i+1} S_i + \frac{1}{i+1} H(\hat{\theta}_i, X_{i+1}),$$

3. *Compute $\hat{\theta}_{i+1}$ using Equation (16).*

The sequence $(\hat{\theta}_i)_i$ converges almost surely to θ^* by applying Theorem 6.1. Once the convergence of $(\hat{\theta}_i)_i$ is established, the convergence of the sequence $(S_i)_i$ to $\mathbb{E}(Z)$ follows from Theorem 3.1. Moreover, it ensues from Theorem 3.2 that the sequence $(S_i)_i$ also satisfies an optimal central limit theorem

$$\sqrt{n}(S_n - \mathbb{E}(Z)) \xrightarrow[n \rightarrow \infty]{law} \mathcal{N}(0, \text{Var}(H(\theta^*, X))).$$

For $(S_i)_i$ to converge, the sequence $(\hat{\theta}_i)_i$ has to be adapted to the filtration $\sigma(X_1, \dots, X_i)_i$. Therefore, one has to rewrite Equation (16) in a way that makes $\hat{\theta}_i$ adapted. Note that the convergence rates of the sequences $(S_i)_i$ observed in Algorithms 5.2 and 5.1 are the same. Of course, Algorithm 5.2 can also be implemented using Equation (15) instead of (16).

6 Application to the Gaussian random vector framework

In this section, we follow the ideas of Arouna [2] and extend them to possibly reduce the dimension of the parameter θ such that the stochastic algorithm used does not require too much computation time. The start $*$ notation stands for the transpose operator.

6.1 Presentation of the problem

We consider a d' -multidimensional local volatility model in which each asset is supposed to be driven by the following dynamics under the risk neutral measure.

$$dS_t^i = S_t^i(r dt + \sigma(t, S_t^i) \cdot dW_t^i), \quad S_0^i = s^i.$$

$W = (W^1, \dots, W^{d'})^*$ is a vector of correlated standard Brownian motions. The covariance structure of the Brownian motions is given by $\langle W, W \rangle_t = \Gamma t$ where Γ is a definite positive matrix with a diagonal filled with ones. In our numerical examples, we take $\Gamma_{ij} = \mathbf{1}_{\{i=j\}} + \rho \mathbf{1}_{\{i \neq j\}}$ with $\rho \in (\frac{-1}{d'-1}, 1)$ to ensure that the matrix Γ is positive definite. The function σ is the local volatility function, r is the instantaneous interest rate and the vector $(s^1, \dots, s^{d'})$ is the vector of the spot values. In this model, we want to price path-dependent options whose payoffs can be written as a function of $(S_t, t \leq T)$. Hence, the price is given by the expectation $e^{-rT} \mathbb{E}(\psi(S_t, t \leq T))$. Most of the time, this expectation must be computed by Monte Carlo methods and one has to consider an approximation of $\psi(S_t, t \leq T)$ on a time grid $0 = t_0 < t_1 < \dots < t_N = T$. Then, the quantity of interest becomes

$$e^{-rT} \mathbb{E}[\hat{\psi}(S_{t_0}, S_{t_1}, \dots, S_{t_N})].$$

The discretisation of the asset S can for instance be obtained using an Euler scheme, which means that the function $\hat{\psi}$ can be expressed in terms of the Brownian increments or equivalently using a random normal vector. These remarks finally turn the original pricing problem into the computation of an expectation of the form $\mathbb{E}[\phi(G)]$ where G is a standard normal random vector in $\mathbb{R}^{N \times d'}$ and $\phi : \mathbb{R}^{N \times d'} \mapsto \mathbb{R}$ is a measurable and integrable function. Using Equation (5), we have for all $\theta \in \mathbb{R}^d$,

$$\mathbb{E}(\psi(G)) = \mathbb{E} \left(\phi(G + A\theta) e^{-A\theta \cdot G - \frac{|A\theta|^2}{2}} \right), \quad (17)$$

where A is $d \times Nd'$ matrix. The particular choice $d = Nd'$ and $A = I_d$ corresponds to Equation (5). When $d' = 1$, the choice $d = 1$ and $A = (\sqrt{t_1}, \sqrt{t_2 - t_1}, \dots, \sqrt{t_N - t_{N-1}})^*$ corresponds to adding a linear drift to the one dimensional standard Brownian motion W and we recover the Cameron-Martin formula. The interest of adding the matrix A is that it enables to reduce the dimension of the parameter θ which speeds up the optimisation algorithm. In particular, it is quite common to have a basket of 40 stocks and around 100 time steps which leads to $Nd' = 4000$. This is far to large for an optimisation problem and the matrix A enables to keep the value of d reasonable. From our numerical experiments, we advise to choose $d \geq d'$.

Transformation (17) actually relies on an importance sampling change of measure. Other strategies may be applicable such as stratification for instance as it is explained by Glasserman et al. in [6]. It ensues from Proposition 2.1, that the second moment

$$v(\theta) + \mathbb{E}(\psi(G))^2 = \mathbb{E} \left(\psi(G + A\theta)^2 e^{-2A\theta \cdot G - |A\theta|^2} \right) = \mathbb{E} \left(\phi(G)^2 e^{-A\theta \cdot G + \frac{|A\theta|^2}{2}} \right)$$

is strongly convex, infinitely differentiable and

$$\nabla v(\theta) = \mathbb{E} \left(A^*(A\theta - G) \phi(G)^2 e^{-A\theta \cdot G + \frac{|A\theta|^2}{2}} \right).$$

Letting $U(\theta, X) = A^*(A\theta - X) \phi(X)^2 e^{-A\theta \cdot X + \frac{|A\theta|^2}{2}}$ with X a normal random vector in $\mathbb{R}^{Nd'}$ fits in the framework of Section 4 and Equation (15) provides a pragmatic way of approximating the optimal parameter θ^* .

The next section gives convergence results for Algorithm 5.2 for the particular case of adaptive importance sampling for normal random vectors.

6.2 Convergence results

The following result defines the framework of applicability of the adaptive importance sampling methodology for normal random vectors. From Proposition 2.1 and Theorems 4.3, 3.1 and 3.2, we can deduce the following result.

Theorem 6.1. *If there exists $\varepsilon > 0$ such that $\mathbb{E}(\phi(G)^{4+\varepsilon}) < \infty$ then, the sequence $(\theta_n)_n$ converges a.s. to θ^* for any increasing sequence of compact sets $(K_j)_j$ satisfying (14) and the adaptive estimator S_n is asymptotically normal with optimal limiting variance $v(\theta^*)$.*

Remark 6.2. *Proposition 6.1 extends the result of [1, Theorem 4]. Our result is valid for any increasing sequences of compact sets $(K_j)_j$ satisfying (14) whereas Arouna needed a condition on the compact sets to ensure the convergence of $(\theta_n)_n$. The only condition required is some integrability on the payoff function and nothing has to be checked along the algorithm paths, which is a great improvement from a practical point of view.*

For the vast majority of payoff functions commonly used, the assumptions of Theorem 6.1 are always satisfied.

6.3 Numerical results

In this section, the quantity ‘‘Var MC’’ denotes the variance of the crude Monte Carlo estimator computed on-line on a single run of the algorithm. The variance denoted ‘‘Var ADIS’’ is the variance of the ADIS algorithm computed using the on-line estimator given by Theorem 3.3.

Basket options We consider options with payoffs of the form $(\sum_{i=1}^d \omega^i S_T^i - K)_+$ where $(\omega^1, \dots, \omega^d)$ is a vector of algebraic weights. The strike value K can be taken negative to deal with Put like options. The results of Table 1 must be considered together with the extra

ρ	K	Price	Var MC	Var ADIS
0.1	45	7.21	12.24	2.24
	55	0.57	1.83	0.79
0.2	50	3.29	13.53	2.26
0.5	45	7.65	43.25	6.00
	55	1.90	14.10	1.76
0.9	45	8.26	69.47	9.29
	55	2.82	30.87	3.90

Table 1: Basket option in dimension $d = 40$ with $r = 0.05$, $T = 1$, $S_0^i = 50$, $\sigma^i = 0.2$, $\omega^i = \frac{1}{d}$ for all $i = 1, \dots, d$ and $n = 10000$.

computation cost of the ADIS procedure. When a crude Monte Carlo procedure takes 0.27 CPU seconds, the ADIS algorithm require 0.65 CPU seconds, which means that it takes 2.5 times more time for the same sample size. The extra computation cost induced by the use of the ADIS algorithm is a factor of 2.5, whereas it reduced the variance at least by 6. The computation time required to achieve a given precision is then reduced by approximately $6/2.5$. Figure 1 has been obtained by running the ADIS algorithm 100 000 independently and it does show that the ADIS estimator is convergent and asymptotically normal as proved in Theorem 3.2. We have also computed the variance of the ADIS estimator using independent runs and found 2.21 compared to 2.26 for the on-line computed variance. This illustrates the conclusion of Theorem 3.3.

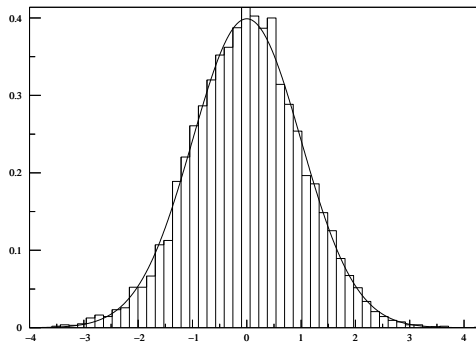


Figure 1: Limiting distribution of the ADIS algorithm for the option of Table 1 with $\rho = 0.2$ and $K = 50$.

Barrier Basket Options We consider basket options in dimension I with a discrete barrier on each asset. For instance, if we consider a Down and Out Call option, the payoff writes down $(\sum_{i=1}^I \omega^i S_T^i - K)_+ \mathbf{1}_{\{\forall i \leq I, \forall j \leq N, S_{t_j}^i \geq L^i\}}$ where $\omega = (\omega^1, \dots, \omega^I)$ is a vector of positive weights, $L = (L^1, \dots, L^I)$ is the vector of barriers, $K > 0$ the strike value and $t_N = T$. We consider

one time step per month, which means that for an option with maturity time $T = 2$, the number of time steps is $N = 24$. From now on, we fix $I = 5$. By choosing the matrix A given by $A_{(j-1)I+i,i} = \sqrt{t_j - t_{j-1}}$ (with the convention $t_0 = 0$) for $j = 1, \dots, N$ and $i = 1, \dots, I$, all the other coefficients of A being zero, the effective dimension of the importance sampling parameter is $I = 5$ rather than $IN = 120$.

K	Price	Var MC	Var ADIS	Var ADIS reduced
45	2.37	22.46	2.94	2.90
50	1.18	10.97	1.25	0.86
55	0.52	4.76	0.82	0.53

Table 2: Down and Out Call option in dimension $I = 5$ with $\sigma = 0.2$, $S_0 = (50, 40, 60, 30, 20)$, $L = (40, 30, 45, 20, 10)$, $\rho = 0.3$, $r = 0.05$, $T = 2$, $\omega = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $n = 100\,000$.

First, we note from Table 2 that the reduced and non-reduced ADIS algorithm achieve almost the same variance reduction. Actually, it is even advisable to reduce the size of the importance sampling parameter to reduce the noise in the stochastic approximation and therefore in the adaptive Monte Carlo estimator. If we measure the CPU times of the different estimator, we respectively find 4.2, 7.5, 14.5 seconds for the crude Monte Carlo estimator, the reduced ADIS estimator and the non reduced ADIS estimator. The non reduced algorithm takes twice the CPU time of the reduced one which in turn takes twice the CPU time of a crude Monte Carlo computation. The reduced algorithm reduces the variance by a factor little less than 10. When put together with the extra computational cost, it brings out an overall gain of 5. We have run the ADIS algorithm 100 000 times independently to obtain Figure 2 which rather well illustrates Theorem 6.1. As in the previous example, the variance of the ADIS estimator computed on independent runs is 0.87 compared to 0.6 for the on-line estimation of the variance given by Theorem 3.3.

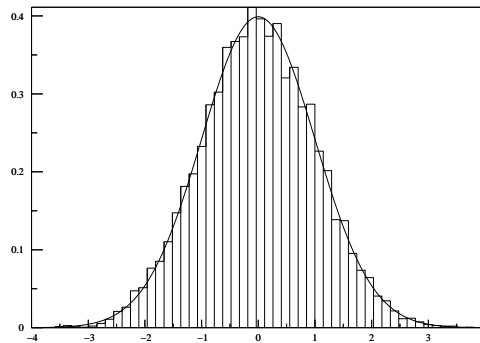


Figure 2: Limiting distribution of the reduced ADIS algorithm for the option of Table 2 with $K = 50$

7 Conclusion

The randomly truncated stochastic algorithm described by Equation (15) provides a convenient and easily implementable way of approximating the optimal drift parameter θ^* . The theoretical results obtained in this work apply for all the payoff functions commonly used in finance. The only real drawback of the adaptive methods relying on stochastic approximation comes from the difficulty to tune the gain sequence governing the stochastic algorithm, even though averaging stochastic algorithm are considerably less sensitive to the proper choice of the gain sequence. To encounter the fine tuning of the algorithm, Jourdain and Lelong [9] have recently suggested to use deterministic optimisation techniques coupled with sample approximation, but their technique can not be implemented in an adaptive manner as we did in Algorithm 5.2.

References

- [1] B. Arouna. Robbins-Monro algorithms and variance reduction in finance. *The Journal of Computational Finance*, 7(2), Winter 2003/2004.
- [2] Bouhari Arouna. Adaptive Monte Carlo method, a variance reduction technique. *Monte Carlo Methods Appl.*, 10(1):1–24, 2004.
- [3] H.F. Chen and Y.M. Zhu. *Stochastic Approximation Procedure with randomly varying truncations*. Scientia Sinica Series, 1986.
- [4] M. Duflo. *Random Iterative Models*. Springer-Verlag Berlin and New York, 1997.
- [5] Paul Glasserman. *Monte Carlo methods in financial engineering*, volume 53 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2004. , Stochastic Modelling and Applied Probability.
- [6] Paul Glasserman, Philip Heidelberger, and Perwez Shahabuddin. Asymptotically optimal importance sampling and stratification for pricing path-dependent options. *Math. Finance*, 9(2):117–152, 1999.
- [7] P. Hall and C. C. Heyde. *Martingale limit theory and its application*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. Probability and Mathematical Statistics.
- [8] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer-Verlag Berlin, 1987.
- [9] B. Jourdain and J. Lelong. Robust adaptive importance sampling for normal random vectors. *Annals of Applied Probability*, 19(5):1687–1718, 2009.
- [10] Reiichiro Kawai. An importance sampling method based on the density transformation of Lévy processes. *Monte Carlo Methods Appl.*, 12(2):171–186, 2006.
- [11] Reiichiro Kawai. Adaptive monte carlo variance reduction for Lévy processes with two-time-scale stochastic approximation. *Methodology and Computing in Applied Probability*, 10(2):199–223, 2008.

- [12] Reiichiro Kawai. Optimal importance sampling parameter search for Lévy processes via stochastic approximation. *SIAM Journal on Numerical Analysis (to appear)*, 2009.
- [13] Sujin Kim and Shane G. Henderson. Adaptive control variates for finite-horizon simulation. *Math. Oper. Res.*, 32(3):508–527, 2007.
- [14] Harold J. Kushner and Jichuan Yang. Stochastic approximation with averaging of the iterates: Optimal asymptotic rate of convergence for general processes. *SIAM Journal on Control and Optimization*, 31(4):1045–1062, 1993.
- [15] H.J. Kushner and G.G. Yin. *Stochastic Approximation and Recursive Algorithms and Applications*. Springer-Verlag New York, second edition, 2003.
- [16] J. Lelong. *Etude asymptotique des algorithmes stochastiques et calcul du prix des options Parisiennes*. PhD thesis, Ecole Nationale des Ponts et Chausées, <http://tel.archives-ouvertes.fr/tel-00201373/fr/>, 2007.
- [17] J. Lelong. Almost sure convergence of randomly truncated stochastic algorithms under verifiable conditions. *Statistics & Probability Letters*, 78(16), 2008.
- [18] V. Lemaire and G. Pagès. *Unconstrained Recursive Importance Sampling*. <http://hal.archives-ouvertes.fr/hal-00293466/fr/>, July 2008.
- [19] D. Lépingle. Sur le comportement asymptotique des martingales locales. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*, volume 649 of *Lecture Notes in Math.*, pages 148–161. Springer, Berlin, 1978.
- [20] Mariane Pelletier. Asymptotic almost sure efficiency of averaged stochastic algorithms. *SIAM J. Control Optim.*, 39(1):49–72 (electronic), 2000.
- [21] B. T. Polyak and A. B. Juditsky. Acceleration of stochastic approximation by averaging. *SIAM J. Control Optim.*, 30(4):838–855, 1992.
- [22] Rolando Rebolledo. Central limit theorems for local martingales. *Z. Wahrsch. Verw. Gebiete*, 51(3):269–286, 1980.
- [23] Herbert Robbins and Sutton Monro. A stochastic approximation method. *Ann. Math. Statistics*, 22:400–407, 1951.
- [24] Ward Whitt. Proofs of the martingale FCLT. *Probab. Surv.*, 4:268–302, 2007.